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A New and Stronger Central Sets Theorem

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Abstract. Furstenberg's original *Central Sets Theorem* applied to *central* subsets of $\mathbb{N}$ and finitely many specified sequences in $\mathbb{Z}$. In this form it was already strong enough to derive some very strong combinatorial consequences, such as the fact that a central subset of $\mathbb{N}$ contains solutions to all partition regular systems of homogeneous equations. Subsequently the Central Sets Theorem was extended to apply to arbitrary semigroups and countably many specified sequences. In this paper we derive a new version of the Central Sets Theorem for arbitrary semigroups S which applies to *all* sequences in S at once. We show that the new version is strictly stronger than the original version applied to the semigroup $(\mathbb{R},+)$. And we show that the noncommutative versions are strictly increasing in strength.

1. Introduction

In [3] Furstenberg defined a *central* subset of the set $\mathbb{N}$ of positive integers in terms of some notions from topological dynamics. He showed that if $\mathbb{N}$ is partitioned into finitely many classes, one of these classes contains a central set. Then he proved the following theorem. (For any set X, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.)

1.1 The Original Central Sets Theorem (Furstenberg). *Let $l \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,l\}$, let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Let C be a central subset of $\mathbb{N}$. Then there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) for all n, $\max H_n < \min H_{n+1}$ and

(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1,2,\ldots,l\}$, $\sum_{n\in F}\left(a_n + \sum_{t\in H_n} y_{i,t}\right) \in C$.

Proof. [3, Proposition 8.21]. □

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He pointed out that an immediate consequence is that whenever $\mathbb{N}$ is divided into finitely many classes, and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ is given, one of the classes must contain arbitrarily long arithmetic progressions with the increment $d \in FS(\langle x_n\rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$. (To see this, let $l \in \mathbb{N}$ and for $i \in \{0, 1, \ldots, l\}$ let $y_{i,n} = i \cdot x_n$. Pick a cell of the partition which contains a central set C and pick $\langle a_n\rangle_{n=1}^{\infty}$ and $\langle H_n\rangle_{n=1}^{\infty}$ as guaranteed by the Central Sets Theorem. Now throw away all but the first term of each sequence. Let $d = \sum_{t\in H_1} x_t$. Then for $i \in \{0, 1, \ldots, l\}$, $a_1 + id = a_1 + \sum_{t\in H_1} y_{i,t} \in C$.) Furstenberg also used central sets to prove Rado's Theorem [7] by showing that any central subset of $\mathbb{N}$ contains solutions to all partition regular systems of homogeneous linear equations.

Subsequently, after looking at an early draft of the paper [4] by Furstenberg and Katznelson which derived Ramsey Theoretic results using idempotents in enveloping semigroups, Vitaly Bergelson had the idea that one might be able to derive the conclusion of the Central Sets Theorem for a set $C \subseteq \mathbb{N}$ which had an idempotent in the smallest ideal of $\beta\mathbb{N}$ in its closure. (Here $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$. We shall present a brief introduction to its structure later in this section.) He was right. This suggested the following definition which makes sense in any semigroup.

1.2 Definition. Let S be a discrete semigroup and let C be a subset of S. Then C is *central* if and only if there is an idempotent p in the smallest ideal of βS such that $p \in c\ell C$.

In [1] it was shown, with the assistance of B. Weiss, that a subset C of $\mathbb{N}$ is central according to Definition 1.2 if and only if C is central according to Furstenberg's original definition. Furstenberg's original definition extends naturally to an arbitrary semigroup and in [8] Hong-ting Shi and Hong-wei Yang showed that this extended definition is equivalent to that of Definition 1.2.

In [2], the Central Sets Theorem was extended to arbitrary semigroups. The version for commutative semigroups extended Theorem 1.1 by allowing the choice of the sequence which was used to vary as n varied. (We shall deal with noncommutative versions later.) For purposes of comparison with the noncommutative versions we introduce the following notation.

1.3 Definition. Let $(S, +)$ be a commutative semigroup, let $a \in S$, let $H \in \mathcal{P}_f(\mathbb{N})$, and let $\langle y_t\rangle_{t=1}^{\infty}$ be a sequence in S. Then $x(a, H, \langle y_i\rangle_{t=1}^{\infty}) = a + \sum_{t\in H} y_t$.

With this notation conclusion (2) of Theorem 1.1 becomes "for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n\in F} x(a_n, H_n, \langle y_{i,t}\rangle_{t=1}^{\infty}) \in C$."

1.4 Theorem. *Let $(S,+)$ be a commutative semigroup. Let $l \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,l\}$, let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S. Let C be a central subset of S. Then there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in S and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*
(1) for all n, $\max H_n < \min H_{n+1}$ and
(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $f : F \to \{1,2,\ldots,l\}$, $\sum_{n\in F} x(a_n, H_n, \langle y_{f(n),t}\rangle_{t=1}^{\infty}) \in C$.

Proof. [2, Corollary 2.10]. □

The alert reader may have noticed that in Theorem 1.1 C is central in $\mathbb{N}$ while the sequences $\langle y_{i,n}\rangle_{n=1}^{\infty}$ are allowed to come from $\mathbb{Z}$. It is a fact, which follows from [6, Exercise 4.3.5 and Theorem 1.65], that any set central in $(\mathbb{N},+)$ is also central in $(\mathbb{Z},+)$, so Theorem 1.1 does follow from Theorem 1.4.

In [6] we extended the Central Sets Theorem further by dealing with countably many sequences at a time. The straightforward extension of Theorem 1.4 to countably many sequences (in which conclusion (2) would read "for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $f : F \to \mathbb{N}$, $\sum_{n\in F} x(a_n, H_n, \langle y_{f(n),t}\rangle_{t=1}^{\infty}) \in C$") is not valid. One can see this because it would easily imply that any central set in $\mathbb{N}$, and thus one cell of any finite partition of $\mathbb{N}$, would contain infinite arithmetic progressions. One needs to restrict oneself to dealing with finitely many sequences at one time, so we use the following set of functions. Given sets X and Y, we write ${}^{X}Y$ for the set of functions from X to Y.

1.5 Definition. $\Phi = \{f \in {}^{\mathbb{N}}\mathbb{N} : \text{for all } n \in \mathbb{N},\ f(n) \leq n\}$.

1.6 Theorem. *Let $(S,+)$ be a commutative semigroup and for each $i \in \mathbb{N}$, let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S. Let C be a central subset of S. Then there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in S and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*
(1) for all n, $\max H_n < \min H_{n+1}$ and
(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $f \in \Phi$, $\sum_{n\in F} x(a_n, H_n, \langle y_{f(n),t}\rangle_{t=1}^{\infty}) \in C$.

Proof. [6, Theorem 14.11]. □

In this paper we prove an extension of the Central Sets Theorem for commutative semigroups which applies to all sequences in S at once and we prove the corresponding extension for the Central Sets Theorem for noncommutative semigroups.

In Section 2 we shall derive the new commutative version. We shall also show that there exist commutative semigroups, including $(\mathbb{R},+)$, in which the conclusion of Theorem 1.4 is strictly stronger than the obvious generalization of Theorem 1.1 to arbitrary commutative semigroups.

In Section 3 we shall derive the new noncommutative version and investigate those members of βS all of whose members satisfy the new Central Sets Theorem.

In Section 4 we shall show that in the free semigroup on ω_1 generators the new Central Sets Theorem is strictly stronger than the noncommutative version of Theorem 1.6. We shall also show in that section that in the free semigroup on $\mathfrak{c}$ generators, the noncommutative version of Theorem 1.6 is strictly stronger than the noncommutative version of Theorem 1.4 which is in turn strictly stronger than the noncommutative version of Theorem 1.1.

We now present a very brief review of basic facts about $(\beta S, \cdot)$. For additional information see [6].

Given a discrete semigroup $(S, \cdot)$ we take the points of the Stone-Čech compactification βS of S to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$ and the set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (and a basis for the closed sets) of βS. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. In particular, the operation $\cdot$ on βS extends the operation $\cdot$ on S. If the operation is denoted by $+$, then $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$. The reader should be warned however, that even if S is commutative, βS seldom is. In particular the algebraic centers of $(\beta\mathbb{N}, \cdot)$ and $(\beta\mathbb{N}, +)$ are both equal to $\mathbb{N}$.

With this operation, $(\beta S, \cdot)$ is a compact Hausdorff right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. A nonempty subset I of a semigroup T is a *left ideal* provided $T \cdot I \subseteq I$, a *right ideal* provided $I \cdot T \subseteq I$, and a *two sided ideal* (or simply an *ideal*) provided it is both a left ideal and a right ideal.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T) = \bigcup\{L : L \text{ is a minimal left ideal of } T\} = \bigcup\{R : R \text{ is a minimal right ideal of } T\}$. Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is a *minimal* idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal.

Thus a subset C of S is central if and only if it is a member of a minimal idempotent of βS.

2. The new commutative Central Sets Theorem

As with the older versions, the new Central Sets Theorem for commutative semigroups is a consequence of the general result for all semigroups. However, the commutative version is much simpler to state, and so we present its derivation separately.

We present a nearly self contained proof, relying only on a few basic facts about compact right topological semigroups. We do this to make clear the simplicity of the proof of the new Central Sets Theorem. We begin with the following special case of Theorem 1.4. A subset C of S is *piecewise syndetic* if and only if $\overline{C}\cap K(\beta S)\neq\emptyset$. In particular any central set is piecewise syndetic.

As the referee pointed out, the following theorem is an immediate consequence of the corresponding result which does not require that $\min H > m$. (One may simply delete the first m terms of each sequence.) However, we need this version, and it is no harder to prove than the superficially more restricted version.

2.1 Theorem. *Let $(S,+)$ be a commutative semigroup and let $l\in\mathbb{N}$. For each $i\in\{1,2,\ldots,l\}$, let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S. Let C be a piecewise syndetic subset of S and let $m\in\mathbb{N}$. There exist $a\in S$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that $\min H>m$ and for each $i\in\{1,2,\ldots,l\}$, $x(a,H,\langle y_{i,t}\rangle_{t=1}^{\infty})\in C$.*

Proof. Let $Y=\times_{t=1}^{l}\beta S$. Then by [6, Theorem 2.22] Y is a compact right topological semigroup and if $s\in\times_{t=1}^{l}S$, then λ_s is continuous. For $i\in\mathbb{N}$, let

$$I_i=\big\{\big(x(a,H,\langle y_{1,t}\rangle_{t=1}^{\infty}),\ldots,x(a,H,\langle y_{l,t}\rangle_{t=1}^{\infty})\big):a\in S\,,\ H\in\mathcal{P}_f(\mathbb{N}),\text{ and }\ \min H>i\big\}$$

and let $E_i=I_i\cup\{(a,a,\ldots,a):a\in S\}$.

Let $E=\bigcap_{i=1}^{\infty}\overline{E_i}$ and let $I=\bigcap_{i=1}^{\infty}\overline{I_i}$. We claim that E is a subsemigroup of Y and I is an ideal of E. To this end, let $p,q\in E$. We show that $p+q\in E$ and if either $p\in I$ or $q\in I$, then $p+q\in I$. Let U be an open neighborhood of $p+q$ and let $i\in\mathbb{N}$. Since ρ_q is continuous, pick a neighborhood V of p such that $V+q\subseteq U$. Pick $x\in E_i\cap V$ with $x\in I_i$ if $p\in I$. If $x\in I_i$ so that $x=\big(x(a,H,\langle y_{1,t}\rangle_{t=1}^{\infty}),\ldots,x(a,H,\langle y_{l,t}\rangle_{t=1}^{\infty})\big)$ for some $a\in S$ and some $H\in\mathcal{P}_f(\mathbb{N})$ with $\min H>i$, let $j=\max H$. Otherwise, let $j=i$. Since λ_x is continuous, pick a neighborhood W of q such that $x+W\subseteq U$. Pick $y\in E_j\cap W$ with $y\in I_j$ if $q\in I$. Then $x+y\in E_i\cap U$ and if either $p\in I$ or $q\in I$, then $x+y\in I_i\cap U$.

By [6, Theorem 2.23] $K(Y)=\times_{t=1}^{l}K(\beta S)$. Pick $p\in K(\beta S)\cap\overline{C}$. Then $\overline{p}=(p,p,\ldots,p)\in K(Y)$. We claim that $\overline{p}\in E$. To see this, let U be a neighborhood of $\overline{p}$, let $i\in\mathbb{N}$, and pick $A_1,A_2,\ldots,A_l\in p$ such that $\times_{t=1}^{l}\overline{A_t}\subseteq U$. Pick $a\in\bigcap_{t=1}^{l}A_t$. Then $\overline{a}=(a,a,\ldots,a)\in U\cap E_i$. Thus $\overline{p}\in K(Y)\cap E$ and consequently $K(Y)\cap E\neq\emptyset$.

Then by [6, Theorem 1.65], we have that $K(E) = K(Y) \cap E$ and so $\overline{p} \in K(E) \subseteq I$. Then $I_m \cap \times_{t=1}^{l} C \neq \emptyset$ so pick $z \in I_m \cap \times_{t=1}^{l} C$ and pick $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H > m$ such that $z = \big(x(a, H, \langle y_{1,t}\rangle_{t=1}^{\infty}), \ldots, x(a, H, \langle y_{l,t}\rangle_{t=1}^{\infty})\big)$. □

The following is the new Central Sets Theorem for commutative semigroups.

2.2 Theorem. *Let $(S, +)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$, the set of sequences in S. Let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

(2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $\langle y_{i,n}\rangle_{n=1}^{\infty} \in G_i$, one has $\sum_{i=1}^{m} x(\alpha(G_i), H(G_i), \langle y_{i,t}\rangle_{t=1}^{\infty}) \in C$.

Proof. Pick a minimal idempotent p of βS such that $C \in p$. Let

$$C^\star = \{x \in C : -x + C \in p\} .$$

Since $p + p = p$, $C^\star \in p$. Also by [6, Lemma 4.14], if $x \in C^\star$, then $-x + C^\star \in p$.

We define $\alpha(F) \in S$ and $H(F) \in \mathcal{P}_f(\mathbb{N})$ for $F \in \mathcal{P}_f(\mathcal{T})$ by induction on $|F|$ satisfying the following inductive hypotheses:

(1) if $\emptyset \neq G \subsetneq F$, then $\max H(G) < \min H(F)$ and

(2) if $n \in \mathbb{N}$, $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n = F$, and $\langle f_i\rangle_{i=1}^{n} \in \times_{i=1}^{n} G_i$, then $\sum_{i=1}^{n} x(\alpha(G_i), H(G_i), f_i) \in C^\star$.

Assume first that $F = \{f\}$. Pick by Theorem 2.1 $a \in S$ and $L \in \mathcal{P}_f(\mathbb{N})$ such that $x(a, L, \langle f(t)\rangle_{t=1}^{\infty}) \in C^\star$. Let $\alpha(\{f\}) = a$ and $H(\{f\}) = L$.

Now assume that $|F| > 1$ and $\alpha(G)$ and $H(G)$ have been defined for all proper subsets G of F. Let $K = \bigcup\{H(G) : \emptyset \neq G \subsetneq F\}$ and let $m = \max K$. Let

$$M = \{\textstyle\sum_{i=1}^{n} x(\alpha(G_i), H(G_i), f_i) : n \in \mathbb{N}\,, \emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n \subsetneq F\,, \text{ and } \langle f_i\rangle_{i=1}^{n} \in \times_{i=1}^{n} G_i\}\,.$$

Then M is finite and by hypothesis (2), $M \subseteq C^\star$. Let $B = C^\star \cap \bigcap_{x \in M} (-x + C^\star)$. Then $B \in p$ so pick by Theorem 2.1 $a \in S$ and $L \in \mathcal{P}_f(\mathbb{N})$ such that $\min L > m$ and for each $f \in F$, $x(a, L, \langle f(t)\rangle_{t=1}^{\infty}) \in B$. Let $\alpha(F) = a$ and $H(F) = L$.

Since $\min L \geq m$ we have that hypothesis (1) is satisfied. To verify hypothesis (2), let $n \in \mathbb{N}$, let $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n = F$, and let $\langle f_i\rangle_{i=1}^{n} \in \times_{i=1}^{n} G_i$ If $n = 1$, then $\sum_{i=1}^{n} x(\alpha(G_i), H(G_i), f_1) = x(a, L, f_1) \in B \subseteq C^\star$. So assume that $n > 1$ and let $y = \sum_{i=1}^{n-1} x(\alpha(G_i), H(G_i), f_i)$. Then $y \in M$ so $x(a, L, f_n) \in B \subseteq (-y + C^\star)$ and thus $\sum_{i=1}^{n} x(\alpha(G_i), H(G_i), f_i) = y + x(a, L, f_n) \in C^\star$ as required. □

As a simple application, we present the following corollary which is not directly derivable by a single application of Theorem 1.6. The point of the corollary is that an arithmetic progression A is chosen which "works" for every length k and every sequence $\langle y_n\rangle_{n=1}^\infty$.

2.3 Corollary. *Let C be a central subset of $\mathbb{N}$, let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$, and let $l\in\mathbb{N}$. There exist $a\in\mathbb{N}$ and $d\in FS(\langle x_n\rangle_{n=1}^\infty)$ such that $A=\{a+d,a+2d,\ldots,a+ld\}\subseteq C$ and whenever $\langle y_n\rangle_{n=1}^\infty$ is a sequence in $\mathbb{N}$ and $k\in\mathbb{N}$ there will exist $b\in\mathbb{N}$ and $c\in FS(\langle y_n\rangle_{n=1}^\infty)$ such that $B=\{b+c,b+2c,\ldots,b+kc\}\subseteq C$ and $A+B\subseteq C$.*

Proof. Pick functions α and H as guaranteed by the new Central Sets Theorem. Let $F=\{\langle x_n\rangle_{n=1}^\infty,\langle 2x_n\rangle_{n=1}^\infty,\ldots,\langle lx_n\rangle_{n=1}^\infty\}$. Let $a=\alpha(F)$ and let $d=\sum_{t\in H(F)}x_t$. Given $\langle y_n\rangle_{n=1}^\infty$ and k, let $G=F\cup\{\langle y_n\rangle_{n=1}^\infty,\langle 2y_n\rangle_{n=1}^\infty,\ldots,\langle ky_n\rangle_{n=1}^\infty\}$. Let $b=\alpha(G)$ and let $c=\sum_{t\in H(G)}y_t$. □

Honesty compels us to admit that we could have derived Corollary 2.3 by two applications of Theorem 1.6, or even of Theorem 1.1, by first producing $a\in\mathbb{N}$ and $d\in FS(\langle x_n\rangle_{n=1}^\infty)$ such that $A=\{a+d,a+2d,\ldots,a+ld\}\subseteq C^\star$ and then applying Theorem 1.1 to the central set $\bigcap_{t=1}^{l}(-(a+d)+C^\star)$.

Notice that Theorem 1.6 is an easy consequence of Theorem 2.2. To see this, notice that one can assume that the sequences in the statement of Theorem 1.6 are distinct. Then given such sequences, for each $n\in\mathbb{N}$, let $F_n=\{\langle y_{1,t}\rangle_{t=1}^\infty,\langle y_{2,t}\rangle_{t=1}^\infty,\ldots,\langle y_{n,t}\rangle_{t=1}^\infty\}$ and let $a_n=\alpha(F_n)$ and $H_n=H(F_n)$.

We cannot prove that Theorem 2.2 is strictly stronger than Theorem 1.6 or even Theorem 1.4. (In Section 4 we will show that the corresponding noncommutative versions are indeed strictly increasing in strength.) We can, however, show that Theorem 1.4 is strictly stronger than the obvious generalization of Theorem 1.1 to arbitrary commutative semigroups which we state now.

2.4 Theorem. *Let $(S,+)$ be a commutative semigroup. Let $l\in\mathbb{N}$ and for each $i\in\{1,2,\ldots,l\}$, let $\langle y_{i,n}\rangle_{n=1}^\infty$ be a sequence in S. Let C be a central subset of S. Then there exist sequences $\langle a_n\rangle_{n=1}^\infty$ in S and $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) for all n, $\max H_n<\min H_{n+1}$ and

(2) for all $F\in\mathcal{P}_f(\mathbb{N})$ and all $i\in\{1,2,\ldots,l\}$, $\sum_{n\in F}x(a_n,H_n,\langle y_{i,t}\rangle_{t=1}^\infty)\in C$.

The following semigroup contains much of the known algebraic structure of $\beta\mathbb{N}$ and occurs as a subsemigroup of βS for many semigroups S. (See [6, especially Section 6.1].)

2.5 Definition. $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$.

We shall need the following technical lemma. Recall that $\omega = \mathbb{N} \cup \{0\}$ is the first infinite ordinal.

2.6 Lemma. *Let $(S,+)$ be a commutative semigroup with identity 0 and no other idempotents. Let $m \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,m\}$, let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S. Assume that $\psi : \omega \xrightarrow[\text{onto}]{\text{1-1}} S$, $\psi(0) = 0$, and the restriction of $\widetilde{\psi}$ to $\mathbb{H}$ is an injective homomorphism, where $\widetilde{\psi} : \beta\omega \to \beta S$ is the continuous extension of ψ. Assume further that all idempotents of $\beta S \setminus S$ are in $\widetilde{\psi}[\mathbb{H}]$. Then for each $r \in \mathbb{N}$, there exists $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > r$ and for all $i \in \{1,2,\ldots,m\}$, $\psi^{-1}(\sum_{t\in H} y_{i,t}) \in 2^r \cdot \omega$.*

Proof. Consider the semigroup $(\mathcal{P}_f(\mathbb{N}), \cup)$ and denote the extended operation in $\beta\mathcal{P}_f(\mathbb{N})$ by $\uplus$. (We cannot follow our usual custom of denoting the extended operation by the same symbol as used for the original semigroup since $p \cup q$ already means something.) For each $n \in \mathbb{N}$ let $B_n = \{H \in \mathcal{P}_f(\mathbb{N}) : \min H > n\}$. Then by [6, Theorem 4.20], $\mathcal{B} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathcal{P}_f(\mathbb{N})} B_n$ is a subsemigroup of $(\beta\mathcal{P}_f(\mathbb{N}), \uplus\ )$ so pick an idempotent $p \in \mathcal{B}$. For each $i \in \{1,2,\ldots,m\}$ define $\theta_i : \mathcal{P}_f(\mathbb{N}) \to S$ by $\theta_i(H) = \sum_{t\in H} y_{i,t}$.

Now let $i \in \{1,2,\ldots,m\}$ be given. By [6, Theorem 4.21], if $\widetilde{\theta}_i : \beta\mathcal{P}_f(\mathbb{N}) \to \beta S$ is the continuous extension of θ_i, then the restriction of $\widetilde{\theta}_i$ to $\mathcal{B}$ is a homomorphism. Consequently, $\widetilde{\theta}_i(p)$ is either 0 or an idempotent in $\widetilde{\psi}[\mathbb{H}]$. Thus $\widetilde{\psi}^{-1}\big(\widetilde{\theta}_i(p)\big)$ is either 0 or is an idempotent in $\mathbb{H}$. In any event, $\overline{2^r \cdot \omega}$ is a neighborhood of $\widetilde{\psi}^{-1}\big(\widetilde{\theta}_i(p)\big)$ in $\beta\omega$ so pick $A_i \in p$ such that $\widetilde{\psi}^{-1}\big(\widetilde{\theta}_i[\,\overline{A_i}\,]\big) \subseteq \overline{2^r \cdot \omega}$.

Pick $H \in B_r \cap \bigcap_{i=1}^{m} A_i$. Then $\min H > r$ and for each $i \in \{1,2,\ldots,m\}$,

$$\psi^{-1}(\textstyle\sum_{t\in H} y_{i,t}) = \psi^{-1}\big(\theta_i(H)\big) \in 2^r \cdot \omega\,. \qquad \square$$

As noted before the proof of [6, Theorem 7.28] the word "metrizable" is not really needed in the following theorem.

2.7 Theorem. *For each $\iota < \mathfrak{c}$ let $(S_\iota,+)$ be a semigroup containing $(\omega,+)$ with $|S_\iota| \leq \mathfrak{c}$. Assume further that either $S_0 = \omega$ or S_0 is a countably infinite group which can be mapped into a compact metrizable group by an injective homomorphism. Let $S = \bigoplus_{\iota<\mathfrak{c}} S_\iota$. Then there is a subset A of S which satisfies the conclusion of Theorem 2.4 but does not satisfy the conclusion of Theorem 1.4.*

Proof. Given $\sigma < \mathfrak{c}$ define $e(\sigma) \in S$ by $e(\sigma)(\sigma) = 1$ and $e(\sigma)(\iota) = 0$ if $\iota \neq \sigma$. We shall use two notions of "support" in this proof. For $x \in S$, $\mathrm{supp}(x) = \{\sigma < \mathfrak{c} : x(\sigma) \neq 0\}$. For $x \in \mathbb{N}$, $\mathrm{supp}_2(x) \in \mathcal{P}_f(\omega)$ is defined by $x = \sum_{t\in\mathrm{supp}_2(x)} 2^t$ and $\mathrm{supp}_2(0) = \emptyset$.

If $S_0 = \omega$, let $\psi : \omega \to \omega$ be the identity. If S_0 is a countably infinite group which can be mapped into a compact metrizable group by an injective homomorphism, then by [6, Theorem 7.28] we may pick $\psi : \omega \xrightarrow[\text{onto}]{\text{1-1}} S$ such that $\psi(0) = 0$, the restriction of $\widetilde{\psi}$ to $\mathbb{H}$ is an injective homomorphism, and all idempotents of $\beta S \setminus S$ are in $\widetilde{\psi}[\mathbb{H}]$. In any event the hypotheses of Lemma 2.6 are satisfied.

Notice that $|S| = \mathfrak{c}$ and so, if $\mathcal{T} = {}^{\mathbb{N}}S$, we have that $|\mathcal{P}_f(\mathcal{T})| = \mathfrak{c}$. Enumerate $\mathcal{P}_f(\mathcal{T})$ as $\langle F_\sigma \rangle_{\sigma<\mathfrak{c}}$ and for each $\sigma < \mathfrak{c}$, let $m(\sigma) = |F_\sigma|$. Write

$$F_\sigma = \left\{ \langle y_{\sigma,i,t} \rangle_{t=1}^{\infty} : i \in \{1, 2, \ldots, m(\sigma)\} \right\}.$$

Let $\{E_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets and define $\theta : \mathbb{N} \to \mathbb{N}$ by $n \in E_{\theta(n)}$. Let $D = \{\sigma < \mathfrak{c} : \sigma \text{ is a limit ordinal}\}$ and choose $\gamma : \mathfrak{c} \xrightarrow{\text{1-1}} D$ such that for all $\sigma < \mathfrak{c}$, $\sup\left(\bigcup_{i=1}^{m(\sigma)} \bigcup_{t=1}^{\infty} \mathrm{supp}(y_{\sigma,i,t})\right) < \gamma(\sigma)$.

We choose inductively for $\sigma < \mathfrak{c}$ sequences $\langle a_{\sigma,n} \rangle_{n=1}^{\infty}$ in S and $\langle H_{\sigma,n} \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ as follows. Let $\sigma < \mathfrak{c}$ be given and assume that $\langle a_{\tau,n} \rangle_{n=1}^{\infty}$ and $\langle H_{\tau,n} \rangle_{n=1}^{\infty}$ have been chosen for all $\tau < \sigma$. Choose $k_1 \in E_{m(\sigma)}$ and choose $H_{\sigma,1} \in \mathcal{P}_f(\mathbb{N})$ such that for each $i \in \{1, 2, \ldots, m(\sigma)\}$, 2^{k_1+1} divides $\psi^{-1}\big(\sum_{t\in H_{\sigma,1}} \pi_0(y_{\sigma,i,t})\big) = \psi^{-1}\big(\pi_0(\sum_{t\in H_{\sigma,1}} y_{\sigma,i,t})\big)$ which one can do by Lemma 2.6. Choose $k_2 \in E_{m(\sigma)}$ such that $\psi^{-1}\big(\pi_0(\sum_{t\in H_{\sigma,1}} y_{\sigma,i,t})\big) < 2^{k_2}$ for each $i \in \{1, 2, \ldots, m(\sigma)\}$ and $k_2 > k_1$. (The last inequality is redundant unless $\pi_0(\sum_{t\in H_{\sigma,1}} y_{\sigma,i,t}) = 0$ for each $i \in \{1, 2, \ldots, m(\sigma)\}$, which is possible.) Let $a_{\sigma,1} = \psi(2^{k_1} + 2^{k_2}) \cdot e(0) + e(\gamma(\sigma) + 1)$.

Now let $n \in \mathbb{N}$ and assume that $a_{\sigma,n}$ and $H_{\sigma,n}$ have been chosen. Pick $k_{2n+1} \in E_{m(\sigma)}$ such that $k_{2n+1} > k_{2n}$ and pick $H_{\sigma,n+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{\sigma,n+1} > \max H_{\sigma,n}$ and $2^{k_{2n+1}+1}$ divides $\psi^{-1}\big(\pi_0(\sum_{t\in H_{\sigma,n+1}} y_{\sigma,i,t})\big)$ for each $i \in \{1, 2, \ldots, m(\sigma)\}$. Pick $k_{2n+2} \in E_{m(\sigma)}$ such that $k_{2n+2} > k_{2n+1}$ and $\psi^{-1}\big(\pi_0(\sum_{t\in H_{\sigma,n+1}} y_{\sigma,i,t})\big) < 2^{k_{2n+2}}$ for each $i \in \{1, 2, \ldots, m(\sigma)\}$. Let $a_{\sigma,n+1} = \psi(2^{k_{2n+1}} + 2^{k_{2n+2}}) \cdot e(0) + e(\gamma(\sigma) + n + 1)$.

Now let for each $\sigma < \mathfrak{c}$, $A_\sigma = \bigcup_{i=1}^{m(\sigma)} FS\big(\big\langle x(a_{\sigma,n}, H_{\sigma,n}, \langle y_{\sigma,i,t} \rangle_{t=1}^{\infty}) \big\rangle_{n=1}^{\infty}\big)$ and let $A = \bigcup_{\sigma<\mathfrak{c}} A_\sigma$.

Observe now that if $\sigma < \mathfrak{c}$ and $x \in A_\sigma$, then

(1) $\theta\Big(\min \mathrm{supp}_2 \psi^{-1}\big(\pi_0(x)\big)\Big) = \theta\Big(\max \mathrm{supp}_2 \psi^{-1}\big(\pi_0(x)\big)\Big) = m(\sigma)$ and

(2) there exist $i \in \{1, 2, \ldots, m(\sigma)\}$ and $G \in \mathcal{P}_f(\mathbb{N})$ such that $x = \sum_{n\in G} x(a_{\sigma,n}, H_{\sigma,n}, \langle y_{\sigma,i,t} \rangle_{t=1}^{\infty})$ where

  (a) $\gamma(\sigma) < \max \mathrm{supp}(x) < \gamma(\sigma) + \omega$ and

  (b) $\mathrm{supp}(x) \cap (\gamma(\sigma), \gamma(\sigma) + \omega) = \{\gamma(\sigma) + n : n \in G\}$.

We have directly that A satisfies the conclusion of Theorem 2.4. Suppose that A satisfies the conclusion of Theorem 1.4. Let $f_1(n) = e(0)$ and $f_2(n) = 2 \cdot e(0)$ for each $n \in \mathbb{N}$. Pick sequences $\langle a_n \rangle_{n=1}^\infty$ in S and $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and whenever $K \in \mathcal{P}_f(\mathbb{N})$ and $g : K \to \{1,2\}$,

$$\textstyle\sum_{n \in K} x(a_n, H_n, f_{g(n)}) \in A\,.$$

For $r \in \mathbb{N} \setminus \{1\}$ and $l \in \{2,3,\ldots,r\}$, let

$$b(l,r) = \textstyle\sum_{n=2}^{l} x(a_n, \sum_{t \in H_n}, f_1) + \sum_{n=l+1}^{r+1} x(a_n, H_n, f_2)$$

and note that for each $l \in \{2,3,\ldots,r-1\}$, $\pi_0\big(b(l,r)\big) > \pi_0\big(b(l+1,r)\big)$.

Now let $B = \Big\{1,2,\ldots,\max \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(x(a_1,H_1,f_1)\big)\Big)\Big\}$. We claim that $\Big\{\theta\left(\max \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(b(l,r)\big)\Big)\right) : r \in \mathbb{N} \setminus \{1\} \text{ and } l \in \{2,3,\ldots,r\}\Big\} \subseteq \theta[B]$.

To see this, let $r \in \mathbb{N} \setminus \{1\}$ and $l \in \{2,3,\ldots,r\}$ be given. If

$$\min \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(b(l,r)\big)\Big) \leq \max \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(x(a_1,H_1,f_1)\big)\Big)\,,$$

then $\min \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(b(l,r)\big)\Big) \in B$ so

$$\theta\left(\max \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(b(l,r)\big)\Big)\right) = \theta\left(\min \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(b(l,r)\big)\Big)\right) \in \theta[B]\,.$$

So assume that $\min \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(b(l,r)\big)\Big) > \max \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(x(a_1,H_1,f_1)\big)\Big)$ and let $x = x(a_1,H_1,f_1) + b(l,r)$. Then $x \in A_\sigma$ for some σ so

$$\begin{aligned}
\theta\left(\max \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(b(l,r)\big)\Big)\right) &= \theta\Big(\max \mathrm{supp}_2 \psi^{-1}\big(\pi_0(x)\big)\Big) \\
&= \theta\Big(\min \mathrm{supp}_2 \psi^{-1}\big(\pi_0(x)\big)\Big) \\
&= \theta\left(\min \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(x(a_1,H_1,f_1)\big)\Big)\right) \in \theta[B]\,.
\end{aligned}$$

Now let $k = \max \theta[B]$ and let $r = k+2$. For each $l \in \{2,3,\ldots,r\}$ and any ι with $0 < \iota < \mathfrak{c}$, $\pi_\iota\big(b(l,r)\big) = \pi_\iota(\sum_{n=2}^{r+1} a_n)$ which is independent of l. Thus by observation (2) there exist $\sigma < \mathfrak{c}$ and $G \in \mathcal{P}_f(\mathbb{N})$ such that for each $l \in \{2,3,\ldots,r\}$ there is some $i \in \{1,2,\ldots,m(\sigma)\}$ such that $b(l,r) = \sum_{n \in G} x(a_{\sigma,n}, H_{\sigma,n}, \langle y_{\sigma,i,t} \rangle_{t=1}^\infty)$. Further, since $b(2,r) \in A_\sigma$, we have that $m(\sigma) = \theta\left(\max \mathrm{supp}_2 \psi^{-1}\Big(\pi_0\big(b(2,r)\big)\Big)\right) \leq k$. But we have seen that

$$\pi_0\big(b(2,r)\big) > \pi_0\big(b(3,r)\big) > \ldots > \pi_0\big(b(r,r)\big)$$

so $|\big\{b(l,r) : l \in \{2,3,\ldots,r\}\big\}| = k+1$ while

$$\big|\big\{ \textstyle\sum_{n \in G} x(a_{\sigma,n}, H_{\sigma,n}, \langle y_{\sigma,i,t} \rangle_{t=1}^\infty) : i \in \{1,2,\ldots,m(\sigma)\}\big\}\big| \leq m(\sigma) \leq k\,.$$

This contradiction completes the proof. □

2.8 Corollary. *There is a subset of* $(\mathbb{R},+)$ *which satisfies the conclusion of Theorem* 2.4 *but does not satisfy the conclusion of Theorem* 1.4.

Proof. $(\mathbb{R},+)$ is isomorphic to $\bigoplus_{\iota<\mathfrak{c}}\mathbb{Q}$ and $(\mathbb{Q},+)$ can be mapped into the circle group $\mathbb{T}$ by an injective homomorphism. □

We ask the following question in the broadest terms, but we do not know the answer for $S=\mathbb{N}$ or $S=\mathbb{R}$.

2.9 Question. *Do there exist a commutative semigroup $(S,+)$ and a subset C of S establishing that the conclusions of Theorems* 1.4, 1.6, *and* 2.2 *are not all equivalent.*

We shall see in Section 4 that the noncommutative versions of these theorems are strictly increasing in strength.

3. Rich sets, strongly rich sets, and the new noncommutative Central Sets Theorem

As is customary, we use multiplicative notation for a not necessarily commutative semigroup. The versions of the noncommutative Central Sets Theorem are more complicated because the translates a_n or $\alpha(F)$ must be split into several parts. That is the function of the notion $\mathcal{I}_m$ which we introduce now.

3.1 Definition. For $m\in\mathbb{N}$, $\mathcal{I}_m=\{\big(H(1),H(2),\ldots,H(m)\big)$: each $H(j)\in\mathcal{P}_f(\mathbb{N})$ and for any $j\in\{1,2,\ldots,m-1\}$, $\max H(j)<\min H(j+1)\}$.

The following is the version of the noncommutative Central Sets Theorem given in [6]. In a noncommutative semigroup, by $\prod_{t\in F}x_t$ we mean the product taken in increasing order of indices.

3.2 Theorem. *Let $(S,\cdot)$ be a semigroup, let C be a central subset of S, and for each $l\in\mathbb{N}$, let $\langle y_{l,i}\rangle_{i=1}^\infty$ be a sequence in S. There exist sequences $\langle m(n)\rangle_{n=1}^\infty$, $\langle a(n)\rangle_{n=1}^\infty$, and $\langle H(n)\rangle_{n=1}^\infty$ such that*

(1) for each $n\in\mathbb{N}$, $m(n)\in\mathbb{N}$, $a(n)\in S^{m(n)+1}$, $H(n)\in\mathcal{I}_{m(n)}$, and $\max H(n)\big(m(n)\big)<\min H(n+1)(1)$, and

(2) for each $f\in\Phi$, and each $F\in\mathcal{P}_f(\mathbb{N})$,

$$\prod_{n\in F}\Big(\prod_{j=1}^{m(n)}\big(a(n)(j)\cdot\prod_{t\in H(n)(j)}y_{f(n),t}\big)\Big)\cdot a(n)(m(n)+1)\in C\,.$$

Proof. [6, Theorem 14.15]. □

In [5] it was shown that central sets were not the only sets satisfying the conclusion of the Central Sets Theorem (Theorem 1.4) in commutative semigroups. Sets satisfying the conclusion of Theorem 1.4 were called *rich* and it was shown that any *quasi-central* set, i.e., a set which is a member of an idempotent in the closure of the smallest ideal, is rich. Further it was shown that in $(\mathbb{N},+)$ there are quasi-central sets which are not central and there are rich sets which are not quasi-central.

In this section, we extend the notion of *rich* to arbitrary semigroups, and introduce the notion of *strongly rich*. The new stronger Central Sets Theorem (Corollary 3.10) is the assertion that any central set is strongly rich. We show that there is a closed two sided ideal J of βS such that a set is strongly rich if and only if it is a member of an idempotent in J.

We introduce some special notation. The notation does not reflect all of the variables upon which it depends.

3.3 Definition. Let $(S,\cdot)$ be a semigroup.

(a) $\mathcal{T} = {}^{\mathbb{N}}S$.

(b) $\mathcal{Y} = {}^{\mathbb{N}}\mathcal{T}$.

(c) Given $m \in \mathbb{N}$, $a \in S^{m+1}$, $H \in \mathcal{I}_m$, and $f \in \mathcal{T}$,

$$x(m,a,H,f) = \Big(\prod_{j=1}^{m} \big(a(j)\cdot \prod_{t\in H(j)} f(t)\big)\Big)\cdot a(m+1)\,.$$

(d) Given $Y = \big\langle \langle y_{l,t}\rangle_{t=1}^{\infty}\big\rangle_{l=1}^{\infty} \in \mathcal{Y}$ and $A \subseteq S$, A is a *J_Y-set* if and only if for all $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $H \in \mathcal{I}_m$ such that $\min H(1) \geq n$ and for all $l \in \{1,2,\ldots,n\}$, $x(m,a,H,\langle y_{l,t}\rangle_{t=1}^{\infty}) \in A$.

(e) $A \subseteq S$ is a *J-set* if and only if for each $F \in \mathcal{P}_f(\mathcal{T})$ and each $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $H \in \mathcal{I}_m$ such that $\min H(1) \geq n$ and for each $f \in F$, $x(m,a,H,f) \in A$.

(f) Given $Y \in \mathcal{Y}$, $J_Y = \{p \in \beta S : \text{for all } A \in p,\ A \text{ is a } J_Y\text{-set}\}$.

(g) $J = \{p \in \beta S : \text{for all } A \in p,\ A \text{ is a } J\text{-set}\}$.

(h) $A \subseteq S$ is *rich* iff for each $Y = \big\langle \langle y_{l,i}\rangle_{i=1}^{\infty}\big\rangle_{l=1}^{\infty} \in \mathcal{Y}$, there exist sequences $\langle m(n)\rangle_{n=1}^{\infty}$, $\langle a(n)\rangle_{n=1}^{\infty}$, and $\langle H(n)\rangle_{n=1}^{\infty}$ such that

  (1) for each $n \in \mathbb{N}$, $m(n) \in \mathbb{N}$, $a(n) \in S^{m(n)+1}$, $H(n) \in \mathcal{I}_{m(n)}$, and $\max H(n)\big(m(n)\big) < \min H(n+1)(1)$, and

  (2) for each $f \in \Phi$, and each $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n\in F} x(m(n),a(n),H(n),\langle y_{f(n),t}\rangle_{t=1}^{\infty}) \in A$.

(i) $A \subseteq S$ is *strongly rich* if and only if there exist $m : \mathcal{P}_f(\mathcal{T}) \to \mathbb{N}$, $\alpha \in \times_{F\in\mathcal{P}_f(\mathcal{T})} S^{m(F)+1}$, and $H \in \times_{F\in\mathcal{P}_f(\mathcal{T})} \mathcal{I}_{m(F)}$ such that

  (1) if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F)\big(m(F)\big) < \min H(G)(1)$ and

(2) whenever $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$, and for each $i \in \{1, 2, \ldots, n\}$, $\langle y_{i,t} \rangle_{t=1}^{\infty} \in G_i$, one has $\prod_{i=1}^{n} x\big(m(G_i), \alpha(G_i), H(G_i), \langle y_{i,t} \rangle_{t=1}^{\infty}\big) \in A$.

We omit the routine proof of the following theorem.

3.4 Theorem. *Let S be a semigroup and let $A \subseteq S$. Then A is a J-set if and only if for each $Y \in \mathcal{Y}$, A is a J_Y-set. In particular $J = \bigcap_{Y \in \mathcal{Y}} J_Y$.*

3.5 Theorem. *For each $Y \in \mathcal{Y}$, J_Y is a closed two sided ideal of βS. Consequently J is a closed two sided ideal of βS and so $c\ell K(\beta S) \subseteq J$.*

Proof. Let $Y \in \mathcal{Y}$. By Theorem 3.2 any idempotent in $K(\beta S)$ is in J_Y and thus $J_Y \neq \emptyset$. If $p \in \beta S \setminus J_Y$, pick $A \in p$ such that A is not a J_Y-set. Then $\overline{A}$ is a neighborhood of p missing J_Y. Thus J_Y is closed.

Now let $p \in J_Y$ and let $q \in \beta S$. To see that $p \cdot q \in J_Y$, let $A \in p \cdot q$ and let $n \in \mathbb{N}$. Let $B = \{z \in S : z^{-1}A \in q\}$. Then $B \in p$ so pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $H \in \mathcal{I}_m$ such that $\min H(1) \geq n$ and for all $l \in \{1, 2, \ldots, n\}$, $x(m, a, H, \langle y_{l,t} \rangle_{t=1}^{\infty}) \in B$. Pick $z \in \bigcap_{l=1}^{n} x(m, a, H, \langle y_{l,t} \rangle_{t=1}^{\infty})^{-1}A$. Define $b \in S^{m+1}$ by, for $t \in \{1, 2, \ldots, m+1\}$,

$$b(t) = \begin{cases} a(t) & \text{if } t \leq m \\ a(m+1) \cdot z & \text{if } t = m+1\,. \end{cases}$$

Then for all $l \in \{1, 2, \ldots, n\}$, $x(m, b, H, \langle y_{l,t} \rangle_{t=1}^{\infty}) \in A$.

To see that $q \cdot p \in J_Y$, let $A \in q \cdot p$ and let $n \in \mathbb{N}$. Let $B = \{z \in S : z^{-1}A \in p\}$. Then $B \in q$ and is therefore nonempty so pick $z \in B$. Pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $H \in \mathcal{I}_m$ such that $\min H(1) \geq n$ and for all $l \in \{1, 2, \ldots, n\}$, $x(m, a, H, \langle y_{l,t} \rangle_{t=1}^{\infty}) \in z^{-1}A$. Define $b \in S^{m+1}$ by, for $t \in \{1, 2, \ldots, m+1\}$,

$$b(t) = \begin{cases} z \cdot a(1) & \text{if } t = 1 \\ a(t) & \text{if } t \geq 2\,. \end{cases}$$

Then for all $l \in \{1, 2, \ldots, n\}$, $x(m, b, H, \langle y_{l,t} \rangle_{t=1}^{\infty}) \in A$. □

We note now a strong relationship between rich sets and the ideals J_Y and between strongly rich sets and the ideal J.

3.6 Theorem. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then A is rich if and only if for every $Y \in \mathcal{Y}$ there is an idempotent $p \in J_Y \cap \overline{A}$.*

Proof. In the case S is commutative, this is [5, Corollary 2.11]. The adjustments to the proof needed for the general case can be deduced from the proof of Theorem 3.8, which we present in full detail. □

We shall need the following lemma from [6].

3.7 Lemma. *Let P be a set, let $(D,\leq)$ be a directed set, and let $(S,\cdot)$ be a semigroup. Let $\langle T_i\rangle_{i\in D}$ be a decreasing family of nonempty subsets of S such that for each $i\in D$ and each $x\in T_i$ there is some $j\in D$ such that $x\cdot T_j\subseteq T_i$. Let $\mathbf{Q}=\bigcap_{i\in D} c\ell_{\beta S}T_i$. Then $\mathbf{Q}$ is a compact subsemigroup of βS. Let $\langle E_i\rangle_{i\in D}$ and $\langle I_i\rangle_{i\in D}$ be decreasing families of nonempty subsets of $\times_{t\in P}S$ with the following properties:*

(a) for each $i\in D$, $I_i\subseteq E_i\subseteq \times_{t\in P}T_i$,

(b) for each $i\in D$ and each $x\in I_i$ there exists $j\in D$ such that $x\cdot E_j\subseteq I_i$, and

(c) for each $i\in D$ and each $x\in E_i\backslash I_i$ there exists $j\in D$ such that $x\cdot E_j\subseteq E_i$ and $x\cdot I_j\subseteq I_i$.

Let $Y=\times_{t\in P}\beta S$, let $E=\bigcap_{i\in D} c\ell_Y E_i$, and let $I=\bigcap_{i\in D} c\ell_Y I_i$. Then E is a subsemigroup of $\times_{t\in P}\mathbf{Q}$ and I is an ideal of E. If, in addition, either

(d) for each $i\in D$, $T_i=S$ and $\{a\in S:\overline{a}\notin E_i\}$ is not piecewise syndetic, or

(e) for each $i\in D$ and each $a\in T_i$, $\overline{a}\in E_i$,

then given any $p\in K(\mathbf{Q})$, one has $\overline{p}\in E\cap K(\times_{t\in P}\mathbf{Q})=K(E)\subseteq I$.

Proof. [6, Lemma 14.9] □

As the referee has observed, only the "sufficiency" portion of the following theorem (which is the part with the easier proof) is needed for the corollaries that follow.

3.8 Theorem. *Let $(S,\cdot)$ be a semigroup and let $A\subseteq S$. Then A is strongly rich if and only if there is an idempotent $p\in J\cap\overline{A}$.*

Proof. Sufficiency. Pick $p=p\cdot p\in J\cap\overline{A}$. Recall from the proof of Theorem 2.2 that $A^\star=\{x\in A:x^{-1}A\in p\}$ and if $x\in A^\star$, then $x^{-1}A^\star\in p$. We define $m(F)$, $\alpha(F)$ and $H(F)$ for $F\in\mathcal{P}_f(\mathcal{T})$ by induction on $|F|$ so that

(1) if $\emptyset\neq G\subsetneq F$, then $\max H(G)\big(m(G)\big)<\min H(F)(1)$ and

(2) whenever $n\in\mathbb{N}$, $\emptyset\neq G_1\subsetneq G_2\subsetneq\ldots\subsetneq G_n=F$ and $\tau\in\times_{i=1}^n G_i$, then $\prod_{i=1}^n x\big(m(G_i),\alpha(G_i),H(G_i),\tau(i)\big)\in A^\star$

Assume first that $F=\{f\}$. Then $A^\star$ is a J-set so pick $m(F)\in\mathbb{N}$, $\alpha(F)\in S^{m(F)+1}$, and $H(F)\in\mathcal{I}_{m(F)}$ such that $x\big(m(F),\alpha(F),H(F),f\big)\in A^\star$

Now assume that $|F|>1$ and that $m(G)$, $\alpha(G)$, and $H(G)$ have been defined for all proper subsets G of F. For $\emptyset\neq G\subsetneq F$, let $l(G)=\max H(G)\big(m(G)\big)$ and let

$k = \max\{l(G) : \emptyset \neq G \subsetneq F\} + 1$. Let

$$M = \Big\{ \prod_{i=1}^{n} x\big(m(G_i), \alpha(G_i), H(G_i), \tau(i)\big) : \\ \emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n \subsetneq F \text{ and } \tau \in \times_{i=1}^{n} G_i \Big\}.$$

Then M is a finite subset of $A^\star$ so $B = A^\star \cap \bigcap_{b\in M} b^{-1}A^\star \in p$ and so B is a J-set. Pick $m(F) \in \mathbb{N}$, $\alpha(F) \in S^{m(F)+1}$, and $H(F) \in \mathcal{I}_{m(F)}$ such that $\min H(F)(1) \geq k$ and for each $f \in F$, $x\big(m(F), \alpha(F), H(F), f\big) \in B$.

Hypothesis (1) is satisfied directly. To verify hypothesis (2), let $n \in \mathbb{N}$, let $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n = F$, and let $\tau \in \times_{i=1}^{n} G_i$. If $n = 1$, then

$$x\big(m(G_1), \alpha(G_1), H(G_1), \tau(i)\big) \in B \subseteq A^\star,$$

so assume that $n > 1$. Let $b = \prod_{i=1}^{n-1} x\big(m(G_i), \alpha(G_i), H(G_i), \tau(i)\big)$. Then $b \in M$ so $x\big(m(G_n), \alpha(G_n), H(G_n), \tau(i)\big) \in B \subseteq b^{-1}A^\star$ so $\prod_{i=1}^{n} x\big(m(G_i), \alpha(G_i), H(G_i), \tau(i)\big) \in A^\star$ as required.

Necessity. Pick

$$m : \mathcal{P}_f(\mathcal{T}) \to \mathbb{N}, \ \alpha \in \times_{F\in\mathcal{P}_f(\mathcal{T})} S^{m(F)+1}, \text{ and } H \in \times_{F\in\mathcal{P}_f(\mathcal{T})} \mathcal{I}_{m(F)}$$

as guaranteed by the fact that A is strongly rich. For $F \in \mathcal{P}_f(\mathcal{T})$ define

$$T_F = \Big\{ \prod_{i=1}^{n} x\big(m(F_i), \alpha(F_i), H(F_i), \tau(i)\big) : n \in \mathbb{N}, \\ \text{each } F_i \in \mathcal{P}_f(\mathcal{T}), \ F \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_n, \text{ and } \tau \in \times_{i=1}^{n} F_i \Big\}.$$

Note that if $F, G \in \mathcal{P}_f(\mathcal{T})$, then $T_{F\cup G} \subseteq T_F \cap T_G$, so $\mathbf{Q} = \bigcap_{F\in\mathcal{P}_f(\mathcal{T})} \overline{T_F} \neq \emptyset$. We claim that Q is a subsemigroup of βS. For this it suffices by [6, Theorem 4.20] to show that for all $F \in \mathcal{P}_f(\mathcal{T})$ and all $u \in T_F$, there is some $G \in \mathcal{P}_f(\mathcal{T})$ such that $u \cdot T_G \subseteq T_F$. So let $F \in \mathcal{P}_f(\mathcal{T})$ and $u \in T_F$ be given. Pick strictly increasing $\langle F_i \rangle_{i=1}^{n}$ in $\mathcal{P}_f(\mathcal{T})$ such that $F \subsetneq F_1$ and $u = \prod_{i=1}^{n} x\big(m(F_i), \alpha(F_i), H(F_i), \tau(i)\big)$. Then $u \cdot T_{F_n} \subseteq T_F$.

Now we claim that $K(\mathbf{Q}) \subseteq \overline{A} \cap J$ so that any idempotent in $K(\mathbf{Q})$ establishes the theorem. We have that each $T_F \subseteq \overline{A}$ so $\mathbf{Q} \subseteq \overline{A}$. Let $p \in K(\mathbf{Q})$. We need to show that $p \in J$, so let $B \in p$. We shall show that B is a J-set. So let $F \in \mathcal{P}_f(\mathcal{T})$ and $k \in \mathbb{N}$ be given. We shall produce $v \in \mathbb{N}$, $c \in S^{v+1}$, and $M \in \mathcal{I}_v$ such that $\min M(1) \geq k$ and for each $f \in F$, $x(v, c, M, f) \in B$. Note that we can assume that $|F| \geq k$ so if $G \in \mathcal{P}_f(\mathcal{T})$ and $F \subseteq G$, then $\min H(G)(1) \geq k$.

We shall apply Lemma 3.7 with $P = F$ and $D = \{G \in \mathcal{P}_f(\mathcal{T}) : F \subseteq G\}$. Note that $\mathbf{Q} = \bigcap_{G\in D} \overline{T_G}$ as in Lemma 3.7. For $G \in D$ we shall define a subset I_G of $\times_{f\in F} S$ as follows. Let $w \in \times_{f\in F} S$. Then $w \in I_G$ if and only if there is some $n \in \mathbb{N}\setminus\{1\}$ such that

(1) there exist disjoint nonempty sets C_1 and C_2 such that $\{1,2,\ldots,n\} = C_1 \cup C_2$,

(2) there exist strictly increasing $\langle G_i\rangle_{i=1}^n$ in $\mathcal{P}_f(\mathcal{T})$ with $G \subsetneq G_1$, and

(3) there exists $\tau \in \times_{i\in C_1} G_i$,

such that for each $f \in F$, if $\gamma_f \in \times_{i=1}^n G_i$ is defined by

$$\gamma_f(i) = \begin{cases} \tau(i) & \text{if } i \in C_1 \\ f & \text{if } i \in C_2 \end{cases}$$

then $w(f) = \prod_{i=1}^n x\big(m(G_i), \alpha(G_i), H(G_i), \gamma_f(i)\big)$.

For $G \in D$, let $E_G = I_G \cup \{\overline{b} : b \in T_G\}$.

We claim that $\langle E_G\rangle_{G\in D}$ and $\langle I_G\rangle_{G\in D}$ satisfy statements (a), (b), (c), and (e) of Lemma 3.7. Statements (a) and (e) hold trivially.

To verify (b), let $G \in D$ and let $w \in I_G$. Pick n, C_1, C_2, $\langle G_i\rangle_{i=1}^n$ and τ as guaranteed by the fact that $w \in I_G$. We claim that $w \cdot E_{G_n} \subseteq I_G$. So let $z \in E_{G_n}$.

Assume first that $z = \overline{b}$ for some $b \in T_{G_n}$. Pick $n' \in \mathbb{N}$, strictly increasing $\langle F_i\rangle_{i=1}^{n'}$ in $\mathcal{P}_f(\mathcal{T})$ with $G_n \subsetneq F_1$, and $\tau' \in \times_{i=1}^{n'} F_i$ such that

$$b = \prod_{i=1}^{n'} x\big(m(F_i), \alpha(F_i), H(F_i), \tau'(i)\big)\,.$$

Let $C_1'' = C_1 \cup \{n+1, n+2, \ldots, n+n'\}$ and for $i \in \{1,2,\ldots,n+n'\}$, let

$$L_i = \begin{cases} G_i & \text{if } i \leq n \\ F_{i-n} & \text{if } i > n\,. \end{cases}$$

Define $\tau'' \in \times_{i\in C_1''} L_i$ by, for $i \in C_1''$,

$$\tau''(i) = \begin{cases} \tau(i) & \text{if } i \leq n \\ \tau'(i-n) & \text{if } i > n\,. \end{cases}$$

Then $n+n'$, C_1'', C_2, $\langle L_i\rangle_{i=1}^{n+n'}$, and τ'' establish that $w \cdot z \in I_G$.

Now assume that $z \in I_{G_n}$. Pick n', C_1', C_2', $\langle F_i\rangle_{i=1}^{n'}$ and τ' as guaranteed by the fact that $z \in I_{G_n}$. Let $C_1'' = C_1 \cup \{n+i : i \in C_1'\}$, let $C_2'' = C_2 \cup \{n+i : i \in C_2'\}$, and for $i \in \{1,2,\ldots,n+n'\}$ let

$$L_i = \begin{cases} G_i & \text{if } i \leq n \\ F_{i-n} & \text{if } i > n\,. \end{cases}$$

Define $\tau'' \in \times_{i\in C_1''} L_i$ by, for $i \in C_1''$,

$$\tau''(i) = \begin{cases} \tau(i) & \text{if } i \leq n \\ \tau'(i-n) & \text{if } i > n\,. \end{cases}$$

Then $n+n'$, C_1'', C_2'', $\langle L_i\rangle_{i=1}^{n+n'}$, and τ'' establish that $w\cdot z\in I_G$.

To verify (c) let $G\in D$ and let $w\in E_G\setminus I_G$. Pick $b\in T_G$ such that $w=\overline{b}$. Pick $n\in\mathbb{N}$, strictly increasing $\langle G_i\rangle_{i=1}^n$ in $\mathcal{P}_f(\mathcal{T})$ with $G\subsetneq G_1$, and $\tau\in\times_{i=1}^n G_i$ such that $b=\prod_{i=1}^n x\big(m(G_i),\alpha(G_i),H(G_i),\tau(i)\big)$. Then as above one has that $w\cdot E_{G_n}\subseteq E_G$ and $w\cdot I_{G_n}\subseteq I_G$.

We then have by Lemma 3.7 that $\overline{p}\in I=\bigcap_{G\in D}\overline{I_G}$. Now $\times_{f\in F}\overline{B}$ is a neighborhood of $\overline{p}$ so pick $w\in I_F\cap\times_{f\in F}\overline{B}$. Pick n, C_1, C_2, $\langle G_i\rangle_{i=1}^n$, and $\tau\in\times_{i\in C_1}G_i$ as guaranteed by the fact that $w\in I_F$. Let $r=|C_2|$ and let $h_1,h_2,\ldots,h_r$ be the elements of C_2 listed in increasing order. Let $v=\sum_{i=1}^r m(G_{h_i})$. If $h_1=1$, let $c(1)=\alpha(G_1)(1)$. If $h_1>1$, let

$$c(1)=\textstyle\prod_{i=1}^{h_1-1}\Big(x\big(m(G_i),\alpha(G_i),H(G_i),\tau(i)\big)\Big)\cdot\alpha(G_{h_1})(1)\,.$$

For $1<j\leq m(G_{h_1})$ let $c(j)=\alpha(G_{h_1})(j)$ and for $1\leq j\leq m(G_{h_1})$ let $M(j)=H(G_{h_1})(j)$.

Now let $s\in\{1,2,\ldots,r-1\}$ and let $u=\sum_{i=1}^s m(G_{h_i})$. If $h_{s+1}=h_s+1$ let $c(u+1)=\alpha(G_{h_s})(m(G_{h_s})+1)\cdot\alpha(G_{h_{s+1}})(1)$. If $h_{s+1}>h_s+1$, let

$$c(u+1)=\alpha(G_{h_s})(m(G_{h_s})+1)\cdot\Big(\textstyle\prod_{i=h_s+1}^{h_{s+1}-1}x\big(m(G_i),\alpha(G_i),H(G_i),\tau(i)\big)\Big)\cdot\alpha(G_{h_{s+1}})(1)\,.$$

And for $u<j\leq\sum_{i=1}^{s+1}m(G_{h_i})$, let $M(j)=H(G_{h_{s+1}})(j-u)$.

If $h_r=n$, let $c(v+1)=\alpha(G_n)(m(G_n)+1)$. If $h_r<n$, let $c(v+1)=\alpha(G_{h_r})(m(G_{h_r})+1)\cdot\prod_{i=h_r+1}^n\Big(x\big(m(G_i),\alpha(G_i),H(G_i),\tau(i)\big)\Big)$. Then $c\in S^{v+1}$, $M\in\mathcal{I}_v$ such that $\min M(1)\geq k$, and for each $f\in F$, $x(v,c,M,f)\in B$ as required. □

3.9 Corollary. *Let $(S,\cdot)$ be a semigroup. Every quasi-central subset of S is strongly rich.*

Proof. Theorems 3.5 and 3.8. □

We isolate the following corollary with a full statement of the conclusion because it is the new Central Sets Theorem.

3.10 Corollary. *Let $(S,\cdot)$ be a semigroup and let C be a central subset of S. There exist $m:\mathcal{P}_f(\mathcal{T})\to\mathbb{N}$, $\alpha\in\times_{F\in\mathcal{P}_f(\mathcal{T})}S^{m(F)+1}$, and $H\in\times_{F\in\mathcal{P}_f(\mathcal{T})}\mathcal{I}_{m(F)}$ such that*

(1) if $F,G\in\mathcal{P}_f(\mathcal{T})$ and $F\subsetneq G$, then $\max H(F)\big(m(F)\big)<\min H(G)(1)$ and

(2) whenever $n\in\mathbb{N}$, $G_1,G_2,\ldots,G_n\in\mathcal{P}_f(\mathcal{T})$, $G_1\subsetneq G_2\subsetneq\ldots\subsetneq G_n$, and for each $i\in\{1,2,\ldots,n\}$, $\langle y_{i,t}\rangle_{t=1}^\infty\in G_i$, one has $\prod_{i=1}^n x\big(m(G_i),\alpha(G_i),H(G_i),\langle y_{i,t}\rangle_{t=1}^\infty\big)\in C$.

Proof. A central set is quasi-central. □

Theorem 4.4 below gives an example of a rich set in the free semigroup on ω_1 generators which is not strongly rich.

By Theorem 3.8, the example of [5, Theorem 5.5] of a subset of $\mathbb{N}$ which is rich and not quasi-central is in fact strongly rich. By Theorems 3.6 and 3.8 the example given in Theorem 4.4 of a subset of the free semigroup on ω_1 generators is a member of an idempotent in J_Y for each $Y \in \mathcal{Y}$ but is not a member of any idempotent in J.

4. Strength of the versions of the Central Sets Theorem in noncommutative semigroups

Each of Theorems 1.1 and 1.4 have natural noncommutative versions which we now state. They are, of course, each corollaries of Theorem 3.2.

4.1 Theorem. *Let S be a semigroup, let $Z \in \mathcal{P}_f(\mathcal{T})$, and let C be a central subset of S. There exist sequences $\langle m(n)\rangle_{n=1}^{\infty}$, $\langle a(n)\rangle_{n=1}^{\infty}$, and $\langle H(n)\rangle_{n=1}^{\infty}$ such that*

(1) for each $n \in \mathbb{N}$, $m(n) \in \mathbb{N}$, $a(n) \in S^{m(n)+1}$, $H(n) \in \mathcal{I}_{m(n)}$, and $\max H(n)\big(m(n)\big) < \min H(n+1)(1)$, and

(2) for each $f \in Z$ and each $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n\in F} x(m(n), a(n), H(n), f) \in A$.

4.2 Theorem. *Let S be a semigroup, let $Z \in \mathcal{P}_f(\mathcal{T})$, and let C be a central subset of S. There exist sequences $\langle m(n)\rangle_{n=1}^{\infty}$, $\langle a(n)\rangle_{n=1}^{\infty}$, and $\langle H(n)\rangle_{n=1}^{\infty}$ such that*

(1) for each $n \in \mathbb{N}$, $m(n) \in \mathbb{N}$, $a(n) \in S^{m(n)+1}$, $H(n) \in \mathcal{I}_{m(n)}$, and $\max H(n)\big(m(n)\big) < \min H(n+1)(1)$, and

(2) for each $F \in \mathcal{P}_f(\mathbb{N})$ and each $f : F \to Z$, $\prod_{n\in F} x\big(m(n), a(n), H(n), f(n)\big) \in A$.

We now show that Theorems 4.1, 4.2, 3.2, and Corollary 3.10 are strictly increasing in strength. For the following, recall that any ordinal is the set of its predecessors. In particular, the cardinal ω_1 is the set of countable ordinals.

4.3 Theorem. *Let S be the free semigroup on the alphabet $\mathfrak{c}$. There exist subsets A and B of S such that A satisfies the conclusion of Theorem 4.1 but not that of Theorem 4.2 and B satisfies the conclusion of Theorem 4.2 but not that of Theorem 3.2.*

Proof. Enumerate $\mathcal{P}_f(\mathcal{T})$ as $\langle Z_\sigma\rangle_{\sigma<\mathfrak{c}}$. Choose an injective mapping $\sigma \mapsto z_\sigma$ from $\mathfrak{c}$ to $\mathfrak{c} \setminus \omega$ such that if $f \in Z_\sigma$, $n \in \mathbb{N}$, and δ occurs in $f(n)$, then $\delta < z_\sigma$.

Let $A_\sigma = \{\prod_{n\in F} {z_\sigma}^{2^n-1} f(n) z_\sigma : F \in \mathcal{P}_f(\mathbb{N})$ and $f \in Z_\sigma\}$ and let $A = \bigcup_{\sigma<\mathfrak{c}} A_\sigma$. To see that A satisfies the conclusion of Theorem 4.1, let $Z \in \mathcal{P}_f(\mathcal{T})$ be given and pick $\sigma < \mathfrak{c}$ such that $Z = Z_\sigma$. For each $n \in \mathbb{N}$, let $m(n) = 1$, $a(n) = ({z_\sigma}^{2^n-1}, z_\sigma)$, and $H(n) = (\{n\})$. Then for $n \in \mathbb{N}$ and $f \in Z$, $x(m(n), a(n), H(n), f) = {z_\sigma}^{2^n-1} f(n) z_\sigma$, so for each $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n\in F} x(m(n), a(n), H(n), f) \in A$.

Now suppose that A satisfies the conclusion of Theorem 4.2. Let $g_1(n) = 1$ and $g_2(n) = 2$ for all $n \in \mathbb{N}$ and let $Z = \{g_1, g_2\}$. Pick sequences $\langle m(n)\rangle_{n=1}^\infty$, $\langle a(n)\rangle_{n=1}^\infty$, and $\langle H(n)\rangle_{n=1}^\infty$ as guaranteed for Z. Pick $\sigma < \mathfrak{c}$ such that $x(m(1), a(1), H(1), g_1) \in A_\sigma$. Pick $r \in \mathbb{N}$ such that $2^{r-1} > |Z_\sigma|$. Words in A_σ begin and end with z_σ. Therefore, given $f : \{1, 2, \ldots, r\} \to \{g_1, g_2\}$ with $f(1) = g_1$, there exist $F \in \mathcal{P}_f(\mathbb{N})$ and $h \in Z_\sigma$ such that $\prod_{i=1}^r x\big(m(i), a(i), H(i), f(i)\big) = \prod_{n\in F} {z_\sigma}^{2^n-1} h(n) z_\sigma$. Let d be the number of occurrences of z_σ in $\prod_{i=1}^r \prod_{j=1}^{m(i)+1} a(i)(j)$. Then $d = \sum_{n\in F} 2^n$ so F does not depend on f. But there are 2^{r-1} distinct products of the form $\prod_{i=1}^r x\big(m(i), a(i), H(i), f(i)\big)$ where $f : \{1, 2, \ldots, r\} \to \{g_1, g_2\}$ and $f(1) = g_1$ while there are only at most $|Z_\sigma|$ distinct products of the form $\prod_{n\in F} {z_\sigma}^{2^n-1} h(n) z_\sigma$ for $h \in Z_\sigma$, a contradiction.

Now let $B_\sigma = \{\prod_{n\in F} {z_\sigma}^{2^n-1} f(n)(n) z_\sigma : F \in \mathcal{P}_f(\mathbb{N})$ and $f : F \to Z_\sigma\}$ and let $B = \bigcup_{\sigma<\mathfrak{c}} B_\sigma$. To see that B satisfies the conclusion of Theorem 4.2, let $Z \in \mathcal{P}_f(\mathcal{T})$ be given and pick $\sigma < \mathfrak{c}$ such that $Z = Z_\sigma$. For each $n \in \mathbb{N}$, let $m(n) = 1$, $a(n) = ({z_\sigma}^{2^n-1}, z_\sigma)$, and $H(n) = (\{n\})$. Then for $n \in \mathbb{N}$ and $f : F \to Z$, $x\big(m(n), a(n), H(n), f(n)\big) = {z_\sigma}^{2^n-1} f(n)(n) z_\sigma$, so for each $F \in \mathcal{P}_f(\mathbb{N})$,

$$\prod_{n\in F} x\big(m(n), a(n), H(n), f(n)\big) \in B\,.$$

To see that B does not satisfy the conclusion of Theorem 3.2, for each $l, n \in \mathbb{N}$ let $g_l(n) = l$. Suppose we have sequences $\langle m(n)\rangle_{n=1}^\infty$, $\langle a(n)\rangle_{n=1}^\infty$, and $\langle H(n)\rangle_{n=1}^\infty$ such that

(1) for each $n \in \mathbb{N}$, $m(n) \in \mathbb{N}$, $a(n) \in S^{m(n)+1}$, $H(n) \in \mathcal{I}_{m(n)}$, and $\max H(n)\big(m(n)\big) < \min H(n+1)(1)$, and

(2) for each $f \in \Phi$, and each $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n\in F} x(m(n), a(n), H(n), g_{f(n)}) \in B$.

Pick $\sigma < \mathfrak{c}$ such that $b = x(m(1), a(1), H(1), g_1) \in B_\sigma$. Pick $r \in \mathbb{N}$ such that $r > |Z_\sigma|$. For $i \in \{1, 2, \ldots, r\}$ let $c_i = x(m(r), a(r), H(r), g_i)$. Then for each $i \in \{1, 2, \ldots, r\}$, $c_i \in B$ and $bc_i \in B$. Since b begins with z_σ and each element of B begins and ends with the same letter, one has that c_i ends with z_σ and therefore $c_i \in B_\sigma$. Assume that $H(r) = (L_1, L_2, \ldots, L_{m(r)})$ and for $j \in \{1, 2, \ldots, m(r)\}$, let $l_j = |L_j|$. Then for each $i \in \{1, 2, \ldots, r\}$, $c_i = a(r)(1) i^{l_1} a(r)(2) i^{l_2} \cdots i^{l_{m(r)}} a(r)(m(r)+1)$. Let d be the number of occurrences of z_σ in $\prod_{j=1}^{m(r)+1} a(r)(j)$. If $d = \sum_{n\in F} 2^n$, then for each $i \in \{1, 2, \ldots, r\}$, $c_i = \prod_{n\in F} {z_\sigma}^{2^n-1} h_i(n)(n) z_\sigma$ for some $h_i : F \to Z_\sigma$.

We have that z_σ occurs in $a(r)(1)$ and in $a(r)(m(r)+1)$. Let j be the least member of $\{2,3,\ldots,m(r)+1\}$ such that z_σ occurs in $a(r)(j)$. Then $a(r)(1) = u{z_\sigma}^{2^n-1}v$ and $a(r)(j) = wz_\sigma y$ where $n \in F$, v and w are possibly empty words over the letters less than z_σ, u and y are possibly empty words over the letters less than or equal to z_σ, and u is either empty or ends in a single occurrence of z_σ. (Recall that if $g \in Z_\sigma$, $n \in \mathbb{N}$, and δ occurs in $g(n)$, then $\delta < z_\sigma$.)

Thus for each $i \in \{1,2,\ldots,r\}$,

$$ {z_\sigma}^{2^n-1}h_i(n)(n)z_\sigma = {z_\sigma}^{2^n-1}vi^{l_1}a(r)(2)\cdots a(r)(j-1)i^{l_{j-1}}wz_\sigma\,. $$

Therefore there are r distinct values for $h_i(n)(n)$, while each $h_i(n) \in Z_\sigma$, a contradiction.

□

4.4 Theorem. *Let S be the free semigroup on the alphabet ω_1. There is a subset C of S such that satisfies the conclusion of Theorem* 3.2*, but not that of Corollary* 3.10.

Proof. For each $\lambda < \omega_1$, let S_λ denote the free semigroup on $\{\iota < \omega_1 : \iota \leq \lambda\}$, regarded as a subsemigroup of S. Let $C = \bigcup_{\lambda<\omega_1} \lambda S_\lambda$. So C is the set of words s in S whose first letter is greater than or equal to any other letter in s. We observe that, for each $\lambda < \omega_1$, λS_λ is central in S_λ, because it is a right ideal of S_λ. If $Y = \big\langle \langle y_{i,t}\rangle_{t=1}^\infty \big\rangle_{i=1}^\infty \in \mathcal{Y}$, we can choose $\lambda < \omega_1$ such that $\{y_{i,t} : i,t \in \mathbb{N}\} \subseteq S_\lambda$. It follows from Theorem 3.2 applied to the semigroup S_λ, that C satisfies the conclusion of Theorem 3.2.

We claim that C does not satisfy the conclusion of Corollary 3.10. To see this, suppose that, on the contrary, there exist functions m, α and H satisfying the conclusion of this theorem. Choose $F = \{f\} \in \mathcal{P}_f(\mathcal{T})$. Then choose $\lambda < \omega_1$ such that $x(m(F),\alpha(F),H(F),f) \in \lambda S_\lambda$. Choose μ satisfying $\lambda < \mu < \omega_1$. Put $g = \langle \mu,\mu,\mu,\ldots\rangle$ and put $G = \{f,g\} \in \mathcal{P}_f(\mathcal{T})$.

We are assuming that $s = x(m(F),\alpha(F),H(F),f)x(m(G),\alpha(G),H(G),g) \in \nu S_\nu$ for some $\nu < \omega_1$. This implies that the first letter of s is ν and hence that $\nu = \lambda$, because the first letter of s is equal to the first letter of $x(m(F),\alpha(F),H(F),f)$. However, μ occurs in s and hence $\mu \leq \lambda$, a contradiction. □

Notice that none of our examples involve a countable semigroup S.

4.5 Question. *Do there exist a countable semigroup S and a subset C of S satisfying the conclusion of one of Theorems* 4.1, 4.2*, or* 3.2*, but not of one or all of the stronger statements?*

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