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Multiply partition regular matrices

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Abstract

Let A be a finite matrix with rational entries. We say that A is *doubly image partition regular* if whenever the set $\mathbb{N}$ of positive integers is finitely coloured, there exists $\vec{x}$ such that the entries of $A\vec{x}$ are all the same colour (or *monochromatic*) and also, the entries of $\vec{x}$ are monochromatic. Which matrices are doubly image partition regular?

More generally, we say that a pair of matrices (A, B), where A and B have the same number of rows, is *doubly kernel partition regular* if whenever $\mathbb{N}$ is finitely coloured, there exist vectors $\vec{x}$ and $\vec{y}$, each monochromatic, such that $A\vec{x} + B\vec{y} = \vec{0}$. (So the case above is the case when B is the negative of the identity matrix.) There is an obvious sufficient condition for the pair (A, B) to be doubly kernel partition regular, namely that there exists a positive rational c such that the matrix $M = \begin{pmatrix} A & cB \end{pmatrix}$ is kernel partition regular. (That is, whenever $\mathbb{N}$ is finitely coloured, there exists monochromatic $\vec{x}$ such that $M\vec{x} = \vec{0}$.) Our aim in this paper is to show that this sufficient condition is also necessary. As a consequence we have that a matrix A is doubly image partition regular if and only if there is a positive rational c such that the matrix $\begin{pmatrix} A & -cI \end{pmatrix}$ is kernel partition regular, where I is the identity matrix of the appropriate size.

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We also prove extensions to the case of several matrices.

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1. Introduction

Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is said to be *kernel partition regular* (abbreviated KPR) if and only if whenever $r \in \mathbb{N}$ and $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$, there exists some $\vec{x} \in \mathbb{N}^v$ such that φ is constant on the entries of $\vec{x}$ and $A\vec{x} = \vec{0}$. In the standard "chromatic" terminology, φ is said to be an *r-colouring* of $\mathbb{N}$ and $\vec{x}$ is said to be *monochromatic*. If r is not specified, one may say simply that $\mathbb{N}$ is finitely coloured by φ. The question of which matrices are KPR was solved in 1933 by Richard Rado [5].

Definition 1.1. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_v$ be the columns of $\vec{A}$. Then A satisfies the *columns condition* if and only if there exists $m \in \{1, 2, \ldots, v\}$ and a partition $\{I_1, I_2, \ldots, I_m\}$ of $\{1, 2, \ldots, v\}$ such that

(1) $\sum_{i \in I_1} \vec{c}_i = \vec{0}$ and

(2) for each $t \in \{2, 3, \ldots, m\}$, if any, $\sum_{i \in I_t} \vec{c}_i$ is a linear combination of $\{\vec{c}_i : i \in \bigcup_{j=1}^{t-1} I_j\}$.

Theorem 1.2 (Rado's Theorem)**.** *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is kernel partition regular if and only if A satisfies the columns condition.*

For example, the fact that the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ satisfies the columns condition (with $I_1 = \{1, 3\}$ and $I_2 = \{2\}$) shows that whenever $\mathbb{N}$ is finitely coloured, there exist x_1, x_2, and x_3, all the same colour, with $x_1 + x_2 = x_3$, which is Schur's Theorem [6].

As another example, the length 4 version of van der Waerden's Theorem [7] says that whenever $\mathbb{N}$ is finitely coloured, there exist $a, d \in \mathbb{N}$ such that $\{a, a+d, a+2d, a+3d\}$ is monochromatic. Letting $x_1 = a$, $x_2 = a + d$, $x_3 = a + 2d$, $x_4 = a + 3d$, and $x_5 = d$, the fact that the matrix

$$\begin{pmatrix} -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix}$$

satisfies the columns condition (with $I_1 = \{1,2,3,4\}$ and $I_2 = \{5\}$) shows that one can get monochromatic $\vec{x}$ with $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = x_5$. That is, one gets a four term arithmetic progression with the terms and the common difference all the same colour.

We remark that the above two examples were already known when Rado's Theorem was proved. The importance of Rado's Theorem is that it reduces the question of whether or not a given matrix is kernel partition regular to a finite computation. For example, the fact that the matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

satisfies the columns condition (with $I_1 = \{1,4,5,7\}$, $I_2 = \{2,6\}$, and $I_3 = \{3\}$) established the previously unknown extension of Schur's Theorem that whenever $\mathbb{N}$ is finitely coloured, there must exist x_1, x_2, and x_3 with $\{x_1, x_2, x_3, x_1+x_2, x_1+x_3, x_2+x_3, x_1+x_2+x_3\}$ monochromatic. (It is easy to similarly establish extensions for any finite number of terms.)

We now turn to the other key notion of partition regularity.

Definition 1.3. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with rational entries. Then A is *image partition regular* (abbreviated IPR) if and only if whenever $\mathbb{N}$ is finitely coloured, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are monochromatic.

Notice that the applications of Rado's Theorem cited above are very naturally stated in terms of image partition regular matrices. Specifically, Schur's Theorem, the length 4 version of van der Waerden's Theorem, and the three term extension of Schur's Theorem are the assertions that the following three matrices are image partition regular:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

In view of the fact that many problems are very naturally stated as questions about image partition regularity, it is surprising that IPR matrices were not characterized until 1993 [2]. Among the characterizations obtained then were two that used the columns condition, and were therefore computable. Several other characterizations have been obtained since then. (See [3, Theorem 2.10] and [4, Theorem 15.24].)

It is very natural to ask the following question about image partition regular matrices. When can one insist that not only are the entries of $A\vec{x}$ all the same colour, but also that the entries of $\vec{x}$ are all the same colour, though not necessarily of the same colour as the entries of $A\vec{x}$? It is this question which motivates the current paper.

There are some finite matrices over $\mathbb{Q}$ which can be seen at a glance to have this property. These are the matrices which have no zero rows and have the property that, for some positive natural number c, the first non-zero entry in each row is equal to c. (Any image of such a matrix consists of some of the elements of an (m, p, c)-set. By [1, Satz 2.2], given $(m, p, c) \in \mathbb{N}^3$ and given any finite colouring of $\mathbb{N}$ one can always find a monochromatic (m, p, c)-set. The elements of such an (m, p, c)-set include $\{cx_1, cx_2, \ldots, cx_m\}$ where the (m, p, c)-set is generated by $\{x_1, x_2, \ldots, x_m\}$.)

There are also very simple IPR matrices which do not have this property. The diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ provides an example, as can be seen by mapping each positive integer to the starting position (mod 2) of its base 2 expansion.

Definition 1.4. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is *doubly image partition regular* (abbreviated doubly IPR) if and only if whenever $\mathbb{N}$ is finitely coloured, there exists monochromatic $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x}$ is monochromatic.

It is easy to see (or see below) a sufficient condition. Suppose that we can insist that, for some positive rational c, we actually have that all the entries of $c\vec{x}$ are the same colour as the entries of $A\vec{x}$; then it follows that A is doubly IPR. One of our main aims in this paper is to show that this sufficient condition is also necessary.

The following very simple fact relates the notion of doubly IPR to kernel partition regularity. Given $n \in \mathbb{N}$ we denote the $n \times n$ identity matrix by I_n. We have that if A is a $u \times v$ matrix with entries from $\mathbb{Q}$ then A is doubly

IPR if and only if whenever $\mathbb{N}$ is finitely coloured, there exist monochromatic $\vec{x} \in \mathbb{N}^v$ and monochromatic $\vec{y} \in \mathbb{N}^u$ such that $A\vec{x} - I_u\vec{y} = \vec{0}$.

This fact in turn motivates the following definition.

Definition 1.5. Let $u, v, w \in \mathbb{N}$. Let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and let B be a $u \times w$ matrix with entries from $\mathbb{Q}$ Then (A, B) is *doubly kernel partition regular* (abbreviated doubly KPR) if and only if whenever $\mathbb{N}$ is finitely coloured, there exist monochromatic $\vec{x}$ and $\vec{y}$ such that $A\vec{x}+B\vec{y} = \vec{0}$.

So a matrix A is doubly IPR if and only if the pair $(A, -I)$ is doubly KPR. A key idea in our proof of the characterisation of doubly IPR is to shift our attention from this 'asymmetrical' case of $(A, -I)$ and to consider instead the more general question of when (A, B) is doubly KPR. Again, it turns out (see Lemma 2.1 below) that if there is a positive rational c such that the matrix $\begin{pmatrix} A & cB \end{pmatrix}$ is KPR then (A, B) is doubly KPR. We will show that this sufficient condition is in fact necessary.

More generally, we make the following definition.

Definition 1.6. Let $u, k, v_1, v_2, \ldots, v_k \in \mathbb{N}$ with $k \geq 2$. For $t \in \{1, 2, \ldots, k\}$, let A_t be a $u \times v_t$ matrix with entries from $\mathbb{Q}$. Then $(A_1, A_2, \ldots, A_k)$ is *multiply kernel partition regular* (abbreviated multiply KPR) if and only if whenever $\mathbb{N}$ is finitely coloured, there exist for each $t \in \{1, 2, \ldots, k\}$, monochromatic $\vec{x}_t$ such that $A_1\vec{x}_1 + A_2\vec{x}_2 + \ldots + A_k\vec{x}_k = \vec{0}$.

Section 2 of this paper consists of a proof in Theorem 2.2 of the fact that $(A_1, A_2, \ldots, A_k)$ is multiply KPR if and only if there exist positive rationals $c_2, c_3, \ldots, c_k$ such that the matrix $\begin{pmatrix} A_1 & c_2A_2 & c_3A_3 & \ldots & c_kA_k \end{pmatrix}$ is KPR. Section 3 consists of derivation of some consequences of this fact, including the fact that the $u \times v$ matrix A is doubly IPR if and only if there is some positive rational c such that $\begin{pmatrix} cI_v \\ A \end{pmatrix}$ is IPR.

We conclude this introduction with the following simple fact which we will use a couple of times.

Lemma 1.7. *Let $u, v, n \in \mathbb{N}$ and let A be a KPR $u \times v$ matrix with rational entries. Then whenever $\mathbb{N}$ is finitely coloured, there exists monochromatic $\vec{x} \in (n\mathbb{N})^v$ such that $A\vec{x} = \vec{0}$.*

Proof. Let φ be a finite colouring of $\mathbb{N}$ and define a colouring ψ of $\mathbb{N}$ by $\psi(x) = \varphi(nx)$. Pick $\vec{y} \in \mathbb{N}^v$ which is monochromatic with respect to ψ such that $A\vec{y} = \vec{0}$. Let $\vec{x} = n\vec{y}$. Then $\vec{x}$ is monochromatic with respect to φ and $A\vec{x} = \vec{0}$. □

2. Characterising multiply kernel partition regular matrices

We begin with the easy half of the main theorem, Theorem 2.2. We write $\mathbb{Q}^+$ for the set of positive rationals.

Lemma 2.1. *Let $u, k, v_1, v_2, \ldots, v_k \in \mathbb{N}$ with $k \geq 2$. For $t \in \{1, 2, \ldots, k\}$, let A_t be a $u \times v_t$ matrix with entries from $\mathbb{Q}$. If there exist $c_2, c_3, \ldots, c_k \in \mathbb{Q}^+$ such that $\begin{pmatrix} A_1 & c_2A_2 & c_3A_3 & \ldots & c_kA_k \end{pmatrix}$ is KPR, then $(A_1, A_2, \ldots, A_k)$ is multiply KPR.*

Proof. Assume that $c_2, c_3, \ldots, c_k \in \mathbb{Q}^+$ and $\begin{pmatrix} A_1 & c_2A_2 & c_3A_3 & \ldots & c_kA_k \end{pmatrix}$ is KPR. Let φ be a finite colouring of $\mathbb{N}$ and let ψ be a finite colouring of $\mathbb{N}$ with the property that if $\psi(x) = \psi(y)$ then

(1) $\varphi(x) = \varphi(y)$ and

(2) if $t \in \{2, 3, \ldots, k\}$ and c_tx and c_ty are integers, then $\varphi(c_tx) = \varphi(c_ty)$.

For each $t \in \{1, 2, \ldots, k\}$, pick $m_t, n_t \in \mathbb{N}$ such that $c_t = \frac{m_t}{n_t}$ and let $n = \prod_{t=2}^{k} n_t$. Pick by Lemma 1.7 $\vec{z} \in (n\mathbb{N})^{v_1+v_2+\ldots+v_k}$ which is monochromatic with respect to ψ such that $\begin{pmatrix} A_1 & c_2A_2 & c_3A_3 & \ldots & c_kA_k \end{pmatrix}\vec{z} = \vec{0}$. For each $t \in \{1, 2, \ldots, k\}$, pick $\vec{x}_t \in (n\mathbb{N})^{v_t}$ such that

$$\vec{z} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_k \end{pmatrix}.$$

Then the entries of $\vec{x}_1$ are all the same colour with respect to φ and given $t \in \{2, 3, \ldots, k\}$, since the entries of $\vec{x}_t$ are in $n\mathbb{N}$, we have that the entries of $c_t\vec{x}_t$ are all the same colour with respect to φ. And $A_1\vec{x}_1 + A_2c_2\vec{x}_2 + \ldots + A_kc_k\vec{x}_k = \vec{0}$. □

The rest of this section will be devoted to a proof of the converse of Lemma 2.1. This proof is somewhat complicated, so we will first present an informal description of the ideas of the proof for the case $k = 2$ (where we have a given doubly KPR pair (A, B)).

There are three key ingredients, two of which have appeared in other papers and one of which is new.

(1) The 'start base p' colouring. This is used in [2].

(2) Simple facts about linear spans and positive cones being closed sets. Again, these have been used in [2].

(3) Looking at linear spans for 'all parts of the partition at once'. This will be explained below, and it is the 'new ingredient'.

Let us fix some notation. The columns of A are $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_v$ and the columns of B are $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_w$.

For a large positive integer p (not necessarily prime), we colour the naturals by first two digits and start position from the left (mod 2), all in the base p expansion. So for example if s is 67100200 and t is 3040567 then s gets colour $(67, 1)$ and t gets colour $(30, 0)$. So we have $2p(p-1)$ colours.

For this colouring, there are monochromatic vectors $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_v \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_w \end{pmatrix}$ with $A\vec{x} + B\vec{y} = \vec{0}$. Say all the entries of $\vec{x}$ start with the two digits d, where d is between 1 and p (this is just for convenience of writing later on) – so for example the above s would have $d = 6 + \frac{7}{p}$ and the t would have $d = 3$. And say all the entries of $\vec{y}$ start with the two digits e.

We have an ordered partition of the index set of the columns of A union the index set of the columns of B, according to which of the x_i and y_i start furthest to the left, which next furthest, and so on. We want to look at each set in the partition as its part in A and its part in B. So we have a partition $D \cup D' \cup D'' \cup \dots$ of the columns of A, and a partition $E \cup E' \cup E'' \cup \dots$ of the columns of B, such that (and here note that we are allowed to have one of D or E empty but not both, and one of D' or E' empty but not both, etc.):

(a) all the x_i for $i \in D$ and all the y_i for $i \in E$ start in the same place as each other;

(b) all the x_i for $i \in D'$ and all the y_i for $i \in E'$ start in the same place as each other, and this place is to the right of the start-place for the D, E terms by an even number of positions;

(c) and so on.

For infinitely many p, this ordered partition (strictly speaking, this pair of ordered partitions) is the same and from now on we will always assume that our p is chosen from this infinite set.

We write $\vec{s}(D)$ for the sum of the columns of A indexed by D, and also $\vec{s}(E)$ for the sum of the columns of B indexed by E. And similarly for $\vec{s}(D')$ etc.

Consider the equation $A\vec{x} + B\vec{y} = \vec{0}$. This says that the sum of all $x_i a_i$ plus the sum of all $y_i b_i$ is zero. If we consider dividing this by a suitable power of p, and using the fact that anything that starts to the right of the x_i in D actually starts at least two places to the right, we see that $d\vec{s}(D) + e\vec{s}(E) + \vec{\delta} = \vec{0}$, where $\vec{\delta}$ denotes a certain sum of the columns of $\begin{pmatrix} A & B \end{pmatrix}$, each with a coefficient that is at most $\frac{1}{p}$.

Now, normally one would proceed by saying that this equation tells us that $\vec{s}(D)$ plus a multiple of $\vec{s}(E)$ equals $(-1/d)\vec{\delta}$, hence the vector $\vec{s}(D)$ is arbitrarily close to the positive cone on the vector $\vec{s}(E)$ (namely the set of all non-negative real multiples of the vector $\vec{s}(E)$). But positive cones are closed, hence in fact $\vec{s}(D)$ is a non-positive multiple of $\vec{s}(E)$. This would give us a first sum of columns of $\begin{pmatrix} A & cB \end{pmatrix}$ that is zero.

However, instead of that, we will stick with that equation, for each fixed p, which is

$$d\vec{s}(D) + e\vec{s}(E) + \vec{\delta} = \vec{0}\,. \tag{1}$$

Now let us consider $\sum x_i a_i + \sum y_i b_i = 0$ when we divide by a different power of p, to focus on D' and E'. We would get a term $d\vec{s}(D') + e\vec{s}(E')$, and a smaller contribution from columns not in D, D', E, E' as well as the terms from D' and E' below the two most significant digits (with coefficients at most $\frac{1}{p}$), and also an unknown contribution from the x_i and y_i that start to the left of where we are, namely the x_i from D and the y_i from E.

So we have:

$d\vec{s}(D') + e\vec{s}(E') + \vec{\delta}' = \vec{v}$ for some $\vec{v}$ in the linear span of the columns of D and E. Write this span as $\mathrm{span}(D, E)$.

In other words:

$$d\vec{s}(D') + e\vec{s}(E') + \vec{\delta}' \tag{2}$$

belongs to $\mathrm{span}(D, E)$.

Next time we obtain:

$$d\vec{s}(D'') + e\vec{s}(E'') + \vec{\delta}'' \tag{3}$$

belongs to $\mathrm{span}(D, D', E, E')$. Continue in the same fashion. (We recall that this is for one fixed p. If we vary p, we will be changing d and e and so on.)

We are now ready for the new ingredient. We do not wish to perform any limiting in equation (1) or (2). Rather, we want to look inside a product space. Let's say that the columns of our matrices live in V (namely $\mathbb{R}^u$). So as to keep the notation manageable, let us assume that our partitions are into 3 parts: so we have D, D', D'' (but there is no D''') and same for E, E', E''. We now take the product of these equations. So, still for fixed p, in the space $V \times V \times V$ we have, combining (1),(2), and (3):

$$d(\vec{s}(D), \vec{s}(D'), \vec{s}(D'')) + e(\vec{s}(E), \vec{s}(E'), \vec{s}(E''))$$

is very close to the set $\{\vec{0}\} \times \mathrm{span}(D, E) \times \mathrm{span}(D, D', E, E')$.

Note that this latter set, say L, is the linear span of a certain set of vectors (such as each vector $(\vec{0}, \vec{a}_i, \vec{0})$ for $i \in D$). Which we can, if we wish, also view as the positive cone on (i.e. the non-negative linear combinations of) a certain finite set of vectors (namely the vectors we have just mentioned and their negatives).

Dividing by d, we see that $-(\vec{s}(D), \vec{s}(D'), \vec{s}(D''))$ is arbitrarily close to the positive cone on $L \cup \{(\vec{s}(E), \vec{s}(E'), \vec{s}(E''))\}$. But positive cones (on finite sets of vectors) are closed sets, so, letting p tend to infinity, we conclude that: $-(\vec{s}(D), \vec{s}(D'), \vec{s}(D''))$ is in the positive cone on $L \cup \{(\vec{s}(E), \vec{s}(E'), \vec{s}(E''))\}$.

In other words, there exists a nonnegative rational c (switching from reals to rationals, which is fine as all coefficients are rationals in our matrices) such that:

$(\vec{s}(D), \vec{s}(D'), \vec{s}(D'')) + c(\vec{s}(E), \vec{s}(E'), \vec{s}(E''))$ belongs to L.

Case 1: c is positive. In this case, looking at what L is, we see that $\begin{pmatrix} A & cB \end{pmatrix}$ satisfies the columns condition where the first block is $D \cup E$, then $D' \cup E'$, then $D'' \cup E''$.

Case 2: $c = 0$. This ought to be a trivial case, but in fact we do not know how to eliminate it directly. Rather, let us return to where we divided by d, and instead divide by e. In other words, we switch the roles of A and B. We obtain that for some nonnegative rational c' we have $(\vec{s}(E), \vec{s}(E'), \vec{s}(E'')) + c'(\vec{s}(D), \vec{s}(D'), \vec{s}(D''))$ belongs to L. Again, if c' is nonzero, we are done. So the only case left is when $c' = 0$. This tells us that the point $(\vec{s}(E), \vec{s}(E'), \vec{s}(E''))$ also belongs to L. But now it follows that for *any* positive rational c at all (indeed, any nonzero c) the matrix $\begin{pmatrix} A & cB \end{pmatrix}$ satisfies the columns condition.

Now we present a more formal version of the proof.

Theorem 2.2. *Let $u, k, v_1, v_2, \ldots, v_k \in \mathbb{N}$ with $k \geq 2$. For $t \in \{1, 2, \ldots, k\}$, let A_t be a $u \times v_t$ matrix with entries from $\mathbb{Q}$. Then $(A_1, A_2, \ldots, A_k)$ is multiply KPR if and only if there exist $c_2, c_3, \ldots, c_k \in \mathbb{Q}^+$ such that*

$$\begin{pmatrix} A_1 & c_2A_2 & c_3A_3 & \ldots & c_kA_k \end{pmatrix}$$

is KPR.

Proof. The sufficiency is Lemma 2.1. We shall prove the necessity.

For $p \in \mathbb{N} \setminus \{1\}$ define $g_p : \mathbb{N} \to \omega$ by $g_p(x) = \max\{t \in \omega : p^t \leq x\}$. Define $\tau_p : \omega \times \mathbb{N} \to \{0, 1, \ldots, p-1\}$ by $x = \sum_{j=0}^{g_p(x)} \tau_p(j, x)p^j$, letting $\tau_p(j, x) = 0$ if $j > g_p(x)$. Define a finite colouring γ_p of $\mathbb{N}$ so that for $x, y \in \mathbb{N}$, $\gamma_p(x) = \gamma_p(y)$ if and only if

(1) $g_p(x) \equiv g_p(y) \pmod{2}$,

(2) $\tau_p(g_p(x), x) = \tau_p(g_p(y), y)$, and

(3) $\tau_p(g_p(x) - 1, x) = \tau_p(g_p(y) - 1, y)$.

For $p \in \mathbb{N} \setminus \{1\}$ and $t \in \{1, 2, \ldots, k\}$, pick $\vec{x}_{t,p} \in \mathbb{N}^{v_t}$ such that $\vec{x}_{t,p}$ is monochromatic with respect to γ_p and

$$A_1\vec{x}_{1,p} + A_2\vec{x}_{2,p} + \ldots + A_k\vec{x}_{k,p} = \vec{0}\,.$$

Pick $m_p \in \mathbb{N}$, $\mu_p(1) > \mu_p(2) > \ldots > \mu_p(m_p)$, and, for each $t \in \{1, 2, \ldots, k\}$, pick pairwise disjoint sets $I_{t,p}(1), I_{t,p}(2), \ldots I_{t,p}(m_p)$ such that

(1) for each $t \in \{1, 2, \ldots, k\}$, $\bigcup_{i=1}^{m_p} I_{t,p}(i) = \{t\} \times \{1, 2, \ldots, v_t\}$,

(2) for each $i \in \{1, 2, \ldots, m_p\}$, $\bigcup_{t=1}^{k} I_{t,p}(i) \neq \emptyset$, and

(3) for each $i \in \{1, 2, \ldots, m_p\}$ and each $(t, j) \in \bigcup_{s=1}^{k} I_{s,p}(i)$, $g_p(x_{t,p,j}) = \mu_p(i)$.

Pick an infinite set $P \subseteq \mathbb{N}$, $m \in \mathbb{N}$, and for each $t \in \{1, 2, \ldots, k\}$ and each $i \in \{1, 2, \ldots, m\}$, $I_t(i)$, such that for each $p \in P$, $m_p = m$, and for each $t \in \{1, 2, \ldots, k\}$ and each $i \in \{1, 2, \ldots, m\}$, $I_{t,p}(i) = I_t(i)$.

By reordering the columns of each A_t, and correspondingly reordering the entries of each $\vec{x}_{t,p}$, we may presume that we have for each $t \in \{1, 2, \ldots, k\}$, $0 = \alpha_t(0) \leq \alpha_t(1) \leq \ldots \leq \alpha_t(m) = v_t$ such that for each $i \in \{1, 2, \ldots, m\}$

and each $t \in \{1,2,\ldots,k\}$, $I_t(i) = \{(t,j) : \alpha_t(i-1) < j \le \alpha_t(i)\}$. Thus, if $p \in P$, $i \in \{1,2,\ldots,m\}$, $t \in \{1,2,\ldots,k\}$, and $\alpha_t(i-1) < j \le \alpha_t(i)$, then $g_p(x_{t,p,j}) = \mu_p(i)$. After the reordering, denote the columns of A_t by $\vec{a}_{t,1}, \vec{a}_{t,2}, \ldots, \vec{a}_{t,v_t}$.

For $i \in \{1,2,\ldots,m\}$, let $J(i) = \bigcup_{t=1}^k I_t(i)$ and note that $\{J(1), J(2), \ldots, J(k)\}$ is a partition of the indices of the columns of $\begin{pmatrix} A_1 & A_2 & A_3 & \ldots & A_k \end{pmatrix}$.

For each $i \in \{1,2,\ldots,m\}$ and each $t \in \{1,2,\ldots,k\}$, let

$$\vec{s}_t(i) = \sum_{j=\alpha_t(i-1)+1}^{\alpha_t(i)} \vec{a}_{t,j}$$

and let $\vec{S}_t = (\vec{s}_t(1), \vec{s}_t(2), \ldots, \vec{s}_t(m))$. For each $p \in P$ and $t \in \{1,2,\ldots,k\}$, let

$$d_{t,p} = \tau_p(g_p(x_{t,p,1}), x_{t,p,1}) + \frac{1}{p}\tau_p(g_p(x_{t,p,1}) - 1, x_{t,p,1})\,.$$

Note that for any $j \in \{1,2,\ldots,v_t\}$,

$$d_{t,p} = \tau_p(g_p(x_{t,p,j}), x_{t,p,j}) + \frac{1}{p}\tau_p(g_p(x_{t,p,j}) - 1, x_{t,p,j})\,,$$

because $\vec{x}_{t,p}$ is monochromatic with respect to γ_p.

Note that, given $i \in \{1,2,\ldots,m\}$, $t \in \{1,2,\ldots,k\}$, and $\alpha_t(i-1) < j \le \alpha_t(i)$, we have that $x_{t,p,j} = p^{\mu_p(i)} d_{t,p} + \sum_{l=0}^{\mu_p(i)-2} \tau_p(l, x_{t,p,j}) p^l$. For $p \in P$ and $i \in \{1,2,\ldots,m\}$, define

$$\begin{aligned} s\vec{m}_p(i) = \sum_{t=1}^k \Big(& \sum_{j=\alpha_t(i-1)+1}^{\alpha_t(i)} \vec{a}_{t,p} \sum_{l=0}^{\mu_p(i)-2} \tau_p(l, x_{t,p,j}) p^{l-\mu_p(i)} + \\ & \sum_{j=\alpha_t(i)+1}^{v_t} \vec{a}_{t,p} x_{t,p,j} p^{-\mu_p(i)} \Big) \end{aligned}.$$

Note that if $M = \max\{||\vec{a}_{t,j}|| : t \in \{1,2,\ldots,k\} \text{ and } j \in \{1,2,\ldots,v_t\}\}$, then for each $i \in \{1,2,\ldots,m\}$, $||s\vec{m}_p(i)|| \le \frac{M}{p}\sum_{t=1}^k v_t$ because $g_p(x_{t,p,j}) \le \mu_p(i) - 2$ if $j > \alpha_t(i)$.

For the next three paragraphs, let $p \in P$ be fixed. We have that

$$\sum_{t=1}^k \sum_{j=1}^{v_t} x_{t,p,j} \vec{a}_{t,j} = \vec{0}\,.$$

Thus dividing by $p^{\mu_p(1)}$ we have $\sum_{t=1}^k d_{t,p}\vec{s}_t(1) + s\vec{m}_p(1) = \vec{0}$.

Now let $i \in \{1,2,\ldots,m\}$. Dividing by $\mu_p(i)$, we have

$$\sum_{t=1}^{k}\sum_{j=1}^{\alpha_t(i-1)} \vec{a}_{t,j}x_{t,p,j}p^{-\mu_p(i)} + \sum_{t=1}^{k} d_{t,p}\vec{s}_t(i) + s\vec{m}_p(i) = \vec{0}\,.$$

Thus $-\vec{s}_1(1) - \frac{1}{d_{1,p}}s\vec{m}_p(1) = \sum_{t=2}^k \frac{d_{t,p}}{d_{1,p}}\vec{s}_t(1)$ and for $i \in \{2,3,\ldots,m\}$,

$$-\vec{s}_1(i) - \frac{1}{d_{1,p}}s\vec{m}_p(i) = \sum_{t=1}^{k}\sum_{j=1}^{\alpha_t(i-1)} \frac{x_{t,p,j}}{d_{1,p}}p^{-\mu_p(i)}\vec{a}_{t,j} + \sum_{t=2}^{k}\frac{d_{t,p}}{d_{1,p}}\vec{s}_t(i)\,.$$

For $i \in \{2,3,\ldots,m\}$, let

$$\begin{aligned} C_i = \{\vec{w} \in \times_{\delta=1}^m \mathbb{R}^u : \ & \vec{w}_i \in \{\vec{a}_{t,j} : t \in \{1,2,\ldots,k\} \text{ and } \\ & \qquad j \in \{1,2,\ldots,\alpha_t(i-1)\}\} \\ & \text{and if } \delta \in \{1,2,\ldots,m\}\setminus\{i\}, \text{ then } \vec{w}_\delta = \vec{0}\}\,. \end{aligned}$$

Let K be the positive cone of $\bigcup_{i=2}^m C_i$, that is, all linear combinations of members of $\bigcup_{i=2}^m C_i$ with non-negative real coefficients. Notice that for $(\vec{w}_1,\vec{w}_2,\ldots,\vec{w}_m) \in \times_{\delta=1}^m \mathbb{R}^u$, $(\vec{w}_1,\vec{w}_2,\ldots,\vec{w}_m) \in K$ if and only if $\vec{w}_1 = \vec{0}$ and for each $i \in \{2,3,\ldots,m\}$, $\vec{w}_i$ is a linear combination with non-negative coefficients of $\{\vec{a}_{t,j} : (t,j) \in \bigcup_{l=1}^{i-1} J(l)\}$.

Let L be the positive cone of $\{\vec{S}_2,\vec{S}_3,\ldots,\vec{S}_k\} \cup \bigcup_{i=2}^m C_i$. We then have that for each $p \in P$, $-\vec{S}_1 - \frac{1}{d_{1,p}}(s\vec{m}_p(1), s\vec{m}_p(2),\ldots,s\vec{m}_p(m)) \in L$. Now L is closed in $\times_{\delta=1}^m \mathbb{R}^u$ and for each $p \in P$, $||(s\vec{m}_p(1), s\vec{m}_p(2),\ldots,s\vec{m}_p(m))|| \leq \frac{mM}{p}\sum_{t=1}^k v_t$. Therefore $-\vec{S}_1 \in L$. And since all entries of all of the vectors generating L are rational, in fact $-\vec{S}_1$ is a linear combination of members of $\{\vec{S}_2,\vec{S}_3,\ldots,\vec{S}_k\} \cup \bigcup_{i=2}^m C_i$ with all coefficients non-negative rational numbers. (See, for example, [4, Lemma 15.23].) Thus there exist non-negative rational numbers $b_{1,2}, b_{1,3},\ldots,b_{1,k}$ such that $-\vec{S}_1 - \sum_{t=2}^k b_{1,t}\vec{S}_t \in K$. Letting $b_{1,1} = 1$, we have $-\sum_{t=1}^k b_{1,t}\vec{S}_t \in K$.

Similarly, for each $r \in \{2,3,\ldots,k\}$ there exist non-negative rationals $b_{r,1}, b_{r,2},\ldots,b_{r,k}$ with $b_{r,r} = 1$ such that $-\sum_{t=1}^k b_{r,t}\vec{S}_t \in K$.

Thus we have $-\sum_{r=1}^k\sum_{t=1}^k b_{r,t}\vec{S}_t \in K$ so $-\sum_{t=1}^k\sum_{r=1}^k b_{r,t}\vec{S}_t \in K$. Since each $b_{r,t} \geq 0$ and $b_{r,r} = 1$, we have for each t that $\sum_{r=1}^k b_{r,t} \geq 1$. For $t \in \{2,3,\ldots,k\}$, let

$$c_t = \frac{\sum_{r=1}^k b_{r,t}}{\sum_{r=1}^k b_{r,1}}\,.$$

Then $-\vec{S}_1 - \sum_{t=2}^{k} c_t\vec{S}_t \in K$ and in fact is a linear combination with non-negative rational coefficients of members of $\bigcup_{i=2}^{m} C_i$. Recalling the description of what it means to be in K, we see that $\begin{pmatrix} A_1 & c_2A_2 & c_3A_3 & \dots & c_kA_k \end{pmatrix}$ satisfies the columns condition with column partition $\{J(1), J(2), \dots, J(m)\}$. $\square$

Notice the amusing fact that the proof of Theorem 2.2 establishes that the columns condition is satisfied with the sum of each set of columns a linear combination of the previous columns using no positive coefficients at all.

3. Some corollaries

An immediate corollary of Theorem 2.2 is the following computable characterisation of doubly IPR. (It is computable because on only needs to see whether there is some $b \in \mathbb{Q}^+$ such that $\begin{pmatrix} A & -bI_u \end{pmatrix}$ satisfies the columns condition.)

Corollary 3.1. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is doubly IPR if and only if there exists $b \in \mathbb{Q}^+$ such that the matrix $\begin{pmatrix} A & -bI_u \end{pmatrix}$ is KPR.*

Proof. We know that A is doubly IPR if and only if the pair $(A, -I_u)$ is doubly KPR, so Theorem 2.2 applies. $\square$

One of the characterisations of image partition regularity is the following.

Theorem 3.2. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is IPR if and only if there exist $b_1, b_2, \dots, b_v \in \mathbb{Q}^+$ such that the matrix*

$$\begin{pmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ 0 & 0 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_v \\ & & & & \\ & & A & & \end{pmatrix}$$

is IPR.

Proof. [3, Theorem 2.10]. $\square$

We show now, as a corollary to Theorem 2.2, that A is doubly IPR if and only if one can choose $b_1 = b_2 = \ldots = b_v$ in Theorem 3.2.

Corollary 3.3. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is doubly IPR if and only if there exists $b \in \mathbb{Q}^+$ such that the matrix*

$$\begin{pmatrix} bI_v \\ A \end{pmatrix}$$

is IPR.

Proof. Using Corollary 3.1, we show that for $b \in \mathbb{Q}^+$, $\begin{pmatrix} A & -bI_u \end{pmatrix}$ is KPR if and only if $\begin{pmatrix} bI_v \\ A \end{pmatrix}$ is IPR. So let $b \in \mathbb{Q}^+$ be given.

For sufficiency, let $\mathbb{N}$ be finitely coloured and pick $\vec{x} \in \mathbb{N}^v$ such that $\vec{z} = \begin{pmatrix} bI_v \\ A \end{pmatrix} \vec{x}$ is monochromatic. Then $\vec{z} = \begin{pmatrix} b\vec{x} \\ A\vec{x} \end{pmatrix}$ so $\begin{pmatrix} A & -bI_u \end{pmatrix}\vec{z} = bA\vec{x} - bA\vec{x} = \vec{0}$.

For necessity, pick $m, n \in \mathbb{N}$ such that $b = \frac{m}{n}$. By Lemma 1.7, pick monochromatic $\vec{z} \in (m\mathbb{N})^{v+u}$ such that $\begin{pmatrix} A & -bI_u \end{pmatrix}\vec{z} = \vec{0}$. Pick $\vec{w} \in (m\mathbb{N})^v$ and $\vec{y} \in (m\mathbb{N})^u$ such that $\vec{z} = \begin{pmatrix} \vec{w} \\ \vec{y} \end{pmatrix}$ and let $\vec{x} = \frac{1}{b}\vec{w}$. Since the entries of $\vec{w}$ are multiples of m, $\vec{x} \in \mathbb{N}^v$. Since $A\vec{w} - b\vec{y} = \vec{0}$ we have $b\vec{y} = A\vec{w} = bA\vec{x}$ so $\vec{y} = A\vec{x}$. Therefore

$$\begin{pmatrix} bI_v \\ A \end{pmatrix} \vec{x} = \begin{pmatrix} b\vec{x} \\ A\vec{x} \end{pmatrix} = \begin{pmatrix} \vec{w} \\ \vec{y} \end{pmatrix} = \vec{z}.$$

$\square$

We briefly consider the following question. If the entries of A are integers and A is doubly IPR, must there exist a positive integer b such that $\begin{pmatrix} A & -bI_u \end{pmatrix}$ is KPR? In fact, this need not be the case.

Example 3.4. *There is a 2×3 matrix A which is doubly IPR but such that there does not exist a positive integer b such that $\begin{pmatrix} A & -bI_u \end{pmatrix}$ is KPR.*

Proof. Let $A = \begin{pmatrix} 4 & -4 & 2 \\ 5 & -5 & 3 \end{pmatrix}$. Then the matrix $\begin{pmatrix} 4 & -4 & 2 & -\frac{1}{2} & 0 \\ 5 & -5 & 3 & 0 & -\frac{1}{2} \end{pmatrix}$ satisfies the columns condition (with $I_1 = \{1, 2\}$, $I_2 = \{3, 5\}$ and $I_3 = \{4\}$) so by Corollary 3.1, A is doubly IPR.

The only value of b other than $b = \frac{1}{2}$ for which $\begin{pmatrix} 4 & -4 & 2 & -b & 0 \\ 5 & -5 & 3 & 0 & -b \end{pmatrix}$ satisfies the columns condition is $b = -2$. $\square$

However, if we demand that no nonempty set of columns of A sums to $\vec{0}$, we do get the desired result.

Corollary 3.5. *Let $u, v \in \mathbb{N}$ and let A be a doubly IPR $u \times v$ matrix with entries from $\mathbb{Z}$. If no nonempty set of columns of A sum to $\vec{0}$, then there exists a positive integer b such that $\begin{pmatrix} A & -bI_u \end{pmatrix}$ is KPR.*

Proof. By Corollary 3.1, pick a positive rational b such that $\begin{pmatrix} A & -bI_u \end{pmatrix}$ is KPR and pick m and I_1, $I_2, \ldots, I_m$ as guaranteed by the columns condition. Now I_1 is not contained in $\{1, 2, \ldots, v\}$ so pick $t \in \{1, 2, \ldots, u\}$ such that $v + t \in I_1$. Then $b = \sum_{\{1,\ldots,v\} \cap I_1} a_{t,j}$ and is therefore an integer. $\square$

We conclude by relating the property of being multiply KPR to central subsets of $\mathbb{N}$. If S is any discrete space, its Stone-Čech compactification βS can be regarded as the set of ultrafilters on S, with the topology defined by choosing the sets of the form $\overline{A} = \{p \in \beta S : A \in p\}$, where A denotes a subset of S, as a base for the open sets. The semigroup operation of S can be extended to βS in such a way that βS becomes a compact right topological semigroup with the property that for every $s \in S$ the mapping $x \mapsto sx$ from βS to itself, is continuous. Any compact right topological semigroup has a smallest ideal which contains an idempotent. An idempotent of this kind is called *minimal*, and a subset of S which is a member of a minimal idempotent is called *central*. These sets have very rich combinatorial properties. The reader is referred to [4] for further information.

We regard $\beta\mathbb{N}$ as a semigroup, with the semigroup operation $+$ being the extension of addition on $\mathbb{N}$. We also regard $\mathbb{N}$ as embedded in $\mathbb{Q}$ and $\beta\mathbb{N}$ as embedded in $\beta\mathbb{Q}_d$, where $\mathbb{Q}_d$ is the set $\mathbb{Q}$ with the discrete topology. Hence, if $c \in \mathbb{Q}$ and $p \in \beta\mathbb{N}$, $cp \in \beta\mathbb{Q}_d$ is defined by the fact that the operation of multiplication on $\mathbb{Q}$ extends to $\beta\mathbb{Q}_d$, and the map $p \mapsto cp$ is continuous.

Definition 3.6. A finite matrix over $\mathbb{Q}$ is said to be a *first entries* matrix if no row is identically zero, the first non-zero entry of each row is positive and the first non-zero entries of two different rows are equal if they occur in the same column. A first entries matrix is said to be *unital* if the first non-zero entry of each row is 1.

The three matrices presented after Definition 1.3 are all unital first entries matrices.

Theorem 3.7. *Let $u, k, v_1, v_2, \ldots, v_k \in \mathbb{N}$ with $k \geq 2$. Let p be a minimal idempotent in $\beta\mathbb{N}$. For each $t \in \{1, 2, \ldots, k\}$, let A_t be a $u \times v_t$ matrix with entries from $\mathbb{Q}$. Then $(A_1, A_2, \ldots, A_k)$ is multiply KPR if and only if there exist minimal idempotents $p_1, p_2, \ldots, p_k$ in $\beta\mathbb{N}$, with $p = p_1$, with the following property: given members $C_1, C_2, \ldots, C_k$ of $p_1, p_2, \ldots, p_k$ respectively, there exists $\vec{x}_t \in C^{v_t}$ for each $t \in \{1, 2, \ldots, k\}$ such that $c_t\vec{x}_t \subseteq C_t^{v_t}$ and $\sum_{t=1}^{k} A_t c_t \vec{x}_t = \vec{0}$.*

Proof. The condition stated is obviously sufficient for $(A_1, A_2, \ldots, A_k)$ to be multiply KPR because, given any finite colouring of $\mathbb{N}$, every element of $\beta\mathbb{N}$ has a member which is monochrome.

To prove that it is necessary, assume that $(A_1, A_2, \ldots, A_k)$ is multiply KPR. By Theorem 2.2, there exist $c_1, c_2, c_3, \ldots, c_k \in \mathbb{Q}^+$, with $c_1 = 1$, such that $A = \begin{pmatrix} c_1A_1 & c_2A_2 & c_3A_3 & \ldots & c_kA_k \end{pmatrix}$ is KPR. For each $t \in \{1, 2, \ldots, k\}$, let $p_t = c_t p$. By [4, Lemma 5.19.2], p_t is also a minimal idempotent in $\beta\mathbb{N}$. Let $C_t \in p_t$ for each $t \in \{1, 2, \ldots, k\}$.

Let $v = v_1 + v_2 + \ldots + v_k$. Since A satisfies the columns condition, there exists $m \in \mathbb{N}$ and a $v \times m$ unital first entries matrix G over $\mathbb{Q}$ such that $AG = O$. We can write G in block form as $G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_t \end{pmatrix}$, where, for each $t \in \{1, 2, \cdots, k\}$, G_t is a $v_t \times m$ matrix over $\mathbb{Q}$. Let $C = \bigcap_{t=1}^{k} c_t^{-1} C_t$. Since $C \in p_1$, C is a central subset of $\mathbb{N}$. By [3, Lemma 2.8], there exists $\vec{x} \in \mathbb{N}^m$ such that all the entries of $G\vec{x}$ are in C. Put $\vec{x}_t = G_t\vec{x}$ for each $t \in \{1, 2, \ldots, k\}$. Then all the entries of $\vec{x}_t$ are in C, and so all the entries of $c_t\vec{x}_t$ are in C_t. Furthermore, $\sum_{t=1}^{k} A_t c_t \vec{x}_t = \sum_{t=1}^{k} c_t A_t G_t \vec{x} = AG\vec{x} = \vec{0}$. $\square$

In a similar vein, the following characterisation of doubly IPR matrices follows very easily from Corollary 3.3. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix over $\mathbb{Q}$. Then A is doubly IPR if and only if, for every minimal idempotent $p \in \beta\mathbb{N}$, there exists a minimal idempotent $q \in \beta\mathbb{N}$ such that, whenever $B \in p$ and $C \in q$, there exists $\vec{x} \in B^v$ satisfying $A\vec{x} \in C^u$.

References

[1] W. Deuber, *Partitionen und lineare Gleichungssysteme*, Math. Zeit. **133** (1973), 109-123.

[2] N. Hindman and I. Leader, *Image partition regularity of matrices*, Comb. Prob. and Comp. **2** (1993), 437-463.

[3] N. Hindman, I. Leader, and D. Strauss) *Image partition regular matrices – bounded solutions and preservation of largeness*, Discrete Math. **242** (2002), 115-144.

[4] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, 2nd edition*, Walter de Gruyter & Co., Berlin, 2012.

[5] R. Rado, *Studien zur Kombinatorik*, Math. Zeit. **36** (1933), 242-280.

[6] I. Schur, *Über die Kongruenz* $x^m + y^m = z^m (\text{mod } p)$, Jahresbericht der Deutschen Math. -Verein. **25** (1916), 114-117.

[7] B. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wiskunde **19** (1927), 212-216.