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Left Large Subsets of Free Semigroups and Groups that are not Right Large

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Abstract

There are several notions of size for subsets of a semigroup S that originated in topological dynamics and are of interest because of their combinatorial applications as well as their relationship to the algebraic structure of the Stone-Čech compactification βS of S. Among these notions are *thick* sets, *central* sets, *piecewise syndetic* sets, *IP* sets, and Δ sets. Two related notions, namely C sets and J sets, arose in the study of combinatorial applications of the algebra of βS.

If the semigroup is noncommutative, then all of these notions have both left and right versions. In any semigroup, a left thick set must be a right J set (and of course a right thick set must be a left J set). We show here that for free semigroups and groups, this is the only relationship that must hold. Specifically, we show that for any free semigroup or free group on more than one generator, there is a set which satisfies all of the left versions of these notions and none of the right versions except J. We also show that for the free semigroup on countably many generators, there is a left J set which is not a right J set.

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1 Introduction

In [4] Furstenberg introduced the notion of *central* subsets of the set $\mathbb{N}$ of positive integers and proved the original version of the *Central Sets Theorem*. He proved there [4, Theorem 8.22], among other things, that any central subset of $\mathbb{N}$ contains solutions to any system of homogeneous linear equations with rational coefficients that satisfy Rado's *columns condition*. Since he also showed [4, Theorem 8.8] that in any finite partition of $\mathbb{N}$, one cell must contain a central set, he obtained Rado's Theorem [9] as a corollary. Many other strong combinatorial properties are satisfied by any central set. See [6, Chapter 15].

Furstenberg's definition of central can be generalized naturally to an arbitrary semigroup. It was subsequently discovered that there is a very simple characterization of central sets in terms of the algebraic structure of the Stone-Čech compactification of the discrete semigroup S. (See the notes to Chapters 14 and 19 of [6] for a detailed history of this discovery.)

We pause now to briefly describe this structure. (For the results presented in this paper, one does not need to know anything about the structure of βS. However, it is essential for much of the motivation. The reader who wants to see the details can find more information than she could possibly want in [6].)

Let $(S, \cdot)$ be a discrete semigroup. The operation can be extended to the Stone-Čech compactification βS of S so that $(\beta S, \cdot)$ becomes a right topological semigroup (meaning that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$) with S contained in its topological center (meaning that for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\lambda_x(q) = x \cdot q$).

As does any compact Hausdorff right topological semigroup, βS has a smallest two sided ideal $K(\beta S)$ which is the union of all of the minimal right ideals of βS and is also the union of all of the minimal left ideals of βS. The intersection of any minimal right ideal with any minimal left ideal is a group. In particular, there are idempotents in $K(\beta S)$. We take the algebraic characterization of central sets as the definition.

Definition 1.1. Let S be a semigroup. A subset A of S is *central* if and only if there is an idempotent in $K(\beta S) \cap c\ell_{\beta S} A$.

We now present the other notions with which we are concerned that originated in relation to dynamical systems. Given any set X, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X. Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$, $FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$, where the products $\prod_{t \in F} x_t$ are computed in increasing order of indices. Also, given a subset A of a subsemigroup S and $t \in S$, $t^{-1}A = \{s \in S : ts \in A\}$ and $At^{-1} = \{s \in S : st \in A\}$. (Note that if S is a group, these agree with the usual meanings.)

Definition 1.2. Let S be a semigroup and let $A \subseteq S$.

(a) The set A is *thick* if and only if for each $F \in \mathcal{P}_f(S)$ there is some $x \in S$ such that $Fx \subseteq A$.

(b) The set A is an IP set if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

(c) The set A is a Δ set if and only if there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that whenever $n<m$ in $\mathbb{N}$, one has $x_m \in x_n \cdot A$.

(d) The set A is *piecewise syndetic* if and only $\big(\exists H \in \mathcal{P}_f(S)\big)\big(\forall F \in \mathcal{P}_f(S)\big)(\exists x \in S)(Fx \subseteq \bigcup_{t\in H} t^{-1}A)$.

Notice that in $(\mathbb{N},+)$, a set A is a Δ set if and only if there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $\{x_m - x_n : n < m\} \subseteq A$. (Thus "$\Delta$" comes from "difference".)

We have mentioned (but not stated) the original Central Sets Theorem. What is currently the most general version follows. It is due to Johnson in [7]. In this we write for $m \in \mathbb{N}$,

$$\mathcal{J}_m = \big\{\big(t(1),t(2),\ldots,t(m)\big) \in \mathbb{N}^m : t(1) < t(2) < \ldots < t(m)\big\}\,.$$

We also write ${}^{\mathbb{N}}S$ for the set of sequences in S.

Theorem 1.3 (Central Sets Theorem)**.** *Let S be a semigroup and let A be a central subset of S. Then there exist*

$$m : \mathcal{P}_f({}^{\mathbb{N}}S) \to \mathbb{N},\ \alpha \in \times_{F\in\mathcal{P}_f({}^{\mathbb{N}}S)} S^{m(F)+1},\ \textit{and}\ \tau \in \times_{F\in\mathcal{P}_f({}^{\mathbb{N}}S)} \mathcal{J}_{m(F)}$$

such that

(1) if $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and $F \subsetneq G$, then $\tau(F)\big(m(F)\big) < \tau(G)(1)$ and

(2) whenever $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$, and for each $i \in \{1,2,\ldots,n\}$, $f_i \in G_i$, one has

$$\begin{aligned}\textstyle\prod_{i=1}^n \alpha(G_i)(1) f_i\big(\tau(G_i)(1)\big)\alpha(G_i)(2) f_i\big(\tau(G_i)(2)\big)\cdots \\ \alpha(G_i)\big(m(G_i)\big) f_i\big(\tau(G_i)\big(m(G_i)\big)\big)\alpha(G_i)(m(G_i)+1) \in A\,.\end{aligned}$$

Many of the strong combinatorial properties that are satisfied by any central set are a consequence of the Central Sets Theorem. This motivates the following definition.

Definition 1.4. Let S be a semigroup and let $A \subseteq S$. Then A is a C set if and only if A satisfies the conclusion of the Central Sets Theorem.

Intimately related to C sets are the much more simply described J sets.

Definition 1.5. Let S be a semigroup and let $A \subseteq S$. Then A is a J set if and only if for each $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F$, $a(1)f\big(t(1)\big)a(2)f\big(t(2)\big)\cdots a(m)f\big(t(m)\big)a(m+1) \in A$.

Definition 1.6. Let S be a semigroup. Then $J(S) = \{p \in \beta S :$ for all $A \subseteq S$, if $p \in c\ell_{\beta S} A$, then A is a J set$\}$.

By [6, Theorem 14.15.1] A is a C set if and only if there is an idempotent $p \in J(S) \cap c\ell_{\beta S} A$ and by [6, Theorem 14.14.4] $J(S)$ is a two sided ideal of βS, which therefore contains $K(\beta S)$. And by [6, Corollary 4.41] $c\ell_{\beta S} K(\beta S) = \{p \in \beta S :$ for all $A \subseteq S$, if $p \in c\ell_{\beta S} A$, then A is piecewise syndetic$\}$. Thus C sets stand in a relationship to J sets which is very similar to the relationship between central sets and piecewise syndetic sets. And one immediately sees that all central sets are piecewise syndetic. By [6, Theorem 4.40], if A is piecewise syndetic, then $K(\beta S) \cap c\ell_{\beta S} A \neq \emptyset$ and therefore $J(S) \cap c\ell_{\beta S} A \neq \emptyset$ so A is piecewise syndetic.

Trivially all central sets are C sets and since A is an IP set if and only if there is an idempotent in $c\ell_{\beta S} A$ by [6, Theorem 5.12], one has that all C sets are IP sets. By [6, Theorem 4.48], A is thick if and only if there is a left ideal of βS contained $c\ell_{\beta S} A$. Since each left ideal in a compact Hausdorff right topological semigroup contains a minimal left ideal, one has that each thick set is central. Finally, if $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$ and for each $n \in \mathbb{N}$, $y_n = \prod_{t=1}^{n} x_t$, then whenever $n < m$ one has $y_m = y_n \cdot \prod_{t=n+1}^{m} x_t$ so each IP set is a Δ set. Thus we have established that all of the implications in Figure 1 are valid, where "PS" abbreviates "piecewise syndetic". See [5] for examples in $(\mathbb{N}, +)$ showing that none of the missing implications holds.

$$\begin{tikzcd}
 & \text{thick} \arrow[d] & \\
 & \text{central} \arrow[dl] \arrow[dr] & \\
PS \arrow[d] & & C \arrow[d] \\
J & & IP \arrow[d] \\
 & & \Delta
\end{tikzcd}$$

Figure 1: Implications among the notions

All of the properties listed are partition regular except "thick". That is if A has the specified property and it is divided into finitely many pieces, then one of them has that same property. This is immediate for all of them except Δ sets from the fact that the property is determined by points of βS which can be viewed as ultrafilters on S. For Δ sets, it is an immediate consequence of Ramsey's Theorem.

Even sets quite far down the heirarchy satisfy strong combinatorial properties. See [5, page 6] for the proof that any J set in $(\mathbb{N}, +)$ contains arbitrarily long arithmetic progressions with increment chosen from the finite sums of any prespecified sequence.

All of the discussion until now has assumed that we constructed the operation on βS so that $(\beta S, \cdot)$ is a right topological semigroup. One could equally well do the extension so that $(\beta S, \cdot)$ is left topological. (In fact, that used to be the customary choice of the first author of this paper.) Had we done so, the notions that we defined would have been different. For the rest of this paper we will refer to the notions that we have defined as "right thick", "right central", etc. The changes to the definitions necessitated are obvious. For example, A is left thick if and only if for each $F \in \mathcal{P}_f(S)$, there is some $x \in S$ such that $xF \subseteq A$.

The definition for left central reads the same as for right central, but the smallest ideals for the right and left operations can be quite different. In [3], El-Mabhouh, Pym, and Strauss showed that if S is the free semigroup on countably many generators, then there is a subsemigroup of βS with respect to the left operation none of whose members is a product of two elements with respect to the left operation. In [1], Anthony showed that if S is the free semigroup on two generators, there is a subset of S which is left piecewise syndetic but not right piecewise syndetic, and consequently that there are points in $K(\beta S)$ with respect to the left operation that are not in the closure of $K(\beta S)$ with respect to the right operation. She also showed there that in any semigroup, the closure of $K(\beta S)$ with respect to the left operation meets the smallest ideal with respect to the right operation. On the other hand, in [2], Burns proved that if S is either the free group or the free semigroup on two generators, then the smallest ideals with respect to the two operations are disjoint.

In [8] it was shown that if S is the free semigroup on countably many generators, then there is a left piecewise syndetic set which is not right piecewise syndetic (and as a consequence there are points in $J(S)$ with respect to the left operation which are not in $J(S)$ with respect to the right operation). We will present this result in Section 3.

We began the current investigation trying to construct in the free semigroup on two generators a left C set which is not a right C set. We also realized that because of the available algebraic tools, it was probably easier to construct a left central set which is not a right C set. (If one can construct a right or left ideal as an intersection of closures of subsets of S, any of these subsets will be central with respect to the appropriate operation.) We eventually succeeded with a rather complicated construction and showed that the set was in fact left thick. Since left thick sets are easy to construct (as will be seen in the next section) it turns out that our constructions were greatly simplified by asking the right question, namely whether there are left thick sets that are neither right piecewise syndetic nor right Δ sets.

We refer throughout to sets with left properties not having right properties. Of course the corresponding results with left and right interchanged are valid.

We conclude this introduction by showing that one cannot construct a left thick set which is not a right J set.

Lemma 1.7. *Let S be any semigroup and let A be a left thick subset of S. Then A is a right J set.*

Proof. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, let $H = \{f(1) : f \in F\}$, and pick $z \in S$. (For us, semigroups are necessarily nonempty.) Then $Hz \in \mathcal{P}_f(S)$ so pick $x \in S$ such that $xHz \subseteq A$. Let $m = 1$, let $a(1) = x$, let $a(2) = z$, and let $t(1) = 1$. Then for each $f \in F$, $a(1)f\big(t(1)\big)a(2) \in A$. $\square$

2 Left thick but neither right piecewise syndetic nor right Δ

We show in this section that for any cardinal $\kappa > 1$, if S is either the free semigroup or free group on κ generators, then there is a subset of S which is left thick but neither right piecewise syndetic nor a right Δ set. In fact, we will show that the set does not satisfy a property even weaker than being a right Δ set.

Lemma 2.1. *Let S be a left cancellative semigroup and let $A \subseteq S$. Then A is a right Δ set if and only if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that whenever $n \leq m$ in $\mathbb{N}$, $\prod_{t=n}^{m} x_t \in A$.*

Proof. Sufficiency. Pick such a sequence. For each $n \in \mathbb{N}$, let $y_n = \prod_{t=1}^{n} x_t$. If $n < m$, then $y_m = y_n \cdot \prod_{t=n+1}^{m} x_t \in y_n \cdot A$.

Necessity. Pick a sequence $\langle y_n \rangle_{n=1}^{\infty}$ such that whenever $n < m$ in $\mathbb{N}$, $y_m \in y_n \cdot A$. For each $n \in \mathbb{N}$, pick $x_n \in A$ such that $y_{n+1} = y_n \cdot x_n$. By induction on k, one sees that if $n, k \in \mathbb{N}$, then $y_{n+k} = y_n \cdot \prod_{t=n}^{n+k} x_t$. Now given $n < m$ in $\mathbb{N}$ pick $a \in A$ such that $y_m = y_n \cdot a$. Since also $y_m = y_n \cdot \prod_{t=n}^{m} x_t$, one has by left cancellation that $\prod_{t=n}^{m} x_t = a \in A$. $\square$

Definition 2.2. Let S be a left cancellative semigroup and let $A \subseteq S$. Then A is a *right weak* Δ set if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in A such that for each $n \in \mathbb{N}$, $x_n \cdot x_{n+1} \in A$.

We will consider separately the free groups and semigroups on infinitely many generators (which are the easier cases). The reason these cases are easier is that the number of elements is the same as the number of finite subsets.

If G is a free group and x, y are elements of G we write $x^\frown y$ for the concatenation of x and y. The assertion that $xy = x^\frown y$ is the assertion that the rightmost letter of x and the leftmost letter of y are not inverses of each other.

Theorem 2.3. *Let κ be an infinite cardinal and let G be the free group on the generators $\langle a_\sigma \rangle_{\sigma<\kappa}$. Enumerate $\mathcal{P}_f(G)$ as $\langle F_\sigma \rangle_{\sigma<\kappa}$. For $\sigma < \kappa$ let*

$$\tau(\sigma) = \max\{\delta < \kappa : (\exists w \in F_\sigma)(a_\delta \textit{ or } a_\delta^{-1} \textit{ occurs in } w)\} + 1\,.$$

Let $A = \bigcup_{\sigma<\kappa} a_{\tau(\sigma)} \cdot F_\sigma$. Then

(a) A is left thick;

(b) A is not right piecewise syndetic; and

(c) A is not a right weak Δ set.

Proof. (a) This is trivial.

(b) Suppose we have $H \in \mathcal{P}_f(G)$ such that for every $F \in \mathcal{P}_f(G)$, there is some $x \in G$ with $Fx \subseteq \bigcup_{t \in H} t^{-1}A$. Let

$$\mu = \max\{\delta < \kappa : (\exists w \in H)(a_\delta \text{ or } a_\delta^{-1} \text{ occurs in } w)\} + 1\,.$$

Let $F = \{a_\mu, a_\mu^{-1}\}$ and pick $x \in G$ such that $Fx \subseteq \bigcup_{t \in H} t^{-1}A$. Since x does not begin with both a_μ and a_μ^{-1}, pick $j \in \{1, -1\}$ such that $a_\mu^j x = a_\mu^j {}^\frown x$. Pick $t \in H$ such that $ta_\mu^j x \in A$. By the choice of μ, $ta_\mu^j x = t^\frown a_\mu^j {}^\frown x$. But the first letter of any member of A has index strictly greater than the index of any other letter, a contradiction.

(c) Suppose that A is a right weak Δ set and pick a sequence $\langle x_n \rangle_{n=1}^\infty$ in A such that for each $n \in \mathbb{N}$, $x_n \cdot x_{n+1} \in A$. For each $n \in \mathbb{N}$, pick $\mu(n) < \kappa$ such that $x_n \in a_{\tau(\mu(n))} \cdot F_{\mu(n)}$. There are no infinite decreasing sequences in κ so pick n such that $\tau\big(\mu(n)\big) \leq \tau\big(\mu(n+1)\big)$. Pick $y \in F_{\mu(n)}$ and $z \in F_{\mu(n+1)}$ such that $x_n = a_{\tau(\mu(n))} \cdot y$ and $x_{n+1} = a_{\tau(\mu(n+1))} \cdot z$. Now the largest δ such that a_δ or a_δ^{-1} occurs in y is less than $\tau\big(\mu(n)\big) \leq \tau\big(\mu(n+1)\big)$ so $x_n \cdot x_{n+1} = a_{\tau(\mu(n))} {}^\frown y^\frown a_{\tau(\mu(n+1))} {}^\frown z$. Since $\tau\big(\mu(n)\big) \leq \tau\big(\mu(n+1)\big)$, $x_n \cdot x_{n+1} \notin A$, a contradiction. □

The proof of the following theorem is nearly identical to the proof of Theorem 2.3, so we omit it.

Theorem 2.4. *Let κ be an infinite cardinal and let S be the free semigroup on the generators $\langle a_\sigma \rangle_{\sigma<\kappa}$. Enumerate $\mathcal{P}_f(S)$ as $\langle F_\sigma \rangle_{\sigma<\kappa}$. For $\sigma < \kappa$ let*

$$\tau(\sigma) = \max\{\delta < \kappa : (\exists w \in F_\sigma)(a_\delta \textit{ occurs in } w)\} + 1\,.$$

Let $A = \bigcup_{\sigma<\kappa} a_{\tau(\sigma)} \cdot F_\sigma$. Then

(a) A is left thick;

(b) A is not right piecewise syndetic; and

(c) A is not a right weak Δ set.

In the following theorem we let $l(w)$ be the length of the word w.

Theorem 2.5. *Let $k \in \mathbb{N} \setminus \{1\}$ and let S be the free semigroup on the k generators $\langle a_n \rangle_{n=0}^{k-1}$. Let $A = \{a_0^v a_1 s : v \in \mathbb{N}\,, s \in S, \text{ and } l(s) < v\}$. Then*

(a) A is left thick;

(b) A is not right piecewise syndetic; and

(c) A is not a right weak Δ set.

Proof. (a) Let $H \in \mathcal{P}_f(S)$ and let $v = \max\{l(s) : s \in H\}+1$. Then $a_0^y a_1 H \subseteq A$.

(b) Suppose we have $H \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$, there is some $x \in S$ with $Fx \subseteq \bigcup_{t\in H} t^{-1}A$. Let $m = \max\{l(t) : t \in H\}$ and let $F = \{a_1^{m+1}\}$. Pick $x \in S$ such that $Fx \subseteq \bigcup_{t\in H} t^{-1}A$. Pick $x \in S$ and $t \in H$ such that $ta_1^{m+1}x \in A$. Pick $v \in \mathbb{N}$ and $s \in S$ with $l(s) < v$ such that $ta_1^{m+1}x = a_0^v a_1 s$. Then t begins with a_0^v so $v \le l(t)$. Thus $2l(t) + 2 \le l(t) + m + 1 + l(x) = v + 1 + l(s) < 2v + 1 \le 2l(t) + 1$, a contradiction.

(c) Suppose that A is a right weak Δ set and pick a sequence $\langle x_n\rangle_{n=1}^\infty$ in A such that for each $n \in \mathbb{N}$, $x_n \cdot x_{n+1} \in A$. For each $n \in \mathbb{N}$ pick $v(n) \in \mathbb{N}$ and $s(n) \in S$ such that $l\big(s(n)\big) < v(n)$ and $x_n = a_0^{v(n)} a_1 s(n)$. Since there are no infinite decreasing sequences in $\mathbb{N}$, pick n such that $v(n) \le v(n+1)$. Then $x_n \cdot x_{n+1} = a_0^{v(n)} a_1 s(n) a_0^{v(n+1)} a_1 s(n+1)$ and $l\big(s(n)a_0^{v(n+1)}a_1 s(n+1)\big) > v(n+1) \ge v(n)$ so $x_n \cdot x_{n+1} \notin A$, a contradiction. $\square$

In the following, $\omega = \mathbb{N} \cup \{0\}$ is the first infinite cardinal.

Theorem 2.6. *Let $k \in \mathbb{N} \setminus \{1\}$ and let G be the free group on the k generators $\langle a_n\rangle_{n=0}^{k-1}$. Enumerate $\mathcal{P}_f(G)$ as $\langle F_n\rangle_{n=0}^\infty$. For each $n < \omega$, let*

$$\tau(n) = \max\{l(s) : s \in F_n\} + 1$$

and let $A = \bigcup_{n=0}^\infty a_0^{\tau(n)} a_1^{\tau(n)} a_0^{\tau(n)} F_n$. Then

(a) A is left thick;

(b) A is not right piecewise syndetic; and

(c) A is not a right weak Δ set.

Proof. (a) This is trivial.

(b) Suppose we have $H \in \mathcal{P}_f(G)$ such that for every $F \in \mathcal{P}_f(G)$, there is some $x \in G$ with $Fx \subseteq \bigcup_{t\in H} t^{-1}A$. Let $m = \max\{l(t) : t \in H\}$ and let $F = \{a_0^{-2m}, a_1^{-2m}\}$. Pick $x \in S$ such that $Fx \subseteq \bigcup_{t\in H} t^{-1}A$. Pick $j \in \{0,1\}$ such that $a_j^{-2m}x = a_j^{-2m} {}^\frown x$. Pick $t \in H$ such that $ta_j^{-2m}x \in A$ and pick $n < \omega$ and $y \in F_n$ such that $ta_j^{-2m}x = a_0^{\tau(n)} a_1^{\tau(n)} a_0^{\tau(n)} y$. Now $l(t) \le m$ so $ta_j^{-2m} = s^\frown a_j^{-r}$ for some $s \in G$ with $l(s) \le l(t)$ and some $r \ge m$. Also $l(y) < \tau(n)$ so $a_0^{\tau(n)} y = a_0^v {}^\frown w$ for some $w \in G$ with $l(w) \le l(y)$ and some $v > 0$. Thus $s^\frown a_j^{-r} {}^\frown x = a_0^{\tau(n)} {}^\frown a_1^{\tau(n)} {}^\frown a_0^v {}^\frown w$.

From the right hand side, the first occurrence of a_j^{-1} is in w so s begins as $a_0^{\tau(n)} {}^\frown a_1^{\tau(n)} {}^\frown a_0^v$ and w ends as $a_j^{-r} {}^\frown x$. Therefore $2\tau(n) + v \le l(s) \le l(t) \le m$ and $m + 1 \le r + 1 \le l(w) \le l(y) < \tau(n) < m$, a contradiction.

(c) Suppose that A is a right weak Δ set and pick a sequence $\langle x_n\rangle_{n=1}^\infty$ in A such that for each $n \in \mathbb{N}$, $x_n \cdot x_{n+1} \in A$. For each $n \in \mathbb{N}$ pick $\mu(n) \in \omega$ such that $x_n \in a_0^{\tau(\mu(n))} a_1^{\tau(\mu(n))} a_0^{\tau(\mu(n))} F_{\mu(n)}$ and pick n such that $\tau\big(\mu(n)\big) \le \tau\big(\mu(n+1)\big)$. Pick $y \in F_{\mu(n)}$ and $z \in F_{\mu(n+1)}$ such that $x_n = a_0^{\tau(\mu(n))} a_1^{\tau(\mu(n))} a_0^{\tau(\mu(n))} y$ and $x_{n+1} = a_0^{\tau(\mu(n+1))} a_1^{\tau(\mu(n+1))} a_0^{\tau(\mu(n+1))} z$.

Now $l(y) < \tau\big(\mu(n)\big) \leq \tau\big(\mu(n+1)\big)$ and $l(z) < \tau\big(\mu(n+1)\big)$ so there exist d and h in G with $l(d) \leq l(y)$ and $l(h) \leq l(z)$ and there exist c, f, and g in $\mathbb{N}$ such that

$$x_n \cdot x_{n+1} = a_0^{\tau(\mu(n))} \frown a_1^{\tau(\mu(n))} a_0^c \frown d \frown a_0^f \frown a_1^{\tau(\mu(n+1))} \frown a_0^g \frown h\,.$$

Also for some $m < \omega$ and some $p \in G$ with $l(p) < \tau(m)$, $x_n \cdot x_{n+1} = a_0^{\tau(m)} a_1^{\tau(m)} a_0^{\tau(m)} p$. Pick $u > 0$ and $v \in G$ such that $l(v) \leq l(p)$ and $a_0^{\tau(m)} p = a_0^u \frown v$. Then $x_n \cdot x_{n+1} = a_0^{\tau(m)} \frown a_1^{\tau(m)} \frown a_0^u \frown v$. Then $x_n x_{n+1}$ begins with $a_0^{\tau(m)} \frown a_1$ and $x_n x_{n+1}$ begins with $a_0^{\tau(\mu(n))} \frown a_1$ so $\tau(m) = \tau\big(\mu(n)\big)$ and thus $a_0^c \frown d \frown a_0^f \frown a_1^{\tau(\mu(n+1))} \frown a_0^g \frown h = a_0^u \frown v$. Thus the first occurrence of a_1 is in v so $\tau\big(\mu(n+1)\big) + g + l(h) \leq l(v) \leq l(p) < \tau(m) = \tau\big(\mu(n)\big)$, a contradiction. $\square$

3 Left J but not right J

By virtue of Lemma 1.7, the sets produced in Section 2 are necessarily both left J sets and right J sets. In this section we present the (considerably more complicated) construction of left J sets in the free semigroup on countably many generators that are not right J sets.

Throughout this section we will let S be the free semigroup on the generators $\langle a_n \rangle_{n=1}^{\infty}$. (It is convenient for us to index the letters by $\mathbb{N}$ rather than ω.)

In attempting to build a left J set which is not a right J set, the basic idea is the same as in the previous section with left thick sets. That is, one wants to put only enough things into the set to just make it be a left J set. However, the issue is complicated by the fact that ${}^{\mathbb{N}}S$, and thus $\mathcal{P}_f({}^{\mathbb{N}}S)$, has uncountably many members, so one cannot handle the elements of $\mathcal{P}_f({}^{\mathbb{N}}S)$ one at a time. The following simple lemma allows us to only worry about countably many things.

Lemma 3.1. *Let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$. There is an infinite subset C of $\mathbb{N}$ such that for all $f, g \in F$, either $(\forall n \in C)\big(f(n) = g(n)\big)$ or $(\forall n \in C)\big(f(n) \neq g(n)\big)$.*

Proof. Let $k = |F|$. We may assume $k > 1$. Let $l = \begin{pmatrix} k \\ 2 \end{pmatrix}$, and enumerate the pairs in F as $\langle P_i \rangle_{i=1}^{l}$ and let each $P_i = \{f_i, g_i\}$. Pick infinite $C_1 \subseteq \mathbb{N}$ such that either $(\forall n \in C_1)\big(f_1(n) = g_1(n)\big)$ or $(\forall n \in C_1)\big(f_1(n) \neq g_1(n)\big)$. Inductively, given $i < l$ and C_i choose infinite $C_{i+1} \subseteq C_i$ such that either $(\forall n \in C_{i+1})\big(f_{i+1}(n) = g_{i+1}(n)\big)$ or $(\forall n \in C_{i+1})\big(f_{i+1}(n) \neq g_{i+1}(n)\big)$. $\square$

Definition 3.2. (1) $\mathcal{M} = \{f : (\exists r \in \mathbb{N})(f : \{1, 2, \ldots, r\} \to S)\}$.

(2) Define $\psi : \mathcal{P}_f(\mathcal{M}) \to \mathbb{N}$ by, for $H \in \mathcal{P}_f(\mathcal{M})$,
$\psi(H) = \max\{n \in \mathbb{N} : (\exists f \in H)\big(a_n \text{ occurs in } f(1)\big)\}$.

(3) $\mathcal{F} = \big\{H \in \mathcal{P}_f(\mathcal{M}) : (\forall f \in H)(\mathrm{dom}(f) = \{1, 2, \ldots, \psi(H)\})$ and
$(\forall f, g \in H)\big(f \neq g \Rightarrow (\forall t \in \{1, 2, \ldots, \psi(H)\})\big(f(t) \neq g(t)\big)\big)\big\}$.

Lemma 3.3. *There is an injective function $\delta : \mathcal{F} \to 2\mathbb{N}$ such that if $H \in \mathcal{F}$, $f \in H$, $t \in \{1, 2, \ldots, \psi(H)\}$, $n \in \mathbb{N}$, and n occurs in $f(t)$, then $n < \delta(H)$.*

Proof. Since $\mathcal{F}$ is countable, one may enumerate $\mathcal{F}$ and construct the function δ inductively. $\square$

Definition 3.4. Let δ be as in Lemma 3.3.

$$A = \{a_{\delta(H)} h\big(\psi(H)\big) a_{\delta(H)} h(\psi(H) - 1) \cdots a_{\delta(H)} h(1) a_{\delta(H)} : H \in \mathcal{F} \text{ and } h \in H\}\,.$$

The set A will remain fixed for the remainder of this section.

Theorem 3.5. *A is a left J set.*

Proof. We need to show that for each $G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ there exist $m \in \mathbb{N}$, $\alpha \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for all $f \in G$,

$$\alpha(m+1) f\big(t(m)\big) \alpha(m) f\big(t(m-1)\big) \cdots \alpha(2) f\big(t(1)\big) \alpha(1) \in A\,.$$

So let $G \in \mathcal{P}_f({}^{\mathbb{N}}S)$. By Lemma 3.1, pick an infinite subset C of $\mathbb{N}$ such that for all $f, g \in G$, either $(\forall n \in C)\big(f(n) = g(n)\big)$ or $(\forall n \in C)\big(f(n) \neq g(n)\big)$.

Enumerate C in increasing order as $\langle c_i \rangle_{i=1}^{\infty}$. Let

$$m = \max\{n \in \mathbb{N} : (\exists f \in G)(a_n \text{ occurs in } f(c_1)\}\,.$$

For $f \in G$, define $h_f : \{1, 2, \ldots, m\} \to S$ by, for $i \in \{1, 2, \ldots, m\}$, $h_f(i) = f(c_i)$.

Let $H = \{h_f : f \in G\}$ and observe that $\psi(H) = m$ so that $H \in \mathcal{F}$. Define $\alpha \in S^{m+1}$ by $\alpha(1) = \alpha(2) = \ldots = \alpha(m+1) = a_{\delta(H)}$ and define $t \in \mathcal{J}_m$ by $t(i) = c_i$ for $i \in \{1, 2, \ldots, m\}$. Let $f \in G$. Then $h_f \in H$ and

$$\begin{aligned}&\alpha(m+1) f\big(t(m)\big) \alpha(m) f\big(t(m-1)\big) \cdots \alpha(2) f\big(t(1)\big) \alpha(1) = \\ &a_{\delta(H)} h_f(m) a_{\delta(H)} h_f(m-1) \cdots a_{\delta(H)} h_f(1) a_{\delta(H)} \in A\,.\end{aligned}$$

$\square$

Theorem 3.6. *A is not a right J set.*

Proof. Define f, g in ${}^{\mathbb{N}}S$ by $f(n) = a_{4n} a_{4n+1}$ and $g(n) = a_{4n} a_{4n+1} a_{4n+2} a_{4n+3}$ and let $F = \{f, g\}$. Suppose that A is a right J set and pick $m \in \mathbb{N}$, $\alpha \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that

$$\begin{aligned} x &= \alpha(1) f\big(t(1)\big) \alpha(2) \cdots \alpha(m) f\big(t(m)\big) \alpha(m+1) \in A \text{ and} \\ y &= \alpha(1) g\big(t(1)\big) \alpha(2) \cdots \alpha(m) g\big(t(m)\big) \alpha(m+1) \in A\,.\end{aligned}$$

Since x and y are in A, pick $H, K \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $h \in H$, and $k \in K$ such that

$$\begin{aligned} x &= a_{\delta(H)} h\big(\psi(H)\big) a_{\delta(H)} h(\psi(H) - 1) \cdots a_{\delta(H)} h(1) a_{\delta(H)} \text{ and} \\ y &= a_{\delta(K)} k\big(\psi(K)\big) a_{\delta(K)} k(\psi(K) - 1) \cdots a_{\delta(K)} k(1) a_{\delta(K)}\,.\end{aligned}$$

Since x and y both begin with the word $\alpha(1)$, we have that $a_{\delta(H)} = a_{\delta(K)}$. Since δ is injective, we then have $H = K$. Let $r = \psi(H)$. Then we have

$$\begin{array}{rcl} x & = & a_{\delta(H)}h(r)a_{\delta(H)}h(r-1)\cdots a_{\delta(H)}h(1)a_{\delta(H)} \text{ and} \\ y & = & a_{\delta(H)}k(r)a_{\delta(H)}k(r-1)\cdots a_{\delta(H)}k(1)a_{\delta(H)} \,. \end{array}$$

Note that $h \neq k$ since the lengths of x and y are different and therefore, for each $t \in \{1, 2, \ldots, r\}$, $h(t) \neq k(t)$.

Now $\delta(H)$ is greater than any n such that there is some $t \in \{1, 2, \ldots, r\}$ such that a_n occurs in $k(t)$. Therefore $\delta(H)$ is the largest n such that a_n occurs in y. Thus $\delta(H)$ is not any of $4n$, $4n+1$, or $4n+2$. Also, $\delta(H)$ is even so $\delta(H) \neq 4n+3$. Thus the only occurrences of $a_{\delta(H)}$ in y are in $\alpha(1), \alpha(2), \ldots, \alpha(m+1)$. Consequently it is also true that the only occurrences of $a_{\delta(H)}$ in x are in $\alpha(1), \alpha(2), \ldots, \alpha(m+1)$.

Now we claim that $r \leq m$. To see this, define

$$\gamma : \{1, 2, \ldots, r+1\} \to \{1, 2, \ldots, m+1\}$$

by $\gamma(i) = j$ if and only the i^{th} occurrence of $a_{\delta(H)}$ occurs in $\alpha(j)$. We claim that γ is injective so that $r \leq m$. Suppose that γ is not injective. Then there are some $i \in \{1, 2, \ldots, r+1\}$ and some $j \in \{1, 2, \ldots, m+1\}$ such that $\gamma(i) = \gamma(i+1) = j$. This means that the i^{th} and $(i+1)^{\text{th}}$ occurrences of $a_{\delta(H)}$ occur in $\alpha(j)$. Thus there exist some $u, v \in \{\emptyset\} \cup S$ such that $\alpha(j) = ua_{\delta(H)}h\big((r+1)-i\big)a_{\delta(H)}v = ua_{\delta(H)}k\big((r+1)-i\big)a_{\delta(H)}v$. We conclude that $h\big((r+1)-i\big) = k\big((r+1)-i\big)$. This is a contradiction.

Now $h \in H$ and $r = \psi(H) \geq \max\{n : a_n \text{ occurs in } h(1)\}$. In x, the last occurrence of $a_{\delta(H)}$ must occur at the end of $\alpha(m+1)$ and the second to last occurrence of $a_{\delta(H)}$ must occur in $\alpha(t)$ for some $t \in \{1, 2, \ldots, m\}$. Therefore there are some $u, v \in \{\emptyset\} \cup S$ such that $h(1) = uf\big(t(m)\big)v = ua_{4t(m)}a_{4t(m)+1}v$. Since $a_{4t(m)}$ occurs in $h(1)$, we then have $4t(m) \leq r$. Since t is increasing, $m \leq t(m)$ and thus $4m \leq r \leq m$, a contradiction. $\square$

We conclude our results by showing that A does not satisfy either the right or left versions of any of the properties in Figure 1 except being a left J set. Of course we know that A is not right piecewise syndetic since it is not a right J set.

Theorem 3.7. *(a) A is not a right weak δ set.*

(b) A is not a left weak δ set.

(c) A is not left piecewise syndetic.

Proof. We will do the proof for (a) and (b) at once by showing that there do not exist x and y in A such that $xy \in A$. Suppose we have such. Pick H, K, and R in $\mathcal{F}$, $h \in H$, $k \in K$, and $r \in R$ such that

$$\begin{array}{rcl} x & = & a_{\delta(H)}h\big(\psi(H)\big)a_{\delta(H)}h(\psi(H)-1)\cdots a_{\delta(H)}h(1)a_{\delta(H)} \,, \\ y & = & a_{\delta(K)}k\big(\psi(K)\big)a_{\delta(K)}k(\psi(K)-1)\cdots a_{\delta(K)}k(1)a_{\delta(K)}, \text{ and} \\ xy & = & a_{\delta(R)}r\big(\psi(R)\big)a_{\delta(R)}r(\psi(R)-1)\cdots a_{\delta(R)}r(1)a_{\delta(R)} \,. \end{array}$$

From the first letter of x and the first letter of xy we conclude that $\delta(H) = \delta(R)$ so $H = R$. From the last letter of y and the last letter of xy we conclude that $\delta(K) = \delta(R)$ so $K = R$. But then on the one hand there are $2\psi(H) + 2$ occurrences of $a_{\delta(H)}$ in xy while on the other hand there are only $\psi(H)+1$ such occurrences.

(c) Suppose we have $G \in \mathcal{P}_f(S)$ such that for all $F \in \mathcal{P}_f(S)$ there exists $x \in S$ with $xF \subseteq \bigcup_{t\in G} At^{-1}$. Let $\mathcal{R} = \{H \in \mathcal{F} : (\exists t \in G)(t \text{ ends in } a_{\delta(H)})\}$. Let $n = \max\{\delta(H) : H \in \mathcal{R}\} + 1$ and let $F = \{a_n\}$. Pick $x \in S$ and $t \in G$ such that $xa_nt \in A$. Pick $H \in \mathcal{F}$ and $h \in H$ such that

$$xa_nt = a_{\delta(H)}h\big(\psi(H)\big)a_{\delta(H)}h(\psi(H)-1)\cdots a_{\delta(H)}h(1)a_{\delta(H)}\,.$$

Then $H \in \mathcal{R}$ so $n > \delta(H)$ and thus a_n occurs in $h(t)$ for some $t \in \mathrm{dom}(h)$. Therefore $\delta(H) > n$, a contradiction. $\square$

Our construction in this section seems heavily dependent on having a countable infinity of generators for S.

Question 3.8. *Let S be a free semigroup on some number of generators other than 1 or ω. Is there a left J set in S which is not a right J set?*

Recall that we showed in Lemma 1.7 that in any semigroup, any left thick set must be a right J set.

Question 3.9. *Do there exist a semigroup S and a left central subset of S which is not a right J set?*

Question 3.10. *Do there exist a semigroup S and a left piecewise syndetic subset of S which is not a right J set?*

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