This paper was published in *Journal of Combinatorial Theory (Series A)* **113** (2006), 1219-1242. To the best of my knowledge this is the final version as it was submitted to the publisher. – NH

Multiplicative Structures in Additively Large Sets

Mathias Beiglböck[1]

Vitaly Bergelson[2]

Neil Hindman[2]

and

Dona Strauss

Abstract. Previous research extending over a few decades has established that multiplicatively large sets (in any of several interpretations) must have substantial additive structure. We investigate here the question of how much multiplicative structure can be found in additively large sets. For example, we show that any translate of a set of finite sums from an infinite sequence must contain all of the initial products from another infinite sequence. And, as a corollary of a result of Renling Jin, we show that if A and B have positive upper Banach density, then $A + B$ contains all of the initial products from an infinite sequence. We also show that if a set has a complement which is not additively piecewise syndetic, then any translate of that set is both additively and multiplicatively large in several senses.

We investigate whether a subset of $\mathbb{N}$ with bounded gaps – a *syndetic* set – must contain arbitrarily long geometric progressions. We believe that we establish that this is a significant open question.

1. Introduction

It has been known since 1979 [17, Theorem 2.6] that whenever the set $\mathbb{N}$ of positive integers is partitioned into finitely many parts, one of those parts must contain a sequence with all of its finite sums (without repetition) and another sequence with all of its finite products. And it was shown in 1990 [5, Corollary 5.5] that one cell of any finite partition of $\mathbb{N}$ must be both additively and multiplicatively *central* (a notion defined in Definition 1.4 below). In particular, in addition to the finite sums and products mentioned above, it must contain arbitrarily long arithmetic progressions and arbitrarily long geometric progressions. (See [19, Part III] for much more information about the kinds of combinatorial structures guaranteed to any central set.)

[1]This author thanks the Austrian Science Foundation FWF for its support through Project no. S8312

[2]These authors acknowledge support received from the National Science Foundation (USA) via grants DMS-0345350 and DMS-0243586 respectively.

As we noted, we take $\mathbb{N}$ to be the set of positive integers. We write ω for $\mathbb{N} \cup \{0\}$, the set of nonnegative integers.

There is a long history in Ramsey Theory of asking, when one knows that some cell of any finite partition of $\mathbb{N}$ must contain certain structures, whether being large in any of several senses is good enough to guarantee the presence of those structures. For example, van der Waerden's Theorem [24] says that whenever $\mathbb{N}$ is partitioned into finitely many cells, one of these must contain arbitrarily long arithmetic progressions; Szemerédi's Theorem [23], says that any subset of $\mathbb{N}$ with positive upper density must contain arbitrarily long arithmetic progressions.

Sets which are large in any of several multiplicative senses must have substantial additive structure. For example, it is not hard to show that any set which is *piecewise syndetic* in $(\mathbb{N}, \cdot)$ (a notion defined in Definition 1.2 below) must contain arbitrarily long arithmetic progressions. (We shall present a short proof of this fact after Definition 1.2.) More recently in [3] and [4], several results have been obtained about additional structure that must be present in any multiplicatively large set. For example [3, Theorem 1.5] says that any *multiplicatively large* set (in a sense that we will not define here but which is implied by piecewise syndeticity in $(\mathbb{N}, \cdot)$) must contain a configuration of the form $\big\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\big\}$ for each $k \in \mathbb{N}$. Additional results in this direction can be found in [3].

We are concerned in this paper with the reverse question: how much multiplicative structure can be guaranteed in any additively large set? With one possible exception where we do not know, the answer is "not much". This undoubtedly is related to the fact that multiplication distributes over addition, and not the other way around. (For example, if an arithmetic progression is multiplied by a constant, the result is an arithmetic progression.)

We shall be concerned with several notions of largeness. Among these are various notions of density which we define next. The notions $\overline{d}$ and $\underline{d}$ are referred to as *upper asymptotic density* and *lower asymptotic density* respectively, d^* is referred to as *upper Banach density* and $\overline{\ell d}$ and $\underline{\ell d}$ are referred to as *upper logarithmic density* and *lower logarithmic density* respectively. If the limits involved in Definition 1.1(c) and (g) exist, then d and ℓd are referred to as *asymptotic density* and *logarithmic density* respectively.

1.1 Definition. Let $A \subseteq \mathbb{N}$.

(a) $\overline{d}(A) = \limsup\limits_{n\to\infty} \dfrac{|A \cap \{1,2,\ldots,n\}|}{n}$.

(b) $\underline{d}(A) = \liminf\limits_{n\to\infty} \dfrac{|A \cap \{1,2,\ldots,n\}|}{n}$.

(c) If $\lim_{n\to\infty} \frac{|A\cap\{1,2,\ldots,n\}|}{n}$ exists, then $d(A) = \lim_{n\to\infty} \frac{|A\cap\{1,2,\ldots,n\}|}{n}$.

(d) $d^*(A) = \sup\{\alpha :$ for each $m \in \mathbb{N}$ there exist $k,n \in \mathbb{N}$ such that $n > m$ and $\frac{|A\cap\{k+1,k+2,\ldots,k+n\}|}{n} \geq \alpha\}$.

(e) $\overline{\ell d}(A) = \limsup_{n\to\infty} \frac{\sum\{\frac{1}{a} : a\in A\cap\{1,2,\ldots,n\}\}}{\log n}$.

(f) $\underline{\ell d}(A) = \liminf_{n\to\infty} \frac{\sum\{\frac{1}{a} : a\in A\cap\{1,2,\ldots,n\}\}}{\log n}$.

(g) If $\lim_{n\to\infty} \frac{\sum\{\frac{1}{a} : a\in A\cap\{1,2,\ldots,n\}\}}{\log n}$ exists, then

$$\ell d(A) = \lim_{n\to\infty} \frac{\sum\{\frac{1}{a} : a\in A\cap\{1,2,\ldots,n\}\}}{\log n}.$$

Relations among the notions defined above are displayed in Lemma 2.1 below.

Other notions of largeness with which we shall be concerned originated in the study of topological dynamics and make sense in any semigroup. Five of these have simple elementary descriptions and we introduce them now. The sixth, *central*, is most simply defined in terms of the algebraic structure of βS, the Stone-Čech compactification of the discrete semigroup S, which we shall describe shortly. Given a (not necessarily commutative) semigroup $(S,+)$, a subset A of S, and $x\in S$, we let $-x+A = \{y\in S : x+y\in A\}$. Given a sequence $\langle x_n\rangle_{n=1}^\infty$ in S, $FS(\langle x_n\rangle_{n=1}^\infty) = \{\sum_{n\in F} x_n : F\in\mathcal{P}_f(\mathbb{N})\}$ where $\mathcal{P}_f(\mathbb{N})$ is the set of finite nonempty subsets of $\mathbb{N}$ and the sums are taken in increasing order of indices. If the operation is denoted by $\cdot$, we write $FP(\langle x_n\rangle_{n=1}^\infty) = \{\prod_{n\in F} x_n : F\in\mathcal{P}_f(\mathbb{N})\}$ and let $x^{-1}A = \{y\in S : x\cdot y\in A\}$. If $m\in\mathbb{N}$, then $FP(\langle x_n\rangle_{n=1}^m) = \{\prod_{n\in F} x_n : \emptyset \neq F\subseteq\{1,2,\ldots,m\}\}$.

1.2 Definition. Let $(S,+)$ be a semigroup and let $A\subseteq S$.

(a) A is *thick* if and only if whenever $F\in\mathcal{P}_f(S)$ there exists $x\in S$ such that $F+x\subseteq A$.

(b) A is *syndetic* if and only if there exists $G\in\mathcal{P}_f(S)$ such that $S=\bigcup_{t\in G} -t+A$.

(c) A is *piecewise syndetic* if and only if there exists $G\in\mathcal{P}_f(S)$ such that $\bigcup_{t\in G} -t+A$ is thick.

(d) A is an *IP-set* if and only if there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq A$.

(e) A is a Δ-*set* if and only if there is a sequence $\langle s_n\rangle_{n=1}^\infty$ in S such that whenever $n<m$ one has $s_m\in s_n+A$.

If S can be embedded in a group, then a subset A of S is a Δ-set if and only if there is a sequence $\langle s_n\rangle_{n=1}^{\infty}$ in S such that $\{-s_n+s_m : n<m\}\subseteq A$, so A is a *difference set*.

Notice that each of the notions thick and syndetic imply piecewise syndetic and that any thick set is an IP-set. (To verify the latter assertion, having chosen $\langle x_t\rangle_{t=1}^{n}$, pick $a\in S$ and pick $y\in S$ such that $\big(\{a\}\cup(FS(\langle s_n\rangle_{n=1}^{\infty})+a)\big)+y\subseteq A$ and let $x_{n+1}=a+y$.) Also, any IP-set is a Δ-set. (Given $FS(\langle x_n\rangle_{n=1}^{\infty})\subseteq A$, let for each n, $s_n=\sum_{t=1}^{n}x_t$.) It is easy to construct examples in $(\mathbb{N},+)$ showing that no other implications among these notions is valid in general.

In any semigroup, the properties "piecewise syndetic", "IP-set", and "Δ-set" are partition regular in the sense that whenever any set with one of those properties is divided into finitely many classes, one of these classes must also have that property. (See [6, Section 2].) Also the properties "$\overline{d}(A)>0$" and "$\overline{\ell d}(A)>0$" are partition regular.

In $(\mathbb{N},+)$ thick sets are those that contain arbitrarily long intervals, syndetic sets are those with bounded gaps, and piecewise syndetic sets are those with a fixed bound b and arbitrarily long intervals within which the gaps are bounded by b.

As we promised, we now present a simple proof of the existence of additive structure in multiplicatively large sets.

1.3 Theorem. *Let A be piecewise syndetic in $(\mathbb{N},\cdot)$. Then A contains arbitrarily long arithmetic progressions.*

Proof. Pick $G\in\mathcal{P}_f(\mathbb{N})$ such that $\bigcup_{t\in G}t^{-1}A$ is thick in $(\mathbb{N},\cdot)$ and let $r=|G|$. Let $l\in\mathbb{N}$ and pick by van der Waerden's Theorem some $n\in\mathbb{N}$ such that whenever $\{1,2,\ldots,n\}$ is r-colored, there is a monochrome length l arithmetic progression. Pick $x\in\mathbb{N}$ such that $\{1,2,\ldots,n\}\cdot x\subseteq\bigcup_{t\in G}t^{-1}A$. Then $\{1,2,\ldots,n\}\subseteq\bigcup_{t\in G}(tx)^{-1}A$ so pick $t\in G$ and $a,d\in\mathbb{N}$ such that $\{a,a+d,\ldots,a+(l-1)d\}\subseteq(tx)^{-1}A$. Then $\{txa,txa+txd,\ldots,txa+(l-1)txd\}\subseteq A$. $\square$

We now present a brief review of basic facts about $(\beta S,+)$. For additional information and any unfamiliar terminology encountered see [19].

Given a discrete semigroup $(S,+)$ we take the points of the Stone-Čech compactification βS of S to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A\subseteq S$, $\overline{A}=\{p\in\beta S : A\in p\}$ and the set $\{\overline{A} : A\subseteq S\}$ is a basis for the open sets (and a basis for the closed sets) of βS. Given $p,q\in\beta S$ and $A\subseteq S$, $A\in p+q$ if and only if $\{x\in S : -x+A\in q\}\in p$. Be cautioned that even if $(S,+)$ is commutative, $(\beta S,+)$ is not likely to be commutative.

With this operation, $(\beta S, +)$ is a compact Hausdorff right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q + p$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x + q$ is continuous. A subset I of a semigroup T is a *left ideal* provided $T + I \subseteq I$, a *right ideal* provided $I + T \subseteq I$, and a *two sided ideal* (or simply an *ideal*) provided it is both a left ideal and a right ideal.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T) = \bigcup\{L : L$ is a minimal left ideal of $T\} = \bigcup\{R : R$ is a minimal right ideal of $T\}$. Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is a *minimal* idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is member of the smallest ideal.

A subset of S is an IP-set if and only if it is a member of some idempotent in βS. (See, for example, [19, Theorem 5.12].)

The notion of *central* subsets of $(\mathbb{N}, +)$ was introduced by H. Furstenberg in [14] using notions from topological dynamics. His definition was shown in [5] (with the assistance of B. Weiss) to be equivalent to the following definition when restricted to $(\mathbb{N}, +)$.

1.4 Definition. Let S be a semigroup and let $A \subseteq S$. Then A is *central* if and only if there is a minimal idempotent p of βS such that $A \in p$.

A central set is in particular a piecewise syndetic IP-set. Given a minimal idempotent p and a finite partition of S, one cell must be a member of p, hence at least one cell of any finite partition of S must be central. Central sets are fundamental to the Ramsey Theoretic applications of the algebra of βS.

In 1936 Davenport and Erdős [10] showed that any set with positive upper logarithmic density (in particular any set with positive asymptotic density) contains all products of initial segments of some sequence $\langle y_n \rangle_{n=1}^{\infty}$ in $\mathbb{N} \setminus \{1\}$. (We present a simplified version of their proof as Theorem 2.8.) It is a result of Besicovitch [8] that positive upper asymptotic density is not enough for this result. We present as Theorem 2.13 an extension of Besicovitch's result showing that there exist sets with upper asymptotic density arbitrarily close to 1 that do not contain all products of initial segments of some sequence $\langle y_n \rangle_{n=1}^{\infty}$ in $\mathbb{N} \setminus \{1\}$.

In Section 2 we show that any translate of an additive IP-set contains all products of initial segments of some sequence $\langle y_n \rangle_{n=1}^{\infty}$ but that translates of additive Δ sets need not. We show also that neither the Davenport-Erdős result nor the translate of IP-set result is stronger than the other. We present in this section two corollaries of earlier results. As a corollary of a result of Ahlswede, Khachatrian, and Sárközy (and of Szemerédi's Theorem) we show that if $\overline{\ell d}(A) > 0$, then there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N} \setminus \{1\}$ such that for all $m \in \mathbb{N}$, there exist arbitrarily long arithmetic progressions in A such that if a is any term in such a progression, then $a \cdot FP(\langle x_n \rangle_{n=1}^m) \subseteq A$. And as a corollary to a result of Jin, we show that whenever $d^*(A) > 0$ and $d^*(B) > 0$, $A + B$ contains all products of initial segments of some sequence. More quantitative issues and other related concepts are treated in [13].

In 1996 Brown and Gordon [9] showed that given k, any set with sufficiently large upper asymptotic density contains a length k geometric progression, but that this density must be quite large. In Section 3 we improve slightly on these results and show that thick sets with quite large upper asymptotic density need not contain length 3 geometric progressions. We have not been able to determine whether any additively syndetic set must contain arbirarily long geometric progressions or even length 3 geometric progressions.

In a surprising result we show that sets of the kind considered by Brown and Gordon (obtained by restricting the powers of certain primes occuring in the prime factorizations) cannot be additively piecewise syndetic. For example, it is a consequence of Theorem 3.9 that the set of $x \in \mathbb{N}$ such that the prime factorization of x contains no term of the form p^{100} is not additively piecewise syndetic. (Terms of the form p^k with $k > 100$ would be allowed.) As a consequence of the final result mentioned in the abstract, this says that if B is the set of numbers whose prime factorization does include some p^{100}, then for any $t \in \mathbb{Z}$, $t + B$ is both additively and multiplicatively central.

In Section 4 we address (but do not answer) the question of whether any additively syndetic set must contain arbitrarily long geometric progressions. We present there some very strong consequences of an affirmative answer to this question.

2. Initial products in certain additively large sets

By the *initial products* of a sequence $\langle y_n \rangle_{n=1}^\infty$ in $\mathbb{N} \setminus \{1\}$, we mean $\{\prod_{i=1}^n y_i : n \in \mathbb{N}\}$. We begin by displaying some simple facts about upper and lower asymptotic density and logarithmic density and the relations among them.

2.1 Lemma. *Let A and B be subsets of* $\mathbb{N}$.

(a) $\overline{d}(A) + \overline{d}(B) \geq \overline{d}(A \cup B)$

(b) $\overline{\ell d}(A) + \overline{\ell d}(B) \geq \overline{\ell d}(A \cup B)$

(c) $\overline{d}(A) = 1 - \underline{d}(\mathbb{N} \setminus A)$

(d) $\overline{\ell d}(A) = 1 - \underline{\ell d}(\mathbb{N} \setminus A)$

(e) $\underline{d}(A) \leq \underline{\ell d}(A)$

(f) $\overline{d}(A) \geq \overline{\ell d}(A)$

(g) $d^*(A) \geq \overline{d}(A)$

(h) If $d(A)$ exists, so does $\ell d(A)$ and $\ell d(A) = d(A)$.

Proof. Conclusions (a) through (d) and (g) can be routinely checked.

To verify (e) let $\alpha = \underline{\ell d}(A)$ and suppose that $\underline{d}(A) > \alpha$. Pick δ such that $\underline{d}(A) > \delta > \alpha$. Pick $k \in \mathbb{N}$ such that whenever $x \in \mathbb{N}$ and $x \geq k$ one has $|A \cap \{1, 2, \ldots, x\}| > \delta x$. Enumerate A in order as $\langle a_n \rangle_{n=1}^{\infty}$. Then if $n \geq \delta k$ one has $a_n \leq \frac{n}{\delta}$. Let $m = \lceil \delta k \rceil$. For any $n \in \mathbb{N}$,

$$\begin{aligned} \textstyle\sum \left\{ \frac{1}{a} : a \in A \cap \{1, 2, \ldots, \lfloor \frac{n}{\delta} \rfloor\} \right\} &\geq \textstyle\sum_{t=m}^{n} \frac{1}{a_t} \\ &\geq \textstyle\sum_{t=m}^{n} \frac{\delta}{t} \\ &> \delta \cdot (\log(n+1) - \log m) \end{aligned}$$

so

$$\frac{\sum \left\{ \frac{1}{a} : a \in A \cap \{1, 2, \ldots, \lfloor \frac{n}{\delta} \rfloor\} \right\}}{\log \lfloor \frac{n}{\delta} \rfloor} > \frac{\delta \cdot (\log(n+1) - \log m)}{\log n - \log \delta} \to \delta$$

so $\underline{\ell d}(A) \geq \delta$, a contradiction.

Conclusion (f) follows from conclusions (c), (d), and (e), and conclusion (h) follows from conclusions (e) and (f). $\square$

We now set out to establish a result of Davenport and Erdős, namely that any set with positive upper logarithmic density contains the initial products from some sequence. We do this as a favor to the reader. Their original proof [10, Theorem 2] used a theorem of Hardy and Littlewood to obtain a lemma which they used in the proof of the theorem. They subsequently in [11] provided an elementary proof of the same lemma. We present here as Lemma 2.7 only as much of that lemma as is needed to complete the proof. Each of the following five lemmas is well known by many people, and there are probably a few people who know them all well.

2.2 Lemma. *Let P be any property which may be possessed by subsets of $\mathbb{N}$. Assume that when A has property P, there is some $c \in A \setminus \{1\}$ such that $(A \cap \mathbb{N}c) \setminus \{c\}$ has property P. Then any A with property P contains the initial products from some sequence in $\mathbb{N} \setminus \{1\}$.*

Proof. Choose $y_1 \in A \setminus \{1\}$ such that $(A \cap \mathbb{N}y_1) \setminus \{y_1\}$ has property P. Inductively assume that we have chosen $\langle y_i \rangle_{i=1}^m$ such that for each $n \in \{1, 2, \ldots, m\}$, $\prod_{i=1}^n y_i \in A$ and $(A \cap \mathbb{N} \prod_{i=1}^n y_i) \setminus \{\prod_{i=1}^n y_i\}$ has property P. Let $B = (A \cap \mathbb{N} \prod_{i=1}^m y_i) \setminus \{\prod_{i=1}^m y_i\}$ and pick $c \in B$ such that $(B \cap \mathbb{N}c) \setminus \{c\}$ has property P. Let $y_{m+1} = \dfrac{c}{\prod_{i=1}^m y_i}$. □

2.3 Definition. Let $\langle p_n \rangle_{n=1}^\infty$ enumerate the primes in order. For $l, k \in \mathbb{N}$ with $l \leq k$, $N_{l,k} = \{\prod_{i=l}^k p_i^{\alpha_i} : \text{for each } i \in \{l, l+1, \ldots, k\}, \alpha_i \in \omega\}$ and $N_k = N_{1,k}$.

2.4 Lemma. *Let $l, k \in \mathbb{N}$ with $l \leq k$. Then $\sum_{n \in N_{l,k}} \frac{1}{n} = \prod_{i=l}^k \frac{p_i}{p_i - 1}$.*

Proof. We proceed by induction on k. If $k = l$, then $\sum_{n \in N_{l,l}} \frac{1}{n} = \sum_{t=0}^\infty \left(\frac{1}{p_l}\right)^t = \frac{p_l}{p_l - 1}$. Now let $k \geq l$ and assume that $\sum_{n \in N_{l,k}} \frac{1}{n} = \prod_{i=l}^k \frac{p_i}{p_i - 1}$. Then $\sum_{n \in N_{l,k+1}} \frac{1}{n} = \sum_{t=0}^\infty \left(\frac{1}{p_{k+1}}\right)^t \cdot \left(\sum_{n \in N_{l,k}} \frac{1}{n}\right) = \sum_{t=0}^\infty \left(\frac{1}{p_{k+1}}\right)^t \cdot \prod_{i=l}^k \frac{p_i}{p_i - 1} = \prod_{i=l}^{k+1} \frac{p_i}{p_i - 1}$. □

2.5 Lemma. *Let $k \in \mathbb{N}$. Then $\lim\limits_{L \to \infty} \overline{\ell d}(\bigcup\{\mathbb{N}m : m \in N_k \text{ and } m \geq L\}) = 0$.*

Proof. By Lemma 2.1(f) it suffices to show that

$$\lim_{L \to \infty} \overline{d}(\bigcup\{\mathbb{N}m : m \in N_k \text{ and } m \geq L\}) = 0\,.$$

To see this, let $L, x \in \mathbb{N}$. Then for each $m \in N_k$ with $m \geq L$, $|\mathbb{N}m \cap \{1, 2, \ldots, x\}| \leq \frac{x}{m}$ so

$$\frac{|\bigcup\{\mathbb{N}m : m \in N_k \text{ and } m \geq L\} \cap \{1, 2, \ldots, x\}|}{x} \leq \sum\{\tfrac{1}{m} : m \in N_k \text{ and } m \geq L\}\,.$$

By Lemma 2.4, $\sum_{m \in N_k} \frac{1}{m}$ converges, so $\lim\limits_{L \to \infty} \sum\{\frac{1}{m} : m \in N_k \text{ and } m \geq L\} = 0$ as required. □

In [11], Davenport and Erdős say that the following lemma is well known. We thank Mate Wierdl for supplying us with the proof.

2.6 Lemma. *There is a positive $C \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $\sum_{n \in N_k} \frac{1}{n} \leq C \log p_k$.*

Proof. By [16, Theorem 427] there exists $B \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $\sum_{i=1}^k \frac{1}{p_i} < \log\log p_k + B$. Also, given p_i, $\log\left(\frac{p_i}{p_{i-1}}\right) = \log\left(1 + \frac{1}{p_i - 1}\right) < \frac{1}{p_i - 1} = \frac{1}{p_i} + \frac{1}{p_i(p_i - 1)}$ so $\sum_{i=1}^k \log\left(\frac{p_i}{p_i - 1}\right) < \sum_{i=1}^k \frac{1}{p_i} + \sum_{i=1}^k \frac{1}{p_i(p_i - 1)} < \sum_{i=1}^k \frac{1}{p_i} + 1$. Thus

$$\begin{aligned}\textstyle\sum_{n\in N_k}\frac{1}{n} &= \textstyle\prod_{i=1}^k\left(\frac{p_i}{p_i-1}\right)\\ &= \exp\left(\textstyle\sum_{i=1}^k\log\left(\frac{p_i}{p_i-1}\right)\right)\\ &< \exp\left(\textstyle\sum_{i=1}^k\frac{1}{p_i}+1\right)\\ &< \exp(\log\log p_k + B + 1)\\ &= e^{B+1}\log p_k\end{aligned}$$

□

2.7 Lemma. *Let A be an infinite set and let $\langle a_n\rangle_{n=1}^\infty$ enumerate A in increasing order. Let $B=\bigcup_{a\in A}\mathbb{N}a$. Then $\lim_{M\to\infty}\overline{\ell d}(B\setminus\bigcup_{n=1}^M\mathbb{N}a_n)=0$.*

Proof. For each $k\in\mathbb{N}$, let $D_k=\dfrac{\sum_{n\in B\cap N_k}\frac{1}{n}}{\sum_{n\in N_k}\frac{1}{n}}$ and let $E_k=B\setminus\bigcup_{a\in A\cap N_k}\mathbb{N}a$. We show first that if $k<m$ and $p_k\le x<p_{m+1}$, then

$$(*)\qquad \textstyle\sum\left\{\frac{1}{n}:n\in E_k\cap\{1,2,\ldots,x\}\right\}\le\sum_{n\in N_m}\frac{1}{n}(D_m-D_k)\,.$$

To this end we first observe that

$$E_k\cap\{1,2,\ldots,x\}\subseteq(B\cap N_m)\setminus\{bn:b\in B\cap N_k\text{ and }n\in N_{k+1,m}\}$$

and $\{bn:b\in B\cap N_k\text{ and }n\in N_{k+1,m}\}\subseteq B\cap N_m$. Observe also by Lemma 2.4 that $(\sum_{n\in N_k}\frac{1}{n})\cdot(\sum_{n\in N_{k+1,m}}\frac{1}{n})=\sum_{n\in N_m}\frac{1}{n}$. Consequently

$$\begin{aligned}\textstyle\sum\left\{\frac{1}{b}:b\in E_k\cap\{1,2,\ldots,x\}\right\} &\le \textstyle\sum_{b\in B\cap N_m}\frac{1}{b}-\sum\{\frac{1}{bn}:b\in B\cap N_k\text{ and }n\in N_{k+1,m}\}\\ &= \textstyle\sum_{b\in B\cap N_m}\frac{1}{b}-\sum_{b\in B\cap N_k}\frac{1}{b}\cdot\left(\sum_{n\in N_{k+1,m}}\frac{1}{n}\right)\\ &= \textstyle\sum_{n\in N_m}\frac{1}{n}\cdot\left(\dfrac{\sum_{n\in B\cap N_m}\frac{1}{n}}{\sum_{n\in N_m}\frac{1}{n}}-\dfrac{\sum_{n\in B\cap N_k}\frac{1}{n}}{\sum_{n\in N_k}\frac{1}{n}}\right)\\ &= \textstyle\sum_{n\in N_m}\frac{1}{n}\cdot(D_m-D_k)\,,\end{aligned}$$

as required. In particular note that if $k<m$, then $D_k\le D_m\le 1$. Let $D=\lim_{k\to\infty}D_k$. Let C be as guaranteed by Lemma 2.6. It now suffices to show that

$$(**)\qquad\text{for every }k\in\mathbb{N}\,,\ \limsup_{M\to\infty}\overline{\ell d}(B\setminus\textstyle\bigcup_{n=1}^M\mathbb{N}a_n)\le C(D-D_k)\,.$$

To this end, let $k\in\mathbb{N}$ and $\epsilon>0$ be given. We show that there is some $N\in\mathbb{N}$ such that for all $M\ge N$, $\overline{\ell d}(B\setminus\bigcup_{n=1}^M\mathbb{N}a_n)<C(D-D_k)+\epsilon$. Pick by Lemma 2.5 some $L\in\mathbb{N}$ such that $\overline{\ell d}(\bigcup\{\mathbb{N}m:m\in N_k\text{ and }m\ge L\})<\epsilon$ and pick $N\in\mathbb{N}$ such that $a_N\ge L$. Let $M\ge N$. Then $B\setminus\bigcup_{m=1}^M\mathbb{N}a_m\subseteq B\setminus\bigcup_{m=1}^N\mathbb{N}a_m\subseteq E_k\cup\bigcup\{\mathbb{N}m:m\in N_k\text{ and }m\ge L\}$. By Lemma 2.1(b) it suffices to show that $\overline{\ell d}(E_k)\le C(D-D_k)$. To this end, let $x>p_{k+1}$

and pick $m \in \mathbb{N}$ such that $p_m \leq x < p_{m+1}$. Then by $(*)$

$$\frac{\sum \left\{ \frac{1}{n} : n \in E_k \cap \{1, 2, \ldots, x\} \right\}}{\log x} \leq \frac{\sum_{n \in N_m} \frac{1}{n}}{\log p_m} (D_m - D_k) \leq C(D - D_k)\,,$$

the last inequality holding by Lemma 2.6 and the fact that $D_m \leq D$. $\square$

It is natural to ask whether Lemma 2.7 holds with $\overline{\ell d}$ replaced by $\overline{d}$. It is a consequence of Theorem 2.13 and the proof of Theorem 2.8 that it does not.

2.8 Theorem (Davenport and Erdős). *Let $A \subseteq \mathbb{N}$ and assume that $\overline{\ell d}(A) > 0$. Then there is a sequence $\langle y_n \rangle_{n=1}^\infty$ in $\mathbb{N} \setminus \{1\}$ such that for each $n \in \mathbb{N}$, $\prod_{i=1}^n y_i \in A$.*

Proof. We may assume that $1 \notin A$. By Lemma 2.2 it suffices to show that there is some $c \in A$ such that $\overline{\ell d}(A \cap \mathbb{N}c) > 0$. Let B and $\langle a_n \rangle_{n=1}^\infty$ be as in the statment of Lemma 2.7 and pick by Lemma 2.7 some $M \in \mathbb{N}$ such that $\overline{\ell d}(B \setminus \bigcup_{m=1}^M \mathbb{N}a_m) < \overline{\ell d}(A)$. Then by Lemma 2.1(b),

$$\begin{aligned} \overline{\ell d}(A) &\leq \textstyle\sum_{m=1}^M \overline{\ell d}(A \cap \mathbb{N}a_m) + \overline{\ell d}(A \setminus \bigcup_{m=1}^M \mathbb{N}a_m) \\ &\leq \textstyle\sum_{m=1}^M \overline{\ell d}(A \cap \mathbb{N}a_m) + \overline{\ell d}(B \setminus \bigcup_{m=1}^M \mathbb{N}a_m) \\ &< \textstyle\sum_{m=1}^M \overline{\ell d}(A \cap \mathbb{N}a_m) + \overline{\ell d}(A) \end{aligned}$$

so there is some m such that $\overline{\ell d}(A \cap \mathbb{N}a_m) > 0$. $\square$

Notice that one cannot ask that $FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A$. In fact one cannot ask that any y_n be in A except y_1. To see this consider $A = 3\mathbb{N} + 2$.

Notice that when one establishes in the proof above that there is some $c \in A$ with $\overline{\ell d}(A \cap \mathbb{N}c) > 0$, one also establishes that there is some $x \in \mathbb{N} \setminus \{1\}$ with $A \cap x^{-1}A \neq \emptyset$. (If $xc \in A$, then $c \in A \cap x^{-1}A$.) In fact the following much stronger result holds.

2.9 Theorem (Ahlswede, Khachatrian, and Sárközy). *Let $A \subseteq \mathbb{N}$ and assume that $\overline{\ell d}(A) > 0$. Then there exists $x \in \mathbb{N} \setminus \{1\}$ such that $\overline{\ell d}(A \cap x^{-1}A) > 0$.*

Proof. [1, Theorem 3]. $\square$

This result in turn has strong consequences.

2.10 Corollary. *Let $A \subseteq \mathbb{N}$ and assume that $\overline{\ell d}(A) > 0$. Then there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N} \setminus \{1\}$ such that for all $m \in \mathbb{N}$, there exist $a \in A$ and $d \in \mathbb{N}$ such that for all $t \in \{0, 1, \ldots, m\}$, $(a + td) \cdot FP(\langle x_n \rangle_{n=1}^m) \subseteq A$.*

Proof. Let $A_1 = A$, pick $x_1 \in \mathbb{N} \setminus \{1\}$ such that $\overline{\ell d}(A_1 \cap {x_1}^{-1} A_1) > 0$, and let $A_2 = A_1 \cap {x_1}^{-1} A_1$. Inductively, given A_n with $\overline{\ell d}(A_n) > 0$, pick $x_n \in \mathbb{N} \setminus \{1\}$ such that $\overline{\ell d}(A_n \cap {x_n}^{-1} A_n) > 0$ and let $A_{n+1} = A_n \cap {x_n}^{-1} A_n$.

One then easily establishes by induction that for each $m \in \mathbb{N}$,

$$A_{m+1} = A \cap \bigcap \{y^{-1}A : y \in FP(\langle x_n \rangle_{n=1}^m)\}\,.$$

Given m, pick by Szemerédi's Theorem [23] $a, d \in \mathbb{N}$ such that $\{a, a+d, \ldots, a+md\} \subseteq A \cap \bigcap \{y^{-1}A : y \in FP(\langle x_n \rangle_{n=1}^m)\}$, using the fact from Lemma 2.1 that $\overline{d}(B) \geq \overline{\ell d}(B)$ for any set B. $\square$

It is natural to ask whether positive upper asymptotic density of a set A is enough to guarantee the existence of an infinite sequence in $\mathbb{N} \setminus \{1\}$ with initial products in A. This question was answered by A. Besicovitch [8] in 1934. He showed that there exist sets with upper density arbitrarily close to $\frac{1}{2}$ that do not contain distinct elements one of which divides the other. We present here as Theorem 2.13 a minor modification of Besicovitch's proof showing that there exist sets with upper density arbitrarily close to 1 that do not contain initial products from any infinite sequence in $\mathbb{N} \setminus \{1\}$. (If $\overline{d}(A) = 1$, then A is thick, so by Corollary 2.17 below, will contain initial products of some infinite sequence.) Necessarily any such example must have $\underline{d}(A) = 0$, because by Lemma 2.1, $\overline{\ell d}(A) \geq \underline{\ell d}(A) \geq \underline{d}(A)$.

2.11 Theorem (Erdős). $\displaystyle\lim_{T \to \infty} \overline{d}\left(\bigcup \{\mathbb{N}m : m \in \mathbb{N} \text{ and } T < m \leq 2T\}\right) = 0.$

Proof. This is in [12] and is presented as [15, Theorem 10, p. 256]. $\square$

2.12 Lemma. *Let $k \in \mathbb{N}$. Then* $\displaystyle\lim_{T \to \infty} \overline{d}\left(\bigcup \{\mathbb{N}m : m \in \mathbb{N} \text{ and } T < m \leq 2^k T\}\right) = 0.$

Proof. This is an immediate consequence of Theorem 2.11 and Lemma 2.1(a). $\square$

2.13 Theorem. *Let $k \in \mathbb{N}$ and let $\epsilon > 0$. Then there exists $A \subseteq \mathbb{N}$ such that $\overline{d}(A) \geq 1 - \frac{1}{2^k} - \epsilon$ and there do not exist $y_1, y_2, \ldots y_{k+1}$ in $\mathbb{N} \setminus \{1\}$ with $y_1 \in A$ and $\prod_{i=1}^{k+1} y_i \in A$.*

Proof. Choose $T_1 \in \mathbb{N}$ such that $\overline{d}\left(\bigcup \{\mathbb{N}m : m \in \mathbb{N} \text{ and } T_1 < m \leq 2^k T_1\}\right) < \dfrac{\epsilon}{2}$ and pick $M_1 \in \mathbb{N}$ such that for all $l > M_1$

$$|\{1, 2, \ldots, l\} \cap \bigcup \{\mathbb{N}m : m \in \mathbb{N} \text{ and } T_1 < m \leq 2^k T_1\}| < \frac{\epsilon}{2} \cdot l\,.$$

Inductively, having chosen T_{t-1} and M_{t-1}, pick $T_t > M_{t-1}$ such that

$$\overline{d}\left(\bigcup \{\mathbb{N}m : m \in \mathbb{N} \text{ and } T_t < m \leq 2^k T_t\}\right) < \frac{\epsilon}{2^t}$$

and pick $M_t > M_{t-1}$ such that for all $l > M_t$

$$|\{1, 2, \ldots, l\} \cap \bigcup \{\mathbb{N}m : m \in \mathbb{N} \text{ and } T_t < m \leq 2^k T_t\}| < \frac{\epsilon}{2^t} \cdot l\,.$$

Let

$$A = \{m \in \mathbb{N} : T_1 < m \leq 2^k T_1\}$$

$$\cup \textstyle\bigcup_{t=2}^{\infty} \left(\{m \in \mathbb{N} : T_t < m \leq 2^k T_t\} \setminus \bigcup_{s=1}^{t-1} \bigcup \{\mathbb{N}m : m \in \mathbb{N} \text{ and } T_s < m \leq 2^k T_s\} \right).$$

Given $t \in \mathbb{N} \setminus \{1\}$, we have that

$$\begin{aligned} |A \cap \{1, 2, \ldots, 2^k T_t\}| &> (2^k - 1) \cdot T_t - \\ &\qquad \textstyle\sum_{s=1}^{t-1} |\{1, 2, \ldots, 2^k T_t\} \cap \bigcup \{\mathbb{N}m : m \in \mathbb{N} \text{ and } T_s < m \leq 2^k T_s\}| \\ &> (2^k - 1) \cdot T_t - \textstyle\sum_{s=1}^{t-1} \dfrac{\epsilon}{2^s} \cdot 2^k \cdot T_t \\ &> (2^k - 1) \cdot T_t - \epsilon \cdot 2^k \cdot T_t \\ &= (1 - \dfrac{1}{2^k} - \epsilon) \cdot 2^k \cdot T_t \end{aligned}$$

and so $\overline{d}(A) \geq 1 - \frac{1}{2^k} - \epsilon$.

Now suppose we have $y_1, y_2, \ldots y_{k+1}$ in $\mathbb{N} \setminus \{1\}$ with $y_1 \in A$ and $\prod_{i=1}^{k+1} y_i \in A$. Pick r such that $T_r < y_1 \leq 2^k T_r$. Then $\prod_{i=1}^{k+1} y_i > 2^k T_r$ so pick $t > r$ such that

$$\textstyle\prod_{i=1}^{k+1} y_i \in \{m \in \mathbb{N} : T_t < m \leq 2^k T_t\} \setminus \bigcup_{s=1}^{t-1} \bigcup \{\mathbb{N}m : m \in \mathbb{N} \text{ and } T_s < m \leq 2^k T_s\}.$$

But $T_r < y_1 \leq 2^k T_r$ and $\prod_{i=1}^{k+1} y_i \in \mathbb{N}y_1$, a contradiction. $\square$

We now show, using a modification of the proof of [15, Theorem 1, p. 244], that the result of Theorem 2.13 is sharp.

2.14 Theorem. *If $k \in \mathbb{N}$, $A \subseteq \mathbb{N}$, and $\overline{d}(A) > 1 - \frac{1}{2^k}$, then there exist $y_1, y_2, \ldots y_{k+1}$ in $\mathbb{N} \setminus \{1\}$ with $\prod_{i=1}^{l} y_i \in A$ for each $l \in \{1, 2, \ldots, k+1\}$.*

Proof. Assume that $\overline{d}(A) > 1 - \frac{1}{2^k}$ and suppose that A does not satisfy the conclusion of the theorem. We show that for all $n \in \mathbb{N}$, $|A \cap \{1, 2, \ldots, n\}| \leq n(1 - \frac{1}{2^k}) + \frac{k^2}{2} + 1$.

Let $B_k = \{t \in \mathbb{N} : 2^{k-1}(2t-1) \leq n\}$ and note that $|B_k| \leq \frac{n}{2^k} + \frac{1}{2}$. For $t \in B_k$, let $C_{k,t} = \{2^i(2t-1) : i \in \omega \text{ and } 2^i(2t-1) \leq n\}$. Then $|A \cap C_{k,1}| \leq k+1$ and for each $t \in B_k \setminus \{1\}$, $|A \cap C_{k,t}| \leq k$. (The reason for the distinction is that no $y_i = 1$.) If $k = 1$, then $A \cap \{1, 2, \ldots, n\} \subseteq \bigcup_{t \in B_k} (A \cap C_{k,t})$ so $|A \cap \{1, 2, \ldots, n\}| \leq \frac{n}{2} + \frac{3}{2}$ as required.

Assume that $k > 1$. For $l \in \{1, 2, \ldots, k-1\}$, let $B_l = \{t \in \mathbb{N} : 2^{l-1}(2t-1) \leq n < 2^l(2t-1)\}$ and note that $|B_l| < \frac{n}{2^{l+1}} + 1$. For $t \in B_l$ let $C_{l,t} = \{2t-1, 2(2t-1), \ldots, 2^{l-1}(2t-1)\}$. Then $|A \cap C_{l,t}| \leq |C_{l,t}| = l$. Now, $\{1, 2, \ldots, n\} = \bigcup_{l=1}^{k} \bigcup_{t \in B_l} C_{l,t}$, so using the easily checked fact that $\sum_{l=1}^{k-1} \frac{l}{2^{l+1}} = 1 - \frac{k+1}{2^k}$, we have

$$
\begin{aligned}
|A \cap \{1,2,\ldots,n\}| &\leq k \cdot |B_k| + 1 + \textstyle\sum_{l=1}^{k-1} l \cdot |B_l| \\
&< k \cdot (\tfrac{n}{2^k} + \tfrac{1}{2}) + 1 + \textstyle\sum_{l=1}^{k-1} l \cdot (\tfrac{n}{2^{l+1}} + 1) \\
&= n\tfrac{k}{2^k} + \tfrac{k}{2} + 1 + n(1 - \tfrac{k+1}{2^k}) + \tfrac{k(k-1)}{2} \\
&= n(1 - \tfrac{1}{2^k}) + \tfrac{k^2}{2} + 1\,.
\end{aligned}
$$
□

The only question left open by Theorems 2.13 and 2.14 is what happens when $\overline{d}(A) = 1 - \frac{1}{2^k}$. According to a footnote on page 244 of [15], Erdős has shown that if $\overline{d}(A) = \frac{1}{2}$, then there exist $y_1, y_2 \in \mathbb{N} \setminus \{1\}$ such that $y_1, y_1y_2 \in A$. The second author has no doubt that the corresponding result holds for all k. That is, he believes that the conclusion of Theorem 2.14 holds when $\overline{d}(A) = 1 - \frac{1}{2^k}$.

Now we turn our attention to showing that translates of sets of finite sums contain initial products of some infinite sequence in $\mathbb{N} \setminus \{1\}$. In the following lemma we use the fact that for any sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and any $n \in \mathbb{N}$, $FS(\langle x_t \rangle_{t=1}^{\infty}) \cap \mathbb{N}n \neq \emptyset$. (To see this, pick $F \subseteq \mathbb{N}$ such that $|F| = n$ and for all $t, k \in F$, $x_t \equiv x_k \pmod{n}$. Then n divides $\sum_{t \in F} x_t$.)

2.15 Lemma. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$, let $a \in \mathbb{Z}$, and let*

$$B = \big(a + FS(\langle x_n \rangle_{n=1}^{\infty})\big) \cap \mathbb{N}\,.$$

Then for all $s \in B$ there exists $t \in \mathbb{N}$ such that $t > s$ and $t \equiv 1 \pmod{s}$ and $st \in B$.

Proof. Let $s \in B$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $s = a + \sum_{n \in F} x_n$, and let $k = \max F + 1$. Pick $w \in FS(\langle x_n \rangle_{n=k}^{\infty}) \cap \mathbb{N}s^2$. Let $u = \frac{w}{s}$ and note that s divides u. Let $t = 1 + u$. Then $st = s + w = a + \sum_{n \in F} x_n + w \in B$. □

2.16 Theorem. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$, let $a \in \mathbb{Z}$, and let*

$$B = \big(a + FS(\langle x_n \rangle_{n=1}^{\infty})\big) \cap \mathbb{N}\,.$$

Then there exists a sequence $\langle y_n \rangle_{n=1}^{\infty}$ in $\mathbb{N} \setminus \{1\}$ such that for each $n \in \mathbb{N}$, $\prod_{i=1}^{n} y_i \in B$ and $y_{n+1} \equiv 1 \pmod{\prod_{i=1}^{n} y_i}$. In particular the terms of $\langle y_n \rangle_{n=1}^{\infty}$ are pairwise relatively prime.

Proof. Trivially $B \setminus \{1\} \neq \emptyset$ so pick $y_1 \in B \setminus \{1\}$. Inductively, given $\langle y_i \rangle_{i=1}^{n}$ with $\prod_{i=1}^{n} y_i \in B$, let $s = \prod_{i=1}^{n} y_i$ and pick t as guaranteed by Lemma 2.15. Let $y_{n+1} = t$. □

2.17 Corollary. *If A is a piecewise syndetic subset of $(\mathbb{N}, +)$, then there exists a sequence $\langle y_n \rangle_{n=1}^{\infty}$ in $\mathbb{N} \setminus \{1\}$ such that for each $n \in \mathbb{N}$, $\prod_{i=1}^{n} y_i \in A$ and $y_{n+1} \equiv 1 \pmod{\prod_{i=1}^{n} y_i}$.*

Proof. Pick $G \in \mathcal{P}_f(\mathbb{N})$ such that $\bigcup_{k \in G} -k + A$ is additively thick, and therefore is an additive IP-set. Whenever an IP-set is partitioned into finitely many parts, one of these parts is an IP-set. (See [14, Proposition 8.13] or [19, Corollary 5.15].) Consequently, there is some $k \in G$ such that $-k + A$ is an additive IP-set. $\square$

A recent result of Renling Jin provides a powerful application of Corollary 2.17.

2.18 Theorem (Jin). *If A and B are subsets of $\mathbb{N}$, $d^*(A) > 0$, and $d^*(B) > 0$, then $A + B$ is piecewise syndetic.*

Proof. [20, Corollary 3]. $\square$

2.19 Corollary. *If A and B are subsets of $\mathbb{N}$, $d^*(A) > 0$, and $d^*(B) > 0$, then there exists a sequence $\langle y_n \rangle_{n=1}^{\infty}$ in $\mathbb{N} \setminus \{1\}$ such that for each $n \in \mathbb{N}$, $\prod_{i=1}^{n} y_i \in A + B$ and $y_{n+1} \equiv 1 \pmod{\prod_{i=1}^{n} y_i}$.*

Proof. Theorem 2.18 and Corollary 2.17. $\square$

Notice that the set $A+B$ of Corollary 2.19 need not have positive upper logarithmic density – in fact it need not have positive upper asymptotic density. To see this, consider $A = B = \big\{2^n + t : n \in \mathbb{N} \text{ and } t \in \{1, 2, \ldots, n\}\big\}$.

One could prove Theorem 2.16 using Lemma 2.2, as was done in the proof of Theorem 2.8. But then one would lose the conclusion that the terms of the sequence are pairwise relatively prime. Note that Alexander [2, Theorem 3.10] has proved that if $\overline{\ell d}(A) > 0$ and $\langle k_n \rangle_{n=1}^{\infty}$ is any sequence in $\mathbb{N}$, then there exists a sequence $\langle y_n \rangle_{n=1}^{\infty}$ in $\mathbb{N} \setminus \{1\}$ such that for each $n \in \mathbb{N}$, $\prod_{i=1}^{n} y_i \in A$ and y_{n+1} is composed entirely of primes greater than $(\prod_{i=1}^{n} y_i)^{k_n}$.

We note that neither of Theorems 2.8 or 2.16 is stronger than the other, even if the conclusion about the terms of the sequence $\langle y_n \rangle_{n=1}^{\infty}$ is deleted from Theorem 2.16. On one hand, it is not hard to show that $\overline{\ell d}\big(FS(\langle 4^n \rangle_{n=1}^{\infty})\big) = 0$, so Theorem 2.8 does not imply the weak form of Theorem 2.16. On the other hand, it is a result of Ernst Straus that there exist sets with asymptotic density arbitrarily close to 1 which do not contain $t + FS(\langle x_n \rangle_{n=1}^{\infty})$ for any $t \in \mathbb{Z}$. A version of this result was presented in [18, Theorem 11.6]. Unfortunately, this theorem assumed that $t \in \mathbb{N}$. Perhaps more unfortunately, the proof of [18, Theorem 11.6] was unnecessarily complicated because the author of [18] (who had the excuse of being relatively young at the time) did not understand the proof as Ernst had explained it to him, and converted it to a far too difficult proof. We present here what we believe to be roughly the original proof.

2.20 Theorem (Ernst Straus). *Let $\epsilon > 0$. There exists $A \subseteq \mathbb{N}$ such that $d(A) > 1-\epsilon$ and there do not exist $t \in \mathbb{Z}$ such that $(t + A) \cap \mathbb{N}n \neq \emptyset$ for every $n \in \mathbb{N}$. In particular, there do not exist a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $t \in \mathbb{Z}$ such that $t + FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. Choose a sequence $\langle q_t \rangle_{t=0}^{\infty}$ of primes such that $\frac{1}{q_0} + \sum_{t=1}^{\infty} \frac{2}{q_t} < \epsilon$.

For each $n \in \mathbb{N}$ let $B_n = \bigcup_{t=0}^{n} \big((t + \omega q_t) \cup (-t + \omega q_t)\big)$, let $m_n = \prod_{t=0}^{n} q_t$, and let $b_n = |B_n \cap \{1, 2, \ldots, m_n\}|$. Then $b_n \leq m_n \cdot \left(\frac{1}{q_0} + \sum_{t=1}^{n} \frac{2}{q_t}\right)$ and for each $k \in \mathbb{N}$,

$$\begin{gathered} B_n \cap \{1, 2, \ldots, km_n\} = \\ \big\{x + (s-1)m_n : x \in B_n \cap \{1, 2, \ldots, m_n\} \text{ and } s \in \{1, 2, \ldots, k\}\big\} \cup \\ \big\{lm_n + t : l \in \{1, 2, \ldots, k-1\} \text{ and } t \in \{1, 2, \ldots, n\}\big\} \end{gathered}$$

so $kb_n \leq |B_n \cap \{1, 2, \ldots, km_n\}| \leq kb_n + (k-1)n$.

Let $B = \bigcup_{n=1}^{\infty} B_n$ and let $A = \mathbb{N} \setminus B$. Notice that for each n, $b_{n+1} \geq |B_n \cap \{1, 2, \ldots, m_{n+1}\}| \geq q_{n+1} b_n$ and thus $\dfrac{b_{n+1}}{m_{n+1}} = \dfrac{b_{n+1}}{q_{n+1} m_n} \geq \dfrac{b_n}{m_n}$. Thus $\langle \frac{b_n}{m_n} \rangle_{n=1}^{\infty}$ is a nondecreasing sequence bounded above by $\frac{1}{q_0} + \sum_{t=1}^{\infty} \frac{2}{q_t}$. Let $\alpha = \lim\limits_{n \to \infty} \dfrac{b_n}{m_n}$. Then $\alpha < \epsilon$. We claim that $d(A) = 1 - \alpha$.

To see that $\limsup\limits_{r \to \infty} \dfrac{|A \cap \{1, 2, \ldots, r\}|}{r} \leq 1 - \alpha$, first observe that given $k, n \in \mathbb{N}$,

$$|A \cap \{1, 2, \ldots, km_n\}| \leq km_n - |B_n \cap \{1, 2, \ldots, km_n\}| \leq km_n - kb_n\,.$$

Suppose that $\limsup\limits_{r \to \infty} \dfrac{|A \cap \{1, 2, \ldots, r\}|}{r} > 1 - \alpha$ and let

$$\delta = \limsup_{r \to \infty} \frac{|A \cap \{1, 2, \ldots, r\}|}{r} - (1 - \alpha)\,.$$

Pick $n \in \mathbb{N}$ such that $\frac{b_n}{m_n} > \alpha - \frac{\delta}{3}$ and pick $r > \frac{3m_n}{\delta} + m_n$ such that $\dfrac{|A \cap \{1, 2, \ldots, r\}|}{r} > 1 - \alpha + \frac{2\delta}{3}$. Pick $k \in \mathbb{N}$ such that $km_n \leq r < (k+1)m_n$ and note that $\frac{1}{k} < \frac{m_n}{r - m_n} < \frac{\delta}{3}$. Then

$$\begin{aligned} 1 - \alpha + \tfrac{2\delta}{3} &< \frac{|A \cap \{1, 2, \ldots, r\}|}{r} \\ &\leq \frac{|A \cap \{1, 2, \ldots, km_n\}| + m_n}{km_n} \\ &\leq \frac{km_n - kb_n + m_n}{km_n} \\ &= 1 - \tfrac{b_n}{m_n} + \tfrac{1}{k} \\ &< 1 - \alpha + \tfrac{\delta}{3} + \tfrac{\delta}{3}\,, \end{aligned}$$

a contradiction.

To see that $\liminf_{r\to\infty} \dfrac{|A\cap\{1,2,\ldots,r\}|}{r} \geq 1-\alpha$, first observe that given $k,n\in\mathbb{N}$,

$$\begin{aligned} |A\cap\{1,2,\ldots,km_n\}| &\geq km_n - kb_n - (k-1)n - |(B\setminus B_n)\cap\{1,2,\ldots,km_n\}| \\ &\geq km_n - kb_n - (k-1)n - \textstyle\sum_{t=n+1}^{\infty} \frac{2}{q_t} km_n\,. \end{aligned}$$

Suppose that $\liminf_{r\to\infty} \dfrac{|A\cap\{1,2,\ldots,r\}|}{r} < 1-\alpha$ and let

$$\delta = 1-\alpha-\liminf_{r\to\infty} \frac{|A\cap\{1,2,\ldots,r\}|}{r}\,.$$

Pick $n\in\mathbb{N}$ such that $\frac{n}{m_n} < \frac{\delta}{4}$ and $\sum_{t=n+1}^{\infty}\frac{2}{q_t} < \frac{\delta}{4}$. Pick $r > \frac{4m_n}{\delta}$ such that $\dfrac{|A\cap\{1,2,\ldots,r\}|}{r} < 1-\alpha-\frac{3\delta}{4}$. Pick $k\in\mathbb{N}$ such that $(k-1)m_n < r \leq km_n$. Then

$$\begin{aligned} 1-\alpha-\tfrac{3\delta}{4} &> \frac{|A\cap\{1,2,\ldots,r\}|}{r} \\ &\geq \frac{|A\cap\{1,2,\ldots,km_n\}| - m_n}{km_n} \\ &\geq \frac{km_n - kb_n - (k-1)n - \sum_{t=n+1}^{\infty}\frac{2}{q_t}km_n - m_n}{km_n} \\ &= 1 - \tfrac{b_n}{m_n} - \tfrac{(k-1)n}{km_n} - \textstyle\sum_{t=n+1}^{\infty}\frac{2}{q_t} - \frac{1}{k} \\ &> 1-\alpha-\tfrac{\delta}{4}-\tfrac{\delta}{4}-\tfrac{\delta}{4}\,, \end{aligned}$$

a contradiction.

Now suppose one has $t\in\mathbb{Z}$ such that $(t+A)\cap\mathbb{N}n\neq\emptyset$ for every $n\in\mathbb{N}$. Pick $a\in(-t+A)\cap\mathbb{N}q_{|t|}$. Then $t+a\in B_{|t|}$, a contradiction. $\square$

Notice that the set produced in the proof of Theorem 2.20 is not piecewise syndetic because, as shown in the proof of Corollary 2.17, any piecewise syndetic set contains a translate of an IP-set.

By Lemma 2.1(h), Theorem 2.20 provides examples of sets with logarithmic density arbitrarily close to 1 that do not contain any translate of any IP-set. It is easy to see that if $\overline{d}(A)=1$, then A is thick. Since always $\overline{d}(A)\geq\overline{\ell d}(A)$, the corresponding conclusion applies also to $\overline{\ell d}(A)$. Therefore Theorem 2.20 provides the strongest possible conclusion about logarithmic density.

A crucial part of the proof of Theorem 2.16 is the fact that additive IP-sets meet $\mathbb{N}n$ for every n. This property is also shared by additive Δ-sets. (Given a sequence $\langle s_t\rangle_{t=1}^{\infty}$ with $\{s_m - s_t : t<m\}\subseteq A$ and given $n\in\mathbb{N}$, choose $t<m$ such that $s_t\equiv s_m \pmod{n}$.) Consequently, each Δ-set does contain initial products from some sequence. The question then naturally arises as to whether translates of Δ-sets must

contain initial products of some sequence. We show in Corollary 2.23 that they need not.

2.21 Theorem. *There is a sequence $\langle a_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that for all $l,m,u,v\in\mathbb{N}$ and all $t\in\mathbb{Z}$ if $l\leq m$, $u\leq v$, $0<|t|\leq m$, $(l,m)\neq(u,v)$, and $t+\sum_{i=u}^{v}a_i>0$, then $t+\sum_{i=l}^{m}a_i\nmid t+\sum_{i=u}^{v}a_i$.*

Proof. Let $a_1=7$. Then $|\{a_1+1,a_1+2,\ldots,2a_1-2\}|=5$ and $\big|\big\{l(t+a_1):t\in\{-1,1\}$ and $l\in\{1,2\}\big\}\big|\leq 4$ so pick

$$b_2\in\{a_1+1,a_1+2,\ldots,2a_1-2\}\setminus\big\{l(t+a_1):t\in\{-1,1\}\text{ and }l\in\{1,2\}\big\}\,.$$

Let $c_2=-1+(a_1)^2$ and pick $a_2>49c_2+a_1+11$ such that $a_2\equiv b_2 \bmod c_2$.

Let $n\in\mathbb{N}\setminus\{1\}$ and assume that we have chosen $\langle a_k\rangle_{k=1}^{n}$, $\langle b_k\rangle_{k=2}^{n}$, and $\langle c_k\rangle_{k=2}^{n}$ in $\mathbb{N}$ such that for each $w\in\{2,3,\ldots,n\}$

(1) if $m,v\in\{1,2,\ldots,w\}$, $t\in\{-m,-m+1,\ldots,m-1,m\}\setminus\{0\}$, $u\in\{1,2,\ldots,v\}$, $l\in\{1,2,\ldots,m\}$, $(l,m)\neq(u,v)$, and $t+\sum_{i=u}^{v}a_i>0$, then $t+\sum_{i=l}^{m}a_i\nmid t+\sum_{i=u}^{v}a_i$;

(2) $c_w=\prod_{l=1}^{w-1}\prod_{m=l}^{w-1}\prod_{t=1}^{w-1}(-t^2+(\sum_{i=l}^{m}a_i)^2)$; and

(3) $a_w-\sum_{i=1}^{w-1}a_i>(6w^3+1)c_w+5w+1$.

(4) $a_w\equiv b_w \bmod c_w$.

For $n=2$ hypotheses (2), (3), and (4) hold directly. To verify hypothesis (1) we need to show that

(a) if $t\in\{-1,1\}$, then $t+a_1\nmid t+a_1+a_2$;

(b) if $t\in\{-1,1\}$, then $t+a_1\nmid t+a_2$;

(c) if $t\in\{-2,-1,1,2\}$, then $t+a_2\nmid t+a_1+a_2$; and

(d) if $t\in\{-2,-1,1,2\}$, then $t+a_2\nmid t+a_1$, $t+a_1+a_2\nmid t+a_2$, and $t+a_1+a_2\nmid t+a_1$.

For (d) simply note that $0<t+a_1<t+a_2<t+a_1+a_2$.

For (c) $a_2>a_1+2\geq a_1-t$ so $t+a_2<t+a_1+a_2<2(t+a_2)$.

For (b) suppose $t+a_1\mid t+a_2$. Then $t+a_2\equiv t+b_2 \bmod c_2$ so $t+a_2\equiv t+b_2 \bmod(t+a_1)$ so $t+a_1\mid t+b_2$. But $t+a_1<t+b_2\leq t+2a_1-2\leq 2(t+a_1)$, a contradiction.

For (a) suppose $t+a_1\mid t+a_1+a_2$. Now $t+a_1+a_2\equiv t+a_1+b_2 \bmod c_2$ so $t+a_1+a_2\equiv t+a_1+b_2 \bmod(t+a_1)$ so $t+a_1\mid t+a_1+b_2$. Pick $d\in\mathbb{N}$ such that $d(t+a_1)=t+a_1+b_2$. Then $t+a_1<t+a_1+b_2\leq t+a_1+2a_1-2=t+19\leq 20<4(t+a_1)$ so $d=2$ or $d=3$. Let $l=d-1$. Then $b_2=l(t+a_1)$ which was specifically excluded, a contradiction.

The hypotheses having been verified at $n=2$, we proceed to define a_{n+1}, b_{n+1}, and c_{n+1}. First we let $c_{n+1}=\prod_{l=1}^{n}\prod_{m=l}^{n}\prod_{t=1}^{n}(-t^2+(\sum_{i=l}^{m}a_i)^2)$, as required by (2).

Next, for each $t \in \{-n, -n+1, \ldots, n-1, n\} \setminus \{0\}$ and each $j \in \{1, 2, \ldots, n\}$, let $y_{t,j} = t + \sum_{i=j}^{n} a_i$. Now $|\{\sum_{i=1}^{n} a_i + 2n + 1, \sum_{i=1}^{n} a_i + 2n + 2, \ldots, 2a_n - 3n - 1\}| = a_n - \sum_{i=1}^{n-1} a_i - 5n - 1 > (6n^3 + 1)c_n$ so

$$|\mathbb{N}c_n \cap \{\sum_{i=1}^{n} a_i + 2n + 1, \sum_{i=1}^{n} a_i + 2n + 2, \ldots, 2a_n - 3n - 1\}| \geq (6n^3 + 1)\,.$$

Also,

$$\left|\left\{ly_{t,j} - y_{t,m} : l \in \{1, 2, 3\}\,,\ j, m \in \{1, 2, \ldots, n\}, \text{ and } t \in \{-n, -n+1, \ldots, n-1, n\} \setminus \{0\}\right\}\right| \leq 6n^3$$

so pick

$$b_{n+1} \in \mathbb{N}c_n \cap \{\sum_{i=1}^{n} a_i + 2n + 1, \sum_{i=1}^{n} a_i + 2n + 2, \ldots, 2a_n - 3n - 1\} \setminus \left\{ly_{t,j} - y_{t,m} : l \in \{1, 2, 3\}\,,\ j, m \in \{1, 2, \ldots, n\}, \text{ and } t \in \{-n, -n+1, \ldots, n-1, n\} \setminus \{0\}\right\}.$$

Let $z = c_n \prod_{t=1}^{n} \prod_{j=1}^{n} (y_{t,j} y_{-t,j})$. Pick $a_{n+1} > \sum_{i=1}^{n} a_i + (6(n+1)^3 + 1)c_{n+1} + 5n + 6$ such that $a_{n+1} \equiv b_{n+1} \bmod z$.

Again hypotheses (2), (3), and (4) are satisfied directly. To verify (1), let $m, v \in \{1, 2, \ldots, n+1\}$, $u \in \{1, 2, \ldots, v\}$, $l \in \{1, 2, \ldots, m\}$, $t \in \{-m, -m+1, \ldots, m-1, m\} \setminus \{0\}$, and assume that $(l, m) \neq (u, v)$ and $t + \sum_{i=u}^{v} a_i > 0$. If $m \leq n$ and $v \leq n$ we have by induction hypothesis (1) that $t + \sum_{i=l}^{m} a_i \nmid t + \sum_{i=u}^{v} a_i$.

Also, if $v \leq n$ and $m = n + 1$ we have that $0 < t + \sum_{i=u}^{v} a_i < t + \sum_{i=l}^{m} a_i$ so that $t + \sum_{i=l}^{m} a_i \nmid t + \sum_{i=u}^{v} a_i$.

Thus we may assume that $v = n + 1$. Suppose that $t + \sum_{i=l}^{m} a_i \mid t + \sum_{i=u}^{v} a_i$.

<u>Case 1. $m < n$.</u> Then $t + \sum_{i=l}^{m} a_i \mid c_n$ and $c_n \mid b_{n+1}$ and $a_{n+1} \equiv b_{n+1} \bmod z$ so $a_{n+1} \equiv b_{n+1} \bmod c_n$ so $t + \sum_{i=l}^{m} a_i \mid a_{n+1}$ and thus $t + \sum_{i=l}^{m} a_i \mid t + \sum_{i=u}^{n+1} a_i - a_{n+1}$. Since clearly $t + \sum_{i=l}^{m} a_i \nmid t$ we have that $u \leq n$ and $t + \sum_{i=l}^{m} a_i \mid t + \sum_{i=u}^{n} a_i$. But this violates induction hypothesis (1).

<u>Case 2. $m = n$.</u> Then $t + \sum_{i=l}^{m} a_i = y_{t,l}$. Assume first that $u = n + 1$. Then $y_{t,l} \mid t + a_{n+1}$ and $t + a_{n+1} \equiv t + b_{n+1} \bmod z$ so $t + a_{n+1} \equiv t + b_{n+1} \bmod y_{t,l}$ so $y_{t,l} \mid t + b_{n+1}$. But $y_{t,l} \leq y_{n,1} = n + \sum_{i=1}^{n} a_i < t + b_{n+1} < 2a_n - 2n = 2y_{-n,n} \leq 2y_{t,l}$, a contradiction.

So $u \leq n$. Then $y_{t,l} \mid y_{t,u} + a_{n+1}$ so, as above $y_{t,l} \mid y_{t,u} + b_{n+1}$. Pick $d \in \mathbb{N}$ such that $dy_{t,l} = y_{t,u} + b_{n+1}$. Then $dy_{-n,n} \leq dy_{t,l} = y_{t,u} + b_{n+1} \leq y_{n,1} + b_{n+1} = n + \sum_{i=1}^{n} a_i + b_{n+1} < n + \sum_{i=1}^{n} a_i + 2a_n - 3n < 4y_{-n,n}$ so $d \in \{1, 2, 3\}$. But the possibility that $b_{n+1} = dy_{t,l} - y_{t,n}$ for $d \in \{1, 2, 3\}$ was specifically excluded.

Case 3. $m = n+1$. Pick $d \in \mathbb{N}$ such that $d(t+\sum_{i=l}^{n+1} a_i) = t+\sum_{i=u}^{n+1} a_i$. We cannot have $d = 1$ because if $\sum_{i=l}^{n+1} a_i = \sum_{i=u}^{n+1} a_i$, one would have $l = u$ and thus $(l, m) = (u, v)$. Therefore $d(t + \sum_{i=l}^{n+1} a_i) \geq 2(-n-1+a_{n+1}) > n+1+\sum_{i=u}^{n+1} a_i \geq t+\sum_{i=u}^{n+1} a_i$, a contradiction. $\square$

2.22 Corollary. *There is an additive Δ-set A with the property that for each $t \in \mathbb{Z}\setminus\{0\}$, there does not exist a sequence $\langle y_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}\setminus\{1\}$ such that for all $n \in \mathbb{N}$, $\prod_{i=1}^{n} y_i \in t+A$.*

Proof. Let $\langle a_n\rangle_{n=1}^{\infty}$ be as guaranteed by Theorem 2.21 and let $A = \{\sum_{i=m}^{n} a_i : m, n \in \mathbb{N}$ and $m \leq n\}$. Suppose one has a sequence $\langle y_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}\setminus\{1\}$ such that for all $n \in \mathbb{N}$, $\prod_{i=1}^{n} y_i \in t+A$. Pick n such that $\prod_{i=1}^{n} y_i > t+\sum_{i=1}^{|t|} a_i$. Pick $l \leq m$ and $u \leq v$ such that $\prod_{i=1}^{n} y_i = t+\sum_{i=l}^{m} a_i$ and $\prod_{i=1}^{n+1} y_i = t+\sum_{i=u}^{v} a_i$. This is a contradiction. $\square$

2.23 Corollary. *For each $k \in \mathbb{N}$, there is an additive Δ-set A with the property that for each $t \in \{-k, -k+1, \ldots, k-1, k\}\setminus\{0\}$, there do not exist $y \in t+A$ and $z \in \mathbb{N}\setminus\{1\}$ such that $yz \in t+A$.*

Proof. Let $\langle a_n\rangle_{n=1}^{\infty}$ be as guaranteed by Theorem 2.21 and let $A = \{\sum_{i=m}^{n} a_i : m, n \in \mathbb{N}$ and $k \leq m \leq n\}$. $\square$

3. Geometric progressions in additively large sets

In a paper primarily concerned with arithmetic progressions [21], Rankin showed that if a set A contains no k-term arithmetic progression, then the set $G(A)$ defined below contains no k-term geometric progression, even if one allows the common ratio to be a non integer rational.

3.1 Definition.

(a) For $k \in \mathbb{N}\setminus\{1\}$ and $n \in \mathbb{N}$, $e_k(n) \in \omega$ is the largest power of k that divides n.

(b) Let $A \subseteq \mathbb{N}$. Then $G(A) = \big\{n \in \mathbb{N} : \text{for every prime } p,\ e_p(n) \in A \cup \{0\}\big\}$.

Rankin also showed that for any A, $G(A)$ has density and he showed how to compute it.

Brown and Gordon [9] noted that if A is the set constructed by the greedy algorithm to not contain any length k arithmetic progression (i.e., one puts into A at each stage the first number that does not complete a k-term arithmetic progression), then $G(A)$ is the set constructed by the greedy algorithm to not contain any length k geometric progression. They also showed [9, Theorem 3] that the greedy algorithm produces the largest density of any set of the form $G(A)$ that contains no k-term geometric progression. In the case $k = 3$, the density of this $G(A)$ is roughly .7197.

It is interesting to note that if one restricts oneself to preventing 3-term geometric progressions with integer common ratio, the greedy algorithm produces exactly the same set as when the ratio is allowed to be a non-integer. However, we see one can do better with respect to upper density when one restricts to integer common ratios.

3.2 Theorem. *There is a thick subset A of $(\mathbb{N}, +)$ such that $\overline{d}(A) = \frac{3}{4}$ and there do not exist $c \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ with $\{c, cr, cr^2\} \subseteq A$.*

Proof. Let $a_1 = 1$. Given $n \in \mathbb{N}$, let $b_n = 4a_n - 1$ and let $a_{n+1} = (b_n)^3$. Let $A = \bigcup_{n=1}^{\infty} \{x \in \mathbb{N} : a_n \leq x \leq b_n\}$. It is routine to show that $\overline{d}(A) = \frac{3}{4}$.

Suppose that one has $c \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ with $\{c, cr, cr^2\} \subseteq A$. Pick $n \leq m$ such that $a_n \leq c \leq b_n$ and $a_m \leq cr \leq b_m$. Now $r \geq 2$ so $cr^2 \geq 4c \geq 4a_n > b_n$ so $cr^2 \geq a_{n+1} = (b_n)^3$.

Assume first that $n = m$. Then $a_n r \leq cr \leq b_n < 4a_n$ so $r \leq 3$. But $(b_n)^3 \leq cr^2 = r(cr) \leq 3b_n$, so $(b_n)^2 \leq 3$, a contradiction.

Therefore $n < m$. Now $cr \geq a_m \geq (b_n)^3$ and $c \leq b_n$, so $r \geq (b_n)^2 > 4$ and thus $cr^2 > 4a_m > b_m$, so $cr^2 \geq a_{m+1} = (b_m)^3$. But $cr^2 \leq (cr)^2 \leq (b_m)^2$, a contradiction. $\square$

The set produced in the proof of Theorem 3.2 certainly contains geometric progressions with non integer common ratios. We set out to show next that there exist additively thick sets that do not. In the process we show that their multiplicative structure can be quite limited. While additively thick sets by definition contain translates of any finite set, and by Theorem 2.16 they must contain the initial products from some infinite sequence, we shall show that they need not contain a translate of the pairwise products of any infinite sequence.

3.3 Lemma. *Let $n \in \mathbb{N}$ and let $A = \{n, n+1, n+2, \ldots, n + \lceil \sqrt{n}\, \rceil - 1\}$. There do not exist $a \in A$ and $r \in \mathbb{Q} \setminus \{1\}$ such that $ar \in A$ and $ar^2 \in A$.*

Proof. Suppose we have such a and r. Then $\{ar^2, ar^2\frac{1}{r}, ar^2(\frac{1}{r})^2\} \subseteq A$ so we may presume that $r > 1$.

Let $b = a - n$, $c = ar - n$, and $d = ar^2 - n$. Then $0 \leq b < c < d < \sqrt{n}$. Also $r = \dfrac{n+c}{n+b}$ so $d + n = \dfrac{(n+c)^2}{n+b}$ and thus

$$(*) \qquad n^2 + nb + nd + bd = n^2 + 2nc + c^2$$

so that $bd \equiv c^2 \pmod{n}$.

Now $0 \le bd < n$ and $0 < c^2 < n$ so $|bd - c^2| < n$ and thus $bd = c^2$. From $(*)$ we conclude that $nb + nd = 2nc$ and thus $b^2 + 2bd + d^2 = 4c^2 = 4bd$. Therefore, $(b-d)^2 = 0$, so that $b = d$, a contradiction. $\square$

Observe the contrast between the following lemma and Theorem 2.16.

3.4 Lemma. *There is a thick subset A of $(\mathbb{N}, +)$ such that $\overline{d}(A) = \frac{1}{2}$ and for any $t \in \mathbb{Z}$ and any $a, b, d \in \mathbb{N}$ satisfying $d > 4b > 64a$ and $ad > 2|t|$, ad and bd cannot both be in $t + A$.*

Proof. For $r > 1$ in $\mathbb{Q}$, let $g(r) = \max\{k \in \omega : 2^k \le r\}$. Let $A = \{n \in \mathbb{N} : g(n) = 2^m$ for some $m \in \omega\}$. Clearly A is a thick subset of $(\mathbb{N}, +)$ and $\overline{d}(A) = \frac{1}{2}$.

Suppose that $t \in \mathbb{Z}$ and $a, b, d \in \mathbb{N}$ satisfy $d > 4b > 64a$, $ad > |t|$, and $ad, bd \in t + A$.

We observe that, for any $x, y \in \mathbb{N}$, $g(xy) \in \{g(x) + g(y), g(x) + g(y) + 1\}$. Furthermore, for any $v \in \mathbb{N}$, there is a unique $u \in \mathbb{N}$ such that $u \le v$ and $u + v$ is a power of 2.

Now $2^{g(ad)-1} \le \frac{ad}{2} \le -t + ad \le \frac{3ad}{2} < 2^{g(ad)+2}$ and so

$$g(-t + ad) - (g(a) + g(d)) \in \{-1, 0, 1, 2\}\,.$$

Similarly, $g(-t + bd) - (g(b) + g(d)) \in \{-1, 0, 1, 2\}$. Let $i, j \in \{-1, 0, 1, 2\}$ satisfy $i + g(a) + g(d) = g(-t + ad)$ and $j + g(b) + g(d) = g(-t + bd)$. Since $g(-t + ad)$ and $g(-t + bd)$ are powers of 2 and $i + g(a) < g(d)$ and $j + g(b) \le g(d)$, $i + g(a) = j + g(b)$ and so $g(b) \le g(a) + 3$. Thus $b < 2 \cdot 2^{g(b)} \le 16 \cdot 2^{g(a)} \le 16a$, a contradiction. $\square$

3.5 Theorem. *There is a thick subset A of $(\mathbb{N}, +)$ such that there do not exist $c \in \mathbb{N}$ and $r \in \mathbb{Q} \setminus \{1\}$ with $\{c, cr, cr^2\} \subseteq A$. This set also has the property that there do not exist $t \in \mathbb{Z}$ and integers a, b, d in $\mathbb{N}$, with $d > 4b > 64a$ and $ad > 2|t|$, such that ad and bd are both in A. (In particular there do not exist $t \in \mathbb{Z}$ and an injective sequence $\langle y_n \rangle_{n=1}^{\infty}$ such that $t + y_i y_j \in A$ whenever i and j are distinct elements of $\mathbb{N}$.)*

Proof. Let g be defined as in Lemma 3.4. Let

$$A = \left\{ n \in \mathbb{N} : g(n) = 2^m \text{ for some } m \ge 2 \text{ in } \mathbb{N} \text{ and } n - 2^{g(n)} < \left\lceil \sqrt{2^{g(n)}} \right\rceil \right\}.$$

By Lemma 3.4, it is sufficient to show that there do not exist $c \in \mathbb{N}$ and $r \in \mathbb{Q}$ such that $\{c, cr, cr^2\} \subseteq A$. Suppose, on the contrary, that c and r do exist with these properties. Assume that $g(c) = 2^k$, $g(cr) = 2^m$, and $g(cr^2) = 2^n$. Then $k \le m \le n$ and by Lemma 3.3, $k < n$. We have $g(c) + g(r) \in \{2^m, 2^m - 1\}$ and $g(c) + 2g(r) \in \{2^n, 2^n - 1, 2^n - 2\}$. So $2g(r) = 2^n - 2^k$ or $2g(r) = 2^n - 2^k - 2$.

Assume first that $2g(r) = 2^n - 2^k$, Then $g(r) = 2^{n-1} - 2^{k-1}$ and so $g(r)$ is even, because $k \geq 2$. And hence $g(r) = 2^m - 2^k$. So $2^{k-1} + 2^m = 2^k + 2^{n-1}$. This is a contradiction, because the left hand side is the sum of two distinct powers of 2 and therefore so is the right hand side and $\{k-1, m\} = \{k, n-1\}$.

Now assume that $2g(r) = 2^n - 2^k - 2$. Then $g(r) = 2^{n-1} - 2^{k-1} - 1$ and so $g(r)$ is odd. Consequently $g(r) = 2^m - 2^k - 1$. Again we conclude that $2^{k-1} + 2^m = 2^k + 2^{n-1}$. $\square$

Theorems 3.2 and 3.5 raise the question of whether there is an additively thick set with positive upper density which contains no length three geometric progression where non integer common ratios are allowed. We cannot answer this question. However, we do note that if one takes the union of the set produced in the proof of Theorem 3.5 and the set produced by the greedy algorithm, the resulting set has positive density (equal to that of the set produced by the greedy algorithm) and contains no 9-term geometric progression. (Suppose one has $\{c, cr, cr^2, cr^3, cr^4, cr^5, cr^6, cr^7, cr^8\}$ contained in that set. Color $t \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ according to which part cr^t lies in. Since whenever $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ is partitioned into 2 classes, one must contain a length 3 arithmetic progression, this is a contradiction.)

In [9, Theorem 5], Brown and Gordon show that if $k \geq 3$ and a subset A of $\mathbb{N}$ contains no length k geometric progression then $\overline{d}(A) \leq 1 - \dfrac{1}{2^k} - \dfrac{1}{2 \cdot 5^{k-1}} + \dfrac{1}{2 \cdot 6^{k-1}}$. We can produce a slightly smaller bound, and guarantee that any larger set has a length k geometric progression with common ratio 2. (The proof of [9, Theorem 5] shows that any set with density exceeding their bound has a length k geometric progression with common ratio either 2 or $\frac{5}{3}$.)

3.6 Theorem. *Let $k \in \mathbb{N}$, let $A \subseteq \mathbb{N}$, and assume that A contains no k-term geometric progression with common ratio* 2. *Then* $\overline{d}(A) \leq 1 - \dfrac{1}{2^k - 1}$.

Proof. The result is valid, but boring, if $k = 1$, so we shall assume that $k \geq 2$. As in the proofs of Lemma 3.4 and Theorem 3.5 let for each $n \in \mathbb{N}$, $g(n) = \max\{t \in \omega : 2^t \leq n\}$ and let $B_n = \{k-1, k, k+1, \ldots, g(n)\} \cap (k\mathbb{N} - 1)$. For $m \in B_n$, let $C_{n,m} = \{t \in \mathbb{N} : 2t - 1 \leq \frac{n}{2^m}\}$ and note that $|C_{n,m}| \geq \frac{1}{2} \left\lfloor \frac{n}{2^m} \right\rfloor$.

For $m \in B_n$ and $t \in C_{n,m}$, let

$$D_{n,m,t} = \left\{(2t-1)2^{m-k+1}, (2t-1)2^{m-k+2}, \ldots, (2t-1)2^m\right\}.$$

If $(m, t) \neq (m', t')$, then $D_{n,m,t} \cap D_{n,m',t'} = \emptyset$, each $D_{n,m,t} \subseteq \{1, 2, \ldots, n\}$, and each

$D_{n,m,t} \setminus A \neq \emptyset$. So

$$|\{1,2,\ldots,n\} \setminus A| \geq \textstyle\sum_{m\in B_n} \frac{1}{2}\left\lfloor \frac{n}{2^m} \right\rfloor > \displaystyle\sum_{m\in B_n} \frac{1}{2}\left(\frac{n}{2^m} - 1\right)$$

$$= \textstyle\sum_{m\in B_n} \frac{n}{2^{m+1}} - \frac{1}{2}|B_n| > \sum_{m\in B_n} \frac{n}{2^{m+1}} - \frac{g(n)+1}{2}$$

so

$$\frac{|A \cap \{1,2,\ldots,n\}|}{n} < 1 - \textstyle\sum_{m\in B_n} \frac{1}{2^{m+1}} + \frac{g(n)+1}{2n}\,.$$

As $n \to \infty$, $\sum_{m\in B_n} \frac{1}{2^{m+1}} \to \sum_{t=1}^{\infty} \frac{1}{2^{kt}} = \frac{1}{2^k - 1}$ and $\frac{g(n)+1}{2n} \to 0$. □

Notice that the result of Theorem 3.6 is sharp with respect to common ratio 2 because the set $A = \mathbb{N}\setminus\big(\bigcup_{t=0}^{\infty} 2^{kt+k-1}(2\mathbb{N}-1)\big)$ contains no length k geometric progression with common ratio 2 and $d(A) = 1 - \frac{1}{2^k-1}$.

We have seen that there are thick subsets of $(\mathbb{N},+)$ that contain no length 3 geometric progressions, and thick sets are piecewise syndetic. We do not know (and would very much like to know) whether additively syndetic sets must contain long geometric progressions. We shall see in Theorem 3.9 that, not only is the set produced by the greedy algorithm not syndetic, any set of the form $G(A)$ for any proper subset A of $\mathbb{N}$ is not syndetic.

3.7 Theorem. *Let $Y \subseteq \mathbb{N} \setminus \{1\}$ and for each $k \in Y$, let $X_k \subseteq \mathbb{N}$. Let $B = \big\{n \in \mathbb{N} :$ for each $k \in Y$, $e_k(n) \in X_k \cup \{0\}\big\}$. If B is piecewise syndetic in $(\mathbb{N},+)$, then B is syndetic.*

Proof. Assume B is piecewise syndetic and pick $G \in \mathcal{P}_f(\mathbb{N})$ such that $\bigcup_{t\in G}(-t+B)$ is thick. We shall show that $\mathbb{N} = \bigcup_{t\in G}(-t+B)$. So let $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, $\bigcup_{t\in G}(-t+B)$ contains a block of length $m!+n$ so we may choose $s_m \in \mathbb{N}$ and $t_m \in G$ such that $m!\cdot s_m + n \in -t_m + B$. Pick $t \in F$ such that $\{m \in \mathbb{N} : t_m = t\}$ is infinite. We claim that $n \in -t+B$. To see this, let $k \in Y$ be given and pick $m > k^{e_k(n+t)+1}$ such that $m!\cdot s_m + n + t \in B$. Then $e_k(n+t) = e_k(m!\cdot s_m + n + t) \in \{0\} \cup X_k$. □

3.8 Lemma. *Let $n \in \mathbb{N}$, let $p_1, p_2, \ldots, p_n$ be distinct primes, let $t_1, t_2, \ldots, t_n$ be arbitrary elements in $\mathbb{Z}$ and let $m_1, m_2, \ldots, m_n$ be arbitrary elements in $\mathbb{N}$. There exist $x, r_1, r_2, \ldots, r_n$ in $\mathbb{N}$ such that, for every $i \in \{1,2,\ldots,n\}$, $x + t_i = r_i {p_i}^{m_i}$ and $r_i \equiv 1 \pmod{p_i}$.*

Proof. By the Chinese Remainder Theorem, there exists $y, s_1, s_2, \ldots, s_n$ in $\mathbb{N}$ such that, for every $i \in \{1,2,\ldots,n\}$, $y + t_i = s_i p_i^{m_i}$.

Let $w = \prod_{i=1}^{n} {p_i}^{m_i}$ and for each $i \in \{1,2,\ldots,n\}$, let $q_i = \dfrac{w}{{p_i}^{m_i}}$. We can choose $u_i \in \mathbb{N}$ satisfying $q_i u_i \equiv 1 \pmod{p_i}$. Again using the Chinese Remainder Theorem, we can choose $z \in \mathbb{N}$ such that, for every $i \in \{1, 2, \ldots, n\}$, $s_i u_i + z \equiv u_i \pmod{p_i}$. So $s_i + z q_i \equiv 1 \pmod{p_i}$.

Our claim holds with $x = y + zw$ and $r_i = s_i + z q_i$. $\square$

We find the following theorem surprising. The set P can be as thin among the primes as we please, and each X_p need delete only one member of $\mathbb{N}$.

3.9 Theorem. *Let P be an infinite set of primes and for each $p \in P$ let X_p be a proper subset of $\mathbb{N}$. If $B = \big\{n \in \mathbb{N} : \text{for each } p \in P,\ e_p(n) \in X_p \cup \{0\}\big\}$, then B is not piecewise syndetic.*

Proof. By Theorem 3.7 it suffices to show that B is not syndetic. So suppose one has $G \in \mathcal{P}_f(\mathbb{N})$ such that $\mathbb{N} = \bigcup_{t \in G} (-t + B)$. Enumerate G as $t_1, t_2, \ldots, t_n$. For each $i \in \{1, 2, \ldots, n\}$ pick $p_i \in P$ so that $p_i \neq p_j$ when $i \neq j$ and pick $m_i \in \mathbb{N} \setminus X_{p_i}$.

Pick by Lemma 3.8 $x, r_1, r_2, \ldots, r_n$ in $\mathbb{N}$ such that, for every $i \in \{1, 2, \ldots, n\}$, $x + t_i = r_i {p_i}^{m_i}$ and $r_i \equiv 1 \pmod{p_i}$. Pick $i \in \{1, 2, \ldots, n\}$ such that $x + t_i \in B$. But $x + t_i = r_i {p_i}^{m_i}$ and $p_i \not| r_i$ so $e_{p_i}(x + t_i) = m_i$, a contradiction. $\square$

3.10 Corollary. *Let A be the set produced by the greedy algorithm to avoid any three term geometric progressions. Then A is not piecewise syndetic.*

Proof. By [9, Theorem 2] $A = G(B)$ where B is the set produced by the greedy algorithm to avoid any three term arithmetic progressions. So Theorem 3.9 applies. $\square$

We see now that sets whose complement is not piecewise syndetic have the property that all of their translates are both additively and mutiplicatively central. In particular they are additive and multiplicative IP-sets and contain arbitrarily long geometric and arithmetic progressions.

In the following theorem we take $\beta\mathbb{N}$ to be a subset of $\beta\mathbb{Z}$ by identifying the ultrafilter $p \in \beta\mathbb{N}$ with the ultrafilter $\{A \subseteq \mathbb{Z} : A \cap \mathbb{N} \in p\} \in \beta\mathbb{Z}$.

3.11 Theorem. *Let $A \subseteq \mathbb{N}$ and assume that $\mathbb{N} \setminus A$ is not piecewise syndetic in $(\mathbb{N}, +)$. Then for all $t \in \mathbb{Z}$, $c\ell K(\beta\mathbb{N}, +) \subseteq \overline{(t + A) \cap \mathbb{N}}$ and in particular $(t + A) \cap \mathbb{N}$ is central in $(\mathbb{N}, +)$ and in $(\mathbb{N}, \cdot)$.*

Proof. By [19, Theorem 4.40], $K(\beta\mathbb{N}, +) \subseteq \overline{A}$. Let $t \in \mathbb{Z}$. To see that $c\ell K(\beta\mathbb{N}, +) \subseteq \overline{(t + A) \cap \mathbb{N}}$ it suffices to show that $K(\beta\mathbb{N}, +) \subseteq \overline{(t + A) \cap \mathbb{N}}$. So let $q \in K(\beta\mathbb{N}, +)$. By [19, Exercise 4.3.5], $\beta\mathbb{N} \cap K(\beta\mathbb{Z}, +) \neq \emptyset$ and so by [19, Theorem 1.65], $K(\beta\mathbb{N}, +) = \beta\mathbb{N} \cap K(\beta\mathbb{Z}, +)$. In particular $q \in K(\beta\mathbb{Z}, +)$ and so $-t + q \in K(\beta\mathbb{Z}, +)$ and thus, again using [19, Exercise 4.3.5], $-t+q \in \beta\mathbb{N} \cap K(\beta\mathbb{Z},+) = K(\beta\mathbb{N},+) \subseteq \overline{A}$. Thus $q \in \overline{(t+A)\cap\mathbb{N}}$ as required.

We thus have that every minimal idempotent in $(\beta\mathbb{N},+)$ is a member of $\overline{(t+A)\cap\mathbb{N}}$ and so $(t+A)\cap\mathbb{N}$ is central in $(\mathbb{N},+)$ and, by [19, Corollary 16.26] is also central in $(\mathbb{N},\cdot)$. $\square$

3.12 Corollary. *Let $n \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,n\}$ let Q_i be an infinite set of primes, let $f_i : Q_i \to \mathbb{N}$, and let $m_i \in \mathbb{Z}$. For each $i \in \{1,2,\ldots,n\}$, let $B_i = \{x \in \mathbb{N} :$ for some $q \in Q_i$, $e_q(x) = f_i(q)\}$. Then $\bigcap_{i=1}^n \big((m_i+B_i)\cap\mathbb{N}\big)$ is central in $(\mathbb{N},+)$ and in $(\mathbb{N},\cdot)$.*

Proof. By Theorem 3.9 we have for each $i \in \{1,2,\ldots,n\}$ that $\mathbb{N}\setminus B_i$ is not piecewise syndetic, so by Theorem 3.11 $c\ell K(\beta\mathbb{N},+) \subseteq \bigcap_{i=1}^n \overline{(m_i+B_i)\cap\mathbb{N}}$. Consequently as in the last paragraph of the proof of Theorem 3.11 we see that $\bigcap_{i=1}^n \big((m_i+B_i)\cap\mathbb{N}\big)$ is central in $(\mathbb{N},+)$ and in $(\mathbb{N},\cdot)$. $\square$

4. Additively syndetic sets

In this section we address the question of whether sets which are syndetic in $(\mathbb{N},+)$ must contain arbitrarily long geometric progressions, or even whether they must contain length 3 geometric progressions. We are, unfortunately, not able to answer either of these questions, but we show that an affirmative answer to the first of these questions has very strong consequences.

4.1 Definition. (a) $\mathcal{G} = \{A \subseteq \mathbb{N} : A$ contains arbitrarily long geometric progressions$\}$.

(b) $E = \{p \in \beta\mathbb{N} : p \subseteq \mathcal{G}\}$.

(c) Let ψ be a function with domain $\mathbb{N}$, let $k \in \omega$, and let $l \in \mathbb{N}$. Then $B(k,l,\psi) = \big\{n \in \mathbb{N} : (\forall t \in \{-k,-k+1,\ldots,k-1,k\})(\exists s \in \mathbb{N}\setminus\{1\})(\forall j \in \{0,1,\ldots,l\})\big(\psi(t+(n-t)s^{j-l}) = \psi(n)\big)\big\}$.

(d) Let ψ be a function with domain $\mathbb{N}$, let $k \in \omega$, and let $l \in \mathbb{N}$. Then $C(k,l,\psi) = \big\{n \in \mathbb{N} : (\forall t \in \{-k,-k+1,\ldots,k-1,k\})(\exists s \in \mathbb{N}\setminus\{1\})(\forall j \in \{0,1,\ldots,l\})\big(\psi(t+(n-t)s^{j}) = \psi(n)\big)\big\}$.

Recall that it is an easy consequence of van der Waerden's Theorem that if the union of finitely many sets is a member of $\mathcal{G}$, then one of those sets is in $\mathcal{G}$.

We show in the following theorem that the assertion that every additively syndetic set contains arbitrarily long geometric progressions has strong consequences both in terms of the structure of $\beta\mathbb{N}$ and the kind of configurations that can be guaranteed in one cell of a partition of $\mathbb{N}$.

The next lemma is well known, but we cannot find an explicit statement of it.

4.2 Lemma. *If L is a minimal left ideal of $(\beta\mathbb{N},+)$, then L is a left ideal of $(\beta\mathbb{Z},+)$.*

Proof. Pick an idempotent $p \in L$. Then $L = L + p$ so for all $q \in L$, $q = q + p$. Also, since $\beta\mathbb{N} \setminus \mathbb{N}$ is an ideal of $(\beta\mathbb{N},+)$, $L \subseteq \beta\mathbb{N} \setminus \mathbb{N}$. To see that L is a left ideal of $(\beta\mathbb{Z},+)$ it suffices to let $q \in L$ and show that $\mathbb{Z} + q \subseteq L$. So let $t \in \mathbb{Z}$. Then $t + q = (t + q) + p$ and by [19, Exercise 4.3.5], $t + q \in \beta\mathbb{N}$ so $t + q \in \beta\mathbb{N} + p = L$. $\square$

4.3 Theorem. *The following statements are equivalent.*

(a) Whenever A is syndetic in $(\mathbb{N},+)$, $A \in \mathcal{G}$.

(b) There is a left ideal L of $(\beta\mathbb{N},+)$ such that $L \subseteq E$. In particular there is a minimal idempotent p of $(\beta\mathbb{N},+)$ with $p \in E$.

(c) $\bigcap_{t\in\mathbb{Z}}(t + E) \neq \emptyset$.

(d) For every $k \in \mathbb{N}$, $\bigcap_{t=-k}^{k}(t + E) \neq \emptyset$.

(e) For every $k \in \mathbb{N}$, $\bigcap_{t=0}^{k}(t + E) \neq \emptyset$.

(f) For every $k \in \mathbb{N}$, $\bigcap_{t=0}^{k}(-t + E) \neq \emptyset$.

(g) Whenever $r \in \mathbb{N}$ and $\psi : \mathbb{N} \to \{1, 2, \ldots, r\}$, there is a left ideal L of $(\beta\mathbb{N},+)$ such that $L \subseteq E \cap \bigcap_{k=0}^{\infty}\bigcap_{l=1}^{\infty}\big(\overline{B(k,l,\psi)} \cap \overline{C(k,l,\psi)}\big)$.

(h) Whenever $r, l \in \mathbb{N}$, $k \in \omega$, and $\psi : \mathbb{N} \to \{1, 2, \ldots, r\}$, $B(k,l,\psi)$ is thick.

(i) Whenever $l \in \mathbb{N}$ and $\psi : \mathbb{N} \to \{1, 2\}$, $B(0,l,\psi)$ is thick.

(j) Whenever $r, l \in \mathbb{N}$, $k \in \omega$, and $\psi : \mathbb{N} \to \{1, 2, \ldots, r\}$, $C(k,l,\psi)$ is thick.

(k) Whenever $l \in \mathbb{N}$ and $\psi : \mathbb{N} \to \{1, 2\}$, $C(0,l,\psi)$ is thick.

(l) Whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there is some $i \in \{1, 2, \ldots, r\}$ such that for all $t \in \mathbb{Z}$, $\mathbb{N} \cap (t + A_i) \in \mathcal{G}$.

(m) Whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there is some $i \in \{1, 2, \ldots, r\}$ such that for all $t \in \mathbb{N}$, $t + A_i \in \mathcal{G}$.

Proof. We show first that statements (b) through (f) are equivalent.

To see that (b) implies (c), pick a minimal left ideal L of $(\beta\mathbb{N},+)$ which is contained in E. By Lemma 4.2 L is a left ideal of $(\beta\mathbb{Z},+)$. Therefore, given $t \in \mathbb{Z}$, $-t+p \in L \subseteq E$ so $p \in t + E$.

It is trivial that (c) implies (d) and (d) implies (e). To see that (e) implies (f), let $k \in \mathbb{N}$ and pick $p \in \bigcap_{t=0}^{k}(t + E)$. Then $-k + p \in \bigcap_{t=0}^{k}(-t + E)$.

To see that (f) implies (b), for each $k \in \mathbb{N}$ let $H_k = \bigcap_{t=0}^{k}(-t+E)$. Given $t \in \mathbb{N}$, the function $\lambda_{-t} : \beta\mathbb{Z} \to \beta\mathbb{Z}$ is continuous and so $-t+E$ is closed. Thus $\{H_k : k \in \mathbb{N}\}$ is a set of closed subsets of $\beta\mathbb{N}$ with the finite intersection property and so we may pick $p \in \bigcap_{k=1}^{\infty} H_k$. Then $\mathbb{N}+p \subseteq E$ so $\beta\mathbb{N}+p = c\ell(\mathbb{N}+p) \subseteq E$.

To see that (a) implies (f), let $k \in \mathbb{N}$ and suppose that $\bigcap_{t=0}^{k}(-t+E) = \emptyset$. For each $p \in \beta\mathbb{N}$, pick $t_p \in \{0,1,\ldots,k\}$ such that $t_p + p \notin E$ and pick $B_p \in (t_p+p) \setminus \mathcal{G}$ with $B_p \subseteq \mathbb{N}$ and let $A_p = \mathbb{N} \cap (-t_p + B_p)$. Then $\{\overline{A_p} : p \in \beta\mathbb{N}\}$ is an open cover of $\beta\mathbb{N}$ so pick a finite set $F \subseteq \beta\mathbb{N}$ such that $\beta\mathbb{N} = \bigcup_{p\in F} \overline{A_p}$ and in particular $\mathbb{N} = \bigcup_{p\in F} A_p$. Let $C = \bigcup_{p\in F} B_p$. Then $\mathbb{N} \subseteq \bigcup_{p\in F}(-t_p + C)$ so C is syndetic in $(\mathbb{N},+)$ and thus $C \in \mathcal{G}$. But then some $B_p \in \mathcal{G}$, a contradiction.

We show now that (b) implies (g). Pick a minimal left ideal L of $(\beta\mathbb{N},+)$ such that $L \subseteq E$. By Lemma 4.2 L is a left ideal of $(\beta\mathbb{Z},+)$. Let $k \in \omega$ and $l \in \mathbb{N}$ and suppose we have some p in $L \setminus \big(\overline{B(k,l,\psi)} \cap \overline{C(k,l,\psi)}\big)$. Pick $i \in \{1,2,\ldots,r\}$ such that $\psi^{-1}[\{i\}] \in p$.

For $t \in \{-k,-k+1,\ldots,k-1,k\}$, let $X_t = \{n \in \psi^{-1}[\{i\}] \setminus \big(B(k,l,\psi) \cap C(k,l,\psi)\big) : (\exists s \in \mathbb{N}\setminus\{1\})(\forall j \in \{0,1,\ldots,l\})(\psi(t+(n-t)s^{j-l}) = \psi(t+(n-t)s^j) = i)\}$. We claim that each $X_t \in p$ so suppose instead that we have some $t \in \{-k,-k+1,\ldots,k-1,k\}$ such that $X_t \notin p$. Let $D = \psi^{-1}[\{i\}] \setminus \big(X_t \cup B(k,l,\psi) \cup C(k,l,\psi)\big)$. Then $D \in p$ so $-t+D \in -t+p$ and therefore $-t+p \in E$ so pick $m \in \mathbb{N}$ and $s \in \mathbb{N}\setminus\{1\}$ such that $\{ms^j : j \in \{0,1,\ldots,2l\}\} \subseteq -t+D$. Let $n = t+ms^l$. Then $n \in D$ so $\psi(n) = i$. Also, $m = (n-t)s^{-l}$. Given $j \in \{0,1,\ldots,l\}$, we have that $t+(n-t)s^{j-l} = t+ms^j \in D$ so that $\psi(t+(n-t)s^{j-l}) = i$ and $t+(n-t)s^j = t+ms^{j+l} \in D$ so that $\psi(t+(n-t)s^j) = i$. Thus $n \in X_t$, a contradiction.

Now pick $n \in \bigcap_{t=-k}^{k} X_t$. We claim that $n \in B(k,l,\psi) \cap C(k,l,\psi)$ which will be a contradiction. So let $t \in \{-k,-k+1,\ldots,k-1,k\}$. Since $n \in X_t$, pick $s \in \mathbb{N}\setminus\{1\}$ such that for all $j \in \{0,1,\ldots,l\}$, $\psi(t+(n-t)s^{j-l}) = \psi(t+(n-t)s^j) = i$. This establishes that $n \in B(k,l,\psi) \cap C(k,l,\psi)$ as required.

That (g) implies each of (h) and (j) follows from the fact in [7, Theorem 2.9(c)] that a subset A of $\mathbb{N}$ is thick in $(\mathbb{N},+)$ if and only if there is some left ideal L of $(\beta\mathbb{N},+)$ such that $L \subseteq \overline{A}$.

Trivially (h) implies (i) and (j) implies (k).

To see that (i) implies (a) and (k) implies (a), let A be syndetic in $(\mathbb{N},+)$ and define $\psi : \mathbb{N} \to \{1,2\}$ by $\psi(n) = 1$ if $n \in A$ and $\psi(n) = 2$ if $n \notin A$. Since A is syndetic, it has nonempty intersection with any thick set, so for any $l \in \mathbb{N}$, $A \cap B(0,l,\psi) \neq \emptyset$ and $A \cap C(0,l,\psi) \neq \emptyset$. Given any n in either of those intersections, $\psi(n) = 1$, so there must be a length $l+1$ geometric progression contained in A.

To see that (c) implies (l), pick $p \in \bigcap_{t \in \mathbb{Z}} (t + E)$, let $r \in \mathbb{N}$, and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. Pick $i \in \{1, 2, \ldots, r\}$ such that $A_i \in p$. Then for each $t \in \mathbb{Z}$, $\mathbb{N} \cap (t + A_i) \in p$.

It is trivial that (l) implies (m). To complete the proof, we show that (m) implies (a). So let A be syndetic in $(\mathbb{N}, +)$ and pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $\mathbb{N} \subseteq \bigcup_{k \in F} (-k + A)$. Pick $k \in F$ such that for all $t \in \mathbb{N}$, $t + (-k + A) \in \mathcal{G}$. In particular $k + (-k + A) \in \mathcal{G}$.□

It is easy to see that the set E is a two-sided ideal of $(\beta\mathbb{N}, \cdot)$ and so $c\ell K(\beta\mathbb{N}, \cdot) \subseteq E$. Theorem 4.3 tells us in particular that if every additively syndetic subset of $\mathbb{N}$ contains arbitrarily long geometric progressions, then $K(\beta\mathbb{N}, +) \cap E \neq \emptyset$. In fact under that assumption, given any minimal right ideal R of $(\beta\mathbb{N}, +)$, $R \cap E \neq \emptyset$. On the other hand it has recently been shown [22] that $K(\beta\mathbb{N}, +) \cap c\ell K(\beta\mathbb{N}, \cdot) = \emptyset$.

Notice that statement (j) of Theorem 4.3 has translated geometric progressions with a common starting point among all of the progressions (because $t + (n - t) \cdot \varphi(t)^0 = n$) and statement (h) has translated geometric progressions with a common ending point among all of the progressions.

There is a significant contrast between these two conclusions. On one hand, if $r \in \mathbb{N}$, $k, l \in \mathbb{N}$, and $\psi : \mathbb{N} \to \{1, 2, \ldots, r\}$ is a random coloring of $\mathbb{N}$ then for each $i \in \{1, 2, \ldots, r\}$ and each $n \in \psi^{-1}[\{i\}]$ the probability is 1 that for each $t \in \{-k, -k+1, \ldots, k-1, k\}$ there is some $s \in \mathbb{N} \setminus \{1\}$ with $\{t + (n - t)s^j : j \in \{0, 1, \ldots, l\}\} \subseteq \psi^{-1}[\{i\}]$. On the other hand, if $n \in B(0, l, \psi)$, there must be some $s \in \mathbb{N} \setminus \{1\}$ such that s^l divides n and for each $i \in \{1, 2, \ldots, r\}$ there is no $n \in \psi^{-1}[\{i\}]$ such that the probability that there is some $s \in \mathbb{N} \setminus \{1\}$ with $\{ns^j : j \in \{0, 1, \ldots, l\}\} \subseteq \psi^{-1}[\{i\}]$ is 1.

We have some experimental evidence that the following very weak form of statement (j) may be false:

(*) Whenever $\mathbb{N} = A_1 \cup A_2$ there exists $i \in \{1, 2\}$ such that A_i and $1 + A_i$ both contain length three geometric progressions.

Specifically consider the following version of a greedy algorithm.

(1) Put $1 \in A_1$.

(2) Find the first unassigned n and assign it to A_1.

(3) If there exist $a \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ such that all of a, ar, and ar^2 are in A_1, announce failure and stop.

(4) If there exist $a \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ such that some two of a, ar, and ar^2 are in A_1, assign the third to A_2.

(5) If there exist $a \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ such that all of a, $ar + r - 1$, and $ar^2 + r^2 - 1$ are in A_2, announce failure and stop.

(6) If there exist $a \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ such that some two of a, $ar + r - 1$, and $ar^2 + r^2 - 1$ are in A_2, assign the third to A_1.

(7) If any assignment was made in steps (4) or (6) go to step (3). Otherwise go to step (2).

We have implemented this algorithm for the numbers 1 through 30000 (restricting steps (3) and (4) to $ar^2 \leq 30000$ and steps (5) and (6) to $ar^2 + r^2 - 1 \leq 30000$) and it did not terminate until all numbers had been assigned. That is, it produced A_1 and A_2 whose union is $\{1, 2, \ldots, 30000\}$ such that A_1 contains no three term geometric progression and $1 + A_2$ contains no three term geometric progression. (If $\{b, br, br^2\} \subseteq 1 + A_2$, let $a = b - 1$. Then $\{a, ar + r - 1, ar^2 + r^2 - 1\} \subseteq A_2$.)

References

[1] R. Ahlswede, L. Khachatrian, and A. Sárközy, *On the quotient sequence of sequences of integers*, Acta Arith. **91** (1999), 117-132.

[2] R. Alexander, *Density and multiplicative structure of sets of integers*, Acta Arith. **12** (1967), 321-332.

[3] V. Bergelson, *Multiplicatively large sets and ergodic Ramsey theory*, Israel J. Math. **148** (2005), 23-40.

[4] V. Bergelson, M. Beiglböck, N. Hindman, and D. Strauss, *Some new results in multiplicative and additive Ramsey theory*, manuscript*.

[5] V. Bergelson and N. Hindman, *Nonmetrizable topological dynamics and Ramsey theory*, Trans. Amer. Math. Soc. **320** (1990), 293-320.

[6] V. Bergelson and N. Hindman, *Partition regular structures contained in large sets are abundant*, J. Comb. Theory (Series A) **93** (2001), 18-36.

[7] V. Bergelson, N. Hindman, and R. McCutcheon, *Notions of size and combinatorial properties of quotient sets in semigroups*, Topology Proceedings **23** (1998), 23-60.

[8] A. Besicovitch, *On the density of certain sequences of integers*, Math. Ann. **110** (1934), 336-341.

[9] B. Brown and D. Gordon, *On sequences without geometric progressions*, Mathematics of Computation **65** (1996), 1749-1754.

[10] H. Davenport and P. Erdős, *On sequences of positive integers*, Acta. Arith. **2** (1936), 147-151.

* (Currently available at `http://members.aol.com/nhindman/`.)

[11] H. Davenport and P. Erdős, *On sequences of positive integers*, Indian J. of Math. **15** (1951), 19-24.

[12] P. Erdős, *Note on sequences of integers no one of which is divisible by any other*, J. London Math. Soc. **10** (1935), 125-128.

[13] P. Erdős, A. Sárközi, and E. Szemerédi, *On divisibility properties of sequences of integers*, Number Theory, Colloq. Debrecen 1968, (1970), 35-49 .

[14] H. Furstenberg, *Recurrence in ergodic theory and combinatorical number theory*, Princeton University Press, Princeton, 1981.

[15] H. Halberstam and K. Roth, *Sequences*, Clarendon Press, Oxford, 1966.

[16] G. Hardy and E. Wright, *An introduction to the theory of numbers*, Clarendon Press, Oxford, 1954.

[17] N. Hindman, *Partitions and sums and products of integers*, Trans. Amer. Math. Soc. **247** (1979), 227-245.

[18] N. Hindman, *Ultrafilters and combinatorial number theory*, in Number Theory Carbondale 1979, M. Nathanson ed., Lecture Notes in Math. **751** (1979), 119-184.

[19] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.

[20] R. Jin, *The sumset phenomenon*, Proc. Amer. Math. Soc. **130** (2001), 855-861.

[21] R. Rankin, *Sets of integers containing not more than a given number of terms in arithmetical progression*, Proc. Royal Soc. Edinburgh Section A, **65** (1960/61), 332-344.

[22] D. Strauss, *The smallest ideals of* $\beta\mathbb{N}$ *under addition and multiplication*, Topology and its Applications, to appear.

[23] E. Szemerédi, *On sets of integers containing no* k *elements in arithmetic progression*, Acta. Math. **27** (1975), 199-245.

[24] B. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wiskunde **19** (1927), 212-216.

Mathias Beiglböck
Institute of Discrete Mathematics and Geometry
Vienna University of Technology
Wiedner Hauptstr. 8-10
1040 Wien
AUSTRIA
mathias5656@yahoo.com

Vitaly Bergelson
Department of Mathematics
Ohio State University
Columbus, OH 43210
USA
vitaly@math.ohio-state.edu

Neil Hindman
Department of Mathematics
Howard University
Washington, DC 20059
USA
nhindman@aol.com

Dona Strauss
Department of Pure Mathematics
University of Hull
Hull HU6 7RX
UK
d.strauss@hull.ac.uk