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THE NUMBER OF MINIMAL LEFT AND MINIMAL RIGHT IDEALS IN βS

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Abstract. Given an infinite discrete semigroup S, its Stone-Čech compactification βS has a natural operation extending that of S and making βS into a compact right topological semigroup. As such, βS has a smallest two sided ideal $K(\beta S)$, which is the union of all of the minimal left ideals and is the union of all of the minimal right ideals. It has been known that some weak cancellation assumptions on S guarantee the existence of many minimal left ideals and many minimal right ideals. We present here a couple of new results in that direction, but we are primarily interested in providing information about the existence of a large number of minimal right or minimal left ideals in an arbitrary semigroup (with no cancellation assumptions). For example, we show that for any infinite semigroup S, one of the following three statements holds: (1) S has a finite ideal, in which case $K(\beta S) \subseteq S$ and is finite; (2) βS has at least $2^{\mathfrak{c}}$ minimal left ideals; or (3) βS has at least $2^{\mathfrak{c}}$ minimal right ideals.

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1. Introduction

Given a discrete semigroup S, we take the Stone-Čech compactification βS of S to consist of the ultrafilters on S, identifying a point $s \in S$ with the principal ultrafilter $e(s) = \{A \subseteq S : s \in A\}$. Given a subset A of S we identify $c\ell_{\beta S} A$ with βA.

The operation on S extends to βS (and is usually denoted by the same symbol, or simply by juxtaposition) making βS a compact right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$ the function $\lambda_x : \beta S \to \beta S$ is continuous, where for $q \in \beta S$, $\rho_p(q) = qp$ and $\lambda_x(q) = xq$.

As is true of any compact Hausdorff right topological semigroup or, indeed, of any semigroup which contains a minimal left ideal with an idempotent, βS has a smallest two sided ideal $K(\beta S)$ which is the union of all of the minimal left ideals of βS and is also the union of all of the minimal right ideals of βS. Any two minimal left ideals are isomorphic as are any two minimal right ideals. The intersection of any minimal left ideal with any minimal right ideal is a group and any two such groups are isomorphic. See [8] for an elementary introduction to the topological and algebraic structure of βS.

The structure of the smallest ideal of βS has had substantial combinatorial applications. (See [8, Part III] for many of these.) But in this paper we will be interested only in studying the algebraic structure itself. In particular, we will be interested in determining how many minimal left ideals and how many minimal right ideals can be found in βS for a given infinite semigroup S. Note that the intersection of two left ideals is a left ideal if it is nonempty, so distinct minimal left ideals are disjoint. The corresponding statement is true for right ideals.

The first results in this direction were obtained by C. Chou. In [4], he obtained results about left invariant subsets of βS which immediately imply that, if S is an infinite cancellative semigroup, then βS has at least $2^{\mathfrak{c}}|S|$ minimal left ideals. This was before the algebra of βS had been defined! In [1] J. Baker and P. Milnes proved that $(\beta\mathbb{N}, +)$ has $2^{\mathfrak{c}}$ minimal right ideals. Since $|\beta\mathbb{N}| = 2^{\mathfrak{c}}$, this result is best possible. (When we mention $\beta\mathbb{N}$ without specifying the operation, we will always assume that operation is addition.)

These results were later generalized to semigroups satisfying certain cancellation assumptions. Given a set X we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.

Definition 1.1. Let S be a semigroup.

(1) If $a, b \in S$, then $X_{a,b} = \{x \in S : ax = b\}$.
(2) If $a, b \in S$, then $Y_{a,b} = \{x \in S : xa = b\}$.
(3) A is a *left solution set* if and only if there exist $a, b \in S$ such that $A = X_{a,b}$.
(4) A is a *right solution set* if and only if there exist $a, b \in S$ such that $A = Y_{a,b}$.

Notice that S is left cancellative if and only if each left solution set has at most one member and S is right cancellative if and only if each right solution set has at most one member.

Definition 1.2. Let S be an infinite semigroup and let $\kappa = |S|$.

(1) S is *weakly left cancellative* if and only if each left solution set in S is finite.
(2) S is *weakly right cancellative if* and only if each right solution set in S is finite.
(3) S is *very weakly left cancellative* if and only if the union of fewer than κ left solution sets has cardinality less than κ.
(4) S is *very weakly right cancellative* if and only if the union of fewer than κ right solution sets has cardinality less than κ.

Note that if κ is regular, then S is very weakly left cancellative if and only if every left solution set is smaller than κ. It is a fact [8, Theorem 4.36] that $S^* = \beta S \setminus S$ is an ideal of βS if and only if S is both weakly left cancellative and weakly right cancellative.

Theorem 1.3. *If S is an infinite cancellative semigroup, then βS has at least $2^{\mathfrak{c}}$ minimal right ideals.*

Proof. [8, Corollary 6.41]. $\square$

Theorem 1.4. *If S is a very weakly left cancellative infinite semigroup with cardinality κ, then βS has 2^{2^κ} minimal left ideals.*

Proof. [3, Theorem 1.7]. $\square$

It is well-known that, if S is a left amenable semigroup, then every minimal left ideal in βS is the support of a left invariant mean on S. (See, for example, [5, Proposition 9.16].) Hence, if S is infinite, left amenable and very weakly left cancellative, the cardinality of the set of left invariant means on βS is $2^{2^{|S|}}$.

In [12], Y. Zelenyuk showed that, if G is an infinite abelian group, then βG contains $2^{2^{|G|}}$ minimal right ideals. It is an open question whether this is true for arbitrary infinite groups.

We shall need the following simple and well-known lemma. We include a proof because we do not have a reference.

Lemma 1.5. *Let S be a semigroup which contains a minimal left ideal with an idempotent. Let T be a subsemigroup of S which also contains a minimal left ideal with an idempotent and which meets $K(S)$. Then the following statements hold:*

1. $K(T) = T \cap K(S)$*;*
2. *The minimal left ideals of T are precisely the non-empty sets of the form $T \cap L$, where L denotes a minimal left ideal of S;*
3. *The minimal right ideals of T are precisely the non-empty sets of the form $T \cap R$, where R denotes a minimal right ideal of S;*
4. *If T is an ideal of S, $K(T) = K(S)$.*

Proof. (1) [8, Theorem 1.65].

(2) We note that, for every idempotent $e \in T$, $Se \cap T = Te$. To see this, let $x \in Se \cap T$. Then $x = xe \in Te$. So $Se \cap T \subseteq Te$, and the reverse inclusion is obvious.

Suppose that L is a minimal left ideal of S. If $T \cap L \neq \emptyset$, it is a left ideal in T and therefore contains an idempotent $e \in K(T)$ by [8, Corollary 1.47 and Theorem 1.56]. By (a), $e \in K(S)$. So $L \cap T = Se \cap T = Te$, a minimal left ideal of T.

Now suppose that M is a minimal left ideal of T. Then M contains an idempotent $f \in K(T)$. Since $f \in K(S)$, Sf is a minimal left ideal of S and $Sf \cap T = Tf = M$.

(3) This proof is the right-left switch of the proof of (2).

(4) If T is an ideal of S, then $K(S) \subseteq T$. So $K(T) = T \cap K(S) = K(S)$. □

In Section 2 we provide characterizations of semigroups which have many minimal right or minimal left ideals. Most of these results do not involve cancellation assumptions. Notice that one could not hope to have any cancellation condition as necessary for the existence of many minimal left or right ideals because of the following simple observation (in which T can be as badly behaved as one wishes).

Lemma 1.6. *Let T and M be disjoint semigroups and let $S = T \cup M$. For $x \in T$ and $y \in M$, define $xy = yx = y$. Then S is a semigroup. The minimal left ideals of βS are the minimal left ideals of βM and the minimal right ideals of βS are the minimal right ideals of βM.*

Proof. Since M is an ideal of S, βM is a ideal of βS by [8, Corollary 4.18]. In particular, every minimal left ideal of βS meets βM and every minimal right ideal of βS meets βM. So our claim follows from Lemma 1.5. □

We investigate the condition $K(\beta S) \subseteq S^*$ in Section 3.

If κ is a singular cardinal and one defines $\alpha \vee \delta = \max\{\alpha, \delta\}$, then the semigroup $(\kappa, \vee)$ is not very weakly left cancellative. (For $\alpha < \kappa$, $X_{\alpha,\alpha} = \{\delta < \kappa : \delta \leq \alpha\}$ and so, if A is cofinal in κ, then $\bigcup_{\alpha \in A} X_{\alpha,\alpha} = \kappa$.) In Section 4 of this paper we introduce the notion of *extremely weakly left cancellative* semigroups, a property which is satisfied by $(\kappa, \vee)$, and show that the conclusion of Theorem 1.4 remains valid for such semigroups.

2. Numbers of Minimal Left and Minimal Right Ideals

In contrast with earlier results, most of the results of this section make no use of cancellation assumptions. An important consideration is whether all or some of $K(\beta S)$ lies in S^*, an issue addressed by some of the next few lemmas.

Lemma 2.1. *Let S be an infinite semigroup and let $\omega \leq \kappa \leq |S|$. If $(\forall F \subseteq S)\big(|F| < \kappa \Rightarrow (\exists t \in S)(Ft \cap F = \emptyset)\big)$, then $(\forall F \subseteq S)(|F| < \kappa \Rightarrow |\{t \in S : Ft \cap F = \emptyset\}| \geq \kappa)$.*

Proof. Let $F \subseteq S$ with $|F| < \kappa$, let $G = \{t \in S : Ft \cap F = \emptyset\}$, and suppose that $|G| < \kappa$. Let $H = F \cup G \cup GG$. Then $|H| < \kappa$ so pick $t \in S$ such that $Ht \cap H = \emptyset$. Then $Ft \cap F = \emptyset$ so $t \in G$. But then, $tt \in Ht \cap H$, a contradiction. □

Lemma 2.2. *Let S be an infinite semigroup. The following statements are equivalent.*

(a) *There is a left ideal of βS contained in S^*.*
(b) $\big(\forall F \in \mathcal{P}_f(S)\big)(\exists t \in S)(Ft \cap F = \emptyset)$.
(c) $\big(\forall F \in \mathcal{P}_f(S)\big)(\{t \in S : Ft \cap F = \emptyset\}$ *is infinite*$)$.

Proof. (a) implies (b). Pick a left ideal L of S^* such that $L \subseteq S^*$ and pick $p \in L$. Let $F \in \mathcal{P}_f(S)$. For each $x \in F$, $xp \in S^*$ so $S \setminus F \in xp$ and thus $x^{-1}(S \setminus F) \in p$. Pick $t \in \bigcap_{x \in F} x^{-1}(S \setminus F)$.

That (b) implies (c) follows from Lemma 2.1.

(c) implies (a). Inductively on $|F|$ pick for each $F \in \mathcal{P}_f(S)$, some $t_F \in S \setminus (F \cup \{t_H : \emptyset \neq H \subsetneq F\})$ such that $Ft_F \cap F = \emptyset$. Let $\mathcal{A} = \big\{\{t_F : F \in \mathcal{P}_f(S) \text{ and } H \subseteq F\} : H \in \mathcal{P}_f(S)\big\}$. We claim that $\mathcal{A}$ has the infinite finite intersection property. So let $\mathcal{H}$ be a finite nonempty subset of $\mathcal{P}_f(S)$, let $H = \bigcup \mathcal{H}$ and pick an injective sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $S \setminus H$. For each n, let $F_n = H \cup \{x_1, x_2, \ldots x_n\}$. Then $\{t_{F_n} : n \in \mathbb{N}\}$ is an infinite subset of $\bigcap_{G \in \mathcal{H}} \{t_F : F \in \mathcal{P}_f(S) \text{ and } G \subseteq F\}$. Pick by [8, Corollary 3.14] some $p \in S^*$ such that $\mathcal{A} \subseteq p$.

We claim that $\beta Sp \subseteq S^*$ for which it suffices that $Sp \subseteq S^*$. Let $x \in S$ and suppose that $xp = y \in S$. (That is, $e(x)p = e(y)$.) Then $x^{-1}\{y\} \in p$. Also, $\{t_F : F \in \mathcal{P}_f(S) \text{ and } \{x, y\} \subseteq F\} \in p$. So pick $F \in \mathcal{P}_f(S)$ such that $\{x, y\} \subseteq F$ and $t_F \in x^{-1}\{y\}$. Then $xt_F = y$. So $Ft_F \cap F \neq \emptyset$, a contradiction. □

Lemma 2.3. *Let S be a discrete semigroup. If $I = K(\beta S) \cap S$ is nonempty, then $K(\beta I) = K(\beta S)$.*

Proof. Since I is an ideal of S, it follows from [8, Corollary 4.18] that $\beta I = c\ell(I)$ is an ideal of βS. So our claim follows from Lemma 1.5. □

Conclusion (2) in the following lemma is important because many of the results about the structure of semigroups have that statement as hypothesis.

Lemma 2.4. *Let S be an infinite semigroup and assume that $I = K(\beta S) \cap S \neq \emptyset$.*

(1) *For all $a, b \in I$, aSb is a finite group and $aSb = a\beta Sb$.*
(2) *S has a minimal left ideal with an idempotent.*
(3) *$I = K(S)$.*
(4) *If S has μ minimal left ideals, where μ is an infinite cardinal, then βS has at least $2^{2^{\mu}}$ minimal left ideals and $K(\beta S)$ meets S^*.*

(5) *If S has μ minimal right ideals, where μ is an infinite cardinal, then βS has at least 2^{2^μ} minimal right ideals and $K(\beta S)$ meets S^*.*
(6) *If $K(\beta S) \subseteq S$, then $K(\beta S)$ is finite.*
(7) *Suppose that I has μ_1 minimal left ideals and μ_2 minimal right ideals. Let $\mu = \max\{\mu_1, \mu_2\}$. If μ is infinite, $|K(\beta S)| = |K(\beta I)| = |\beta I| = 2^{2^\mu}$.*

Proof. (1) Let $a, b \in I$. Let $R = a\beta S$ and $L = \beta S b$. Then R is a minimal right ideal of βS and L is a minimal left ideal of βS. By [8, Theorem 1.61], $RL = R \cap L$ and RL is a group. Then $RL = a\beta S\beta S b \subseteq a\beta S b \subseteq R \cap L = RL$. So $a\beta S b$ is a group. Further, $a\beta S b = \lambda_a\big[\rho_b[\beta S]\big]$ so $a\beta S b$ is compact. As a right topological group $a\beta S b$ is homogeneous. Also $a\beta S b$ has an isolated point (for example aab) so all points of $a\beta S b$ are isolated. Since $a\beta S b$ is compact, we have that $a\beta S b$ is finite. Since $a\beta S b = c\ell(aSb)$ and aSb is finite, we have $a\beta S b = aSb$.

(2) Pick $a \in I$. Then Sa is a left ideal of S and aSa is a group which thus has an idempotent. So it suffices to show that Sa is a minimal left ideal of S. Let L be a left ideal of S with $L \subseteq Sa$. By [8, Corollary 4.18], $c\ell(L)$ is a left ideal of βS and $c\ell(L) \subseteq c\ell(Sa) = \beta S a$ and $\beta S a$ is a minimal left ideal of βS since $a \in K(\beta S)$. So $c\ell(L) = \beta S a$. So $Sa \subseteq c\ell(L) \cap S = L$.

(3) By (2) and [8, Theorem 1.65], $K(S) = K(\beta S) \cap S$.

(4) Assume that S has μ minimal left ideals, where μ is an infinite cardinal. Let $\mathcal{L}$ denote the set of minimal left ideals of S. We give $\mathcal{L}$ the discrete topology and define a semigroup operation $*$ on $\mathcal{L}$ by putting $L_1 * L_2 = L_2$. So $(\mathcal{L}, *)$ is a discrete right zero semigroup. Define a homomorphism $f : I \to \mathcal{L}$ by $f(x) = L$ if $x \in L$.

By Lemma 2.3, $K(\beta S) = K(\beta I)$ so it suffices to show that βI has at least 2^{2^μ} minimal left ideals. Now $\widetilde{f} : \beta I \to \beta\mathcal{L}$ is surjective by [8, Exercise 3.4.1] and a homomorphism by [8, Corollary 4.22]. Also $\beta\mathcal{L}$ is a right zero semigroup by [8, Exercise 4.2.2] and $|\beta\mathcal{L}| = 2^{2^\mu}$. For each $p \in \beta\mathcal{L}$, $\widetilde{f}^{-1}[\{p\}]$ is a left ideal of βI. So there are at least 2^{2^μ} pairwise disjoint left ideals of βI.

To see that $K(\beta S)$ meets S^*, choose any $p \in \mathcal{L}^*$. Since $p \in K(\beta\mathcal{L})$, there exists $x \in K(\beta I)$ for which $\widetilde{f}(x) = p$ by [8, Exercise 1.7.3]. This implies that $x \in I^*$ and hence that $x \in S^*$.

(5) Let $\mathcal{R}$ denote the set of minimal right ideals of S and define an operation $*$ on $\mathcal{R}$ making $(\mathcal{R}, *)$ a left zero semigroup. Then by [8, Exercise 4.2.1], $\beta\mathcal{R}$ is a left zero semigroup, so the rest of the proof is a right-left switch of the proof of (4).

(6) Suppose that $K(\beta S) \subseteq S$. By Lemma 1.5(2) and (3), the minimal left ideals of S and βS and the minimal right ideals of S and βS are the same, so in particular $K(\beta S) = K(S) = I$. By (4) and (5), the number of minimal left ideals of S is finite, and so is the number of minimal right ideals of S. So there are only a finite number of groups of the form RL, where R is a minimal right ideal and L is a minimal left ideal of S. By (1), each of these groups is finite. So $K(\beta S) = K(S)$ is finite.

(7) The number of groups of the form RL, where R is a minimal right ideal of I and L is a minimal left ideal of I, is $\mu_1 \cdot \mu_2 = \mu$. Each of these groups is finite. So $|I| = \mu$ and $|\beta I| = 2^{2^\mu}$. Thus $|K(\beta I)| \leq 2^{2^\mu}$. However, by (4) and (5), $|K(\beta S)| \geq 2^{2^\mu}$ and by Lemma 2.3, $K(\beta S) = K(\beta I)$. $\square$

The following lemma is the exact left-right switch of Lemma 2.2, but the proof is more complicated. (This is a common phenomenon because of the lack of symmetry of the continuity.)

Lemma 2.5. *Let S be an infinite semigroup. The following statements are equivalent.*

(a) *There is a right ideal of βS contained in S^*.*
(b) $\big(\forall F \in \mathcal{P}_f(S)\big)(\exists t \in S)(tF \cap F = \emptyset)$.
(c) $\big(\forall F \in \mathcal{P}_f(S)\big)(\{t \in S : tF \cap F = \emptyset\}$ *is infinite*$)$.

Proof. (a) implies (b). Pick a right ideal $R \subseteq S^*$ and pick $p \in R$. Let $F \in \mathcal{P}_f(S)$. Suppose that for all $t \in S$, $tF \cap F \neq \emptyset$. Then $S \subseteq \bigcup_{a\in F} \bigcup_{b\in F} Y_{a,b}$. So pick $a, b \in F$ such that $Y_{a,b} \in p$. Then ρ_a is constantly equal to b on $Y_{a,b}$. So $\rho_a(p) = b$. Thus $b \in p\beta S \subseteq R$, a contradiction.

(b) implies (c). This is the same as the proof that (b) implies (c) in Lemma 2.2.

(c) implies (a). Assume (c) holds and suppose that every minimal right ideal of βS meets S. Let $I = K(\beta S) \cap S$. By Lemma 2.4, for all $a, b \in I$, $a\beta S b = aSb$ is a finite group. For each $F \in \mathcal{P}_f(I)$, let $H_F = \bigcup_{a\in F} \bigcup_{b\in F} aSb$. Then H_F is finite so pick $t_F \in S$ such that

$t_F H_F \cap H_F = \emptyset$. Direct $\mathcal{P}_f(I)$ by inclusion and let p be a cluster point of the net $\langle t_F \rangle_{F \in \mathcal{P}_f(I)}$. (Note that if I is finite, then $p = t_I$.)

We claim that for any $x \in K(\beta S)$, $px \in S^*$, so let $x \in K(\beta S)$ be given and suppose instead that $px = b \in S$. Then $b \in I$. Pick $c \in bSb$. Then $pxc = bc$. Note also that x is in some minimal right ideal R of βS. Pick $a \in R \cap S$. Then $a\beta S = R$ so $x \in a\beta S$. So $xc \in a\beta Sc = aSc$. Since $pxc = bc$, $\rho_{xc}^{-1}[\{bc\}]$ is a neighborhood of p so one may pick $F \in \mathcal{P}_f(I)$ such that $\{a, b, c\} \subseteq F$ and $t_F \in \rho_{xc}^{-1}[\{bc\}]$. So $t_F xc = bc$. Now $xc \in aSc$ so $xc \in H_F$. Also $bc \in bSb$ so $bc \in H_F$. Thus $t_F xc \in t_F H_F \cap H_F$, a contradiction.

Now pick $x \in K(\beta S)$. Given any $y \in \beta S$, $xy \in K(\beta S)$. So $pxy \in S^*$. Therefore, $px\beta S \subseteq S^*$, a contradiction. □

We thank the referee for bringing the following result to our attention. Recall that $U_\kappa(S)$ is the set of κ-uniform ultrafilters on S, that is those ultrafilters all of whose members have cardinality at least κ.

Theorem 2.6. *Let S be an infinite semigroup and let $\kappa = |S|$. Statements (a), (b), (c), and (d) are equivalent and imply statement (e).*

(a) $(\forall F \subseteq S)(|F| < \kappa \Rightarrow |\{t \in S : Ft \cap F = \emptyset\}| = \kappa)$.
(b) $(\forall F \subseteq S)\big(|F| < \kappa \Rightarrow (\exists t \in S \setminus F)(Ft \cap F = \emptyset)\big)$.
(c) $(\forall F \subseteq S)\big(|F| < \kappa \Rightarrow (\exists t \in S)(Ft \cap F = \emptyset)\big)$.
(d) *S is not the union of fewer than κ left solution sets.*
(e) *βS has 2^{2^κ} minimal left ideals contained in $U_\kappa(S)$.*

Proof. Trivially (a) implies (b) and (b) implies (c). That (c) implies (a) follows from Lemma 2.1.

To see that (d) implies (c) let $F \subseteq S$ with $|F| < \kappa$. Pick $t \in S \setminus \bigcup_{a \in F} \bigcup_{b \in F} X_{a,b}$. To see that (c) implies (d), let $H \subseteq S \times S$ with $|H| < \kappa$. Let $F = \pi_1[H] \cup \pi_2[H]$ and pick $t \in S$ such that $Ft \cap F = \emptyset$. Then $t \in S \setminus \bigcup_{(a,b) \in H} X_{a,b}$.

To see that (e) implies (d) enumerate S as $\langle s_\sigma \rangle_{\sigma < \kappa}$. We inductively choose an injective κ-sequence $\langle t_\sigma \rangle_{\sigma < \kappa}$ so that if $\sigma < \delta < \kappa$, then $s_\sigma t_\delta \notin \{s_\mu t_\eta : \mu < \eta < \delta\}$. Let t_0 and t_1 be any distinct members of S. Now let $\delta < \kappa$ and assume that $\langle t_\sigma \rangle_{\sigma < \delta}$ has been chosen. Let $F = \{s_\sigma : \sigma < \delta\} \cup \{s_\mu t_\eta : \mu < \eta < \delta\}$. Pick $t_\delta \in S \setminus \{t_\sigma : \sigma < \delta\}$ such that $Ft_\delta \cap F = \emptyset$.

Let $M = \{p \in U_\kappa(S) : \{t_\delta : \delta < \kappa\} \in p\}$. By [8, Theorem 3.58], $|M| = 2^{2^\kappa}$. We shall show that if $p, q \in M$ and $p \neq q$, then $\beta Sp \cap \beta Sq = \emptyset$. So let p and q be distinct members of M and pick $B \in p$ and $C \in q$ such that $B \cap C = \emptyset$. Let

$$\begin{aligned} D &= \{s_\sigma t_\delta : \sigma < \delta < \kappa \text{ and } t_\delta \in B\} \text{ and let} \\ E &= \{s_\sigma t_\delta : \sigma < \delta < \kappa \text{ and } t_\delta \in C\}\,. \end{aligned}$$

Then $D \cap E = \emptyset$ so it suffices to show that $\beta Sp \subseteq \overline{D}$ and $\beta Sq \subseteq \overline{E}$. The proofs being identical, we establish the former, for which it suffices to show that $Sp \subseteq \overline{D}$. So let $\sigma < \kappa$ be given. Since $p \in U_\kappa(S)$, $\{t_\delta : \delta \leq \sigma\} \notin p$ and therefore $B \cap \{t_\delta : \delta > \sigma\} \in p$ and so $s_\sigma(B \cap \{t_\delta : \delta > \sigma\}) \in s_\sigma p$ and thus $D \in s_\sigma p$.

Finally, we show that for all $p \in M$, $\beta Sp \subseteq U_\kappa(S)$. Let $p \in M$ be given. Since $\beta Sp = c\ell(Sp)$ and $U_\kappa(S)$ is closed, it suffices to show that $Sp \subseteq U_\kappa(S)$. To this end, let $\sigma < \kappa$ and let $A \in s_\sigma p$. We claim that $|A| = \kappa$. Pick $B \in p$ such that $s_\sigma B \subseteq A$. Then $B \cap \{t_\delta : \sigma < \delta < \kappa\} \in p$ and $\lambda_{s_\sigma} : B \cap \{t_\delta : \sigma < \delta < \kappa\} \xrightarrow{1\text{-}1} A$ so $|A| = \kappa$. □

Corollary 2.7. *Let S be an infinite very weakly left cancellative semigroup of cardinality κ. Then βS has 2^{2^κ} minimal left ideals contained in $U_\kappa(S)$.*

Proof. S trivially satisfies statement (d) of Theorem 2.6. □

We now characterize completely when there are many minimal left ideals in a countable semigroup.

Theorem 2.8. *Let S be a countably infinite semigroup. The following statements are equivalent.*

(a) *There is a left ideal of βS contained in S^*.*
(b) $\big(\forall F \in \mathcal{P}_f(S)\big)(\exists t \in S)(Ft \cap F = \emptyset)$.
(c) $\big(\forall F \in \mathcal{P}_f(S)\big)(\{t \in S : Ft \cap F = \emptyset\}$ *is infinite*).
(d) *βS has infinitely many minimal left ideals.*
(e) *βS has $2^\mathfrak{c}$ minimal left ideals.*
(f) *βS has $2^\mathfrak{c}$ minimal left ideals contained in S^*.*

Proof. Statements (a), (b), and (c) are equivalent by Lemma 2.2.

That (c) implies (f) follows from Theorem 2.6. Trivially, (f) implies (e) and (e) implies (d).

(d) implies (b). Pick infinite $M \subseteq \beta S$ such that if p and q are distinct members of M, then $\beta Sp \cap \beta Sq = \emptyset$. Let $F \in \mathcal{P}_f(S)$. Pick $p \in M \setminus F$ such that $F \cap \beta Sp = \emptyset$. (We have that

$$|\{q \in M : \beta Sq \cap F \neq \emptyset\}| \leq |F|.)$$

Let $A = S \setminus F$. Then for each $x \in F$, $xp \in \beta Sp$ so $A \in xp$. Pick $t \in \bigcap_{x\in F} x^{-1}A$. Then $Ft \cap F = \emptyset$. $\square$

We are able to characterize the existence of many minimal right ideals without assuming the countability of S.

Definition 2.9. Let S be a set. Then $\mathcal{R} = \mathcal{R}(S)$ is the set of finite partitions of S.

Theorem 2.10. *Let S be an infinite semigroup. The following statements are equivalent.*

(a) *βS has infinitely many minimal right ideals.*
(b) *Every left ideal of βS is infinite.*
(c) *Some minimal left ideal of βS is infinite.*
(d) *βS has at least $2^{\mathfrak{c}}$ minimal right ideals.*
(e) $(\forall \mathcal{F} \in \mathcal{R})(\exists \mathcal{G} \in \mathcal{R})(\forall A \in \mathcal{G})(\exists B \in \mathcal{F})(\exists s, t \in B)$ $(sA \cap tA = \emptyset)$.
(f) $(\forall \mathcal{F} \in \mathcal{R})(\exists \mathcal{G} \in \mathcal{R})(\forall A \in \mathcal{G})(\exists B \in \mathcal{F})\big(\exists F \in \mathcal{P}_f(B)\big)$ $(\bigcap_{s\in F} sA = \emptyset)$.

Proof. (a) implies (b). Every left ideal has nonempty intersection with every right ideal and distinct minimal right ideals are disjoint.

That (b) implies (c) is trivial, as is the fact that (d) implies (a).

(c) implies (d). [8, Theorem 6.39].

(b) implies (e). Let $\mathcal{F} \in \mathcal{R}$ be given and suppose that

$$(\forall \mathcal{G} \in \mathcal{R})(\exists A \in \mathcal{G})(\forall B \in \mathcal{F})(\forall s, t \in B)(sA \cap tA \neq \emptyset)\,.$$

Let $\mathcal{A} = \{A \subseteq S : (\exists B \in \mathcal{F})(\exists s, t \in B)(s(S \setminus A) \cap t(S \setminus A) = \emptyset)\}$. We claim that $\mathcal{A}$ has the finite intersection property. To see this, let $\mathcal{H} \in \mathcal{P}_f(\mathcal{A})$ and suppose that $\bigcap \mathcal{H} = \emptyset$. Let $\mathcal{G}$ be the partition of S generated by $\{S \setminus A : A \in \mathcal{H}\}$. Pick $D \in \mathcal{G}$ such that

$$(\forall B \in \mathcal{F})(\forall s, t \in B)(sD \cap tD \neq \emptyset)\,.$$

Pick $x \in D$. Since $\bigcap \mathcal{H} = \emptyset$, pick $A \in \mathcal{H}$ such that $x \in S \setminus A$. Then $D \subseteq S \setminus A$. Pick $B \in \mathcal{F}$ and $s, t \in B$ such that $s(S \setminus A) \cap t(S \setminus A) = \emptyset$. Since $D \subseteq S \setminus A$, this is a contradiction.

Since $\mathcal{A}$ has the finite intersection property, pick $p \in \beta S$ such that $\mathcal{A} \subseteq p$. For $B \in \mathcal{F}$, pick $s_B \in B$. Now $\beta S p$ is infinite so Sp is infinite. Pick $t \in S$ such that $tp \notin \{s_B p : B \in \mathcal{F}\}$ and pick $B \in \mathcal{F}$ such that $t \in B$. Since $tp \neq s_B p$, pick $D \in tp \setminus s_B p$ and let $A = t^{-1}D \cap s_B^{-1}(S \setminus D)$. Then $A \in p$ so $(S \setminus A) \notin \mathcal{A}$. So for all $x, y \in B$, $xA \cap yA \neq \emptyset$. In particular, $tA \cap s_B A \neq \emptyset$. But $tA \subseteq D$ and $s_B A \subseteq (S \setminus D)$, a contradiction.

That (e) implies (f) is trivial.

(f) implies (b). Let L be a left ideal of βS and suppose L is finite. Pick $p \in L$. Then Sp is finite. For $q \in Sp$, let

$$B_q = \{s \in S : sp = q\}$$

and let $\mathcal{F} = \{B_q : q \in Sp\}$. Then $\mathcal{F} \in \mathcal{R}$. Pick $\mathcal{G} \in \mathcal{R}$ such that $(\forall A \in \mathcal{G})(\exists B \in \mathcal{F})(\exists F \in \mathcal{P}_f(B))(\bigcap_{s \in F} sA = \emptyset)$. Pick $A \in \mathcal{G} \cap p$. Pick $q \in Sp$ and $F \in \mathcal{P}_f(B_q)$ such that $\bigcap_{s \in F} sA = \emptyset$. For each $s \in F$, $sp = q$ so $\bigcap_{s \in F} sA \in q$, a contradiction. □

Lemma 2.11. *Let S be an infinite semigroup, let L be a left ideal of βS, and let p be an idempotent in L. If L is finite, then there exists $B \in p$ such that for all $a \in B$, $\{s \in S : as = s\} \in p$.*

Proof. Pick $A \in p$ such that $\overline{A} \cap L = \{p\}$. Since $\rho_p(p) \in \overline{A}$, pick $B \in p$ such that $\rho_p[\,\overline{B}\,] \subseteq \overline{A}$ and let $a \in B$. Then $ap \in \overline{A} \cap \beta S p \subseteq \overline{A} \cap L = \{p\}$. So $ap = p$. Since $\lambda_a(p) = p$, we have by [8, Theorem 3.35] that $\{s \in S : as = s\} \in p$. □

Surprisingly, the following theorem with a strong cancellation assumption appears to be new. We remark that the theorem does not hold if right cancellativity is replaced by weak right cancellativity. If $S = (\mathbb{N}, \vee)$, S is weakly cancellative; but S^* is the unique minimal right ideal of βS. (See [8, Exercise 4.1.11].) This contrasts with theorems about the number of minimal left ideals. We shall see in Theorem 4.8 that, if an infinite semigroup S is extremely weakly left cancellative, then βS has $2^{2^{|S|}}$ minimal left ideals.

Theorem 2.12. *Let S be an infinite right cancellative semigroup. Then βS has at least $2^{\mathfrak{c}}$ minimal right ideals.*

Proof. By Theorem 2.10, it suffices to let L be a minimal left ideal of βS and show that L is infinite. Suppose instead that L is finite and pick an idempotent $p \in L$. By Lemma 2.11, pick $B \in p$ such that for all $a \in B$, $\{s \in S : as = s\} \in p$. Then for each $a \in B$, we have by right cancellation that a is a right identity for S. Thus given $t \in S$, $B \subseteq \{a \in S : ta = t\}$ so $tp = t$. That is, ρ_p is the identity on S, hence on βS. So $\beta S = \beta Sp$ so L is infinite. □

Lemma 2.13. *Let S be an infinite semigroup. Then $K(\beta S) \cap S^* \neq \emptyset$ if and only if S does not contain a finite ideal.*

Proof. Necessity. If I is a finite ideal of S, then by [8, Corollary 4.18] $c\ell I$ is an ideal of βS so $K(\beta S) \subseteq c\ell I = I$.

Sufficiency. If $K(\beta S) \subseteq S$, then by Lemma 2.4(6), $K(\beta S)$ is finite. □

A sufficient condition for $K(\beta S)$ meeting S^* is that S^* should be an ideal of βS, so that $K(\beta S) \subseteq S^*$. We have seen that this is true if and only if S weakly cancellative. Of course, weak cancellativity is not necessary for the condition $K(\beta S) \subseteq S^*$, as can be seen by applying Lemma 1.6.

Theorem 2.14. *Let S be an infinite semigroup and assume that $K(\beta S)$ meets S^*. If βS contains a finite left ideal, then any finite minimal left ideal that meets S^* is contained in S^* and there are at least $2^{\mathfrak{c}}$ minimal left ideals in βS.*

Proof. Let L be a finite minimal left ideal L of βS which meets S^*. To see that $L \subseteq S^*$, suppose that there exists $x \in L \cap S$. Then $L = \beta Sx = c\ell(Sx)$. But Sx is finite and so $c\ell(Sx) = Sx \subseteq S$.

Pick an idempotent $p \in L$. Let $\mu = ||p||$ and note that $\mu \geq \omega$. Pick $C \in p$ such that $|C| = \mu$. Pick by Lemma 2.11 some $B \in p$ such that for all $a \in B$, $\{s \in S : as = s\} \in p$. We may presume $B \subseteq C$. For each $a \in B$, let $P_a = \{s \in C : as = s\}$. Let $\mathcal{A} = \{P_a : a \in B\}$. Then $\mathcal{A} \subseteq p$ so $\mathcal{A}$ has the μ-uniform finite intersection property. So by [8, Theorem 3.62],

$$|\{q \in U_\mu(C) : \mathcal{A} \subseteq q\}| = 2^{2^\mu} .$$

If $q \in U_\mu(C)$ and $\mathcal{A} \subseteq q$, then for all $a \in B$, $aq = q$ and therefore $pq = q$. So $|p\beta S| \geq 2^{2^\mu}$. Since $p \in K(\beta S)$, $p\beta S$ is a minimal right ideal. Since each minimal left ideal is finite and $p\beta S \subseteq K(\beta S) = \bigcup\{M : M \text{ is a minimal left ideal}\}$, one has $|\{M : M \text{ is a minimal left ideal}\}| \geq 2^{2^\mu}$. □

The following theorem tells us that except in a trivial situation, there will be many minimal one sided ideals.

Theorem 2.15. *Let S be an infinite semigroup. At least one of the following statements holds.*

(1) *S has a finite ideal, in which case $K(\beta S) \subseteq S$ and is finite.*
(2) *βS has at least $2^{\mathfrak{c}}$ minimal left ideals.*
(3) *βS has at least $2^{\mathfrak{c}}$ minimal right ideals.*

Proof. If some minimal left ideal of βS is infinite we have by Theorem 2.10 that statement (3) holds. So we assume that the minimal left ideals of S are finite.

If $K(\beta S) \subseteq S^*$, then by Theorem 2.14, statement (2) holds. So we may assume $I = K(\beta S) \cap S \neq \emptyset$. If I is finite, then $c\ell(I)$ is an ideal of βS by [8, Corollary 4.18], so $K(\beta S) \subseteq c\ell(I) = I$ so statement (1) holds.

Thus we assume $I = K(\beta S) \cap S$ is infinite. By Lemma 2.4(3), $I = K(S)$ and for any minimal left ideal L of S and any minimal right ideal R of S, $L \cap R$ is finite. Thus there are either infinitely many minimal left ideals of S or infinitely many minimal right ideals of S. Thus by conclusion (4) or (5) of Lemma 2.4, statement (2) or (3) holds. $\square$

Corollary 2.16. *Let S be an infinite semigroup. Either $K(\beta S)$ is a finite subset of S or $|K(\beta S)| \geq 2^{\mathfrak{c}}$.*

Proof. This is an immediate consequence of Theorem 2.15. $\square$

We now show that we can get any finite number of minimal left ideals or minimal right ideals.

We leave the proof of the following lemma as an exercise.

Lemma 2.17. *Let κ be an infinite cardinal and let $S = (\kappa, \vee)$. Let $C = \{p \in \beta S : (\forall A \in p)(A \text{ is cofinal in } \kappa)\}$ and let $p \in C$.*

(1) *For all $q \in \beta S$, $q \vee p = p$.*
(2) *For all $q \in \beta S \setminus C$, $p \vee q = p$.*
(3) *$C = K(\beta S)$.*

Theorem 2.18. *Let κ be an infinite cardinal and let $n \in \mathbb{N}$. There is a semigroup S such that $|S| = \kappa$, $K(\beta S) \subseteq S^*$, and βS has precisely n minimal right ideals.*

Proof. Let L be a left zero semigroup with $|L| = n$ and let $R = (\kappa, \vee)$. Let $S = L \times R$. Then βS can be viewed as $L \times \beta R$ and by Lemma 2.17, $K(\beta S) = L \times C$, where

$$C = \{p \in \beta R : (\forall A \in p)(A \text{ is cofinal in } \kappa)\}.$$

Then $K(\beta S) \subseteq L \times R^* = S^*$.

For $a \in L$, $\{a\} \times \beta R$ is a right ideal of βS so there exists n pairwise disjoint right ideals (and thus at least n minimal right ideals).

Suppose $M_1, M_2, \ldots, M_{n+1}$ are pairwise disjoint right ideals. For $i \in \{1, 2, \ldots, n+1\}$, pick $(a_i, p_i) \in M_i$. Pick $i \neq j$ such that $a_i = a_j$. Pick $(b, q) \in L \times C$.

Then $(a_i, q) = (a_i, p_i)(b, q) \in M_i$ and $(a_i, q) = (a_j, p_j)(b, q) \in M_j$, a contradiction. □

The corresponding result for left ideals is not as strong, since $K(\beta S)$ is not contained in S^*. We shall see in Theorem 2.21, that βS has at least $2^{\mathfrak{c}}$ minimal left ideals if $K(\beta S) \subseteq S^*$.

Theorem 2.19. *Let κ be an infinite cardinal and let $n \in \mathbb{N}$. There is a semigroup S such that $|S| = \kappa$ and βS has precisely n minimal left ideals.*

Proof. Let L be a left zero semigroup with $|L| = \kappa$ and let R be a right zero semigroup with $|R| = n$. Let $S = L \times R$. Then βS can be identified with $\beta L \times R$ and by [8, Exercise 4.2.2] βL is right zero. As in the previous proof, one easily shows that βS has precisely n minimal left ideals. □

Recall that for an ultrafilter p, $||p|| = \min\{|A| : A \in p\}$. Given S with $|S| \geq \kappa$, $U_\kappa(S) = \{p \in \beta S : ||p|| \geq \kappa\}$. An ultrafilter p on S is *uniform* if and only if $||p|| = |S|$.

Theorem 2.20. *Let S be an infinite semigroup of cardinality κ. If $U_\kappa(S)$ contains a left ideal of βS, then βS has 2^{2^κ} minimal left ideals.*

Proof. Suppose that $U_\kappa(S)$ contains a left ideal L of βS and that $p \in L$. Let $\langle F_\alpha \rangle_{\alpha<\kappa}$ be an enumeration of $\mathcal{P}_f(S)$. We claim that we can choose an injective κ-sequence $\langle t_\alpha \rangle_{\alpha<\kappa}$ in S for which the sets $F_\alpha t_\alpha$ with $\alpha < \kappa$ are pairwise disjoint.

To see this, choose any $t_0 \in S$. Then assume that $\beta < \kappa$ and that we have chosen $t_\alpha \in S$ for every $\alpha < \beta$. Let

$$X = \{t_\alpha : \alpha < \beta\} \cup \bigcup\{F_\alpha t_\alpha : \alpha < \beta\}\,.$$

Then $|X| < \kappa$. For every $s \in S$, $sp \in U_\kappa(S)$ so $X \notin sp$ and thus $s^{-1}X \notin p$. Hence $S \setminus s^{-1}X \in p$. We also have $S \setminus X \in p$. It follows that $(S \setminus X) \cap \bigcap\{(S \setminus s^{-1}X : s \in F_\beta\} \neq \emptyset$. So we can choose t_β in this set.

For each $H \in \mathcal{P}_f(S)$, put $A_H = \{t_\alpha : \alpha < \kappa \text{ and } H \subseteq F_\alpha\}$. It is clear that $|A_H| = \kappa$. Furthermore, for every finite number of sets $H_1, H_2, \cdots, H_n$ in $\mathcal{P}_f(S)$, if $K = \bigcup_{i=1}^n H_i$, then $A_K = \bigcap_{i=1}^n A_{H_i}$ and so $|\bigcap_{i=1}^n A_{H_i}| = 2^{2^\kappa}$.It follows from [8, Theorem 3.62] that, if $\mathcal{A} = \{p \in U_\kappa(S) : \big(\forall H \in \mathcal{P}_f(S)\big)(A_H \in p)\}$, we have $|\mathcal{A}| = 2^{2^\kappa}$.

We shall show that, if p and q are distinct elements of $\mathcal{A}$, the left ideals βSp and βSq of βS are disjoint. To see this, let B and C be disjoint subsets of $\{t_\alpha : \alpha < \kappa\}$ for which $B \in p$ and $C \in q$. Let $D = \bigcup\{F_\alpha t_\alpha : t_\alpha \in B\}$ and $E = \bigcup\{F_\alpha t_\alpha : t\alpha \in C\}$. Then $D \cap E = \emptyset$. For every $s \in S$, $A_{\{s\}} \cap B \in p$. So $\{st_\alpha : t_\alpha \in A_{\{s\}} \cap B\} \in sp$. Since this set is contained in D, $sp \in \overline{D}$. So $Sp \subseteq \overline{D}$ and hence $\beta Sp \subseteq \overline{D}$. Similarly, $\beta Sq \subseteq \overline{E}$. Thus $\beta Sp \cap \beta Sq = \emptyset$. □

Theorem 2.21. *Let S be a semigroup for which S^* contains a minimal left ideal of βS. Then βS contains at least $2^{\mathfrak{c}}$ minimal left ideals.*

Proof. Let L be a minimal left ideal of βS contained in S^*. Let $p \in L$ and let $\mu = \min\{\|sp\| : s \in S\}$. We can choose $s \in S$ for which $\|sp\| = \mu$. Put $q = sp$. We claim that for every $t \in S$, $\|tq\| = \mu$. To see this, let $t \in S$. First pick $B \in q$ such that $|B| = \mu$. Then $tB \in tq$ and $|tB| \leq |B|$, so $||tq|| \leq \mu$. On the other hand, $||tq|| = ||tsp|| \geq \mu$.

Let $Q \in q$ with $|Q| = \mu$. Let T denote the subsemigroup of S generated by Q. Then $|T| = \mu$. By Lemma 1.5, $q \in K(\beta T)$. Since $U_\mu(T)$ is closed and $\|tq\| = \mu$ for every $t \in T$, it follows that $\beta Tq \subseteq U_\mu(T)$. So, by Theorem 2.20, βT contains 2^{2^μ} minimal left ideals. By Lemma 1.5, for each minimal left ideal L of βT, there is a minimal left ideal M of βS for which $L = M \cap \beta T$. So βS contains at least 2^{2^μ} minimal left ideals. □

Corollary 2.22. *Let S be an arbitrary semigroup. The number of minimal left ideals of βS is either finite or at least $2^{\mathfrak{c}}$. The cardinality of each minimal right ideal of βS is either finite or at least $2^{\mathfrak{c}}$.*

Proof. First consider the case in which the number of minimal left ideals of βS is less than $2^{\mathfrak{c}}$. Then, by Theorem 2.21, every minimal ideal of βS meets S. We claim that if L is a minimal left ideal of

βS, then there is some $s \in S \cap K(\beta S)$ such that $L = \beta Ss$. Indeed, pick $s \in L \cap S$. Since βSs is a left ideal of βS contained in L, we have $L = \beta Ss$. Now we claim that, if $s, t \in S \cap K(\beta S)$, then $\beta Ss = \beta St$ if and only if $Ss = St$. The sufficiency is immediate since $\beta Ss = c\ell(Ss)$. For the necessity, let $s, t \in S \cap K(\beta S)$ and assume that $\beta Ss = \beta St$. We shall show that $St \subseteq Ss$. Let L be the minimal left ideal of βS with $t \in L$. Then $L = \beta St = \beta Ss$ so $t \in \beta Ss = c\ell(Ss)$. Since t is isolated, this means that $t \in Ss$ so $St \subseteq Ss$. So βS and S have the same number of minimal left ideals. By Lemma 2.4(4), this number must be finite. In this case, each minimal right ideal is finite because each minimal right ideal has a finite intersection with each minimal left ideal, by Lemma 2.4(1).

In the case in which βS has at least $2^{\mathfrak{c}}$ minimal left ideals, each minimal right ideal has cardinality at least $2^{\mathfrak{c}}$ because it intersects each minimal left ideal. □

Corollary 2.23. *Let S be a commutative semigroup. If S does not contain a finite ideal, the number of minimal left ideals of βS is at least $2^{\mathfrak{c}}$.*

Proof. Suppose that the number of minimal left ideals of βS is less than $2^{\mathfrak{c}}$. Then, by Theorem 2.21, $K(\beta S)$ meets S. It follows from Lemma 2.4(4) that the number of minimal left ideals of S is finite. Since the left ideals of S are also the right ideals of S, the number of minimal right ideals of S is finite, and so the number of maximal groups in $K(S)$ is finite. By Lemma 2.4(1), each of these groups is finite. Thus $K(S)$ is finite. □

We conclude this section by discussing the cardinalities of minimal left and minimal right ideals of βS.

Definition 2.24. An *extremally disconnected* is a topological space in which the closure of every open set is open. A *Stonean space* is a compact Hausdorff extremally disconnected space.

Definition 2.25. If X is a topological space, $B(X)$ will denote the Boolean algebra of clopen subsets of X.

Let S denote an arbitrary discrete semigroup. We know that no infinite cardinal less than $2^{\mathfrak{c}}$ can occur as the cardinality of a minimal left ideal of βS because infinite closed subsets of βS contain copies of $\beta\mathbb{N}$. (See for example [8, Theorem 3.59].) The following lemma shows that there are other cardinals which cannot occur as cardinalities of minimal left ideals of βS since the only infinite cardinals that can occur have the form 2^μ. The lemma is based on the duality between Boolean algebras and compact totally disconnected spaces. The reader is referred to [9] for information about this subject.

Lemma 2.26. *Let S be an arbitrary semigroup and let $p \in E(\beta S)$. Let $L = \beta Sp$. Then L is Stonean and, if L is infinite, $|L| = 2^{|B(L)|}$.*

Proof. The fact that L is Stonean follows from [7, Lemma 2], using the well-known fact [6, Exercise 6M(1)] that the Stone-Čech compactification of an extremally disconnected Hausdorff space is Stonean. It then follows from [9, 13.7] that $|L| = 2^{|B(L)|}$. $\square$

Lemma 2.27. *Let G be an infinite discrete group of cardinality κ which can be emebedded in a direct sum of countable groups. If L is a minimal left ideal and R is a minimal right ideal of βG, then $L \cap R$ contains a copy of the free group on 2^{2^κ} generators so $|L| = |R| = |L \cap R| = 2^{2^\kappa}$.*

Proof. [11, Theorem 1.1]. $\square$

The hypotheses of Lemma 2.27 hold for every infinite abelian group, because every group of this kind can be embedded in a direct sum of groups which are finite or copies of $\mathbb{Q}$. It is an open question whether every infinite group G has the property that copies of the free group on $2^{2^{|G|}}$ generators exist in the smallest ideal of βG.

Lemma 2.28. *Let S be an abelian cancellative semigroup and let G denote the group of quotients of S. Then βS contains a left ideal of βG.*

Proof. We recall that G is an abelian group in which every element can be expressed in the form ab^{-1} for some $a, b \in S$. We can regard S as a subsemigroup of G and βS as a subsemigroup of βG.

For each $F \in \mathcal{P}_f(S)$, put $x_F = \prod F$. We order $\mathcal{P}_f(S)$ by inclusion and choose a limit point $p \in \beta S$ of the net $\langle x_F \rangle_{F \in \mathcal{P}_f(S)}$. We claim that $\beta Gp \subseteq \beta S$, for which it suffices, since βS is closed in βG, to show that $Gp \subseteq \beta S$. To this end, let $t \in G$ and pick $a, b \in S$ such that $t = ab^{-1}$. Then $tx_F \in S$ for every $F \in \mathcal{P}_f(S)$ for which $\{b\} \subseteq F$. Since $\{tx_F : F \in \mathcal{P}_f(S) \text{ and } \{b\} \subseteq F\} \in tp$, $tp \in \beta S$. $\square$

Theorem 2.29. *Let S be an infinite abelian cancellative semigroup of cardinality κ. Then βS contains 2^{2^κ} minimal right ideals.*

Proof. Let G denote the group of quotients of S. It was shown in [12, Theorem 1] that βG contains 2^{2^κ} minimal right ideals. By Lemma 2.28, there is a minimal left ideal L of βG for which $L \subseteq \beta S$. For each minimal right ideal R of βG, there is an idempotent $p_R \in R \cap L$. The 2^{2^κ} right ideals of βG of the form $p_R \beta G$ are disjoint. A fortiori, the right ideals $p_R \beta S$ of βS are disjoint. $\square$

Theorem 2.30. *Let S be an infinite abelian cancellative semigroup of cardinality κ. Let L be a minimal left ideal and R a minimal right ideal of βS. Then $L \cap R$ contains a copy of the free group on 2^{2^κ} generators. So $|L| = |R| = |L \cap R| = 2^{2^{|S|}}$.*

Proof. This is immediate from Lemmas 2.27 and 2.28. $\square$

3. The Condition $K(\beta S) \subseteq S^*$

We investigate in this section some results involving the statement that $K(\beta S) \subseteq S^*$ or its negation.

Theorem 3.1. *Let S be an infinite semigroup. Statements* (a) *and* (b) *are equivalent and are implied by statement* (c)*. If $K(\beta S) \cap S \neq \emptyset$, then all three statements are equivalent.*

(a) *βS has fewer than $2^{\mathfrak{c}}$ minimal right ideals.*
(b) *βS has only finitely many minimal right ideals.*
(c) *S is the union of finitely many right solution sets.*

Proof. By Theorem 2.10, statements (a) and (b) are equivalent. Assume statement (c) holds and pick $n \in \mathbb{N}$ and

$$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in S$$

such that $S = \bigcup_{i=1}^n Y_{a_i, b_i}$. We claim that βS has at most n minimal right ideals. Suppose instead that $R_1, R_2, \ldots, R_{n+1}$ are pairwise disjoint right ideals. Pick i such that $R_i \cap \{b_1, b_2, \ldots, b_n\} = \emptyset$. Pick $p \in R$. Pick $i \in \{1, 2, \ldots, n\}$ such that $Y_{a_i, b_i} \in p$. Then $pa_i = b_i$. So $b_i \in R$.

Now assume that $I = K(\beta S) \cap S \neq \emptyset$ and that βS has only finitely many minimal right ideals. By Lemma 2.4(3), $I = K(S)$. We claim that if R is a minimal right ideal of βS and $R \cap S \neq \emptyset$, then $R \cap S$ contains only one minimal right ideal of S. Suppose instead one has $x, y \in R \cap S$ such that $xS \cap yS = \emptyset$. Then $M = M \cap M = x\beta S \cap y\beta S = \overline{xS} \cap \overline{yS} = \emptyset$, a contradiction.

Thus S has only finitely many minimal right ideals. Pick a minimal left ideal L of S. By Lemma 2.4(1), if R is an minimal right ideal of S, then $L \cap R$ is finite, and therefore L is finite. Pick $a \in L$. Then $L = Sa$ so $S = \bigcup_{b \in L} Y_{a,b}$. □

Notice that if $S = (\omega, \vee)$, then statement (b) of Theorem 3.1 holds, but statement (c) does not.

Theorem 3.2. *Let S be an infinite semigroup. Statement* (c) *implies statement* (b) *which implies statement* (a)*. If statement* (d) *holds, then statements* (a), (b), *and* (c) *are equivalent. If S is countable, then statements* (a), (b), *and* (c) *are equivalent and imply statement* (d).

(a) *βS has fewer than $2^{\mathfrak{c}}$ minimal left ideals.*
(b) *βS has only finitely many minimal left ideals.*
(c) *S is the union of finitely many left solution sets.*
(d) *$K(\beta S) \cap S \neq \emptyset$.*

Proof. Trivially (b) implies (a). That (c) implies (b) is established in the same way as the corresponding implication in Theorem 3.1.

Now assume that $I = K(\beta S) \cap S \neq \emptyset$. We shall show that (a) implies (c), so assume that (a) holds. By Lemma 2.4(4), S has only finitely many minimal left ideals. Pick a minimal right ideal R of S. By Lemma 2.4(1), the intersection of R with any minimal left ideal of S is finite, so R is finite. Pick $a \in R$. Then $R = aS$ so $S = \bigcup_{b \in R} X_{a,b}$.

Finally, assume that S is countable. By Theorem 2.8, statements (a) and (b) are equivalent and imply statement (d). Thus by what we have just shown, statement (a) implies statement (c). □

Thick subsets of a semigroup are intimately related to minimal left ideals.

Definition 3.3. Let S be an infinite semigroup and let $A \subseteq S$. Then A is *thick* if and only if for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ such that $Fx \subseteq A$.

Theorem 3.4. *Let S be an infinite semigroup and let $A \subseteq S$. Then A is thick if and only if there is some left ideal of βS contained in $c\ell_{\beta S} A$.*

Proof. [2, Theorem 2.9(c)]. □

Recall that by Lemma 2.2, there is a left ideal of βS contained in S^* if and only if for all $F \in \mathcal{P}_f(S)$, $\{t \in S : Ft \cap F = \emptyset\}$ is infinite. Recall also that a subset A of S is *syndetic* if and only if $S \setminus A$ is not thick. Equivalently, A is syndetic if and only if there is some $F \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in F} t^{-1}A$, where

$$t^{-1}A = \{s \in S : ts \in A\}.$$

And recall that A is *piecewise syndetic* if and only if there is some $F \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in F} t^{-1}A$ is thick.

Theorem 3.5. *Let S be an infinite semigroup. The following statements are equivalent.*

(a) $K(\beta S) \subseteq S^*$.

(b) *For all $F \in \mathcal{P}_f(S)$, $\{t \in S : Ft \cap F = \emptyset\}$ is syndetic.*

Proof. To see that (a) implies (b), assume that (a) holds. Let $F \in \mathcal{P}_f(S)$, let $D = \{x \in S : Fx \cap F = \emptyset\}$, and suppose that D is not syndetic. Pick by [2, Theorem 2.9] a minimal left ideal L of βS such that $\overline{D} \cap L = \emptyset$ and pick $p \in L$. Then $\{x \in S : Fx \cap F \neq \emptyset\} \in p$. That is, $\bigcup_{a \in F} \bigcup_{b \in F} X_{a,b} \in p$. Pick $a, b \in F$ such that $X_{a,b} \in p$. Then $ap = b$ so $b \in L \cap S \subseteq K(\beta S) \cap S$, a contradiction.

To see that (b) implies (a), assume that (b) holds and suppose we have some $b \in K(\beta S) \cap S$. By [8, Theorem 4.40], $\{b\}$ is piecewise syndetic. (Recall that we are identifying the points of S with the principal ultrafilters.) So pick $G \in \mathcal{P}_f(S)$ such that for all $F \in \mathcal{P}_f(S)$ there exists $y \in S$ such that $Fy \subseteq \bigcup_{t \in G} t^{-1}\{b\}$. Let

$$D = \{x \in S : (G \cup \{b\})x \cap (G \cup \{b\}) = \emptyset\}.$$

Then D is syndetic so pick $F \in \mathcal{P}_f(S)$ such that $S = \bigcup_{z \in F} z^{-1}D$. Pick $y \in S$ such that $Fy \subseteq \bigcup_{t \in G} t^{-1}\{b\}$ and pick $z \in F$ such that $zy \in D$. Pick $t \in G$ such that $tzy = b$. Then $tzy \in Gzy \cap \{b\}$, contradicting the fact that $zy \in D$. □

There is a significant contrast between Theorems 2.18 and 2.19. For example, if $\kappa = \omega$, one produces S with $K(\beta S) \subseteq S^*$ such that βS has only one minimal right ideal by taking $S = (\omega, \vee)$, in which case S^* is a right zero semigroup. We conclude this section by showing that one cannot do the corresponding thing to obtain S^* as a left zero semigroup.

Notice that the conditions of the following lemma are not left-right switches of each other. (The first conclusion says that $X_{s,s} \in p$ while the second says that $X_{s,t} \in p$.)

Lemma 3.6. *Let S be a semigroup.*

1. *S^* is a left zero semigroup if and only if* $(\forall p \in S^*)\big(\exists F \in \mathcal{P}_f(S)\big)(\forall s \in S \setminus F)(\{t \in S : st = s\} \in p)$.
2. *S^* is a right zero semigroup if and only if* $(\forall p \in S^*)\big(\exists F \in \mathcal{P}_f(S)\big)(\forall s \in S \setminus F)(\{t \in S : st = t\} \in p)$.

Proof. (1) Sufficiency. Let $p, q \in S^*$ and pick F as guaranteed for p. Then for all $s \in S \setminus F$, λ_s is constantly equal to s on a member of p so $sp = p$. Therefore ρ_p is equal to the identity on a member of q and so $qp = q$.

Necessity. Let $p \in S^*$. By [10, Theorem 2.8], pick $F \in \mathcal{P}_f(S)$ such that $qp = q$ for all $q \in \beta S \setminus F$. Let $s \in S \setminus F$. Then $\{s\}$ is a neighborhood of $\lambda_s(p)$ so there is a member A of p such that $\lambda_s[\,\overline{A}\,] \subseteq \{s\}$.

(2) Sufficiency. Let $p, q \in S^*$ and pick F as guaranteed for p. Then for all $s \in S \setminus F$, λ_s is the identity on a member of p and so $sp = p$. Therefore ρ_p is constantly equal to p on a member of q and so $qp = p$.

Necessity. Let $p \in S^*$. By [10, Theorem 2.5] pick $F \in \mathcal{P}_f(S)$ such that $qp = p$ for all $q \in \beta S \setminus F$. Let $s \in S \setminus F$. Then $\lambda_s(p) = p$ so by [8, Theorem 3.35], $\{t \in S : st = t\} \in p$. □

Theorem 3.7. *Let S be an infinite semigroup for which $K(\beta S)$ meets S^*.*

1. *If S^* is a left zero semigroup, then there is a finite subset F of S such that*
   - (a) *for every $s \in S \setminus F$, $\{t \in S : st \neq s\}$ is finite and*
   - (b) *$\beta S \setminus F$ is the unique minimal left ideal of βS. (In particular, there is no left ideal of βS contained in S^*.)*
2. *If S^* is a right zero semigroup, the minimal left ideals of $K(\beta S)$ are singletons and βS has precisely one minimal right ideal.*

Proof. (1) Pick $p \in S^* \cap K(\beta S)$. Let $L = \beta S p$, a minimal left ideal of βS. Observe that $S^* \subseteq L$. Let $F = \beta S \setminus L = S \setminus L$. By Lemma 3.6(1) pick a finite subset G of S such that for every $s \in S \setminus G$, $\{t \in S : st = s\} \in p$. We claim that $F \subseteq G$, so that F is finite. For this it suffices to show that $S \setminus G \subseteq L$, so let $s \in S \setminus G$. Then λ_s is constantly equal to s on a member of p, so $s = sp \in \beta S p = L$.

To establish (a), let $s \in S \setminus F$. Then $s \in L$. Suppose that $\{t \in S : st \neq s\}$ is infinite and pick $q \in S^*$ such that $\{t \in S : st \neq s\} \in q$. Since $s \in L$ and q is an idempotent in L we have by [8, Lemma 1.30] that $s = sq$. Therefore $\{s\}$ is a neighborhood of sq and therefore $\{t \in S : st = s\} \in q$, a contradiction.

To establish (b), note that L is the only minimal left ideal of βS, since any other minimal left ideal of βS would have to be an infinite subset of $\beta S \setminus L = F$.

(2) Let L be a minimal left ideal of βS which meets S^* and let $p \in L \cap S^*$. By Lemma 3.6, $sp = p$ for all but a finite number of elements s in S. So Sp is finite and $L = c\ell(Sp)$ must also be finite. By Theorem 2.14, $L \subseteq S^*$. So $L = Lp = \{p\}$. Since L intersects every minimal right ideal of βS, βS can have only one minimal right ideal. □

4. Extremely Weakly Left Cancellative Semigroups

We introduce here a cancellation condition which is satisfied by the semigroup $(\kappa, \vee)$ where κ is a singular cardinal and derive some results about semigroups satisfying this condition, including an analogue of Theorem 1.4.

Definition 4.1. Let S be an infinite semigroup and let $\kappa = |S|$. Then S is *extremely weakly left cancellative* if and only if

(1) for all $a \in S$, $|X_{a,a}| < \kappa$ and
(2) if $H \subseteq (S \times S) \setminus \Delta$ and $|H| < \kappa$, then $|\bigcup_{(a,b) \in H} X_{a,b}| < \kappa$.

Notice that in $(\kappa, \vee)$, if $\alpha < \delta$, then $X_{\alpha,\delta} = \{\delta\}$ and $X_{\delta,\alpha} = \emptyset$, so $(\kappa, \vee)$ is extremely weakly left cancellative.

Definition 4.2. Let X be an infinite set. A set $\mathcal{A}$ of subsets of X said to be *almost disjoint* if and only if

(1) for each $A \in \mathcal{A}$, $|A| = |X|$; and
(2) for $A \neq B$ in $\mathcal{A}$, $|A \cap B| < |X|$.

The next theorem is [3, Theorem 2.3].

Theorem 4.3. *Let S be an infinite semigroup which is very weakly cancellative, let $\kappa = |S|$, and let A be a thick subset of S.*

(i) *If there is a family of μ almost disjoint subsets of κ, then there is a family of μ almost disjoint thick subsets of A.*
(ii) *There is a family of κ pairwise disjoint thick subsets of A.*

Note that it was assumed that S was very weakly left and right cancellative. This was used to guarantee that thick subsets have cardinality κ. (In $(\kappa, \vee)$ any cofinal subset is thick.) We are able to drop any right cancellation assumption and replace "very weakly left cancellative" by "extremely weakly left cancellative", if we drop the demand that the almost disjoint sets be found inside an arbitrary thick set.

Theorem 4.4. *Let S be an infinite semigroup which is extremely weakly left cancellative and let $\kappa = |S|$.*

(i) *If there is a family of μ almost disjoint subsets of κ, then there is a family of μ almost disjoint thick subsets of S.*
(ii) *There is a family of κ pairwise disjoint thick subsets of S.*

Proof. (i) Pick by [3, Lemma 2.1(i)] a family $\langle \mathcal{B}_\iota \rangle_{\iota<\mu}$ of almost disjoint subsets of $\mathcal{P}_f(S)$ such that

$$\big(\forall F \in \mathcal{P}_f(S)\big)(\forall \iota < \mu)(\exists G \in \mathcal{B}_\iota)(F \subseteq G)\,.$$

Enumerate $\mathcal{P}_f(S)$ as $\langle F_\sigma \rangle_{\sigma<\kappa}$. We inductively choose a κ sequence $\langle x_\sigma \rangle_{\sigma<\kappa}$ in S such that if $\delta < \sigma < \kappa$, then $F_\sigma x_\sigma \cap F_\delta x_\delta = \emptyset$. To see that we can do this, let $\sigma < \kappa$ and assume $\langle x_\delta \rangle_{\delta<\sigma}$ has been chosen. Let $H = \bigcup_{\delta<\sigma} F_\delta x_\delta$. Then $|H| \leq \max\{\omega, |\sigma|\}$.

We claim $|\bigcup_{u\in F_\sigma} \bigcup_{v\in H} X_{u,v}| < \kappa$. Indeed,

$$\textstyle\bigcup_{u\in F_\sigma} \bigcup_{v\in H} X_{u,v} \subseteq \bigcup_{u\in F_\sigma} X_{u,u} \cup \bigcup_{u\in F_\sigma} \bigcup_{v\in H\setminus\{u\}} X_{u,v}$$

and $|\bigcup_{u\in F_\sigma} X_{u,u}| < \kappa$ because F_σ is finite and

$$\textstyle|\bigcup_{u\in F_\sigma} \bigcup_{v\in H\setminus\{u\}} X_{u,v}| < \kappa\,.$$

Pick $x_\sigma \in (S \setminus \bigcup_{u\in F_\sigma} \bigcup_{v\in H} X_{u,v})$. If $\delta < \sigma$, then $F_\sigma x_\sigma \cap F_\delta x_\delta = \emptyset$.

For $\iota < \mu$, let $D_\iota = \bigcup\{F_\sigma x_\sigma : \sigma < \kappa \text{ and } F_\sigma \in \mathcal{B}_\iota\}$. Then $|D_\iota| = \kappa$.

If $\iota < \gamma < \mu$, then

$$D_\iota \cap D_\gamma = \bigcup\{F_\sigma x_\sigma : \sigma < \kappa \text{ and } F_\sigma \in \mathcal{B}_\iota \cap \mathcal{B}_\gamma\}\,,$$

so $|D_\iota \cap D_\gamma| < \kappa$. Let $\iota < \mu$. Given $F \in \mathcal{P}_f(S)$, pick $F_\sigma \in \mathcal{B}_\iota$ such that $F \subseteq F_\sigma$. Then $F x_\sigma \subseteq F_\sigma x_\sigma \subseteq D_\iota$.

(ii) This is the same argument using [3, Lemma 2.1(ii)]. $\square$

We now turn our attention to extending the following theorem from [3].

We are very interested in members of minimal idempotents because of the substantial combinatorial properties that they posess.

Theorem 4.5. *Let κ be an infinite cardinal and let S be a very weakly left cancellative semigroup with cardinality κ, let p be a minimal idempotent of βS which is uniform, and let $C \in p$.*

(i) *If there is a family of μ almost disjoint subsets of κ, then C contains μ almost disjoint sets each of which is a member of a uniform minimal idempotent in βS.*

(ii) *C contains κ disjoint sets each of which is a member of a uniform minimal idempotent in βS.*

Proof. [3, Theorem 3.3]. □

To show that our extension of Theorem 4.5 is not vacuous, we need a preliminary lemma.

Lemma 4.6. *Let S be an extremely weakly left cancellative semigroup with $|S| = \kappa$. Then $U_\kappa(S)$ is a left ideal of βS.*

Proof. Let $p \in U_\kappa(S)$. Since $\beta S p = c\ell(Sp)$, it suffices to show that $Sp \subseteq U_\kappa(S)$. Let $s \in S$ and let $B \in sp$. Then $s^{-1}B \in p$ so $|s^{-1}B| = \kappa$. Suppose $|B| < \kappa$. Then $s^{-1}B \subseteq X_{s,s} \cup \bigcup_{t \in B \setminus \{s\}} X_{s,t}$. So $|s^{-1}B| < \kappa$, a contradiction. □

Note that $U_\kappa(S)$ need not be an ideal of βS. If S is a right zero semigroup, then S is left cancellative and βS is the unique minimal right ideal. Slightly more esoteric is our canonical example of an extremely weakly left cancellative semigroup which is not very weakly left cancellative, namely $(\kappa, \vee)$ where κ is singular.

We note that by Lemma 4.6, if S is extremely weakly left cancellative, then there are minimal idempotents in $U_\kappa(S)$. We see that the hypothesis of Theorem 4.5 can be weakend.

Theorem 4.7. *Let κ be an infinite cardinal and let S be an extremely weakly left cancellative semigroup with cardinality κ, let p be a minimal idempotent of βS which is uniform, and let $C \in p$.*

(i) *If there is a family of μ almost disjoint subsets of κ, then C contains μ almost disjoint sets each of which is a member of a uniform minimal idempotent in βS.*

(ii) *C contains κ disjoint sets each of which is a member of a uniform minimal idempotent in βS.*

Proof. The proof is nearly identical to the proof of [3, Theorem 3.3]. One only needs to note that when the cancellation assumption is invoked, one is only concerned with finitely many left solution sets of the form $X_{a,a}$. □

We now see that Theorem 1.4 can be extended to extremely weakly left cancellative semigroups.

Theorem 4.8. *If S is an extremely weakly left cancellative semigroup with cardinality κ, then βS has 2^{2^κ} minimal left ideals.*

Proof. Theorem 2.20 and Lemma 4.6. □

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