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LONGER CHAINS OF IDEMPOTENTS IN βG

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Abstract. Given idempotents e and f in a semigroup, $e \leq f$ if and only if $e = fe = ef$. We show that if G is a countable discrete group, p is a right cancelable element of $G^* = \beta G \setminus G$, and λ is a countable ordinal, then there is a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma<\lambda}$ of idempotents in C_p, the smallest compact subsemigroup of G^* with p as a member. We also show that if S is any infinite subsemigroup of a countable group, then any nonminimal idempotent in S^* is the largest element of such a strictly decreasing chain of idempotents. (It had been an open question as to whether there was a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma<\omega+1}$ in $\mathbb{N}^*$.) As other corollaries we show that if S is an infinite right cancellative and weakly left cancellative discrete semigroup, then βS contains a decreasing chain of idempotents of reverse order type λ for every countable ordinal λ and that if S is an infinite cancellative semigroup then the set $U(S)$ of uniform ultrafilters contains such decreasing chains.

1. Introduction

A semigroup $(S, \cdot)$ with a topology is *right topological* if and only if for each $x \in S$, the function $\rho_x : S \to S$ is continuous, where for $y \in S$, $\rho_x(y) = y \cdot x$. In [1, Lemma 1], R. Ellis proved that any compact Hausdorff right topological semigroup contains an idempotent.

If $(S, \cdot)$ is an infinite discrete semigroup, there is a unique extension of the operation to βS making $(\beta S, \cdot)$ a right topological semigroup with S contained in its topological center. (The *topological center* of a right topological semigroup is the set of points x such that λ_x is continuous, where $\lambda_x(y) = x \cdot y$.) The existence of idempotents in βS, especially idempotents in certain subsemigroups of βS, has provided the easiest, and often the first, proof of many results in Ramsey Theory. See [4, Part III] for a multitude of examples of this phenomenon.

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As a compact right topological semigroup, βS has a smallest two sided ideal, $K(\beta S)$, which is the union of all of the minimal left ideals of βS and is also the union of all of the minimal right ideals of βS. The intersection of a minimal left ideal and a minimal right ideal of βS is a group, and any two such groups are isomorphic. Any left ideal contains a minimal left ideal, which is compact, and any right ideal contains a minimal right ideal. Idempotents in $K(\beta S)$ are exactly the idempotents that are minimal with respect to the ordering defined in the abstract.

We take the points of βS to be the ultrafilters on S, identifying the principal ultrafilters with the points of S, and thus pretend that $S \subseteq \beta S$. Given $A \subseteq S$, the closure $\overline{A} = \{p \in \beta S : A \in p\}$. We write $A^* = \overline{A} \setminus S$. Given $p, q \in \beta S$, $A \in pq$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : xy \in A\}$. (So, for example, in the semigroup $(\mathbb{N}, \cdot)$, if A is the set of odd positive integers, then $2^{-1}A = \emptyset$.) We let $U(S)$ be the set of uniform ultrafilters on S. Thus for $p \in \beta S$, $p \in U(S)$ if and only if for every $A \in p$, $|A| = |S|$. We take $\mathbb{N}$ to be the set of positive integers. The first infinite ordinal ω is the set of nonnegative integers. See [4, Part I] for an elementary introduction to the topology and algebra of βS, and see the notes at the end of the chapters for the original references.

In [3] it was shown that any nonminimal idempotent in $(\beta\mathbb{N}, +)$ is part of an infinite decreasing chain of idempotents. That is, if q is a nonminimal idempotent, then there is a sequence of idempotents $\langle q_n \rangle_{n<\omega}$ such that $q_0 = q$ and for each $n \in \omega$, $q_{n+1} < q_n$. We shall show in this paper that the sequence can be extended to $\langle q_\sigma \rangle_{\sigma<\lambda}$ for any countable ordinal λ.

A fundamental tool in our proofs is an analysis of the structure of the smallest compact subsemigroup of $\beta\mathbb{N}$ containing a given member of $\beta\mathbb{N}$.

Definition 1.1. Let S be a compact Hausdorff right topological semigroup and let $p \in S$. Then

$$C_p = \bigcap\{T : T \text{ is a compact subsemigroup of } S \text{ and } p \in T\}\,.$$

Section 2 will consist of preliminary results. In Section 3 we will prove our main theorem dealing with decreasing chains of idempotents in C_p and derive from that several corollaries, including those mentioned in the abstract.

Most of the results in Section 3 deal with cancellative semigroups. In Section 4 we extend some of these results to left cancellative semigroups S which have a right cancelable element in S^*.

Besides the ordering $\leq$ of idempotents in a semigroup, there are transitive and reflexive relations $\leq_L$ and $\leq_R$ defined by $e \leq_L f$ if and only if

$ef = e$ and $e \leq_R f$ if and only if $fe = e$. We write $e <_L f$ when $e \leq_L f$ and it is not the case that $f \leq_L e$. Similarly we write $e <_R f$ when $e \leq_R f$ and it is not the case that $f \leq_R e$. Of course $e \leq f$ if and only if both $e \leq_L f$ and $e \leq_R f$. In [6] it was shown that given any ordinal λ with $|\lambda| \leq \mathfrak{c}$, there exist chains $\langle q_\sigma \rangle_{\sigma<\lambda}$ of idempotents in $\beta\mathbb{N}$ such that $q_\sigma <_L q_\tau$ whenever $\tau < \sigma < \lambda$ and $q_{\sigma+1} < q_\sigma$ for all σ with $\sigma + 1 < \lambda$. In Section 5 we extend this result by showing that for each nonminimal idempotent q in $\beta\mathbb{N}$, there is such a chain with $q_0 = q$.

2. Preliminary results

Given a semigroup $(S, \cdot)$ with identity we will denote the identity of S by 1. Unless otherwise specified, we take the operation on $\mathbb{N}$ and $\mathbb{Z}$ to be addition.

Some of our proofs depend on the existence of elements in the closure of the set of idempotents in a given minimal right ideal of $\beta\mathbb{N}$ that are right cancelable in $\beta\mathbb{N}$. The following lemma guarantees their existence.

Lemma 2.1. *Let R be a minimal right ideal of $\beta\mathbb{N}$. There is an injective sequence $\langle q_n \rangle_{n=1}^\infty$ of idempotents in R such that, if p is an accumulation point of $\langle q_n \rangle_{n=1}^\infty$, then $p \notin \mathbb{Z}^* + \mathbb{Z}^*$. In particular any accumulation point of $\langle q_n \rangle_{n=1}^\infty$ is right cancelable in $\beta\mathbb{Z}$.*

Proof. This is [5, Lemma 3.8]. □

Definition 2.2. If S is a semigroup, then $E(S)$ is the set of idempotents in S.

The following lemma is well known among aficionados. In its proof we use, for the first of many times in this paper, the fact that if p is an idempotent in a semigroup S, then p is a right identity for Sp and a left identity for pS. (If $q = ap$, then $qp = app = ap = q$.)

Lemma 2.3. *Let S and T be compact Hausdorff right topological semigroups, let $h : S \to T$ be a continuous surjective homomorphism.*

1. *If q_1 is an idempotent in T, then there exists $p_1 \in E(S)$ such that $h(p_1) = q_1$.*
2. *If q_1 and q_2 are idempotents in T such that $q_2 < q_1$ and $p_1 \in E(S)$ such that $h(p_1) = q_1$, then there exists $p_2 \in E(S)$ such that $h(p_2) = q_2$ and $p_2 < p_1$.*

Proof. (1) We have that $h^{-1}[\{q_1\}]$ is a compact subsemigroup of S which therefore has an idempotent.

(2) Assume that $p_1 \in E(S)$ such that $h(p_1) = q_1$. If $x \in h^{-1}[\{q_2\}]$, then $h(xp_1) = q_2q_1 = q_2$ so $xp_1 \in h^{-1}[\{q_2\}] \cap Sp_1$ and consequently $h^{-1}[\{q_2\}] \cap Sp_1$ is a compact subsemigroup of S. Pick an idempotent $g \in h^{-1}[\{q_2\}] \cap Sp_1$ and let $p_2 = p_1g$. Then $gp_1 = g$ so $p_2p_1 = p_2$, $p_2p_2 = p_1gp_1g = p_1gg = p_1g = p_2$, and $p_1p_2 = p_1p_1g = p_1g = p_2$. □

Lemma 2.4. *Let S be a compact Hausdorff right topological semigroup and let R be a minimal right ideal of S. If $x \in c\ell E(R)$, then for all $u \in C_x$ and all $v \in R$, $uv = v$.*

Proof. If $p \in E(R)$, then $R = pS$ so for all $v \in R$, $pv = v$. Thus, given $v \in R$, ρ_v is constantly equal to v on $E(R)$ so $xv = v$. Thus

$$\{u \in S : (\forall v \in R)(uv = v)\}$$

is a compact subsemigroup of S with x as a member which therefore contains C_x. □

Of course, in any semigroup S, an element x is *right cancelable* if and only if ρ_x is injective.

Definition 2.5. Let S be an infinite semigroup with identity and let $p \in S^*$.

(a) p is *weakly right cancelable* if and only if there is no $q \in \beta S \setminus \{1\}$ such that $p = qp$.
(b) p is *thin* if and only if there is a function $M : S \to p$ such that $xM(x) \cap yM(y) = \emptyset$ whenever x and y are distinct members of S.
(c) p is *strongly discrete* if and only if p is thin, $p \in U(S)$, and for each $x \in S$, the restriction of λ_x to $M(x)$ is injective.

Note that if $p = qp$, then $1p = 1qp$, so if p is right cancelable, then it is weakly right cancelable.

Lemma 2.6. *Let S be a countable semigroup with identity and let $p \in S^*$. If p is right cancelable in βS, then p is thin. If in addition S is left cancellative, then p is strongly discrete.*

Proof. By [4, Theorem 8.7] we have that for all $A \subseteq S$, there exists $B \subseteq S$ such that $A = \{x \in S : x^{-1}B \in p\}$. Enumerate S as $\langle x_n \rangle_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, pick $B_n \subseteq S$ such that $\{x_n\} = \{x \in S : x^{-1}B_n \in p\}$. Let $M(x_1) = x_1^{-1}B_1$. For $n > 1$, let $M(x_n) = x_n^{-1}B_n \setminus \bigcup_{t=1}^{n-1} x_n^{-1}B_t$. If $t < n$, then $x_tM(x_t) \subseteq B_t$ and $x_nM(x_n) \cap B_t = \emptyset$. □

Lemma 2.7. *Let S be an infinite semigroup with identity and let $p \in S^*$.*

(1) *If p is thin, then p is right cancelable in βS.*

(2) *If p is weakly right cancelable and S is a countable group, then p is strongly discrete.*

Proof. (1) Let $M : S \to p$ be as guaranteed by the definition of thin. Let q and r be distinct members of βS and pick $Q \in q$ and $R \in r$ such that $Q \cap R = \emptyset$. Let $A = \bigcup_{x \in Q} xM(x)$ and let $B = \bigcup_{x \in R} xM(x)$. Then $A \in qp$, $B \in rp$, and $A \cap B = \emptyset$.

(2) By [4, Theorem 8.18] we have that p is right cancelable in βS so Lemma 2.6 applies. □

Definition 2.8. Let κ be an infinite cardinal.

(a) $H_\kappa = \bigoplus_{\sigma<\kappa} \mathbb{Z}_2$.
(b) For $\gamma < \kappa$, let $H_{\kappa,\gamma} = \{x \in H_\kappa : (\forall \sigma < \gamma)\big(x(\sigma)\big) = 0\}$.
(c) $\mathbb{H}_\kappa = \bigcap_{\gamma<\kappa} c\ell_{\beta H_\kappa}(H_{\kappa,\gamma} \setminus \{0\})$.

The set $\mathbb{H} \subseteq \beta\mathbb{N}$ is defined by $\mathbb{H} = \bigcap_{n\in\mathbb{N}} c\ell(2^n\mathbb{N})$. By [4, Theorem 6.27] $\mathbb{H}$ is topologically and algebraically isomorphic to $\mathbb{H}_\omega$. (When we say that sets in right topological semigroups are "topologically and algebraically isomorphic" we mean that there is a function taking one to the other which is both an isomorphism and a homeomorphism.)

Lemma 2.9. *Let κ be an infinite cardinal and let $p \in \mathbb{H}_\kappa$. The following statements are equivalent.*

(1) *p is right cancelable in βH_κ.*
(2) *p is weakly right cancelable.*
(3) *p is thin.*

Proof. That (1) implies (2) is trivial and that (3) implies (1) follows from Lemma 2.7(1).

To see that (2) implies (3), assume that p is weakly right cancelable. We note first that $H_\kappa + p$ is discrete. Indeed, if $a \in H_\kappa$ and

$$a + p \in c\ell\big\{b + p : b \in H_\kappa \setminus \{a\}\big\} = (\beta H_\kappa \setminus \{a\}) + p\,,$$

then pick $q \in \beta H_\kappa \setminus \{a\}$ such that $a + p = q + p$. Then $p = -a + q + p$, and $-a + q \neq 0$. (If $q \in H_\kappa$ this is immediate, and if $q \in H_\kappa^*$, then $-a + q \in H_\kappa^*$ by [4, Corollary 4.33].)

The rest of the proof may be taken verbatim from the proof that (5) implies (6) in [8, Theorem 11.2]. □

Lemma 2.10. *Let S be an infinite semigroup with identity and let $p \in S^*$ be strongly discrete. There is a compact subsemigroup T_p of βS with $p \in T_p$ such that*

(1) *for all* $x \in \beta S \setminus (T_p \cup \{1\})$, $(xT_p) \cap T_p = \emptyset$ *and*
(2) *there is a continuous homomorphism* $\pi_p : T_p \to \beta\mathbb{N}$ *such that* $\pi_p(p) = 1$.

Proof. Let $\mathcal{T}[p]$ be the largest topology on S with respect to which p converges to 1 and λ_a is continuous for each $a \in S$. Let

$$T_p = \{q \in S^* : q \text{ converges to 1 with respect to } \mathcal{T}[p]\}\,.$$

By [8, Lemma 7.1], T_p is a compact subsemigroup of S^*. To verify conclusion (1) suppose we have $x \in \beta S \setminus (T_p \cup \{1\})$ and $q \in T_p$ such that $xq \in T_p$. By [8, Theorem 4.18], $\mathcal{T}[p]$ is zero-dimensional and Hausdorff. Since $x \notin T_p \cup \{1\}$, x does not converge to 1 with respect to $\mathcal{T}[p]$ and thus there is a neighborhood U of 1 such that $U \notin x$. Since $\mathcal{T}[p]$ is zero-dimensional, we may assume U is clopen with respect to $\mathcal{T}[p]$. Let $W = S \setminus U$. Then $W \in x$. We claim that $c\ell_{\beta S}(W)q \subseteq c\ell_{\beta S}(W)$ for which it suffices that $Wq \subseteq c\ell_{\beta S}(W)$, so let $a \in W$. Then W is a neighborhood of $a = \lambda_a(1)$ and λ_a is continuous with respect to $\mathcal{T}[p]$, so pick a neighborhood V of 1 such that $aV \subseteq W$. Then $V \in q$ and $V \subseteq a^{-1}W$ so $a^{-1}W \in q$ and thus $aq \in c\ell_{\beta S}(W)$ as claimed. We thus have that $xq \in c\ell_{\beta S}(W)$ so $U \notin xq$ and thus $xq \notin T_p$.

Conclusion (2) holds by [8, Theorem 7.29]. □

Notice that, since T_p is a compact subsemigroup of βS and $p \in T_p$, we have that $C_p \subseteq T_p$.

Definition 2.11. Let S be an infinite semigroup with identity and let $p \in S^*$ be strongly discrete. Then h_p is the restriction of π_p to C_p.

If $n \in \mathbb{N}$ and q is the sum of p with itself n times, then $h_p(p) = n$. Therefore $\mathbb{N} \subseteq h_p[C_p]$ and consequently $h_p[C_p] = \beta\mathbb{N}$. Observe also that the function h_p is completely determined by the fact that $h_p(p) = 1$. To see this, let $g : C_p \to \beta\mathbb{N}$ be a continuous homomorphism with $g(p) = 1$. Then $\{x \in C_p : g(x) = h_p(x)\}$ is a compact subsemigroup of C_p with p as a member and is therefore equal to C_p.

Lemma 2.12. *Let S be an infinite semigroup with identity and let $p \in S^*$ be strongly discrete. If $x \in C_p$ and $h_p(x)$ is right cancelable in $\beta\mathbb{N}$, then x is weakly right cancelable.*

Proof. Suppose not and pick $u \in \beta S \setminus \{1\}$ such that $x = ux$. Then $x \in (uT_p) \cap T_p$ so by Lemma 2.10(1), $u \in T_p \cup \{1\}$ and since $u \neq 1$, $u \in T_p$. Thus $h_p(x) = \pi_p(u) + \pi_p(x) = \pi_p(u) + h_p(x)$. By [4, Corollary 8.2] (since $1 + h_p(x) = 1 + \pi_p(u) + h_p(x)$)we have $\pi_p(u) \in \mathbb{N}^*$. But then, by [4, Theorem 8.18], $h_p(x)$ is not right cancelable in $\beta\mathbb{N}$. □

Lemma 2.13. *Let S be an infinite semigroup with identity, let $p \in S^*$ be strongly discrete, and let q be an idempotent in C_p such that $h_p(q) \notin K(\beta\mathbb{N})$. There exists $s \in C_p q$ which is weakly right cancelable such that $h_p[C_s] \cap K(\beta\mathbb{N}) = \emptyset$.*

Proof. By [4, Theorem 6.56], choose $y \in \beta\mathbb{N}$ such that $y + h_p(q)$ is right cancelable in $\beta\mathbb{Z}$. Pick $x \in C_p$ such that $h_p(x) = y$ and let $s = xq$. Then $h_p(s) = y + h_p(q)$ so by Lemma 2.12, s is weakly right cancelable. By [4, Theorem 8.57], $C_{h_p(s)} \cap K(\beta\mathbb{Z}) = \emptyset$. By [4, Exercise 4.3.8], $K(\beta\mathbb{Z}) = K(\beta\mathbb{N}) \cup -K(\beta\mathbb{N})$, so $C_{h_p(s)} \cap K(\beta\mathbb{N}) = \emptyset$. Since $h_p^{-1}[C_{h_p(s)}]$ is a compact subsemigroup containing s, we have $h_p[C_s] \subseteq C_{h_p(s)}$. □

Lemma 2.14. *Let S be an infinite semigroup with identity, let $p \in S^*$ be strongly discrete, and let R be a minimal right ideal of C_p. There exists $s \in c\ell E(R)$ which is weakly right cancelable.*

Proof. By [4, Exercise 1.7.3] $h_p[R]$ is a minimal right ideal of $\beta\mathbb{N}$ so by Lemma 2.1, there is an injective sequence $\langle q_n \rangle_{n=1}^{\infty}$ of idempotents in $h_p[R]$ all of whose limit points are right cancelable in $\beta\mathbb{Z}$. We claim that for each $n \in \mathbb{N}$ there is an idempotent $u_n \in R$ such that $h_p(u_n) = q_n$. To see this, let $n \in \mathbb{N}$. Then $\beta\mathbb{N} + q_n$ is a minimal left ideal of $\beta\mathbb{N}$ so $h_p^{-1}[\beta\mathbb{N} + q_n]$ is a left ideal of C_p which contains a minimal left ideal L. Let u_n be the identity of $R \cap L$. Then $h_p(u_n)$ is an idempotent in $h_p[R] \cap (\beta\mathbb{N} + q_n)$, whose only idempotent is q_n. Let s be a limit point of $\langle u_n \rangle_{n=1}^{\infty}$. Then $h_p(s)$ is a limit point of $\langle q_n \rangle_{n=1}^{\infty}$ so is right cancelable in $\beta\mathbb{Z}$. Thus by Lemma 2.12, s is weakly right cancelable in βS. □

Lemma 2.15. *Let S be an infinite semigroup with identity, let q be an idempotent in S^*, let $s \in \beta S q$ be strongly discrete, let λ be an ordinal, and let $\langle u_\sigma \rangle_{\sigma<\lambda}$ be a strictly decreasing sequence of idempotents in C_s. For each $\sigma < \lambda$, let $v_\sigma = q u_\sigma$. Then $\langle v_\sigma \rangle_{\sigma<\lambda}$ is a strictly decreasing sequence of idempotents with $v_0 \leq q$.*

Proof. Note that $C_s \subseteq \beta S q$ so for each $\sigma < \lambda$, $u_\sigma = u_\sigma q$. Given $\sigma < \lambda$, we have $v_\sigma v_\sigma = q u_\sigma q u_\sigma = q u_\sigma u_\sigma = q u_\sigma = v_\sigma$. Now let $\sigma < \tau < \lambda$. Then $v_\sigma v_\tau = q u_\sigma q u_\tau = q u_\sigma u_\tau = q u_\tau = v_\tau$ and $v_\tau v_\sigma = q u_\tau q u_\sigma = q u_\tau u_\sigma = q u_\tau = v_\tau$ so $v_\tau \leq v_\sigma$. We claim that $v_\sigma \neq v_\tau$, so suppose instead $v_\sigma = v_\tau$. Now $C_s s \cup \{s\}$ is a compact semigroup with s as a member, so $C_s \subseteq C_s s \cup \{s\} \subseteq C_s$ so $C_s = C_s s \cup \{s\}$. Since u_σ and u_τ are idempotents, neither is equal to s so pick x_σ and x_τ in C_s such that $u_\sigma = x_\sigma s$ and $u_\tau = x_\tau s$. Since $s \in \beta S q$, pick $r \in \beta S$ such that $s = rq$. Now $q x_\tau r q = q x_\tau s = q u_\tau = v_\tau =$

$v_\sigma = qx_\sigma rq$ so $x_\sigma rqx_\tau rq = x_\sigma rqx_\sigma rq$. That is $u_\sigma u_\tau = u_\sigma u_\sigma$, so $u_\tau = u_\sigma$, a contradiction. □

3. Long strictly decreasing chains

Definition 3.1. Let S be an infinite semigroup with identity and let $p \in S^*$. Then p is *hereditarily strongly discrete* if and only if p is strongly discrete and every $s \in C_p$ which is weakly right cancelable is also strongly discrete.

Notice that by Lemma 2.7(2), if S is a countable group and $p \in S^*$ is strongly discrete, then p is hereditarily strongly discrete.

Lemma 3.2. *Let κ be an infinite cardinal and let $p \in \mathbb{H}_\kappa$ be strongly discrete. Then $C_p \subseteq U(H_\kappa)$ and p is hereditarily strongly discrete.*

Proof. Trivially $\mathbb{H}_\kappa$ is a compact subsemigroup of βH_κ and by [4, Lemma 6.34.3] $U(H_\kappa)$ is a compact subsemigroup (in fact an ideal) of βH_κ. Since $p \in U(H_\kappa)$ by the definition of strongly discrete, we have that $\mathbb{H}_\kappa \cap U(H_\kappa)$ is a compact semigroup with p as a member so $C_p \subseteq \mathbb{H}_\kappa \cap U(H_\kappa)$.

To see that p is hereditarily strongly discrete, let $s \in C_p$ be weakly right cancelable. By Lemma 2.9, s is thin. But also $C_p \subseteq \mathbb{H}_\kappa \cap U(H_\kappa)$ so $s \in U(H_\kappa)$ and (since H_κ is cancellative) s is strongly discrete. □

Definition 3.3. Let S be an infinite semigroup with identity and let λ be an ordinal. $P(\lambda)$ is the following statement. *Given any hereditarily strongly discrete $p \in S^*$ and any idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$, there is a strictly decreasing chain of idempotents $\langle q_\sigma \rangle_{\sigma<\lambda}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$.*

Lemma 3.4. *Let S be an infinite semigroup with identity and let $\lambda > 0$ be an ordinal. Then $P(\lambda) \Rightarrow P(\lambda+1)$.*

Proof. Assume $P(\lambda)$. Let $p \in S^*$ be hereditarily strongly discrete and let q be an idempotent in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$. By Lemma 2.13 pick $s \in C_p q$ which is weakly right cancelable in βS such that $h_p[C_s] \cap K(\beta\mathbb{N}) = \emptyset$. Since $s \in C_p$, s is strongly discrete. Let R be a minimal right ideal of C_s. By Lemma 2.14 pick $t \in c\ell E(R)$ which is weakly right cancelable. Note that $C_t \subseteq C_s \subseteq \beta S q$. By Lemma 2.4 we have that for all $u \in C_t$ and all $v \in R$, $uv = v$. Pick any idempotent w in $C_t \setminus h_t^{-1}[K(\beta\mathbb{N})]$ and choose a strictly decreasing chain $\langle u_\sigma \rangle_{\sigma<\lambda}$ of idempotents in $C_t \setminus h_t^{-1}[K(\beta\mathbb{N})]$ with $u_0 = w$. (We will not use the fact that $u_0 = w$.)

If $\sigma < \tau < \lambda$, then $u_\tau = u_\tau u_\sigma \in C_s u_\sigma$ so $\langle C_s u_\sigma \rangle_{\sigma<\lambda}$ is a nested sequence of closed left ideals of C_s so we may pick a minimal left ideal L of C_s with $L \subseteq \bigcap_{\sigma<\lambda} C_s u_\sigma$. Let u_λ be the identity of $R \cap L$. Let $\sigma < \lambda$. We have

$u_\lambda \in C_s u_\sigma$ so $u_\lambda = u_\lambda u_\sigma$. Also $u_\sigma \in C_t$ and $u_\lambda \in R$, so $u_\lambda = u_\sigma u_\lambda$ and so $u_\lambda \leq u_\sigma$. We need to show that $u_\sigma \neq u_\lambda$. (Of course, if λ is a limit ordinal, this is immediate. But our proof does not depend on λ being a successor.) We shall show that $u_\sigma \notin R$. So suppose instead that $u_\sigma \in R$. Then $u_\sigma \in C_t \cap R$ so $\emptyset \neq C_t \cap R \subseteq C_t \cap K(C_s)$ and thus by [4, Theorem 1.65], $K(C_t) = C_t \cap K(C_s)$ so $u_\sigma \in K(C_t)$. But then $h_t(u_\sigma) \in K(\beta\mathbb{N})$, a contradiction. We have thus established that $\langle u_\sigma \rangle_{\sigma \leq \lambda}$ is a strictly decreasing chain of idempotents in C_s. Recall that $s \in C_p q$ so pick $r \in C_p$ such that $s = rq$.

Now for each $\sigma \leq \lambda$, let $v_\sigma = q u_\sigma$. By Lemma 2.15 we have that $\langle v_\sigma \rangle_{\sigma \leq \lambda}$ is a strictly decreasing chain of idempotents in C_p.

Suppose there is some $\sigma \leq \lambda$ such that $h_p(v_\sigma) \in K(\beta\mathbb{N})$. Then $s u_\sigma \in C_s$ and $s u_\sigma = r q u_\sigma = r v_\sigma$ so $h_p(r v_\sigma) = h_p(r) + h_p(v_\sigma) \in K(\beta\mathbb{N})$ contradicting the fact that $h_p[C_s] \cap K(\beta\mathbb{N}) = \emptyset$.

Note that $v_0 \leq q$. If $v_0 = q$, then we are done, so assume $v_0 < q$ and replace v_0 by q. □

The following is the main result of this paper. Recall that h_p is surjective and therefore there are idempotents in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$.

Theorem 3.5. *Let S be an infinite semigroup with identity and let $\lambda > 0$ be a countable ordinal. Given any hereditarily strongly discrete $p \in S^*$ and any idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$, there is a strictly decreasing chain of idempotents $\langle q_\sigma \rangle_{\sigma < \lambda}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$.*

Proof. We prove by induction that $P(\lambda)$ holds. If $\lambda = 1$, let $q_0 = q$. So assume that $\lambda > 1$ and $P(\alpha)$ holds for all α with $0 < \alpha < \lambda$. If λ is a successor, then $P(\lambda)$ holds by Lemma 3.4, so assume that λ is a limit ordinal. Pick a strictly increasing sequence $\langle \alpha_n \rangle_{n<\omega}$ of ordinals with $\alpha_0 > 0$ such that $\lambda = \sup\{\alpha_n : n < \omega\}$. Let $p \in S^*$ be hereditarily strongly discrete and let q be an idempotent in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$. By $P(\alpha_0 + 1)$ pick a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma \leq \alpha_0}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$.

Now let $n < \omega$ and assume we have chosen $\langle q_\sigma \rangle_{\sigma \leq \alpha_n}$. Let δ be the ordinal such that $\alpha_{n+1} = \alpha_n + \delta$. By $P(\delta + 1)$, pick a strictly decreasing chain $\langle r_\sigma \rangle_{\sigma \leq \delta}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $r_0 = q_{\alpha_n}$. For $0 < \tau \leq \delta$, let $q_{\alpha_n + \tau} = r_\tau$. □

Corollary 3.6. *Let G be a countably infinite group and let $\lambda > 0$ be a countable ordinal. Given any $p \in G^*$ which is right cancelable in βG and any idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$, there is a strictly decreasing chain of idempotents $\langle q_\sigma \rangle_{\sigma < \lambda}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$.*

Proof. Let $p \in G^*$ which is right cancelable in βG. As we have already remarked, p is hereditarily strongly discrete. So Theorem 3.5 applies. □

Corollary 3.7. *Let κ be an infinite cardinal and let $\lambda > 0$ be a countable ordinal. Given any strongly discrete $p \in \mathbb{H}_\kappa$, and any idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$, there is a strictly decreasing chain of idempotents $\langle q_\sigma \rangle_{\sigma<\lambda}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$. This chain of idempotents is contained in $U(H_\kappa) \cap \mathbb{H}_\kappa$.*

Proof. Let $p \in \mathbb{H}_\kappa$ be strongly discrete. By Lemma 3.2, p is hereditarily strongly discrete. So Theorem 3.5 applies. Since $C_p \subseteq U(H_\kappa) \cap \mathbb{H}_\kappa$, the final conclusion holds. □

Of course minimal idempotents by definition do not have any idempotents below them. We see now that for any subsemigroup of a countably infinite group, any nonminimal idempotent has long chains of idempotents below it.

Corollary 3.8. *Let G be a countably infinite group, let $\lambda > 0$ be a countable ordinal, and let S be an infinite subsemigroup of G. For every nonminimal idempotent $q \in S^*$, there is a strictly decreasing chain of idempotents $\langle v_\sigma \rangle_{\sigma<\lambda}$ in S^* such that $v_0 = q$.*

Proof. By [4, Theorem 6.56] pick $r \in S^*$ such that $s = rq$ is right cancelable in βG. By Lemma 2.7(2), s is strongly discrete. Since $\beta\mathbb{N}$ has nonminimal idempotents and $h_s[C_s] = \beta\mathbb{N}$, by Lemma 2.3 there is an idempotent $t \in C_s \setminus h_s^{-1}[K(\beta\mathbb{N})]$. By Corollary 3.6 pick a strictly decreasing sequence $\langle u_\sigma \rangle_{\sigma<\alpha}$ of idempotents in C_s. For each $\sigma < \lambda$, let $v_\sigma = qu_\sigma$. By Lemma 2.15, $\langle v_\sigma \rangle_{\sigma<\alpha}$ is a strictly decreasing sequence of idempotents with $v_0 \leq q$. If $v_0 \neq q$, replace v_0 by q. □

Lemma 3.9. *There is a strongly discrete $p \in \mathbb{H}_\kappa$.*

Proof. For each $\alpha < \kappa$ let a_α be the characteristic function of $\{\alpha\}$ and for each $\gamma < \kappa$, let

$$A_\gamma = \{a_\alpha : \gamma \leq \alpha < \kappa\}\,.$$

By [4, Corollary 3.14] pick $p \in U(H_\kappa)$ such that $\{A_\gamma : \gamma < \kappa\} \subseteq p$. Since for each γ, $A_\gamma \subseteq H_{\kappa,\gamma}$, we have $p \in \mathbb{H}_\kappa$. For $x \in H_\kappa \setminus \{0\}$, let

$$\phi(x) = \max\{\sigma : x(\sigma) \neq 0\}$$

and let $\phi(0) = -1$. Define $M : H_\kappa \to p$ by $M(x) = A_{\phi(x)+1}$. Now let $x \neq y$ in H_κ. We claim that $\big(x + M(x)\big) \cap \big(y + M(y)\big) = \emptyset$. Suppose one has

$z = x + a_\alpha = y + a_\delta$ where $a_\alpha \in M(x)$ and $a_\delta \in M(y)$. Then $\alpha = \phi(z) = \delta$ so $x = y$, a contradiction. □

Corollary 3.10. *Let S be an infinite cancellative semigroup with identity and let λ be a countable ordinal. There is a strictly decreasing chain of idempotents $\langle v_\sigma \rangle_{\sigma<\lambda}$ in $U(S)$.*

Proof. Let $\kappa = |S|$. By [6, Theorem 2.7] S^* contains a copy of $\mathbb{H}_\kappa$. The proof of that theorem produces a subset T of S and a bijective function $\theta : T \to H_\kappa$ with continuous extension $\widetilde{\theta} : c\ell_{\beta S} T \to \beta H_\kappa$. The restriction of $\widetilde{\theta}$ to $\widetilde{\theta}^{-1}[\mathbb{H}_\kappa]$ is a homeomorphism and an isomorphism. And $\widetilde{\theta}[U(T)] = U(H_\kappa)$.

By Lemma 3.9, pick a strongly discrete member p of $\mathbb{H}_\kappa$. Since $\beta\mathbb{N}$ has nonminimal idempotents and $h_p[C_p] = \beta\mathbb{N}$, by Lemma 2.3 there is an idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ so by Corollary 3.7 the copy of $\mathbb{H}_\kappa$ contains a strictly decreasing chain of idempotents $\langle v_\sigma \rangle_{\sigma<\lambda}$ which is contained in $U(S)$. □

The following, which answers[6, Question 3.19], is not an immediate corollary of Corollary 3.6 because there are points $p \in \mathbb{N}^*$ that are right cancelable in $\beta\mathbb{N}$ but not right cancelable in $\beta\mathbb{Z}$. (See [4, Example 8.29].)

Corollary 3.11. *Given any $p \in \mathbb{N}^*$ which is right cancelable in $\beta\mathbb{N}$ and any countable ordinal λ, there is a strictly decreasing chain of idempotents $\langle q_\sigma \rangle_{\sigma<\lambda}$ in C_p.*

Proof. By Theorem 4.10 below (or [4, Exercise 8.5.1(6)]) there is an element $q \in c\ell\{2^n : n \in \mathbb{N}\}$ such that C_q is isomorphic to C_p. By [4, Theorem 8.28], q is right cancelable in $\beta\mathbb{Z}$. By Lemma 2.6 q is strongly discrete and so by Theorem 3.5, there is a strictly decreasing chain of idempotents $\langle q_\sigma \rangle_{\sigma<\lambda}$ in C_q. □

Note that the number of decreasing chains headed by a given nonminimal q is vast. By [4, Theorem 6.56], there are $2^{\mathfrak{c}}$ choices of $r \in \beta\mathbb{N}$ for which $r + q$ is right cancelable in $\beta\mathbb{Z}$ and, for any two different choices r_1 and r_2 among these, the left ideals $\beta\mathbb{N} + r_1 + q$ and $\beta\mathbb{N} + r_2 + q$ are disjoint. So, in defining decreasing chains $\langle q_n \rangle_{n<\omega}$ with $q_0 = q$, one has $2^{\mathfrak{c}}$ choices for q_1. For each of these, there are $2^{\mathfrak{c}}$ choices for q_2, and so on. The chains defined by these choices never intersect, except at q.

We conclude this section by establishing that one can get long chains of idempotents while weakening the cancellation hypotheses on S. Given a set X, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.

Definition 3.12. Let $(S,\cdot)$ be a semigroup and let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in S.

(a) Let $m\in\mathbb{N}$. Then

$$FP(\langle x_n\rangle_{n=m}^\infty)=\{\prod_{t\in F}x_t : F\in\mathcal{P}_f(\mathbb{N}) \text{ and } \min F\geq m\}\,,$$

where the products are taken in increasing order of indices.

(b) The sequence $\langle x_n\rangle_{n=1}^\infty$ has *distinct finite products* if and only if whenever F and H are distinct members of $\mathcal{P}_f(\mathbb{N})$, $\prod_{t\in F}x_t\neq\prod_{t\in H}x_t$.

Lemma 3.13. *Let S be a semigroup and let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in S with distinct finite products. Then $\bigcap_{m=1}^\infty\overline{FP(\langle x_n\rangle_{n=m}^\infty)}$ is topologically and algebraically isomorphic to $\mathbb{H}$.*

Proof. [4, Theorem 6.27]. □

Recall that semigroup S is *weakly left cancellative* provided that for each $a,b\in S$, $\{c\in S : ac=b\}$ is finite.

Lemma 3.14. *Let S be an infinite right cancellative and weakly left cancellative semigroup. There is a sequence $\langle x_n\rangle_{n=1}^\infty$ in S which has distinct finite products.*

Proof. This is a consequence of [4, Lemma 6.31]. □

Corollary 3.15. *Let S be an infinite right cancellative and weakly left cancellative semigroup and let $\lambda>0$ be a countable ordinal. Then βS contains a decreasing chain $\langle q_\sigma\rangle_{\sigma<\lambda}$ of idempotents.*

Proof. By [4, Lemma 6.8] all of the idempotents in $\beta\mathbb{N}$ are in $\mathbb{H}$. By Lemmas 3.13 and 3.14, S contains a copy of $\mathbb{H}$. By Corollary 3.8, $\mathbb{H}$ contains a decreasing chain $\langle q_\sigma\rangle_{\sigma<\lambda}$ of idempotents. □

4. Countable left cancellative semigroups

In this section we show that some of our earlier results can be extended to countable semigroups S for which only left cancellation is assumed, provided there is a right cancelable element of S^*. Theorems 4.10 and 4.12 extend [4, Theorem 8.62] and [4, Theorem 8.57] respectively, wherein the hypothesis on S was that it is a countable group. We observe that any right cancelable element of S^* is a strongly discrete ultrafilter by Lemma 2.6, so that the results of [8, §4.3] apply to it. However, the results of this section have self-contained proofs.

There are left cancellative semigroups S for which there are no right cancelable elements in S^*. For example, if S is a right zero semigroup, βS

is also a right zero semigroup and it contains no right cancelable elements if $|S| > 1$. So we begin by pointing out that it is quite easy to find examples of countable left cancellative semigroups S for which S^* does contain right cancelable elements. One example of a left cancellative semigroup which is not right cancellative but in which S^* has right cancelable elements is the ordinal $\omega\cdot\omega$ under ordinal addition. But that is not an especially interesting example, since the only right cancelable elements of S^* are in ω^*. (If $\{\sigma < \omega\cdot\omega : \omega \leq \sigma\} \in p$, then $1+p = 2+p$.)

Theorem 4.1. *There exist a countable left cancellative semigroup S with identity which is not right cancellative and an infinite subset A of S such that every $p \in A^*$ is right cancelable in βS.*

Proof. Let S be the set of all strictly increasing $f : \mathbb{N} \to \mathbb{N}$ with the property that there exist $m \in \mathbb{N}$ and $k \in \omega$ such that for all $t \geq m$, $f(t) = k + t$. Let the operation on S be composition. It is routine to verify that S is closed under composition. Further, since its members are injective, S is left cancellative. It is not right cancellative (though it is weakly right cancellative). For example, let $f(t) = t+1$ for all t, let $g(1) = 1$, $h(1) = 2$, and $g(t) = h(t) = t+1$ for all $t > 1$. Then $g \circ f = h \circ f$.

For each $f \in S$, let $k(f)$ be the number such that $f(t) = k(f) + t$ for all sufficiently large t and let $m(f) = \min\{t : f(t) = k(f) + t\}$. For $n \in \mathbb{N}$, define $h_n \in S$ by

$$h_n(t) = \begin{cases} t & \text{if } t < n \\ 1+t & \text{if } t \geq n\,. \end{cases}$$

Let $A = \{h_n : n \in \mathbb{N}\}$ and let $p \in A^*$. To see that p is right cancelable in βS, let q and r be distinct members of βS and suppose that $q\circ p = r\circ p$. (As usual, we denote the operation in βS by the same symbol used to denote the operation on S.) Pick disjoint subsets B and C of S such that $B \in q$ and $C \in r$. By [4, Theorem 4.15], $\{f \circ h_n : f \in B \text{ and } n > m(f)\} \in q \circ p$ and $\{g \circ h_n : g \in C \text{ and } n > m(g)\} \in q \circ p$ so the intersection is nonempty. Pick $f \in B$, $g \in C$, $n > m(f)$, and $l > m(g)$ such that $f \circ h_n = g \circ h_l$. Now if $t \geq \max\{n, l\}$, then $f\big(h_n(t)\big) = f(t+1) = k(f)+t+1$ and $g\big(h_l(t)\big) = g(t+1) = k(g)+t+1$, so $k(f) = k(g)$. We claim next that $n = l$, so suppose instead without loss of generality that $n < l$. Then $f\big(h_n(n)\big) = f(n+1) = k(f)+n+1$ while $g\big(h_l(n)\big) = g(n) \leq k(g)+n$, a contradiction.

Now, if $t < n$, then $f(t) = f\big(h_n(t)\big) = g\big(h_n(t)\big) = g(t)$. And, if $t > n$, then $f(t) = f\big(h_n(t-1)\big) = g\big(h_n(t-1)\big) = g(t)$. Finally, $f(n) = k(f)+n = k(g)+n = g(n)$, so $f = g$, a contradiction. □

For the rest of this section we will fix a countable left cancellative semigroup S with identity 1 and $p \in S^*$ which is right cancelable in βS. We begin a construction in S based on a similar construction for countable groups in [4, Section 8.5]. We assume that we have enumerated S as $\langle s_n \rangle_{n=1}^{\infty}$ with $s_1 = 1$. For $s, t \in S$ we write $s \prec t$ if and only if $s = s_i$, $t = s_j$, and $i < j$. Of course, $s \preceq t$ means $s \prec t$ or $s = t$. When we write $FP(\{t \in S : t \preceq b\})$ we mean all products of the form $t_1 t_2 \cdots t_m$ such that $t_1 \prec t_2 \prec \ldots \prec t_m \preceq b\}$.

Definition 4.2. M is a *strongly discrete function for* p if and only if $M : S \to p$, $1 \notin M(x)$ for any x, and if x and y are distinct members of S, then $xM(x) \cap yM(y) = \emptyset$.

Note that by Lemma 2.6, p is strongly discrete, so there exists a strongly discrete function for p. We do not fix a particular strongly discrete function for p because we will use two such functions in the proof of Theorem 4.12.

Definition 4.3. Let M be a strongly discrete function for p, let $k \in \mathbb{N}$, and let $\langle b_1, b_2, \ldots, b_k \rangle \in M(1)^k$. Then $x \in S$ is the *M-product of* $\langle b_1, b_2, \ldots, b_k \rangle$ if and only if

(i) $x = b_1 b_2 \cdots b_k$;
(ii) if $1 \le i < j \le k$, then $b_i \prec b_j$; and
(iii) if $i \in \{2, 3, \ldots, k\}$ and $s \in FP\{t \in S : t \preceq b_{i-1}\}$, then $b_i \in M(s)$ and $sb_i \neq 1$.

We say that *x is an M-product* provided there is some $\langle b_1, b_2, \ldots, b_k \rangle$ such that x is the M-product of $\langle b_1, b_2, \ldots, b_k \rangle$. We note that we require an M-product to satisfy more stringent conditions than in [8, §4.3] wherein the only requirement on $\langle b_1, b_2, \ldots, b_k \rangle$ was that for $t \in \{2, 3, \ldots, k\}$, $b_t \in M(b_1 b_2 \cdots b_{t-1})$. We need the stronger requirements for our proofs.

Definition 4.4. Let M be a strongly discrete function for p. $T_M = \{x : x$ is an M-product$\}$.

Lemma 4.5. *Let M be a strongly discrete function for p. Assume that $s \in S \setminus \{1\}$, x is the M-product of $\langle b_1, b_2, \ldots, b_m \rangle$, y is the M-product of $\langle c_1, c_2, \ldots, c_n \rangle$, $sb_1 \neq 1$, $s \prec b_1$, $b_1 \in M(s)$, and $sx = y$. Then $n > m$ and, if $k = n - m$, then $s = c_1 c_2 \cdots c_k$ and for $i \in \{1, 2, \ldots, m\}$, $b_i = c_{k+i}$.*

Proof. Suppose the conclusion fails and pick a counterexample with $n + m$ as small as possible. If $m = n = 1$, then $sb_1 = c_1$ so $sM(s) \cap 1M(1) \neq \emptyset$ and thus $s = 1$, a contradiction.

Suppose next that $m = 2$ and $n = 1$. Let $u = sb_1$. Then

$$u \in FP\{t \in S : t \preceq b_1\}$$

so $b_2 \in M(u)$ and thus $uM(u) \cap 1M(1) \neq \emptyset$ and so $sb_1 = 1$, a contradiction.

Now assume that $m > 2$ and $n = 1$. Let $u = sb_1 \cdots b_{m-1}$. Then

$$uM(u) \cap 1M(1) \neq \emptyset$$

so $u = 1$. But $sb_1 \cdots b_{m-2} \in FP\{t \in S : t \preceq b_{m-2}\}$ so $(sb_1 \cdots b_{m-2})b_{m-1} \neq 1$.

We thus have that $n > 1$. If $m = 1$, let $v = c_1 c_2 \cdots c_{n-1}$. Then

$$sM(s) \cap vM(v) \neq \emptyset$$

so $s = v$ and by left cancellation, $b_1 = c_n$. Thus with $k = n-1$ the conclusion of the lemma holds.

Finally assume that $m > 1$ and $n > 1$. Let $u = sb_1 \cdots b_{m-1}$ and $v = c_1 c_2 \cdots c_{n-1}$. Then $uM(u) \cap vM(v) \neq \emptyset$ so $u = v$. Let $k = (n-1)-(m-1) = n - m$. By the minimality of $n + m$, we have that $s = c_1 c_2 \cdots c_k$ and for $i \in \{1, 2, \ldots, m-1\}$, $b_i = c_{k+i}$. By left cancellation, $b_m = c_n = c_{k+m}$. □

Lemma 4.6. *Let M be a strongly discrete function for p. Then M-products are unique.*

Proof. Assume that x is the M-product of $\langle d_1, d_2, \ldots, d_m \rangle$ and x is the M-product of $\langle c_1, c_2, \ldots, c_n \rangle$. We need to show that $m = n$ and for $i \in \{1, 2, \ldots, n\}$, $d_i = c_i$. We may assume without loss of generality that $m \leq n$. Suppose first that $m = 1$ and $n > 1$ and let $v = c_1 c_2 \cdots c_{n-1}$. Then $d_1 \in 1M(1) \cap vM(v)$ so $v = 1$. If $n = 2$, then $v = c_1 \neq 1$ because $c_1 \in M(1)$. If $n > 2$, then $v \neq 1$ because $(c_1 c_2 \cdots c_{n-2})c_{n-1} \neq 1$.

Thus we have that $m > 1$. Let $s = d_1$ and for $i \in \{1, 2, \ldots, m-1\}$, let $b_i = d_{1+i}$. Then by Lemma 4.5, if $k = n - (m-1)$, we have $d_1 = c_1 c_2 \cdots c_k$ and for $i \in \{1, 2, \ldots, m-1\}$, $b_i = c_{k+i}$. If $k = 1$, this is the conclusion we are after, so suppose $k > 1$. Let $v = c_1 c_2 \cdots c_{k-1}$. Then $d_1 \in 1M(1) \cap vM(v)$ so $v = 1$. But this yields a contradiction just as in the previous paragraph. □

Definition 4.7. Let M be a strongly discrete function for p. Define $h_M : T_M \to \mathbb{N}$ and $\varphi_M : T_M \to \mathbb{N}$ as follows.

(a) If x is the M-product of $\langle b_1, b_2, \ldots, b_m \rangle$ then $h_M(x) = m$.

(b) If x is the M-product of $\langle b_1, b_2, \ldots, b_m \rangle$ and for $j \in \{1, 2, \ldots, m\}$, $b_j = s_{t(j)}$, then $\varphi_M(x) = \sum_{j=1}^{m} 2^{t(j)}$.

By Lemma 4.6, h_M and φ_M are well defined. We denote by $\widetilde{h_M}$ and $\widetilde{\varphi_M}$ the continuous extensions of these functions taking $\overline{T_M}$ to $\beta\mathbb{N}$.

Definition 4.8. Let M be a strongly discrete function for p and let $n \in \mathbb{N}$.

(a) $T_{M,n} = \{x : x$ is the M-product of $\langle b_1, b_2, \ldots, b_m \rangle$ and for all $s \in FP(\langle s_i \rangle_{i=1}^{n})$, $s \prec b_1$, $sb_1 \neq 1$, and $b_1 \in M(s)\}$.

(b) $T_{M,\infty} = \bigcap_{n=1}^{\infty} \overline{T_{M,n}}$.

Lemma 4.9. *Let M be a strongly discrete function for p. Then $T_{M,\infty}$ is a compact subsemigroup of S^*, $p \in T_{M,\infty}$, the restriction of $\widetilde{h_M}$ to $T_{M,\infty}$ is a homomorphism with $\widetilde{h_M}(p) = 1$, and the restriction of $\widetilde{\varphi_M}$ to $T_{M,\infty}$ is a homomorphism.*

Proof. Since $M(1) \subseteq T_M$ and h_M is constantly equal to 1 on $M(1)$, we have that $\widetilde{h_M}(p) = 1$. To see that $p \in T_{M,\infty}$ (and thus that $T_{M,\infty} \neq \emptyset$), let $n \in \mathbb{N}$. Then $\bigcap\{M(s) : s \in FP(\langle s_i\rangle_{i=1}^n)\} \in p$ and for each $s \in FP(\langle s_i\rangle_{i=1}^n)$, $\{t \in S : t \preceq s\}$ is finite and $\{b \in S : sb = 1\}$ has at most one member. The remainder is a member of p which is contained in $T_{M,n}$.

To see that $T_{M,\infty}$ is a subsemigroup of S^*, the restriction of $\widetilde{h_M}$ to $T_{M,\infty}$ is a homomorphism, and the restriction of $\widetilde{\varphi_M}$ to $T_{M,\infty}$ is a homomorphism, it suffices by [4, Theorems 4.20 and 4.21] to let $n \in \mathbb{N}$, let $x \in T_{M,n}$, and show that there is some $m \in \mathbb{N}$ such that for all $y \in T_{M,m}$, $xy \in T_{M,n}$, $h_M(xy) = h_M(x) + h_M(y)$, and $\varphi_M(xy) = \varphi_M(x) + \varphi_M(y)$.

Let $n \in \mathbb{N}$, let $x \in T_{M,n}$, and pick $\langle b_1, b_2, \ldots, b_k\rangle \in M(1)^k$ such that x is the M-product of $\langle b_1, b_2, \ldots, b_k\rangle$, for all $s \in FP(\langle s_i\rangle_{i=1}^k)$, $s \prec b_1$, $sb_1 \neq 1$, and $b_1 \in M(s)$. Pick $m \in \mathbb{N}$ such that $b_k = s_m$. Let $y \in T_{M,m}$ and pick $\langle c_1, c_2, \ldots, c_l\rangle \in M(1)^l$ such that y is the M-product of $\langle c_1, c_2, \ldots, c_l\rangle$, for all $s \in FP(\langle s_i\rangle_{i=1}^l)$, $s \prec c_1$, $sc_1 \neq 1$, and $c_1 \in M(s)$. Then xy is the M-product of $\langle b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_l\rangle$, $xy \in T_{M,n}$, $h_M(xy) = h_M(x)+h_M(y)$, and $\varphi_M(xy) = \varphi_M(x) + \varphi_M(y)$. □

For the statement of the following theorem, we remind the reader of our standing hypothesis about S and p.

Theorem 4.10. *Let S be a countable left cancellative semigroup with identity and let $p \in S^*$ be right cancelable in βS. There is some $q \in \{2^n : n \in \mathbb{N}\}^*$ such that $C_q \subseteq \beta\mathbb{N}$ is topologically isomorphic to C_p.*

Proof. Since p is strongly discrete, pick a strongly discrete function M for p. By Lemma 4.6, φ_M is injective on T_M so $\widetilde{\varphi_M}$ is injective on $T_{M,\infty}$. By Lemma 4.9, $C_p \subseteq T_{M,\infty}$ and $\widetilde{\varphi_M}$ is a homomorphism on C_p. Let $q = \widetilde{\varphi_M}(p)$. □

Lemma 4.11. *Let M be a strongly discrete function for p. If $x \in \beta S \setminus (T_{M,\infty} \cup \{1\})$, then $xT_{M,\infty} \cap T_{M,\infty} = \emptyset$.*

Proof. Let $x \in \beta S \setminus \{1\}$ and assume that $xT_{M,\infty} \cap T_{M,\infty} \neq \emptyset$. We shall show that $x \in T_{M,\infty}$. Pick $y \in T_{M,\infty}$ such that $xy \in T_{M,\infty}$. Let $n \in \mathbb{N}$. We shall show that $T_{M,n} \in x$.

Now $\{s \in S \setminus \{1\} : s^{-1}T_{M,n} \in y\} \in x$ so it suffices that

$$\{s \in S \setminus \{1\} : s^{-1}T_{M,n} \in y\} \subseteq T_{M,n}\,.$$

So let $s \in S \setminus \{1\}$ such that $s^{-1}T_{M,n} \in y$. Pick $r \in \mathbb{N}$ such that $s = s_r$. Then $T_{M,r} \in y$ so pick $z \in s^{-1}T_{M,n} \cap T_{M,r}$. Pick $l \in \mathbb{N}$ and $\langle c_1, c_2, \ldots, c_l\rangle \in M(1)^l$ such that $sb_1 \cdots b_m$ is the M-product of $\langle c_1, c_2, \ldots, c_l\rangle$ and for all $t \in FP(\langle s_i\rangle_{i=1}^r)$, $t \prec c_1$, $tc_1 \neq 1$, and $c_1 \in M(t)$. By Lemma 4.5, $s = c_1c_2 \cdots c_k$ for some $k \in \mathbb{N}$ so that $s \in T_{M,n}$ as required. □

Again, we remind the reader of our standing hypotheses.

Theorem 4.12. *Let S be a countable left cancellative semigroup with identity and let $p \in S^*$ be right cancelable in βS. Then $C_p \cap K(\beta S) = \emptyset$.*

Proof. Suppose instead that we have some $q \in C_p \cap K(\beta S)$. Pick a strongly discrete function $N : S \to p$. Now $S^* \cap \bigcap_{s\in S} \overline{N(s)}$ is a nonempty G_δ so by [4, Theorem 3.36] pick $r \neq p$ in $S^* \cap \bigcap_{s\in S} \overline{N(s)}$ and pick $B \in r \setminus p$. For $s \in S$, let $M(s) = N(s) \setminus B$. Then M is a strongly discrete function for p.

Let R be the minimal right ideal of βS with $q \in R$ and pick a minimal left ideal L of βS such that $L \subseteq \beta Sr$. Let u be the identity of $L \cap R$. By [4, Corollary 4.33], $L \subseteq S^*$ so $u \in S^*$. Then $uq = q$ and $q \in T_{M,\infty}$ by Lemma 4.9. Thus, by Lemma 4.11, $u \in T_{M,\infty}$. In particular $T_M \in u$.

Now also $u \in \beta Sr$ and since r is strongly discrete (via the function N) and u is an idempotent, $r \neq u$ so there is some $s \in S \setminus \{1\}$ such that $T_M \in sr$. Pick $t \in s^{-1}T_M \cap B \cap N(s)$. Then $st \in T_M$ so pick $k \in \mathbb{N}$ and $\langle b_1, b_2, \ldots, b_k\rangle \in M(1)^k$ such that st is the M-product of $\langle b_1, b_2, \ldots, b_k\rangle$. If $k = 1$, then $st \in sN(s) \cap 1N(1)$, so $s = 1$, a contradiction. Thus $k > 1$. Let $v = b_1b_2 \cdots b_{k-1}$. Then $st \in sN(s) \cap vN(v)$ so $s = v$ and by left cancellation, $t = b_k$. But then $t \in B$ and $b_k \in M(1) = N(1) \setminus B$, a contradiction. □

5. Long $<_L$-chains

Recall that for idempotents e and f in a semigroup, $e <_L f$ if and only if $ef = e$ and it is not the case that $fe = f$. Equivalently, $e <_L f$ if and only if Se is a proper subset of Sf. Using a result from [6] we show here that one can get chains as long as possible below any nonminimal idempotent in $\beta\mathbb{N}$, provided that one weakens the strictly decreasing requirement at limit ordinals to the requirement that if $\sigma < \tau$, then $p_\tau <_L p_\sigma$. We will use the following extension of Lemma 2.3.

Lemma 5.1. *Let S and T be compact Hausdorff right topological semigroups, let $h : S \to T$ be a continuous surjective homomorphism, and let λ*

be an ordinal. Assume that $\langle u_\sigma\rangle_{\sigma<\lambda}$ is a chain of idempotents in T such that $u_\tau <_L u_\sigma$ whenever $\sigma<\tau<\lambda$ and $u_{\sigma+1}<u_\sigma$ whenever $\sigma+1<\lambda$. Then there is a chain $\langle p_\sigma\rangle_{\sigma<\lambda}$ of idempotents in S such that $p_\tau <_L p_\sigma$ whenever $\sigma<\tau<\lambda$, $p_{\sigma+1}<p_\sigma$ whenever $\sigma+1<\lambda$, and $h(p_\sigma)=u_\sigma$ for every $\sigma<\lambda$.

Proof. Pick by Lemma 2.3 an idempotent $p_0\in S$ such that $h(p_0)=u_0$. Assume that $0<\gamma<\lambda$ and we have chosen $\langle p_\sigma\rangle_{\sigma<\gamma}$ as required.

Case 1. $\gamma=\delta+1$ for some δ. Pick by Lemma 2.3 an idempotent $p_\gamma\in S$ such that $h(p_\gamma)=u_\gamma$ and $p_\gamma<p_\delta$.

Case 2. γ is a limit ordinal. We claim that for each $\sigma<\gamma$,

$$(S+p_\sigma)\cap h^{-1}[\{u_\gamma\}]\neq\emptyset\,.$$

To see this, pick $x\in h^{-1}[\{u_\gamma\}]$. Then $x+p_\sigma\in S+p_\sigma$ and $h(x+p_\sigma)=u_\gamma+u_\sigma=u_\gamma$. Consequently, we have that $h^{-1}[\{u_\gamma\}]\cap\bigcap_{\sigma<\gamma}(S+p_\sigma)$ is a compact right topological semigroup so pick $p_\gamma\in h^{-1}[\{u_\gamma\}]\cap\bigcap_{\sigma<\gamma}(S+p_\sigma)$. □

Theorem 5.2. *Let $\lambda>0$ be an ordinal.*

(1) *If there is a chain $\langle p_\sigma\rangle_{\sigma<\lambda}$ of idempotents in $\beta\mathbb{N}$ such that $p_\tau <_L p_\sigma$ whenever $\sigma<\tau<\lambda$, then $|\lambda|\leq\mathfrak{c}$.*
(2) *If $|\lambda|\leq\mathfrak{c}$, then there is a chain $\langle p_\sigma\rangle_{\sigma<\lambda}$ of idempotents in $\beta\mathbb{N}$ such that $p_\tau <_L p_\sigma$ whenever $\sigma<\tau<\lambda$ and $p_{\sigma+1}<p_\sigma$ whenever $\sigma+1<\lambda$.*

Proof. [6, Corollary 3.18]. □

Theorem 5.3. *Let q be a nonminimal idempotent in $\beta\mathbb{N}$ and let $\lambda>0$ be an ordinal such that $|\lambda|\leq\mathfrak{c}$. Then there is a chain $\langle q_\sigma\rangle_{\sigma<\lambda}$ of idempotents in $\beta\mathbb{N}$ such that $q_0=q$, $q_\tau <_L q_\sigma$ whenever $\sigma<\tau<\lambda$, and $q_{\sigma+1}<q_\sigma$ whenever $\sigma+1<\lambda$.*

Proof. Pick by Theorem 5.2 a chain $\langle u_\sigma\rangle_{\sigma<\lambda}$ of idempotents in $\beta\mathbb{N}$ such that $u_\tau <_L u_\sigma$ whenever $\sigma<\tau<\lambda$ and $u_{\sigma+1}<u_\sigma$ whenever $\sigma+1<\lambda$. By [4, Theorem 6.56] pick $r\in\beta\mathbb{N}$ such that $r+q$ is right cancelable in $\beta\mathbb{N}$ and let $p=r+q$. By Lemma 5.1 pick a chain $\langle v_\sigma\rangle_{\sigma<\lambda}$ of idempotents in C_p such that $v_\tau <_L v_\sigma$ whenever $\sigma<\tau<\lambda$, $v_{\sigma+1}<v_\sigma$ whenever $\sigma+1<\lambda$, and $h_p(v_\sigma)=u_\sigma$ for every $\sigma<\lambda$. For $\sigma<\lambda$, let $q_\sigma=q+v_\sigma$. Exactly as in the proof of Lemma 3.4 we see that $q_\tau <_L q_\sigma$ whenever $\sigma<\tau<\lambda$ and $q_{\sigma+1}<q_\sigma$ whenever $\sigma+1<\lambda$. If $q_0=q$, then we are done. Otherwise replace q_0 by q. □

Clearly, Theorem 5.2 implies that $\beta\mathbb{N}$ contains many infinite decreasing chains of principal left ideals. Whether $\beta\mathbb{N}$ contains any infinite increasing chain of semiprincipal left ideals, has been a tantalising open problem for several decades. (See the notes to Chapter 6 of [4] for a discussion of the history of this problem.) It was shown in [2, Corollary 1.8] that $\beta\mathbb{Z}$ contains strictly increasing chains of principal right ideals. (In [2] these were called left ideals because $\beta\mathbb{Z}$ was taken to be left topological rather than right topological.) We show here that one can get such increasing chains generated by idempotents. Recall that the statement that $p <_R q$ means that $p \leq_R q$ and it is not true that $q \leq_R p$ or equivalently that $p + \beta\mathbb{N} \subsetneq q + \beta\mathbb{N}$.

Theorem 5.4. *There is an infinite sequence $\langle p_n \rangle_{n<\omega}$ of idempotents in $\beta\mathbb{N}$ such that $p_n <_R p_{n+1}$ for every $n \in \omega$.*

Proof. We choose an infinite sequence $\langle E_n \rangle_{n<\omega}$ of infinite subsets of ω such that for each n, $E_{n+1} \subseteq E_n$ and $E_n \setminus E_{n+1}$ is infinite. Given $x \in \mathbb{N}$, let $\mathrm{supp}(x)$ be the finite subset of ω such that $x = \sum_{t\in \mathrm{supp}(x)} 2^t$. For each $n \in \omega$ we let

$$H_n = \textstyle\bigcap_{k=0}^{\infty} c\ell_{\beta\mathbb{N}}\{x \in \mathbb{N} : \mathrm{supp}(x) \subseteq E_n \text{ and } \min \mathrm{supp}(x) > k\}\,.$$

For each $n \in \mathbb{N}$ define $\mu_n : \omega \to \omega$ by $\mu_n(x) = \sum_{t\in E_n\cap \mathrm{supp}(x)} 2^t$, where $\sum_{t\in\emptyset} 2^t = 0$. Using [4, Theorem 4.20] we see that each H_n is a compact subsemigroup of $\mathbb{H}$ and it is routine to verify that $H_n \setminus H_{n+1}$ is an ideal of H_n. Using [4, Theorem 4.21] we see that for each $n \in \mathbb{N}$, the restriction of μ_n to $\mathbb{H}$ is a homomorphism. Note also that for all $x \in \omega$ and all $m \leq n$, $\mu_m\big(\mu_n(x)\big) = \mu_n(x) = \mu_n\big(\mu_m(x)\big)$ so $\widetilde{\mu_m} \circ \widetilde{\mu_n} = \widetilde{\mu_n} = \widetilde{\mu_n} \circ \widetilde{\mu_m}$.

Next we claim that if p is a minimal idempotent in H_0 and $m \in \mathbb{N}$, then $\widetilde{\mu_m}(p)$ is a minimal idempotent in H_m. To see this, let $A = \{x \in \mathbb{N} : \mathrm{supp}(x) \cap E_m \neq \emptyset\}$ and let $B = \overline{A} \cap H_0$. Then B is an ideal of H_0 and $\widetilde{\mu_m}[B] = H_m$. Since B is an ideal of H_0, $K(H_0) \subseteq B$ and therefore by [4, Theorem 1.65], $K(B) = K(H_0)$. Then by [4, Exercise 1.7.3], $\widetilde{\mu_m}[K(H_0)] = K(H_m)$ and so $\widetilde{\mu_m}(p) \in K(H_m)$ as required.

Now pick a minimal idempotent q of H_0. Let $r_0 = q$ and inductively for $n \in \mathbb{N}$, let $r_n = \widetilde{\mu_n}(q) + r_{n-1}$. Then $r_n = \widetilde{\mu_n}(q) + \widetilde{\mu_{n-1}}(q) + \ldots + \widetilde{\mu_1}(q) + q$ so if $1 \leq m < n$, then

$$\begin{aligned}\widetilde{\mu_m}(r_n) &= \widetilde{\mu_m}\big(\widetilde{\mu_n}(q)\big) + \widetilde{\mu_m}\big(\widetilde{\mu_{n-1}}(q)\big) + \ldots + \widetilde{\mu_m}\big(\widetilde{\mu_1}(q)\big) + \widetilde{\mu_m}(q)\\ &= \widetilde{\mu_n}(q) + \widetilde{\mu_{n-1}}(q) + \ldots + \widetilde{\mu_m}(q)\end{aligned}$$

since $\widetilde{\mu_m}\big(\widetilde{\mu_t}(q)\big) = \widetilde{\mu_m}(q)$ if $t \leq m$ and $\widetilde{\mu_m}(q)$ is an idempotent.

Consequently if $m < n$ we have $\widetilde{\mu_m}(r_n) + r_{m-1} = r_n$. Now if $m < t \leq n$, then $\widetilde{\mu_t}(q) \in H_t \subseteq H_m$ so $\widetilde{\mu_m}(r_n) \in H_m + \widetilde{\mu_m}(q)$ which is a (compact) minimal left ideal of H_m. Let x be a cluster point of the sequence

$\langle r_n \rangle_{n=1}^{\infty}$. Then for each $m \in \mathbb{N}$, $\widetilde{\mu_m}(x) \in H_m + \widetilde{\mu_m}(q) \subseteq K(H_m)$ and $\widetilde{\mu_m}(x) + r_{m-1} = x$. We then have that $x \in \bigcap_{m=1}^{\infty}(\widetilde{\mu_m}(x) + H_0)$ so, being nonempty, $\bigcap_{m=1}^{\infty}(\widetilde{\mu_m}(x) + H_0)$ is a right ideal of H_0 so by [4, Theorem 2.7] we may pick an idempotent $p_0 \in \bigcap_{m=1}^{\infty}(\widetilde{\mu_m}(x) + H_0)$ which is minimal in H_0. Then for each $m \in \mathbb{N}$, $p_0 \leq_R \widetilde{\mu_m}(x)$ and so $\widetilde{\mu_m}(p_0) \leq_R \widetilde{\mu_m}(x)$. We have seen that $\widetilde{\mu_m}(p_0) \in K(H_m)$ and $\widetilde{\mu_m}(x) \in K(H_m)$ so $\widetilde{\mu_m}(p_0) + H_m = \widetilde{\mu_m}(x) + H_m$ and therefore $\widetilde{\mu_m}(x) \leq_R \widetilde{\mu_m}(p_0)$ and thus $p_0 \leq_R \widetilde{\mu_m}(p_0)$. Thus if $k < m$, $\widetilde{\mu_k}(p_0) \leq_R \widetilde{\mu_k}\big(\widetilde{\mu_m}(p_0)\big) = \widetilde{\mu_m}(p_0)$. For each $n \in \mathbb{N}$, let $p_n = \widetilde{\mu_n}(p_0)$. We then have that for each $n \in \omega$, $p_n \leq_R p_{n+1}$. To complete the proof we need to show that it is not the case that $p_{n+1} \leq_R p_n$. We have noted that $H_n \setminus H_{n+1}$ is an ideal of H_n so $K(H_n) \cap H_{n+1} = \emptyset$. If we had $p_{n+1} = p_n + p_{n+1}$, we would have $p_{n+1} \in K(H_n)$. □

We conclude by listing some open questions. In [9, Corollary 5] it was shown that if G is a countable discrete group, then Martin's Axiom implies that there is an idempotent in $p \in G^*$ which is minimal and $\leq_L$-maximal. (That is, there does not exist q such that $p <_L q$.) In particular, p is minimal and maximal with respect to $<$.

Questions 5.5. (1) Is there a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma < \omega_1}$ of idempotents in $\beta\mathbb{N}$?

(2) Can one show in ZFC that there is a minimal idempotent q in $\beta\mathbb{N}$ which is also maximal?

(3) Can it be shown in ZFC that maximal idempotents in $\beta\mathbb{N}$ exist?

(4) Are all semigroups of the form C_q, where $q \in \{2^n : n \in \mathbb{N}\}^*$, isomorphic?

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