This paper was published in Topology and its Applications 213 (2016),
199-211. To the best of my knowledge, this is the final version as it was submitted to the publisher.- NH Separating Linear Expressions in the Stone-Čech Compactification of Direct Sums

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#### Abstract

A finite sequence $\vec{a}=\left\langle a_{i}\right\rangle_{i=1}^{m}$ in $\mathbb{Z} \backslash\{0\}$ is compressed provided $a_{i} \neq a_{i+1}$ for $i<m$. It is known that if $\vec{a}=\left\langle a_{i}\right\rangle_{i=1}^{m}$ and $\vec{b}=\left\langle b_{i}\right\rangle_{i=1}^{k}$ are compressed sequences in $\mathbb{Z} \backslash\{0\}$, then there exist idempotents $p$ and $q$ in $\beta \mathbb{Q}_{d} \backslash\{0\}$ such that $a_{1} p+a_{2} p+\ldots+a_{m} p=b_{1} q+b_{2} q+\ldots+b_{k} q$ if and only if $\vec{b}$ is a rational multiple of $\vec{a}$. In fact, if $\vec{b}$ is not a rational multiple of $\vec{a}$, then there is a partition of $\mathbb{Q} \backslash\{0\}$ into two cells, neither of which is a member of $a_{1} p+a_{2} p+\ldots+a_{m} p$ and a member of $b_{1} q+b_{2} q+\ldots+b_{k} q$ for any idempotents $p$ and $q$ in $\beta \mathbb{Q}_{d} \backslash\{0\}$. (Here $\beta \mathbb{Q}_{d}$ is the Stone-Cech compactification of the set of rational numbers with the discrete topology.)

In this paper we extend these results to direct sums of $\mathbb{Q}$. As a corollary, we show that if $\vec{b}$ is not a rational multiple of $\vec{a}$ and $G$ is any torsion free commutative group, then there do not exist idempotents $p$ and $q$ in $\beta G_{d} \backslash\{0\}$ such that $a_{1} p+a_{2} p+\ldots+a_{m} p=b_{1} q+b_{2} q+\ldots+b_{k} q$. We also show that for direct sums of finitely many copies of $\mathbb{Q}$ we can separate the corresponding


[^0]Milliken-Taylor systems, with a similar but weaker result for the direct sum of countably many copies of $\mathbb{Q}$.

Key words: Stone-Čech Compactification, linear expressions, Milliken-Taylor systems
2010 MSC: 54D35, 22A15, 05D10

## 1. Introduction

We are investigating the following two questions in this paper. (We will describe the operation on $\beta G_{d}$ and the action of $\mathbb{Z}$ on $\beta G_{d}$ later in this introduction.)

Question 1.1. Which commutative groups $(G,+)$ have the property that whenever $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ are compressed sequences in $\mathbb{Z} \backslash\{0\}$ which are not rational multiples of each other, there do not exist idempotents $p$ and $q$ in $\beta G_{d} \backslash\{0\}$ such that $a_{1} p+a_{2} p+\ldots+a_{m} p=b_{1} q+b_{2} q+\ldots+b_{k} q$ ?

Question 1.2. Which commutative groups $(G,+)$ have the property that whenever $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ are compressed sequences in $\mathbb{Z} \backslash\{0\}$ which are not rational multiples of each other, there is a partition of $G \backslash\{0\}$ into finitely many cells such that there do not exist idempotents $p$ and $q$ in $\beta G_{d} \backslash\{0\}$ and one cell of the partition which is a member of both $a_{1} p+a_{2} p+\ldots+a_{m} p$ and $b_{1} q+b_{2} q+\ldots+b_{k} q$ ?

The motivation for these questions comes from the ability to separate Milliken-Taylor systems. Given a set $X$, we write $\mathcal{P}_{f}(X)$ for the set of finite nonempty subsets of $X$

Definition 1.3. Let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ be a compressed sequence in $\mathbb{Z} \backslash\{0\}$ and let $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ be a sequence in a commutative group $(G,+)$. The MillikenTaylor system generated by $\vec{a}$ and $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ is

$$
\begin{aligned}
M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right)= & \left\{\sum_{i=1}^{m} a_{i} \cdot \sum_{n \in F_{i}} x_{n}: F_{1}, F_{2}, \ldots, F_{m} \in \mathcal{P}_{f}(\mathbb{N})\right. \\
& \text { with } \left.\max F_{i}<\min F_{i+1} \text { for } i<m\right\} .
\end{aligned}
$$

Milliken-Taylor systems are partition regular. That is, given $\vec{a}$, if a commutative group $G$ is partitioned into finitely many cells, then there is one cell which contains $\operatorname{MT}\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right)$ for some sequence $\vec{x}$. (In the alternative coloring terminology common in Ramsey Theory, if $G$ is finitely colored there is a sequence $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ such that $\operatorname{MT}\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right)$ is monochromatic.) The MillikenTaylor systems are so named because the fact that they are partition regular is an immediate consequence of the Milliken-Taylor Theorem ([4, Theorem 2.2] and [5, Lemma 2.2]).

The relationship between Milliken-Taylor systems and linear expressions in $\beta G_{d}$ is given by the following theorem.

Theorem 1.4. Let $G$ be a commutative group, let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ be $a$ compressed sequence in $\mathbb{Z} \backslash\{0\}$, and let $A \subseteq G$. There is a sequence $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ in $G$ such that $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq A$ if and only if there is an idempotent $p \in \beta G_{d}$ such that $A \in a_{1} p+a_{2} p+\ldots+a_{m} p$.

Proof. [3, Theorem 1.5].
Given a compressed sequence $\vec{a}$ in $\mathbb{Z} \backslash\{0\}$, there is a matrix $M$ such that $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right)$ is the set of entries of $M \vec{x}$, where $\vec{x}=\left(\begin{array}{c}x_{1} \\ x_{2} \\ \vdots\end{array}\right)$. These matrices are examples of image partition regular matrices and were some of the first known examples of infinite image partition regular matrices. Finite image partition regular matrices with rational entries have the property that given any finite partition of $\mathbb{N}$, there is one cell which contains an image of all of these matrices. (See [2, Theorem 15.24].) By way of contrast, there is the following theorem.

Theorem 1.5. Let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$. The following statements are equivalent.
(a) Whenever $\mathbb{Z} \backslash\{0\}$ is finitely colored, there exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $\mathbb{Z} \backslash\{0\}$ such that $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ is monochromatic.
(b) Whenever $\mathbb{Q} \backslash\{0\}$ is finitely colored, there exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $\mathbb{Q} \backslash\{0\}$ such that $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ is monochromatic.
(c) There exist idempotents $p$ and $q$ in $\beta \mathbb{Z} \backslash\{0\}$ such that $a_{1} p+a_{2} p+\ldots+a_{m} p=$ $b_{1} q+b_{2} q+\ldots+b_{k} q$.
(d) There exist idempotents $p$ and $q$ in $\beta \mathbb{Q}_{d} \backslash\{0\}$ such that $a_{1} p+a_{2} p+\ldots+$ $a_{m} p=b_{1} q+b_{2} q+\ldots+b_{k} q$.
(e) Whenever $\mathbb{Z} \backslash\{0\}$ is finitely colored, there exist idempotents $p$ and $q$ in $\beta \mathbb{Z} \backslash\{0\}$ and a color class which is a member of both $a_{1} p+a_{2} p+\ldots+a_{m} p$ and $b_{1} q+b_{2} q+\ldots+b_{k} q$.
(f) Whenever $\mathbb{Q} \backslash\{0\}$ is finitely colored, there exist idempotents $p$ and $q$ in $\beta \mathbb{Q}_{d} \backslash\{0\}$ and a color class which is a member of both $a_{1} p+a_{2} p+\ldots+a_{m} p$ and $b_{1} q+b_{2} q+\ldots+b_{k} q$.
(g) The sequence $\vec{b}$ is a rational multiple of $\vec{a}$.

Proof. The equivalence of $(a)$ with $(e)$ and the equivalence of $(b)$ with $(f)$ follow from Theorem 1.4. It is trivial that $(c)$ implies $(e)$ and that $(d)$ implies $(f)$. The fact that $(g)$ implies both $(c)$ and $(d)$ follows from the fact that if $p$ is an idempotent in $\beta \mathbb{Z}$ and $\alpha \in \mathbb{Q} \backslash\{0\}$, then $\alpha p$ is also an idempotent in $\beta \mathbb{Z}$. (For
the details of this argument see [2, Lemma 15.23.2].) Finally, the fact that (a) implies $(g)$ follows from [1, Theorem 3.1] and the fact that $(b)$ implies $(g)$ is $[3$, Theorem 4.3].

We utilize the algebraic structure of $\beta G_{d}$, the Stone-Čech compactification of $G_{d}$, where $(G,+)$ is a commutative group and the subscript indicates that we are giving $G$ the discrete topology. We take the points of $\beta G_{d}$ to be the ultrafilters on $G$, with the points of $G$ being identified with the principal ultrafilters. Given $A \subseteq G, \bar{A}=\left\{p \in \beta G_{d}: A \in p\right\}$. The operation + on $G$ extends to an operation on $\beta G_{d}$, also denoted by + , so that $\left(\beta G_{d},+\right)$ is a right topological semigroup (meaning that for each $p \in \beta G_{d}$, the function $\rho_{p}: \beta G_{d} \rightarrow \beta G_{d}$ is continuous, where $\rho_{p}(q)=q+p$ ) with $G$ contained in its topological center (meaning that for each $x \in G$, the function $\lambda_{x}: \beta G_{d} \rightarrow \beta G_{d}$ is continuous, where $\left.\lambda_{x}(q)=x+q\right)$. Given $p$ and $q$ in $\beta G_{d}$ and $A \subseteq G, A \in p+q$ if and only if $\{x \in S:-x+A \in q\} \in p$. We let $G^{*}=\beta G_{d} \backslash G$. Then $G^{*}$ is a compact subsemigroup - in fact a two sided ideal - of $\beta G_{d}$. As does any compact right topological semigroup, $G^{*}$ has idempotents. See [2, Part I] for an elementary introduction to the algebraic structure of $\beta G_{d}$.

The reader should be cautioned that, even though we denote the operation on $\beta G_{d}$ by + , the operation is not commutative. In fact, by [2, Theorem 6.54], the center of $\left(\beta G_{d},+\right)$ is $G$.

Given $a \in \mathbb{Z} \backslash\{0\}$ and $x \in G$, we let $a x$ have its usual meaning - that is the sum of $x$ with itself $a$ times if $a>0$ and the inverse of $(-a) x$ if $a<0$. If $a \in \mathbb{Z} \backslash\{0\}$ and $p \in G^{*}$, we define $a p=\widetilde{l}_{a}(p)$ where $l_{a}: G \rightarrow G$ is defined as $l_{a}(x)=a x$ and $\widetilde{l}_{a}: \beta G_{d} \rightarrow \beta G_{d}$ is its continuous extension. Thus, for example, if $p \in G^{*}$, then $2 p$ does not mean $p+p$. (In $\beta \mathbb{Z}$, by [2, Theorem 13.18], there is no $p \in \mathbb{Z}^{*}$ such that $2 p=p+p$.) For each $A \subseteq S$ and each $a \in \mathbb{Z} \backslash\{0\}, A \in a p$ if and only if $a^{-1} A \in p$, where $a^{-1} A=\{x \in G: a x \in A\}$.

In Section 2 we prove that if $\kappa$ is any cardinal greater than 0 , and $T=$ $\bigoplus_{\sigma<\kappa} \mathbb{Q}$, then we can separate linear expressions of the form $a_{1} p+a_{2} p+\ldots+a_{m} p$ and $b_{1} q+b_{2} q+\ldots+b_{k} q$ whenever $\vec{b}$ is not a rational multiple of $\vec{a}$ and $p$ and $q$ are idempotents living at infinity. And we derive some consequences of this fact, including the fact mentioned in the abstract that for any torsion free commutative group $G$, there do not exist idempotents $p$ and $q$ in $\beta G_{d} \backslash\{0\}$ with $a_{1} p+a_{2} p+\ldots+a_{m} p=b_{1} q+b_{2} q+\ldots+b_{k} q$. The results of Section 2 show that we can separate Milliken-Taylor systems generated by strongly increasing sequences, that is sequences with the maximum of support of each term less than the minimum of support of the succeding term.

In Section 3 we derive some results involving separating Milliken-Taylor systems. If $\kappa$ is finite, these are the strongest possible. If $\kappa=\omega$, we show that one can prevent three Milliken-Taylor systems from ending up in the same cell of a partition of $T$.

## 2. Separating linear expressions in $\boldsymbol{\beta T}$

We shall use coloring terminology in this section. A finite coloring of a set $X$ is a function with finite range whose domain is $X$. If $\psi$ is a finite coloring of $X$, then $D$ is a color class of $\psi$ if and only if there is some $i$ in the range of $\psi$ such that $D=\psi^{-1}[\{i\}]$.

The following easy lemma is presumably well known.
Lemma 2.1. Let $F$ be a finite subset of $\mathbb{Q} \backslash\{0\}$. There is a finite coloring $\psi$ of $\mathbb{Q} \backslash\{0\}$ such that, if $x, y \in \mathbb{Q} \backslash\{0\}, a, b \in F, \psi(x)=\psi(y)$, and $a \neq b$, then $\psi(a x)=\psi(a y)$, but $\psi(a x) \neq \psi(b y)$.

Proof. Let $H=\{|a|: a \in F\}$. We first note that it suffices to get a finite coloring $\mu$ of $\mathbb{Q}^{+}=\{x \in \mathbb{Q}: x>0\}$ such that if $x, y \in \mathbb{Q}^{+}, a, b \in H, \mu(x)=\mu(y)$, and $a \neq b$, then $\mu(a x)=\mu(b x)$ but $\mu(a x) \neq \mu(b y)$. For then one can define a coloring $\psi$ of $\mathbb{Q} \backslash\{0\}$ so that for $x, y \in \mathbb{Q} \backslash\{0\}, \psi(x)=\psi(y)$ if and only if either (1) $x>0, y>0$, and $\mu(x)=\mu(y)$ or $(2) x<0, y<0$, and $\mu(-x)=\mu(-y)$.

Let $P$ be the set of primes that occur in the prime factorization of any member of $H$ and let $k$ be the largest integer such that there is some $p \in P$ such that $p^{k}$ or $p^{-k}$ occurs in the prime factorzation of some member of $H$. Define a finite coloring $\mu$ of $\mathbb{Q}^{+}$so that for $x, y \in \mathbb{Q}^{+}, \mu(x)=\mu(y)$ if and only if for all $p \in P$, if $p^{r}$ and $p^{l}$ are the powers of $p$ in the prime factorizations of $x$ and $y$ respectively, then $r \equiv l(\bmod 2 k+1)$.

Definition 2.2. Let $\kappa>0$ be an ordinal and let $T=\bigoplus_{\sigma<\kappa} \mathbb{Q}$. For $x \in T$, $\operatorname{supp}(x)=\left\{\sigma<\kappa: x_{\sigma} \neq 0\right\}$. For $x \in T \backslash\{0\}, \alpha(x)=\min \operatorname{supp}(x)$ and $\delta(x)=\max \operatorname{supp}(x)$. For $x, y \in T \backslash\{0\}$, we write $x \ll y$ if and only if $\delta(x)<\alpha(y)$. An idempotent $r \in \beta T_{d}$ is strongly increasing if $r$ has a member $R \subseteq T \backslash\{0\}$ with the property that, for every $x \in R,\{y \in R: x \ll y\} \in r$. A sequence $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ in $T \backslash\{0\}$ is strongly increasing if and only if for each $n \in \mathbb{N}$, $x_{n} \ll x_{n+1}$.

We shall assume throughout that $T$ has the discrete topology. (So we will write $\beta T$ rather than $\beta T_{d}$.) We will use the fact from [2, Lemma 5.11] that if $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ is any sequence in $T$, there is an idempotent in $\bigcap_{l=1}^{\infty} \overline{F S\left(\left\langle x_{n}\right\rangle_{n=l}^{\infty}\right)}$, where $F S\left(\left\langle x_{n}\right\rangle_{n=l}^{\infty}\right)=\left\{\sum_{t \in F} x_{t}: F \in \mathcal{P}_{f}(\{l, l+1, l+2, \ldots\})\right\}$. Note that, if $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ is strongly increasing, every idempotent in $\bigcap_{l=1}^{\infty} \overline{F S\left(\left\langle x_{n}\right\rangle_{n=l}^{\infty}\right)}$ is strongly increasing. (If $p \in \bigcap_{l=1}^{\infty} \overline{F S\left(\left\langle x_{n}\right\rangle_{n=l}^{\infty}\right)}$, then $F S\left(\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right)$ is as required for the definition of strongly increasing idempotent.)

If $p$ is an idempotent in $\beta T$ and $P \in p$, we put $P^{\star}=\{x \in P: x+p \in \bar{P}\}$. Then $P^{\star} \in p$ and by [2, Lemma 4.14], for any $x \in P^{\star},-x+P^{\star} \in p$.

The rest of this section is devoted to the proof of the following theorem and some of its consequences. That proof uses a modification of the gap counting technique used in [1]. The proof of Theorem 2.3 will include the statement and proof of four lemmas.

Theorem 2.3. Let $k, m \in \mathbb{N}$ and let $\kappa$ be an infinite cardinal. Let $\vec{a}=$ $\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$ such
that $\vec{b}$ is not a rational multiple of $\vec{a}$. Let $T=\bigoplus_{\sigma<\kappa} \mathbb{Q}$. There is a finite coloring $\Gamma$ of $T \backslash\{0\}$ such that, if $p$ and $q$ are strongly increasing idempotents in $\beta T$, there is no color class of $\Gamma$ which is a member of both $a_{1} p+a_{2} p+\ldots+a_{m} p$ and $b_{1} q+b_{2} q+\ldots+b_{k} q$.

Proof. Assume without loss of generality that $m \leq k$. Note also that if $m=$ $k=1$, then $\vec{b}$ is a rational multiple of $\vec{a}$, so we may assume that $k>1$. Let $A=\left\{a_{1}, a_{2}, \ldots, a_{m}\right\}$ and let $B=\left\{b_{1}, b_{2}, \ldots, b_{k}\right\}$. By Lemma 2.1, pick a finite set $K$ and $\psi: \mathbb{Q} \backslash\{0\} \rightarrow K$ such that if $x, y \in \mathbb{Q} \backslash\{0\}, a, b \in A \cup B, \psi(x)=\psi(y)$, and $a \neq b$, then $\psi(a x)=\psi(a y)$ but $\psi(a x) \neq \psi(b y)$.

Before continuing with the proof of the theorem, we introduce some notation and prove three lemmas.
Definition 2.4. For $x \in T \backslash\{0\}$, let $G(x)=$

$$
\begin{aligned}
\{(t, u, v): & t \in \operatorname{supp}(x) \backslash\{\delta(x)\} \text { and if } s=\min \{\eta \in \operatorname{supp}(x): \eta>t\} \\
& \text { then } \left.\psi\left(x_{t}\right)=u, \text { and } \psi\left(x_{s}\right)=v\right\}
\end{aligned}
$$

For $x \in T \backslash\{0\}$ and $(u, v) \in K^{2}$, let $G_{u, v}(x)=\{t<\kappa:(t, u, v) \in G(x)\}$.
So $(t, u, v) \in G(x)$ if $t \in \operatorname{supp}(x)$ and $\psi\left(x_{t}\right)=u$ while, for some $s>t$ in $\operatorname{supp}(x), \psi\left(x_{s}\right)=v$ and $x_{w}=0$ whenever $t<w<s$.
Definition 2.5. For every $(u, v) \in K^{2}$ and $x \in T \backslash\{0\}$, define $\varphi_{u, v}(x) \in$ $\{-k,-k+1, \ldots, k-1, k\}$ by $\varphi_{u, v}(x) \equiv\left|G_{u, v}(x)\right|(\bmod 2 k+1)$. We also define $\theta_{1}$ and $\theta_{2}$ mapping $T \backslash\{0\}$ to $K$ by $\theta_{1}(x)=\psi\left(x_{\delta(x)}\right)$ and $\theta_{2}(x)=\psi\left(x_{\alpha(x)}\right)$. Let $\widetilde{\varphi}_{u, v}: \beta(T \backslash\{0\}) \rightarrow\{-k,-k+1, \ldots, k-1, k\}$ denote the continuous extension of $\varphi_{u, v}$ and let $\widetilde{\theta}_{1}: \beta(T \backslash\{0\}) \rightarrow K$ and $\widetilde{\theta}_{2}: \beta(T \backslash\{0\}) \rightarrow K$ be the continuous extensions of $\theta_{1}$ and $\theta_{2}$ respectively.

Note that, if $x, y \in T \backslash\{0\}, i \in\{1,2\}, \theta_{i}(x)=\theta_{i}(y)$, and $c$ and $d$ are distinct members of $A \cup B$, then $\theta_{i}(c y)=\theta_{i}(c x) \neq \theta_{i}(d y)$.
Lemma 2.6. Let $x, y \in T \backslash\{0\}$, let $u, v \in K$, and assume that $x \ll y$. Then $\varphi_{u, v}(x+y) \equiv \varphi_{u, v}(x)+\varphi_{u, v}(y)+h(\bmod 2 k+1)$, where

$$
h= \begin{cases}1 & \text { if } u=\theta_{1}(x) \text { and } v=\theta_{2}(y) ; \\ 0 & \text { otherwise. }\end{cases}
$$

Proof. $G_{u, v}(x+y)=G_{u, v}(x) \cup G_{u, v}(y) \cup H$ where

$$
H=\left\{\begin{array}{cl}
\{\delta(x)\} & \text { if } u=\theta_{1}(x) \text { and } v=\theta_{2}(y) ; \\
\emptyset & \text { otherwise. }
\end{array}\right.
$$

Lemma 2.7. Let $r$ be a strongly increasing idempotent in $\beta T$, let $u, v \in K$, and let $c \in A \cup B$. If $u=\widetilde{\theta}_{1}(c r)$ and $v=\widetilde{\theta}_{2}(c r)$, then $\widetilde{\varphi}_{u, v}(c r)=-1$. Otherwise $\widetilde{\varphi}_{u, v}(c r)=0$.

Proof. Let $E$ be a member of $r$ on which the functions $x \mapsto \theta_{1}(c x), x \mapsto \theta_{2}(c x)$ and $x \mapsto \varphi_{u, v}(c x)$ are constant. Recall that $E^{\star}=\{z \in E:-z+E \in r\} \in r$. Choose $x, y \in E^{\star}$ such that $x+y \in E^{\star}$ and $x \ll y$. Then by Lemma 2.6, $\varphi_{u, v}(c x+c y) \equiv \varphi_{u, v}(c x)+\varphi_{u, v}(c y)+h(\bmod 2 k+1)$ where

$$
h= \begin{cases}1 & \text { if } u=\theta_{1}(c x) \text { and } v=\theta_{2}(c y) \\ 0 & \text { otherwise }\end{cases}
$$

So our claim follows from the fact that $\widetilde{\varphi}_{u, v}(c r)=\varphi_{u, v}(c x)=\varphi_{u, v}(c y)=$ $\varphi_{u, v}(c x+c y)$.

Lemma 2.8. Let $r$ be a strongly increasing idempotent in $\beta$ T. Let $s \in\{2,3, \ldots$, $k\}$, and let $\vec{c}=\left\langle c_{1}, c_{2}, \ldots, c_{s}\right\rangle$ be a compressed sequence in $A \cup B$. Let $R$ denote a member of $r$ on which the functions $\theta_{1}$ and $\theta_{2}$ are constant, as well as all the functions of the form $x \mapsto \varphi_{u, v}\left(c_{i} x\right)$, where $u, v \in K$ and $i \in\{1,2, \ldots, s\}$. Also, let $R$ have the property that, for each $y \in R,\{z \in R: y \ll z\} \in r$. Let $w_{1}, w_{2}, \ldots, w_{s} \in R$ such that $(\forall i \in\{1,2, \ldots, s-1\})\left(w_{i} \ll w_{i+1}\right)$ and let $x=c_{1} w_{1}+c_{2} w_{2}+\ldots+c_{s} w_{s}$. Then, for every $u, v \in K, \varphi_{u, v}(x)>0$ if and only if $u=\theta_{1}\left(c_{i} w_{i}\right)$ and $v=\theta_{2}\left(c_{i+1} w_{i+1}\right)$ for some $i \in\{1,2, \ldots, s-1\}$.
Proof. Let $(u, v) \in K^{2}$ be given. By Lemma 2.6, $\varphi_{u, v}(x) \equiv \sum_{i=1}^{s} \varphi_{u, v}\left(c_{i} w_{i}\right)+$ $h(\bmod 2 k+1)$, where $h=\mid\left\{i \in\{1,2, \ldots, s-1\}: u=\theta_{1}\left(c_{i} w_{i}\right)\right.$ and $v=$ $\left.\theta_{2}\left(c_{i+1} w_{i+1}\right)\right\} \mid$. If there is no $i \in\{1,2, \ldots, s-1\}$ such that $u=\theta_{1}\left(c_{i} w_{i}\right)$ and $v=\theta_{2}\left(c_{i+1} w_{i+1}\right)$, then $h=0$ and by Lemma $2.7, \varphi_{u, v}\left(c_{i} w_{i}\right) \in\{0,-1\}$ for each $i \in\{1,2, \ldots, s\}$.

So assume we have some $i \in\{1,2, \ldots, s-1\}$ such that $u=\theta_{1}\left(c_{i} w_{i}\right)$ and $v=\theta_{2}\left(c_{i+1} w_{i+1}\right)$. We claim that for each $j \in\{1,2, \ldots, s\}, \varphi_{u, v}\left(c_{j} w_{j}\right)=0$. So suppose instead that we have some $j \in\{1,2, \ldots, t\}$ such that $\varphi_{u, v}\left(c_{j} w_{j}\right) \neq 0$. Then by Lemma 2.7, $u=\theta_{1}\left(c_{j} w_{j}\right)$ and $v=\theta_{2}\left(c_{j} w_{j}\right)$. Now $c_{i} \neq c_{i+1}$ so either $c_{i} \neq c_{j}$ or $c_{i+1} \neq c_{j}$. In the first case, $\theta_{1}\left(w_{i}\right)=\theta_{1}\left(w_{j}\right)$ so $u=\theta_{1}\left(c_{i} w_{i}\right) \neq$ $\theta_{1}\left(c_{j} w_{j}\right)=u$, a contradiction. In the second case, $\theta_{2}\left(w_{i+1}\right)=\theta_{2}\left(w_{j}\right)$ so $v=$ $\theta_{2}\left(c_{i+1} w_{i+1}\right) \neq \theta_{2}\left(c_{j} w_{j}\right)=v$, a contradiction.

We now introduce some notation to assist us in our counting of gaps. In the following definition, as well as in the proof of Lemma 2.10, all congruences are $\bmod 2 k+1$. So when we write $|X| \equiv|Y|$ we mean $|X| \equiv|Y|(\bmod 2 k+1)$.
Definition 2.9. Let $x \in T \backslash\{0\}$.
(a) $P(x)=\left\{(u, v) \in K^{2}: \varphi_{u, v}(x) \in\{1,2, \ldots, k\}\right\}$.
(b) $G_{P}(x)=\{(t, u, v) \in G(x):(u, v) \in P(x)\}$.
(c) For $t \in \operatorname{supp}(x), L_{t}(x)=\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G_{P}(x): t^{\prime}<t\right\}$.
(d) For $l \in\{0,1, \ldots, k-1\}$, $S_{l}(x)=\left\{(t, u, v) \in G_{P}(x):\left|L_{t}(x)\right| \equiv l\right\}$.
(e) For $l \in\{0,1, \ldots, k-1\}, T_{l}(x)=\left\{(u, v) \in K^{2}\right.$ : $\left.\left|\left\{t \in \kappa:(t, u, v) \in S_{l}(x)\right\}\right| \equiv 1\right\}$.
(f) For $F \subseteq K^{2}, d \in\{-k,-k+1, \ldots, k-1, k\},(u, v) \in K^{2}$, and $y \in T \backslash\{0\}$,

$$
\begin{aligned}
Y_{F, d, u, v}(y)= & \{(t, u, v) \in G(y): \\
& \left.\mid\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G(y): t^{\prime}<t \text { and }\left(u^{\prime}, v^{\prime}\right) \in F\right\} \mid \equiv d\right\} .
\end{aligned}
$$

(g) For $F \subseteq K^{2}, d \in\{-k,-k+1, \ldots, k-1, k\},(u, v) \in K^{2}$ and $y \in T \backslash\{0\}$, we define $\sigma_{F, d, u, v}(y) \in\{-k,-k+1, \ldots, k-1, k\}$ by

$$
\sigma_{F, d, u, v}(y) \equiv\left|Y_{F, d, u, v}(y)\right| .
$$

Lemma 2.10. Let $r$ be a strongly increasing idempotent in $\beta T$, let $s \in\{1,2, \ldots$, $k\}$, and let $\vec{c}=\left\langle c_{1}, c_{2}, \ldots, c_{s}\right\rangle$ be a compressed sequence in $A \cup B$. Let $R$ denote a member of $r$ on which the functions $\theta_{1}$ and $\theta_{2}$ are constant, the functions $y \mapsto \varphi_{u, v}\left(c_{i} y\right)$ are constant for each $i \in\{1,2, \ldots, s\}$, and the functions $y \mapsto$ $\sigma_{F, d, u, v}\left(c_{i} y\right)$ are constant for each $F \subseteq K^{2}$, each $d \in\{-k,-k+1, \ldots, k-1, k\}$, each $(u, v) \in K^{2}$, and each $i \in\{1,2, \ldots, s\}$. Also, let $R$ have the property that, for each $y \in R,\{z \in R: y \ll z\} \in r$. Choose $w_{1}, w_{2}, \ldots, w_{s} \in R^{\star}$ such that $w_{i} \ll w_{i+1}$ for each $i \in\{1,2, \ldots, s-1\}$ and let $x=c_{1} w_{1}+c_{2} w_{2}+\ldots+c_{s} w_{s}$. Let $l \in\{0,1, \ldots, k-2\}$. If $l \leq s-2$, then $T_{l}(x)=\left\{\left(\theta_{1}\left(c_{l+1} w_{l+1}\right), \theta_{2}\left(c_{l+2} w_{l+2}\right)\right)\right\}$. If $s-2<l$, then $T_{l}(x)=\emptyset$.

Proof. We have by Lemma 2.8 that for $(u, v) \in K^{2},(u, v) \in P(x)$ if and only if there is some $i \in\{1,2, \ldots, s-1\}$ such that $(u, v)=\left(\theta_{1}\left(c_{i} w_{i}\right), \theta_{2}\left(c_{i+1} w_{i+1}\right)\right)$. Therefore, as in the last paragraph of the proof of Lemma 2.8, if $(u, v) \in P(x)$ and $j \in\{1,2, \ldots, s\}$, then $\varphi_{u, v}\left(c_{j} w_{j}\right)=0$.

Let $i \in\{1,2, \ldots, s\}$ be given. Let $t \in \operatorname{supp}\left(c_{i} w_{i}\right)$. Then

$$
\begin{aligned}
L_{t}(x)= & \left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G_{P}(x): t^{\prime}<t\right\} \\
= & \left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G_{P}\left(c_{i} w_{i}\right): t^{\prime}<t\right\} \cup \\
& \bigcup_{j=1}^{i-1}\left(\left\{\left(\delta\left(w_{j}\right), \theta_{1}\left(c_{j} w_{j}\right), \theta_{2}\left(c_{j+1} w_{j+1}\right)\right)\right\} \cup\right. \\
& \left.\bigcup_{\left(u^{\prime}, v^{\prime}\right) \in P(x)}\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right): t^{\prime} \in G_{u^{\prime}, v^{\prime}}\left(c_{j} w_{j}\right)\right\}\right) .
\end{aligned}
$$

(If $i=1$, the above reduces to $L_{t}(x)=\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G_{P}\left(c_{1} w_{1}\right): t^{\prime}<t\right\}$.) Given $\left(u^{\prime}, v^{\prime}\right) \in P(x)$ and $j \in\{1,2, \ldots, i-1\},\left|\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right): t^{\prime} \in G_{u^{\prime}, v^{\prime}}\left(c_{j} w_{j}\right)\right\}\right| \equiv$ $\varphi_{u^{\prime}, v^{\prime}}\left(c_{j} w_{j}\right)$ and $\varphi_{u^{\prime}, v^{\prime}}\left(c_{j} w_{j}\right)=0$, so

$$
\begin{equation*}
\left|L_{t}(x)\right| \equiv\left|\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G_{P}\left(c_{i} w_{i}\right): t^{\prime}<t\right\}\right|+i-1 \tag{*}
\end{equation*}
$$

In particular $\left|L_{t}(x)\right| \equiv l$ if and only if

$$
\left|\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G_{P}\left(c_{i} w_{i}\right): t^{\prime}<t\right\}\right| \equiv l-i+1
$$

Consequently, $(t, u, v) \in G\left(c_{i} w_{i}\right) \cap S_{l}(x)$ if and only if $(t, u, v) \in G\left(c_{i} w_{i}\right)$ and $\mid\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G\left(c_{i} w_{i}\right): t^{\prime}<t\right.$ and $\left.\left(u^{\prime}, v^{\prime}\right) \in P(x)\right\} \mid \equiv l-i+1$.

We still have fixed $i \in\{1,2, \ldots, s\}$. Now fix $(u, v) \in P(x)$. We claim that

$$
\left|\left\{t:(t, u, v) \in G\left(c_{i} w_{i}\right) \cap S_{l}(x)\right\}\right| \equiv 0 .
$$

Define $d \in\{-k,-k+1, \ldots, k-1, k\}$ by $d \equiv l-i+1$. Then

$$
\left|\left\{t:(t, u, v) \in G\left(c_{i} w_{i}\right) \cap S_{l}(x)\right\}\right| \equiv\left|Y_{P(x), d, u, v}\left(c_{i} w_{i}\right)\right| .
$$

So it suffices to show that $\left|Y_{P(x), d, u, v}\left(c_{i} w_{i}\right)\right| \equiv 0$.
We can choose $y \in R^{\star}$ such that $w_{i} \ll y$ and $w_{i}+y \in R^{\star}$. Then the function $\sigma_{P(x), d, u, v}$ is constant on $c_{i} R$, so

$$
\sigma_{P(x), d, u, v}\left(c_{i} w_{i}\right)=\sigma_{P(x), d, u, v}\left(c_{i} y\right)=\sigma_{P(x), d, u, v}\left(c_{i} w_{i}+c_{i} y\right) .
$$

We claim that $\left|Y_{P(x), d, u, v}\left(c_{i} w_{i}+c_{i} y\right)\right| \equiv\left|Y_{P(x), d, u, v}\left(c_{i} w_{i}\right)\right|+\left|Y_{P(x), d, u, v}\left(c_{i} y\right)\right|$. It will follow that $\left|Y_{P(x), d, u, v}\left(c_{i} w_{i}+c_{i} y\right)\right| \equiv 0$ and this will complete the proof of ( $\dagger$ ).

To establish the claim, note that if $(t, u, v) \in G\left(c_{i} w_{i}+c_{i} y\right)$, then $(u, v) \neq$ $\left(\theta_{1}\left(c_{i} w_{i}\right), \theta_{2}\left(c_{i} y\right)\right)$ so $(t, u, v) \in G\left(c_{i} w_{i}+c_{i} y\right)$ if and only if either $(t, u, v) \in$ $G\left(c_{i} w_{i}\right)$ or $(t, u, v) \in G\left(c_{i} y\right)$. (Notice that $(u, v)$ is fixed. We are not claiming that $G\left(c_{i} w_{i}+c_{i} y\right)=G\left(c_{i} w_{i}\right) \cup G\left(c_{i} y\right)$, which is false.) If $(t, u, v) \in G\left(c_{i} w_{i}\right)$ and $\left(u^{\prime}, v^{\prime}\right) \in P(x)$, then

$$
\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G\left(c_{i} w_{i}+c_{i} y\right): t^{\prime}<t\right\}=\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G\left(c_{i} w_{i}\right): t^{\prime}<t\right\} .
$$

If $(t, u, v) \in G\left(c_{i} y\right)$ and $\left(u^{\prime}, v^{\prime}\right) \in P(x)$, then $\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G\left(c_{i} w_{i}+c_{i} y\right)\right.$ : $\left.t^{\prime}<t\right\}=\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G\left(c_{i} y\right): t^{\prime}<t\right\} \cup\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right): t^{\prime} \in G_{u^{\prime}, v^{\prime}}\left(c_{i} w_{i}\right)\right\}$ and $\left|\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right): t^{\prime} \in G_{u^{\prime}, v^{\prime}}\left(c_{i} w_{i}\right)\right\}\right| \equiv \varphi_{u^{\prime}, v^{\prime}}\left(c_{i} w_{i}\right)=0$. Thus $\mid Y_{P(x), d, u, v}\left(c_{i} w_{i}+\right.$ $\left.c_{i} y\right)|\equiv| Y_{P(x), d, u, v}\left(c_{i} w_{i}\right)\left|+\left|Y_{P(x), d, u, v}\left(c_{i} y\right)\right|\right.$ as claimed.

Now $\delta\left(w_{i}\right) \in \operatorname{supp}\left(c_{i} w_{i}\right)$ and

$$
\begin{aligned}
& \left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G_{P}\left(c_{i} w_{i}\right): t^{\prime}<\delta\left(w_{i}\right)\right\}= \\
& G_{P}\left(c_{i} w_{i}\right)=\bigcup_{\left(u^{\prime}, v^{\prime}\right) \in P(x)}\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right): t^{\prime} \in G_{u^{\prime} v^{\prime}}\left(c_{i} w_{i}\right)\right\} .
\end{aligned}
$$

Therefore $\left|\left\{\left(t^{\prime}, u^{\prime}, v^{\prime}\right) \in G_{P}\left(c_{i} w_{i}\right): t^{\prime}<\delta\left(w_{i}\right)\right\}\right| \equiv 0$. Thus by (*), we have $\left|L_{\delta\left(w_{i}\right)}(x)\right| \equiv i-1$ so $\left|L_{\delta\left(w_{i}\right)}(x)\right| \equiv l$ if and only if $i=l+1$. In particular, $\left(\delta\left(w_{i}\right), \theta_{1}\left(c_{i} w_{i}\right), \theta_{2}\left(c_{i+1} w_{i+1}\right)\right) \in S_{l}(x)$ and if $i \neq l+1$, then $\left(\delta\left(w_{i}\right), \theta_{1}\left(c_{i} w_{i}\right), \theta_{2}\left(c_{i+1} w_{i+1}\right)\right) \notin S_{l}(x)$.

Thus, if $l \leq s-2$ and $(u, v)=\left(\theta_{1}\left(c_{l+1} w_{l+1}\right), \theta_{2}\left(c_{l+2} w_{l+2}\right)\right)$, then $\left\{t:(t, u, v) \in S_{l}(x)\right\}=\left\{\delta\left(w_{l+1}\right)\right\} \cup \bigcup_{i=1}^{s}\left\{t:(t, u, v) \in G\left(c_{i} w_{i}\right) \cap S_{l}(x)\right\}$ so by $(\dagger)$, $\left|\left\{t:(t, u, v) \in S_{l}(x)\right\}\right| \equiv 1$. Consequently, $\left(\theta_{1}\left(c_{l+1} w_{l+1}\right), \theta_{2}\left(c_{l+2} w_{l+2}\right)\right) \in T_{l}(x)$.

Finally, if either $l>s-2$ or $(u, v) \neq\left(\theta_{1}\left(c_{l+1} w_{l+1}\right), \theta_{2}\left(c_{l+2} w_{l+2}\right)\right)$, then $\left\{t:(t, u, v) \in S_{l}(x)\right\}=\bigcup_{i=1}^{s}\left\{t:(t, u, v) \in G\left(c_{i} w_{i}\right) \cap S_{l}(x)\right\}$ so by ( $\dagger$ ), $\left|\left\{t:(t, u, v) \in S_{l}(x)\right\}\right| \equiv 0$ and thus $(u, v) \notin T_{l}(x)$.

We now define a finite coloring $\Gamma$ of $T \backslash\{0\}$ by agreeing that $\Gamma(x)=\Gamma(y)$ if and only if
(1) $\theta_{1}(x)=\theta_{1}(y)$,
(2) $\theta_{2}(x)=\theta_{2}(y)$,
(3) $\varphi_{u, v}(x)=\varphi_{u, v}(y)$ for all $(u, v) \in K^{2}$, and
(4) $T_{l}(x)=T_{l}(y)$ for all $l \in\{0,1, \ldots, k-2\}$.

We claim this coloring is as required to complete the proof of Theorem 2.3.
Suppose instead we have strongly increasing idempotents $p$ and $q$ in $\beta T$ and a color class $D$ of $\Gamma$ which is a member of both $a_{1} p+a_{2} p+\ldots+a_{m} p$ and $b_{1} q+b_{2} q+\ldots+b_{k} q$.

Let $P \in p$ and $Q \in q$ be sets on which all of the functions mentioned in the statement of Lemma 2.10 are constant. Suppose also that, for every $y \in P$, $\{z \in P: y \ll z\} \in p$, and for every $y \in Q,\{z \in Q: y \ll z\} \in q$. Pick $w_{1}, w_{2}, \ldots, w_{m}$ in $P^{\star}$ such that $w_{i} \ll w_{i+1}$ for all $i \in\{1,2, \ldots, m-1\}$ and pick $z_{1}, z_{2}, \ldots, z_{k}$ in $Q^{\star}$ such that $z_{i} \ll z_{i+1}$ for all $i \in\{1,2, \ldots, k-1\}$. Let $x=a_{1} w_{1}+a_{2} w_{2}+\ldots+a_{m} w_{m}$ and let $y=b_{1} z_{1}+b_{2} z_{2}+\ldots+b_{k} z_{k}$. By Lemma 2.10, for each $l \in\{0,1, \ldots, m-2\}, T_{l}(x)=\left\{\left(\theta_{1}\left(a_{l+1} w_{l+1}\right), \theta_{2}\left(a_{l+2} w_{l+2}\right)\right)\right\}$ and for each $l \in\{0,1, \ldots, k-2\}, T_{l}(y)=\left\{\left(\theta_{1}\left(b_{l+1} z_{l+1}\right), \theta_{2}\left(b_{l+2} z_{l+2}\right)\right)\right\}$. Also, if $m-2<l \leq k-2$, then $T_{l}(x)=\emptyset$.

We now claim that $m=k$. Indeed, if $m<k$, then by Lemma 2.10, $\emptyset=$ $T_{k-2}(x)=T_{k-2}(y)=\left\{\left(\theta_{1}\left(b_{k-1} z_{k-1}\right), \theta_{2}\left(b_{k} z_{k}\right)\right)\right\}$, a contradiction.

By Lemma 2.10, for each $i \in\{1,2, \ldots, k-1\}, \theta_{1}\left(a_{i} w_{i}\right)=\theta_{1}\left(b_{i} z_{i}\right)$. Also, $\theta_{1}\left(a_{k} w_{k}\right)=\theta_{1}(x)=\theta_{1}(y)=\theta_{1}\left(b_{k} z_{k}\right)$ and thus for each $i \in\{1,2, \ldots, k\}$, $\theta_{1}\left(a_{i} w_{i}\right)=\theta_{1}\left(b_{i} z_{i}\right)$.

We are supposing that $\vec{b}$ is not a rational multiple of $\vec{a}$ so pick the first $s \in\{2,3, \ldots, m\}$ such that $b_{s} / a_{s} \neq b_{1} / a_{1}$. Then

$$
\psi\left(a_{1}\left(w_{s}\right)_{\delta\left(w_{s}\right)}\right)=\theta_{1}\left(a_{1} w_{s}\right)=\theta_{1}\left(a_{1} w_{1}\right)=\theta_{1}\left(b_{1} z_{1}\right)=\theta_{1}\left(b_{1} z_{s}\right)=\psi\left(b_{1}\left(z_{s}\right)_{\delta\left(z_{s}\right)}\right)
$$

so by Lemma 2.1,

$$
\begin{aligned}
& \theta_{1}\left(a_{s} w_{s}\right)=\psi\left(a_{s}\left(w_{s}\right)_{\delta\left(w_{s}\right)}\right)=\psi\left(\frac{a_{s}}{a_{1}} a_{1}\left(w_{s}\right)_{\delta\left(w_{s}\right)}\right) \neq \\
& \psi\left(\frac{b_{s}}{b_{1}} b_{1}\left(z_{s}\right)_{\delta\left(z_{s}\right)}\right)=\psi\left(b_{s}\left(z_{s}\right)_{\delta\left(z_{s}\right)}\right)=\theta_{1}\left(b_{s} z_{s}\right)
\end{aligned}
$$

a contradiction.
Lemma 2.11. Let $r$ be a strongly increasing idempotent in $\beta T$, let $s \in \mathbb{N}$, and let $\left\langle c_{i}\right\rangle_{i=1}^{s}$ be a sequence in $\mathbb{Z} \backslash\{0\}$. Let $R \subseteq T \backslash\{0\}$ be a member of $r$, with the property that for each $y \in R,\{z \in R: y \ll z\} \in r$. Let

$$
\begin{array}{ll}
C=\left\{c_{1} w_{1}+c_{2} w_{2}+\ldots+c_{s} w_{s}:\right. & w_{1}, \ldots, w_{s} \in R \text { and } \\
& \text { for all } \left.\left.i \in\{1,2, \ldots, s-1\}, w_{i} \ll w_{i+1}\right)\right\} .
\end{array}
$$

Then $C \in c_{1} r+c_{2} r+\ldots+c_{s} r$.
Proof. We proceed by induction on $s$. Our claim is clearly true if $s=1$. So assume that $s>1$ and that our claim holds for $s-1$. Let $S \in c_{1} r+c_{2} r+\ldots+c_{s} r$. Since $\left\{y \in T \backslash\{0\}: y+c_{s} r \in \bar{S}\right\} \in c_{1} r+c_{2} r+\ldots+c_{s-1} r$, it follows from our inductive assumption that we can choose $w_{1}, w_{2}, \ldots, w_{s-1} \in R$ such that $c_{1} w_{1}+$ $c_{2} w_{2}+\ldots+c_{s-1} w_{s-1}+c_{s} r \in \bar{S}$ and $w_{i} \ll w_{i+1}$ for every $i<s-2$. We can then choose $w_{s} \in R$ such that $w_{s-1} \ll w_{s}$ and $c_{1} w_{1}+c_{2} w_{2}+\ldots+c_{s-1} w_{s-1}+c_{s} w_{s} \in S$. So $C \cap S \neq \emptyset$.

Corollary 2.12. Let $k, m \in \mathbb{N}$ and let $\kappa$ be an infinite cardinal. Let $\vec{a}=$ $\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$ such that $\vec{b}$ is not a rational multiple of $\vec{a}$. Let $T=\bigoplus_{\sigma<\kappa} \mathbb{Q}$. There is a finite coloring of $T \backslash\{0\}$ such that there do not exist strongly increasing sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $T \backslash\{0\}$ with $\operatorname{MT}\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ monochromatic.

Proof. Let $\Gamma$ be a finite coloring of $T \backslash\{0\}$ as guaranteed by Theorem 2.3 and suppose we have strongly increasing sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $T \backslash\{0\}$ and a color class $D$ of $\Gamma$ such that $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq D$. Pick an idempotent $p \in \bigcap_{l=1}^{\infty} \overline{F S\left(\left\langle x_{n}\right\rangle_{n=l}^{\infty}\right)}$ and an idempotent $q \in \bigcap_{l=1}^{\infty} \overline{F S\left(\left\langle y_{n}\right\rangle_{n=l}^{\infty}\right)}$. As we have observed, $p$ and $q$ are strongly increasing. By Lemma 2.11, with $\left.R=F S\left(\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right)\right), M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \in a_{1} p+a_{2} p+\ldots+a_{m} p$ and, taking $R=$ $F S\left(\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right), M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right) \in b_{1} q+b_{2} q+\ldots+b_{k} q$. This contradicts Theorem 2.3.

Lemma 2.13. Let $(S,+)$ and $(T,+)$ be discrete commutative groups, let $\gamma: T \rightarrow S$ be a homomorphism, and let $\widetilde{\gamma}: \beta T \rightarrow \beta S$ be its continuous extension. Then for all $p, q \in \beta T$ and all $a, b \in \mathbb{Z} \backslash\{0\}, \widetilde{\gamma}(a p+b q)=a \widetilde{\gamma}(p)+b \widetilde{\gamma}(q)$.
Proof. We are claiming that $\widetilde{\gamma} \circ \rho_{b q} \circ \widetilde{l}_{a}$ and $\rho_{b \gamma}(q) \circ \widetilde{l}_{a} \circ \widetilde{\gamma}$ agree on $\beta T$. Since both are continuous functions, it suffices to show that they agree on $T$, so let $x \in T$. We claim $\widetilde{\gamma}(a x+b q)=a \gamma(x)+b \widetilde{\gamma}(q)$. To see that $\widetilde{\gamma} \circ \lambda_{a x} \circ \widetilde{l}_{b}$ and $\lambda_{a \gamma(x)} \circ \widetilde{l}_{b} \circ \widetilde{\gamma}$ agree on $T$ it suffices that they agree on $T$ so let $y \in T$. Then $\gamma(a x+b y)=a \gamma(x)+b \gamma(y)$ as required.

Theorem 2.14. Let $k, m \in \mathbb{N}$ and let $\kappa>0$ be a cardinal. Let $\vec{a}=\left\langle a_{1}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$ such that $\vec{b}$ is not $a$ rational multiple of $\vec{a}$. Let $T=\bigoplus_{\sigma<\kappa} \mathbb{Q}$. Then there do not exist idempotents $p$ and $q$ in $\beta T \backslash\{0\}$ such that $a_{1} p+\ldots+a_{m} p=b_{1} q+\ldots+b_{k} q$.

Proof. Suppose the conclusion fails and let $\kappa$ be the first cardinal for which the conclusion fails. By Theorem 1.5, $\kappa>1$.

Let $n<\min \{\kappa, \omega\}$, let $\pi_{n}: T \rightarrow \mathbb{Q}$ be the projection defined by $\pi_{n}(x)=x_{n}$ and let $\widetilde{\pi}_{n}: \beta T \rightarrow \beta \mathbb{Q}$ be its continuous extension. By Lemma 2.13, $\widetilde{\pi}_{n}(p)$ and $\widetilde{\pi}_{n}(q)$ are idempotents in $\beta \mathbb{Q}$ and $a_{1} \widetilde{\pi}_{n}(p)+\ldots+a_{m} \widetilde{\pi}_{n}(p)=b_{1} \widetilde{\pi}_{n}(q)+\ldots+$ $b_{k} \widetilde{\pi}_{n}(q)$ so by assumption either $\widetilde{\pi}_{n}(p)=0$ or $\widetilde{\pi}_{n}(q)=0$. But by [2, Theorem 4.31], $\mathbb{Q}^{*}$ is a left ideal of $\beta \mathbb{Q}$ so the equation

$$
a_{1} \widetilde{\pi}_{n}(p)+\ldots+a_{m} \widetilde{\pi}_{n}(p)=b_{1} \widetilde{\pi}_{n}(q)+\ldots+b_{k} \widetilde{\pi}_{n}(q)
$$

implies that $\widetilde{\pi}_{n}(p)=\widetilde{\pi}_{n}(q)=0$. Consequently, $\left\{x \in T: x_{n}=0\right\} \in p \cap q$.
If $\kappa<\omega$, one then has that $\bigcap_{n<\kappa}\left\{x \in T: x_{n}=0\right\} \in p \cap q$. That is $p=q=0$, a contradiction. Thus we have that $\kappa \geq \omega$. For each $\sigma<\kappa$, let $C_{\sigma}=\left\{x \in T \backslash\{0\}:(\forall \tau<\sigma)\left(x_{\tau}=0\right)\right\}$. We claim that $p, q \in \bigcap_{\sigma<\kappa} \overline{C_{\sigma}}$. If $\sigma<\omega$, then $C_{\sigma}=\left(\bigcap_{n<\sigma}\left\{x \in T: x_{n}=0\right\}\right) \backslash\{0\}$. So if $\kappa=\omega$, the claim is established.

Assume that $\kappa>\omega$ and let $\omega \leq \sigma<\kappa$. Let $\pi: T \rightarrow \bigoplus_{\tau<\sigma} \mathbb{Q}$ be the natural projection and let $\widetilde{\pi}: \beta T \rightarrow \beta\left(\bigoplus_{\tau<\sigma} \mathbb{Q}\right)$ be its continuous extension. By Lemma 2.13, $\widetilde{\pi}(p)$ and $\widetilde{\pi}(q)$ are idempotents in $\beta\left(\bigoplus_{\tau<\sigma} \mathbb{Q}\right)$ and

$$
a_{1} \widetilde{\pi}(p)+\ldots+a_{m} \widetilde{\pi}(p)=b_{1} \widetilde{\pi}(q)+\ldots+b_{k} \widetilde{\pi}(q)
$$

Let $\mu=|\sigma|$. Then $\bigoplus_{\tau<\sigma} \mathbb{Q}$ and $\bigoplus_{\tau<\mu} \mathbb{Q}$ are isomorphic (and discrete) so $\beta\left(\bigoplus_{\tau<\sigma} \mathbb{Q}\right)$ and $\beta\left(\bigoplus_{\tau<\mu} \mathbb{Q}\right)$ are topologically and algebraically isomorphic. Consequently, by the assumption that $\kappa$ is the smallest cardinal for which the conclusion of the theorem fails, we must have that $\widetilde{\pi}(p)=0$ or $\widetilde{\pi}(q)=0$. And this in turn forces the conclusion that $\widetilde{\pi}(p)=\widetilde{\pi}(q)=0$. So $C_{\sigma} \in p \cap q$.

We thus have that $p, q \in \bigcap_{\sigma<\kappa} \overline{C_{\sigma}}$ as claimed. It follows that $p$ and $q$ are strongly increasing idempotents in $\beta T$. But this contradicts Theorem 2.3.

We get the strongest possible conclusion for preventing equality of linear expressions in direct sums of $\mathbb{Q}$.

Corollary 2.15. Let $k, m \in \mathbb{N}$, let $\kappa>0$ be a cardinal, and let $T=\bigoplus_{\sigma<\kappa} \mathbb{Q}$. Let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$. There exist idempotents $p$ and $q$ in $\beta T \backslash\{0\}$ such that $a_{1} p+\ldots+a_{m} p=$ $b_{1} q+\ldots+b_{k} q$ if and only if $\vec{b}$ is a rational multiple of $\vec{a}$.

Proof. The necessity follows from Theorems 2.14 and 1.5.
For the sufficiency, by Theorem 1.5, there exist idempotents $p$ and $q$ in $\beta \mathbb{Q} \backslash\{0\}$ such that $a_{1} p+\ldots+a_{m} p=b_{1} q+\ldots+b_{k} q$. If one lets

$$
p^{\prime}=\left\{A \subseteq T: \pi_{0}[A] \in p\right\} \text { and } q^{\prime}=\left\{A \subseteq T: \pi_{0}[A] \in q\right\}
$$

it is a routine exercise to show that $p^{\prime}$ and $q^{\prime}$ are idempotents in $\beta T \backslash\{0\}$ such that $a_{1} p^{\prime}+\ldots+a_{m} p^{\prime}=b_{1} q^{\prime}+\ldots+b_{k} q^{\prime}$.

We can provide half of the answer to Question 1.1.
Corollary 2.16. Let $G$ be a torsion free discrete commutative group, let $k, m \in$ $\mathbb{N}$, and let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$ which are not rational multiples of each other. There do not exist idempotents $p$ and $q$ in $\beta G \backslash\{0\}$ such that $a_{1} p+a_{2} p+\ldots+a_{m} p=b_{1} q+b_{2} q+$ $\ldots+b_{k} q$.

Proof. There is a cardinal $\kappa$ such that $G$ can be embedded in $\bigoplus_{\sigma<\kappa} \mathbb{Q}$.

## 3. Separating Milliken-Taylor systems

We have mentioned that Milliken-Taylor systems are partition regular. In fact, a stronger statement holds.
Definition 3.1. Let $(S,+)$ be a commutative semigroup and let $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ be sequences in $S$. Then $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ is a sum subsystem of $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ if and only if there is a sequence $\left\langle F_{n}\right\rangle_{n=1}^{\infty}$ of finite nonempty subsets of $\mathbb{N}$ such that $F_{n}<F_{n+1}$ for each $n$ and $x_{n}=\sum_{t \in F_{n}} y_{t}$ for each $n$.

Theorem 3.2. Let $k, r \in \mathbb{N}$, let $\vec{a}=\left\langle a_{1}, \ldots, a_{k}\right\rangle$ be a compressed sequence in $\mathbb{Z} \backslash\{0\}$, let $(G,+)$ be a commutative group, and let $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ be a sequence in G. If $\operatorname{MT}\left(\vec{a},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)=\bigcup_{i=1}^{r} A_{i}$, then there exist $i \in\{1,2, \ldots, r\}$ and a sum subsystem $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ of $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ such that $\operatorname{MT}\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq A_{i}$.

Proof. [3, Theorem 1.3].
The proof of the following corollary uses a standard technique.
Corollary 3.3. Let $m, k \in \mathbb{N}$ and let $\vec{a}=\left\langle a_{1}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$ such that $\vec{b}$ is not a rational multiple of $\vec{a}$. Then there is a 2 -coloring of $\mathbb{Q} \backslash\{0\}$ with the property that there do not exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $\mathbb{Q} \backslash\{0\}$ with $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ monochromatic.

Proof. Pick by Theorem 1.5, $r \in \mathbb{N}$ and a function $\psi: \psi \backslash\{0\} \rightarrow\{1,2, \ldots, r\}$ such that there do not exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $\mathbb{Q} \backslash\{0\}$ with

$$
M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)
$$

monochromatic. Let

$$
J=\left\{i \in\{1,2, \ldots, r\}:\left(\exists\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right)\left(M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq \psi^{-1}[\{i\}]\right)\right\}
$$

Let $A_{1}=\bigcup_{i \in J} \psi^{-1}[\{i\}]$ and let $A_{2}=\mathbb{Q} \backslash\left(\{0\} \cup A_{1}\right)$. Then by Theorem 3.2, there is no sequence $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ with $M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq A_{1}$ and there is no sequence $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ with $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq A_{2}$.

The following theorem is the only complete answer we can give for separating Milliken-Taylor systems in direct sums.

Theorem 3.4. Let $m, k, r, l \in \mathbb{N}$. Let $(G,+)$ be a commutative group and let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$. Assume that there is an r-coloring of $G \backslash\{0\}$ with the property that there do not exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $G \backslash\{0\}$ with

$$
M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)
$$

monochromatic. Let $T=\bigoplus_{i<l} G$. Then there is an $r$-coloring of $T \backslash\{0\}$ with the property that there do not exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $T \backslash\{0\}$ with $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ monochromatic.
Proof. Let $\psi$ be an $r$-coloring of $G \backslash\{0\}$ with the property that there do not exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $G \backslash\{0\}$ with

$$
M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)
$$

monochromatic. Define an $r$-coloring $\tau$ of $T \backslash\{0\}$ by $\tau(x)=\psi\left(x_{\alpha(x)}\right)$. Suppose we have sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $T$ with

$$
M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)
$$

monochromatic with respect to $\tau$.
By Theorem 3.2, pick $i<l, j<l$, a sum subsystem $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ of $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$, and a sum subsytem $\left\langle w_{n}\right\rangle_{n=1}^{\infty}$ of $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ such that for all $u \in M T\left(\vec{a},\left\langle z_{n}\right\rangle_{n=1}^{\infty}\right)$, $\alpha(u)=i$ and for all $v \in M T\left(\vec{b},\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right), \alpha(v)=j$. Note that no $m$ terms of $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ can have $\left(z_{n}\right)_{i}=0$ or else there would be a member $u$ of $M T\left(\vec{a},\left\langle z_{n}\right\rangle_{n=1}^{\infty}\right)$ with either $u=0$ or $\alpha(u) \neq i$, so we may assume no term of $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ has $\left(z_{n}\right)_{i}=0$, and similarly for $\left\langle w_{n}\right\rangle_{n=1}^{\infty}$. Define sequences $\left\langle z_{n}^{\prime}\right\rangle_{n=1}^{\infty}$ and $\left\langle w_{n}^{\prime}\right\rangle_{n=1}^{\infty}$ in $G$ by $z_{n}^{\prime}=\left(z_{n}\right)_{i}$ and $w_{n}^{\prime}=\left(w_{n}\right)_{j}$. Then $M T\left(\vec{a},\left\langle z_{n}^{\prime}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle w_{n}^{\prime}\right\rangle_{n=1}^{\infty}\right)$ is monochromatic with respect to $\psi$.

Corollary 3.5. Let $m, k, l \in \mathbb{N}$ and let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}\right.$, $\left.\ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$ such that $\vec{b}$ is not a rational multiple of $\vec{a}$. Let $T=\bigoplus_{i<l} \mathbb{Q}$. Then there is a 2 -coloring of $T \backslash\{0\}$ with the property that there do not exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $T \backslash\{0\}$ with $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ monochromatic.

Proof. Corollary 3.3 and Theorem 3.4.
Definition 3.6. Let $\kappa$ be an infinite cardinal, let $T=\bigoplus_{\sigma<\kappa} \mathbb{Q}$, and let $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ be a sequence in $T \backslash\{0\}$.
(a) The sequence $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ is of type $I$ if and only if it is strongly increasing.
(b) The sequence $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ is of type II if and only if there is some $\sigma<\kappa$ such that for all $n, \alpha\left(z_{n}\right)=\sigma$.
(c) The sequence $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ is of type III if and only if there is some $\sigma<\kappa$ such that for all $n, \delta\left(z_{n}\right)=\sigma$.

Lemma 3.7. Let $m \in \mathbb{N}$, let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ be a compressed sequence in $\mathbb{Z} \backslash\{0\}$, let $\kappa$ be an infinite cardinal, and let $T=\bigoplus_{\sigma<\kappa} \mathbb{Q}$.
(1) If $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ is a type II sequence in $T \backslash\{0\}, \alpha\left(z_{n}\right)=\sigma$ for all $n \in \mathbb{N}$, and $H=\{x \in T \backslash\{0\}: \alpha(x)=\sigma\}$, then there is a sum subsystem $\left\langle w_{n}\right\rangle_{n=1}^{\infty}$ of $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ such that $F S\left(\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{a},\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq H$.
(2) If $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ is a type III sequence in $T \backslash\{0\}, \delta\left(z_{n}\right)=\sigma$ for all $n \in \mathbb{N}$, and $H=\{x \in T \backslash\{0\}: \delta(x)=\sigma\}$, then there is a sum subsystem $\left\langle w_{n}\right\rangle_{n=1}^{\infty}$ of $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ such that $F S\left(\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{a},\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq H$.

Proof. The proofs are essentially identical. We shall do the proof for (1). By passing to a subsequence, we may presume that all $\pi_{\sigma}\left(z_{n}\right)$ 's are the same sign and thus $F S\left(\left\langle z_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq H$. By Theorem 3.2, pick a sum subsystem $\left\langle w_{n}\right\rangle_{n=1}^{\infty}$ of $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ such that either $M T\left(\vec{a},\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq H$ or $M T\left(\vec{a},\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq T \backslash H$. We claim that the latter is impossible so suppose that $M T\left(\vec{a},\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq T \backslash H$. Then $a_{1} w_{1}+a_{2} w_{2}+\ldots+a_{m} w_{m} \in T \backslash H$ so $\pi_{\sigma}\left(a_{1} w_{1}+a_{2} w_{2}+\ldots+a_{m} w_{m}\right)=0$. But $\pi_{\sigma}\left(a_{m} w_{m+1}\right) \neq 0$ so $a_{1} w_{1}+a_{2} w_{2}+\ldots+a_{m}\left(w_{m}+w_{m+1}\right) \in H$.

Theorem 3.8. Let $k, m \in \mathbb{N}$ and let $\kappa$ be an infinite cardinal. Let $\vec{a}=$ $\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$ such that $\vec{b}$ is not a rational multiple of $\vec{a}$. Let $T=\bigoplus_{\sigma<\kappa} \mathbb{Q}$. There is a finite coloring of $T \backslash\{0\}$ such that there do not exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $T \backslash\{0\}$ of the same type with $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ monochromatic.

Proof. Let $\Gamma$ be a finite coloring of $T \backslash\{0\}$ as guaranteed for $\vec{a}$ and $\vec{b}$ by Corollary 2.12 and let $\psi$ be a finite coloring of $\mathbb{Q} \backslash\{0\}$ as guaranteed for $\vec{a}$ and $\vec{b}$ by Theorem 1.5. Define a coloring $\tau$ of $T \backslash\{0\}$ by, for $x \in T \backslash\{0\}, \tau(x)=$ $\left(\Gamma(x), \psi\left(x_{\alpha(x)}\right), \psi\left(x_{\delta(x)}\right)\right)$. Suppose that one has sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ of the same type with $\operatorname{MT}\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ monochromatic of color $(i, j, l)$. One has immediately that the sequences are not of type I. Suppose that $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ are both of type II and pick $\sigma$ and $\gamma$ such that for all $n \in \mathbb{N}, \alpha\left(x_{n}\right)=\sigma$ and $\alpha\left(y_{n}\right)=\gamma$. Let $H=\{x \in T \backslash\{0\}: \alpha(x)=\sigma\}$ and let $K=\{x \in T \backslash\{0\}: \alpha(x)=\gamma\}$. By Lemma 3.7 we may pick a sum subsystem $\left\langle w_{n}\right\rangle_{n=1}^{\infty}$ of $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and a sum subsystem $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ of $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ such that $F S\left(\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{a},\left\langle w_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq H$ and $F S\left(\left\langle z_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle z_{n}\right\rangle_{n=1}^{\infty}\right) \subseteq$ $K$. For each $n \in \mathbb{N}$, let $w_{n}^{\prime}=\pi_{\sigma}\left(w_{n}\right)$ and let $z_{n}^{\prime}=\pi_{\gamma}\left(z_{n}\right)$. Then $\psi$ is constantly equal to $j$ on $M T\left(\vec{a},\left\langle w_{n}^{\prime}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle z_{n}^{\prime}\right\rangle_{n=1}^{\infty}\right)$, a contradiction.

Similarly the sequences cannot both be of type III.
Corollary 3.9. Let $m, k, l \in \mathbb{N}$, let $T=\bigoplus_{n<\omega} \mathbb{Q}$, and let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$, $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$, and $\vec{c}=\left\langle c_{1}, c_{2}, \ldots, c_{l}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$, no one of which is a rational multiple of another. There is a finite coloring of $T \backslash\{0\}$ such that there do not exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty},\left\langle y_{n}\right\rangle_{n=1}^{\infty}$, and $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ in $T \backslash\{0\}$ such that $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{c},\left\langle z_{n}\right\rangle_{n=1}^{\infty}\right)$ is monochromatic.
Proof. Any sequence in $T \backslash\{0\}$ has a subsequence of type I or a subsequence of type II, so this is an immediate consequence of Theorem 3.8.

It is a consequence of Corollary 2.16 that we can prevent equality of linear expressions in $\beta \mathbb{R}_{d}$ and its direct sums.
Question 3.10. Let $m, k, l \in \mathbb{N}$ and let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle, \vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$, and $\vec{c}=\left\langle c_{1}, c_{2}, \ldots, c_{l}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$, no one of which is a rational multiple of another. Is there a finite coloring of $\mathbb{R} \backslash\{0\}$ such that there do not exist sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty},\left\langle y_{n}\right\rangle_{n=1}^{\infty}$, and $\left\langle z_{n}\right\rangle_{n=1}^{\infty}$ in $\mathbb{R} \backslash\{0\}$ such that $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{c},\left\langle z_{n}\right\rangle_{n=1}^{\infty}\right)$ is monochromatic?

We conclude by showing that we can separate Milliken-Taylor systems in $(\mathbb{N}, \cdot)$.

Theorem 3.11. Let $\kappa>0$ be a cardinal and let $T=\bigoplus_{\sigma<\kappa} \omega$. Let $m, k, \in \mathbb{N}$ and let $\vec{a}=\left\langle a_{1}, a_{2}, \ldots, a_{m}\right\rangle$ and $\vec{b}=\left\langle b_{1}, b_{2}, \ldots, b_{k}\right\rangle$ be compressed sequences in $\mathbb{Z} \backslash\{0\}$, which are not rational multiples of each other. There is a finite colouring of $T$ with the property that there are no sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $T \backslash\{0\}$ for which $\operatorname{MT}\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ is monochromatic. In particular, the systems $\operatorname{MT}\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right)$ and $M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ can be separated over $(\mathbb{N}, \cdot)$.

Proof. Let $G=\bigoplus_{\sigma<\kappa} \mathbb{Z}$. Define $h: G \rightarrow \mathbb{Z}$ by $h(x)=\sum_{\sigma<\kappa} x_{\sigma}$. Observe that $h$ is a group homomorphism and $h[T \backslash\{0\}]=\omega \backslash\{0\}$. Pick by Theorem 1.5 a finite coloring $\Gamma$ of $\mathbb{Z} \backslash\{0\}$ such that there are no sequences $\left\langle s_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle t_{n}\right\rangle_{n=1}^{\infty}$ in $\mathbb{Z} \backslash\{0\}$ for which $M T\left(\vec{a},\left\langle s_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle t_{n}\right\rangle_{n=1}^{\infty}\right)$ is monochromatic with respect to $\Gamma$. Let $\psi$ be the restriction of $\Gamma \circ h$ to $T \backslash\{0\}$. Then $\psi$ is a finite coloring of $T \backslash\{0\}$. If there were sequences $\left\langle x_{n}\right\rangle_{n=1}^{\infty}$ and $\left\langle y_{n}\right\rangle_{n=1}^{\infty}$ in $T \backslash\{0\}$ with $M T\left(\vec{a},\left\langle x_{n}\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle y_{n}\right\rangle_{n=1}^{\infty}\right)$ monochromatic with respect to $\psi$, then one would have $M T\left(\vec{a},\left\langle h\left(x_{n}\right)\right\rangle_{n=1}^{\infty}\right) \cup M T\left(\vec{b},\left\langle h\left(y_{n}\right)\right\rangle_{n=1}^{\infty}\right)$ monochromatic with respect to $\Gamma$.

The "in particular" conclusion holds because $(\mathbb{N}, \cdot)$ is isomorphic to $\bigoplus_{\sigma<\omega} \omega$.

## References

[1] N. Hindman, I. Leader, and D. Strauss, Separating Milliken-Taylor systems with negative entries, Proc. Edinburgh Math. Soc. 46 (2003), 45-61.
[2] N. Hindman and D. Strauss, Algebra in the Stone-Čech compactification: theory and applications, 2nd edition, Walter de Gruyter \& Co., Berlin, 2012.
[3] N. Hindman and D. Strauss, Separating Milliken-Taylor systems in $\mathbb{Q}$, J. Combinatorics 5 (2014), 305-333.
[4] K. Milliken, Ramsey's Theorem with sums or unions, J. Comb. Theory (Series A) $\mathbf{1 8}$ (1975), 276-290.
[5] A. Taylor, A canonical partition relation for finite subsets of $\omega$, J. Comb. Theory (Series A) 21 (1976), 137-146.


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    ${ }^{1}$ This author acknowledges support received from the National Science Foundation via Grant DMS-1160566.

