Short Note
Infinite compact sets of idempotents in βS

Neil Hindman [*][†] Dona Strauss [‡]

Abstract

We show that if S is a countably infinite right cancellative semigroup and T is an infinite compact set of idempotents in the Stone-Čech compactification βS of S, then T contains an infinite compact left zero semigroup.

Given a discrete semigroup $(S, \cdot)$ we let βS be the Stone-Čech compactification of S with the operation $\cdot$ on βS which makes $(\beta S, \cdot)$ a compact right topological semigroup with S contained in its topological center. If the operation on S is denoted by another symbol such as $+$, we use the same symbol to denote the operation on βS. Because this is a short note, we refer the reader to Chapters 3 and 4 of [2] for basic information about the semigroup βS.

Properties of idempotents in βS play an important role in combinatorics and topological dynamics. (See, for example, [1].) It seems obvious to us that the semigroup $(\beta\mathbb{N}, +)$ cannot contain an infinite compact set of idempotents. Unfortunately, we cannot prove this assertion. Further, we should warn the reader that it was also obvious to us that there could not be an idempotent $p \in \beta\mathbb{N}$ which is both minimal and maximal with respect to the ordering of idempotents wherein $p \leq q$ if and only if $p = p + q = q + p$. But Y. Zelenyuk in [3] gave a ZFC proof that such idempotents exist.

It is known [2, Lemma 9.3] that if S is a countably infinite right cancellative semigroup, then βS does not contain an infinite compact right zero semigroup. It is also known [2, Theorem 6.15.2 and Theorem 6.44] that while any minimal left ideal in $(\beta\mathbb{N}, +)$ has $2^{\mathfrak{c}}$ idempotents which form a left zero semigroup, that semigroup is not compact.

We show in this note that $\beta\mathbb{N}$ contains an infinite compact set of idempotents if and only if it contains an infinite compact left zero subsemigroup. In fact,

*Department of Mathematics, Howard University, Washington, DC 20059, USA.
nhindman@aol.com

†This author acknowledges support received from the National Science Foundation (USA) via Grant DMS-1460023.

‡Department of Pure Mathematics, University of Leeds, Leeds LS2 9J2, UK.
d.strauss@hull.ac.uk

more generally, we have the following theorem.

Theorem 1. *Let S be a countably infinite right cancellative semigroup and assume that T is an infinite compact subset of βS consisting of idempotents. Then T contains an infinite compact left zero semigroup.*

Proof. Let D be the set of right identities for S (which may, of course, be empty). If $D \neq \emptyset$, then D is a left zero semigroup so by [2, Exercise 4.2.1] $\overline{D}$ is a left zero semigroup. Therefore, if $T \cap \overline{D}$ is infinite, we are done. So we will assume that $T \cap \overline{D}$ is finite. Since $\overline{D}$ is open, we may presume that $T \cap \overline{D} = \emptyset$.

Pick a discrete sequence $\langle p_n \rangle_{n=1}^{\infty}$ in T and let $P = \{x_n : n \in \mathbb{N}\}$. (That is to say, the sequence $\langle p_n \rangle_{n=1}^{\infty}$ is injective and P is discrete.) We may assume that $T = c\ell_{\beta S} P$. Note that, since P is discrete and $T = c\ell_{\beta S} P$, we have that $T \setminus P$ is compact. Let $S' = S \setminus D$ and note that if $q \in T$, then $S' \in q$. We claim that if $a \in S'$ and $q \in T$, then $aq \neq q$. So suppose instead that $a \in S'$, $q \in T$, and $aq = q$. Then by [2, Theorem 3.35], $\{x \in S : ax = x\} \in q$ and in particular we may pick some $x \in S$ such that $ax = x$. Then since S is right cancellative, a is a right identity for S, a contradiction.

Next we claim that for all $q \in T \setminus P$, $q \in c\ell_{\beta S}\big((T \cap Tq) \setminus \{q\}\big)$. Suppose instead we have $q \in T \setminus P$ and $A \in q$ such that $\overline{A} \cap \big((T \cap Tq) \setminus \{q\}\big) = \emptyset$. We claim that $q \in c\ell_{\beta S}(P \cap \overline{A}) \cap c\ell_{\beta S}(S'q \cap \overline{A})$. To see this, let $B \in q$. Then $A \cap B \in q$ so $\overline{A \cap B} \cap P \neq \emptyset$, that is $\overline{B} \cap (P \cap \overline{A}) \neq \emptyset$. Also $q = qq$ so $q \in \overline{S'}q = c\ell_{\beta S} S'q$. Therefore $\overline{A \cap B} \cap S'q \neq \emptyset$, that is $\overline{B} \cap (S'q \cap \overline{A}) \neq \emptyset$. Thus $q \in c\ell_{\beta S}(P \cap \overline{A}) \cap c\ell_{\beta S}(S'q \cap \overline{A})$ as claimed.

Thus by [2, Theorem 3.40] we have either

(1) there exist some $q' \in c\ell_{\beta S}(P \cap \overline{A})$ and some $a \in S'$ such that $q' = aq$ or

(2) there exists $n \in \mathbb{N}$ such that $p_n \in \overline{A}$ and $p_n \in c\ell_{\beta S}(S'q \cap \overline{A})$.

In case (1), $q'q = aqq = aq = q'$ and $aq \neq q$ so $q' \in \overline{A} \cap \big((T \cap Tq) \setminus \{q\}\big)$, a contradiction. In case (2) $p_n = rq$ for some $r \in \overline{S'}$ so $p_n q = rqq = rq = p_n$ and $p_n \neq q$ so $p_n \in \overline{A} \cap \big((T \cap Tq) \setminus \{q\}\big)$, a contradiction.

Thus we have established that for all $q \in T \setminus P$, $q \in c\ell_{\beta S}\big((T \cap Tq) \setminus \{q\}\big)$. Next we claim that for all $q \in T \setminus P$, $(T \cap Tq) \setminus P$ is infinite. So let $q \in T \setminus P$. We have that $q \in c\ell_{\beta S}\big((T \cap Tq) \setminus \{q\}\big)$ so $T \cap Tq$ is infinite and closed so $|T \cap Tq| = 2^{\mathfrak{c}}$ and in particular, $T \cap Tq$ is uncountable so $(T \cap Tq) \setminus P$ is infinite as claimed.

Next we claim that there is some $q \in T \setminus P$ which is $\leq_L$-minimal in $T \setminus P$; that is, for all $r \in T \setminus P$, if $r \leq_L q$, then $q \leq_L r$, where $r \leq_L q$ means that $r = r + q$. To see this, let $\mathcal{A} = \{B \subseteq T \setminus P : B \text{ is a chain with respect to } <_L\}$, where $p <_L r$ means that $p \leq_L r$ and it is not the case that $r \leq_L p$. Note that for $p, r \in T$, $p \leq_L r$ if and only if $\beta S p \subseteq \beta S r$. If $p \in T \setminus P$, then $\{p\} \in \mathcal{A}$. By Zorn's Lemma, pick a maximal member B of $\mathcal{A}$. Now $\{(\beta S p) \cap (T \setminus P) : p \in B\}$ is a collection of closed subsets of βS with the finite intersection property, so pick $q \in \bigcap_{p \in B}(\beta S p) \cap (T \setminus P)$. Then $q \leq_L p$ for each $p \in B$. If we had some $r \in T \setminus P$ with $r <_L q$, then $B \cup \{r\}$ would be a larger member of $\mathcal{A}$.

We have that $(T \cap Tq) \setminus P$ is infinite and compact. We claim it is a left zero semigroup. So let $r, s \in (T \cap Tq) \setminus P$. Then $r \in \beta S q$ so $r \leq_L q$ and thus $q \leq_L r$. Consequently $\beta S q \subseteq \beta S r$, and therefore $\beta S r = \beta S q$. Similarly $\beta S s = \beta S q$. Since $\beta S r = \beta S s$ we have $sr = s$ and $rs = r$ as required. □

We close by stating the obvious question.

Question 2. *Does* $(\beta\mathbb{N}, +)$ *contain an infinite compact set of idempotents?*

References

[1] H. Furstenberg and Y. Katznelson, *Idempotents in compact semigroups and Ramsey Theory*, Israel J. Math. **68** (1989), 257-270.

[2] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, second edition*, de Gruyter, Berlin, 2012.

[3] Y. Zelenyuk, *Left maximal idempotents in* G^*, Adv. Math. **262** (2014), 593-603.