SOME NEW EXAMPLES OF INFINITE IMAGE PARTITION REGULAR MATRICES

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Abstract

A matrix A, finite or infinite, is *image partition regular* (over the set $\mathbb{N}$ of positive integers) if and only if, whenever $\mathbb{N}$ is finitely colored, there is a vector $\vec{x}$ of the appropriate size with entries in $\mathbb{N}$ such that all entries of $A\vec{x}$ are the same color (or *monochromatic*). A large number of characterizations of finite matrices that are image partition regular are known. There are no known characterization of infinite image partition regular matrices, and the classes of infinite matrices that are known to be image partition regular have been rather limited; we present a list of those classes of which we are aware. Extending an idea of Patra and Ghosh, we produce several new classes of infinite image partition regular matrices.

1. Introduction

We shall be concerned in this paper with finite or infinite matrices whose entries are rational numbers.

Definition 1.1. A matrix (finite or infinite) is *admissible* if and only if it has entries from $\mathbb{Q}$, no row equal to $\vec{0}$, and finitely many nonzero entries in each row.

We will index the rows and columns of a matrix A by ordinals (here always countable). The first infinite ordinal is $\omega = \mathbb{N} \cup \{0\}$. Recall that an ordinal is the set of its predecessors, so for an ordinal α, the statements $x < \alpha$ and $x \in \alpha$ are synonymous.

If A is an $\alpha \times \delta$ matrix, B is a $\gamma \times \tau$ matrix, and $\mathbf{O}$ represents a matrix with all entries equal to 0 of the appropriate size, then $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is an $(\alpha + \gamma) \times (\delta + \tau)$ matrix.

We will follow the custom of denoting the entries of a matrix by the lower case letter corresponding to the upper case letter which is the name of the matrix.

Definition 1.2. Let α and δ be positive ordinals. An $\alpha \times \delta$ matrix A is *image partition regular* (IPR) if and only if A is admissible and whenever $\mathbb{N}$ is finitely colored, there is some $\vec{x} \in \mathbb{N}^\delta$ such that the entries of $A\vec{x}$ are monochromatic.

Many of the classical results of Ramsey Theory can be naturally viewed as the assertion that certain matrices are image partition regular. For example Schur's Theorem and the length 4 version of van der Waerden's theorem are the assertions that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

are IPR.

The first class of matrices shown to be image partition regular were the matrices that produced Deuber's (m, p, c) *sets* [2]. Deuber used these sets to prove Rado's Conjecture. This was the assertion that if a "large" set was finitely colored, then one of the color classes must be "large", where a large set was defined to be one that contains solutions to every (finite) partition regular system of homogeneous linear equations.

The matrices producing the (m, p, c) sets are special cases of *first entries* matrices.

Definition 1.3. Let $\alpha, \delta \in \mathbb{N}$. An $\alpha \times \delta$ matrix A is a *first entries* matrix if and only if A is admissible and for $i, j < \alpha$, if $k = \min\{t < \delta : a_{i,t} \neq 0\} = \min\{t < \delta : a_{j,t} \neq 0\}$, then $a_{i,k} = a_{j,k} > 0$.

Using Deuber's methods, it was easy to see that all first entries matrices are IPR. The first characterizations of image partition regular matrices were found in [7]. One of these was that a finite matrix A is image partition regular if and only if it has among its images all of the images of some first entries matrix B.

There are now at least a dozen known characterizations of IPR matrices – see [13, Theorem 15.24]. Of these, one that is important from the point of view of this paper is that a finite matrix A is IPR if and only if it has images contained in every *central* subset of $\mathbb{N}$. This has the important consequence that if A and B are finite IPR matrices, then $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is also IPR.

In order to define the notion of *central* that we just mentioned, we need to give a brief introduction to the algebraic structure of $\beta\mathbb{N}$. And we will need to use that structure for most of our proofs in Section 3. In particular, we will use the structures of both $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$, so we will write here about an arbitrary semigroup S.

Given a discrete space S, we take the Stone-Čech compactification of βS of S to be the set of ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets and a basis for the closed sets of βS. If $(S, \cdot)$ is a discrete semigroup, the operation extends to βS so that $(\beta S, \cdot)$ becomes a right topological semigroup with S contained in its topological center. That is, for any $p \in \beta S$, the function $q \mapsto q \cdot p$ from βS to itself is continuous and for any $x \in S$, the function $q \mapsto x \cdot q$ from βS to itself is continuous.

Since $(\beta S, \cdot)$ is a compact Hausdorff right topological semigroup, it has a smallest two sided ideal, $K(\beta S, \cdot)$, which is the union of all of the minimal right ideals of $(\beta S, \cdot)$ and is also the union of all of the minimal left ideals of $(\beta S, \cdot)$. The intersection of any minimal left ideal with any minimal right ideal is a group and any two such groups are isomorphic. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$ where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. Any compact Hausdorff right topological semigroup has an idempotent. Given idempotents p and q in $(\beta S, \cdot)$, $p \leq q$ if and only if $p = p \cdot q = q \cdot p$. An idempotent p is minimal with respect to this ordering if and only if $p \in K(\beta S, \cdot)$. See Part I of [13] for much more information about the structure of βS.

One fact about $(\beta\mathbb{Z}, +)$ and $(\beta\mathbb{Z}, \cdot)$ that we will repeatedly use is that if $p, q \in \beta\mathbb{Z}$ and $a \in \mathbb{Z}$, then by [13, Lemma 13.1], $a(p + q) = ap + aq$.

Definition 1.4. Let $(S, \cdot)$ be a discrete semigroup and let $A \subseteq S$. Then A is *central* if and only if A is a member of some minimal idempotent in $(\beta S, \cdot)$.

In Section 2 of this paper we present, in historical order of their discovery, most of the known examples of infinite image partition regular matrices of which we are aware, concluding with a recent class of IPR matrices discovered by Patra and Ghosh [16]. In Section 3 we present several new classes, many of which involve extensions of the class produced by Patra and Ghosh.

2. Known classes of infinite IPR matrices

For each $l \in \mathbb{N} \setminus \{1\}$ let

$$W_l = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & l \end{pmatrix}.$$

Then van der Waerden's Theorem [18] says that each W_l is IPR. (In the context of this paper this is immediate from the fact that each W_l is a first entries matrix.) It is an easy consequence of van der Waerden's Theorem that given any finite coloring of $\mathbb{N}$, there is one color which contains arbitrarily long arithmetic progressions.

Consequently, the infinite matrix

$$W = \begin{pmatrix} W_2 & \mathbf{O}_3 & \mathbf{O}_3 & \dots \\ \mathbf{O}_4 & W_3 & \mathbf{O}_4 & \dots \\ \mathbf{O}_5 & \mathbf{O}_5 & W_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is IPR, where $\mathbf{O}_m$ is the $m \times 2$ matrix with all entries equal to 0.

The *Finite Sums Matrix* $\mathbf{F}$ is a matrix whose rows are all possible rows with entries from $\{0,1\}$, at least one 1, and only finitely many 1's. (Of course there are many such matrices, differing only by the order of their rows. To be definite, one can specify that the rows occur in lexicographic order.) By the Finite Sums Theorem [6, Theorem 3.1], $\mathbf{F}$ is image partition regular. Given $\vec{x} \in \mathbb{N}^\omega$, the entries of $\mathbf{F}\vec{x}$ are all sums of the form $\sum_{t \in F} x_t$ for $F \in \mathcal{P}_f(\omega)$. (For a set X, we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X.)

Fix an enumeration $\langle B_n \rangle_{n=0}^\infty$ of the finite IPR matrices. For each n, assume that B_n is a $u(n) \times v(n)$ matrix. For each $i \in \mathbb{N}$, let $\vec{0}_i$ be the 0 vector with i entries. Let $\mathbf{D}$ be an $\omega \times \omega$ matrix with all rows of the form $\vec{r}_0{}^\frown \vec{r}_1{}^\frown \vec{r}_2{}^\frown \dots$ where each $\vec{r}_i$ is either $\vec{0}_{v(i)}$ or is a row of B_i, and all but finitely many are $\vec{0}_{v(i)}$. Again, if one wants to be definite, one can specify that the rows of $\mathbf{D}$ occur in lexicographic order.)

It is a consequence of the theorem of [3] and the fact that every finite IPR matrix has its images contained in the images of some (m,p,c) set that $\mathbf{D}$ is image partition regular. Given $\vec{x} \in \mathbb{N}^\omega$, let $\vec{x} = \begin{pmatrix} \vec{x}_0 \\ \vec{x}_1 \\ \vdots \end{pmatrix}$ where each $\vec{x}_i \in \mathbb{N}^{v(i)}$. Then the entries of $\mathbf{D}\vec{x}$ are all sums of the form $\sum_{t \in F} y_t$ where $F \in \mathcal{P}_f(\omega)$ and for $t \in F$, y_t is an entry of $B_t \vec{x}_t$.

As we mentioned in the introduction, we are proceeding in historical order; [18] was published in 1927, [6] was published in 1974, and [3] was published in 1987. Our next example is a whole class of matrices, whose image partition regularity is a consequence of the Milliken-Taylor Theorem obtained independently by Milliken [15] and Taylor [17] and published in 1975 and 1976. So we might seem to be departing from our historical order. However, it was not until the publication of [4] in 1995 that it was noticed that the Milliken-Taylor Theorem guaranteed the existence of a class of infinite partition regular matrices.

Definition 2.1. Let $k \in \omega$ and let $\vec{a} = \langle a_0, a_1, \dots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$.

(a) $c(\vec{a})$ is the sequence obtained from $\vec{a}$ by deleting any term equal to its immediate predecessor.

(b) $\vec{a}$ is said to be *compressed* if and only if $\vec{a} = c(\vec{a})$.

Definition 2.2. Let $\vec{r} \in \mathbb{Z}^\omega$. Then $d(\vec{r})$ is the sequence obtained from $\vec{r}$ by deleting all occurrence of $\vec{0}$.

Definition 2.3. Let $k \in \omega$ and let $\vec{a} = \langle a_0, a_1, \ldots, a_k \rangle$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$. Then $MT(\vec{a})$ is a matrix consisting of all rows $\vec{r}$ with finitely many nonzero entries such that $c\big(d(\vec{r})\big) = \vec{a}$.

Again, if one wishes one can specify that the rows of $MT(\vec{a})$ occur in lexicographic order. In [4] the terms of $\vec{a}$ in the definition of $MT(\vec{a})$ were required to be positive, but the proof of image partition regularity from the Milliken-Taylor Theorem given in [4] can be taken almost verbatim.

As we noted in the introduction, any finite IPR matrix has images in any given central subset of $\mathbb{N}$. By way of contrast, we have the following theorem.

Theorem 2.4. *Let $k, m \in \omega$ and let $\vec{a} = \langle a_0, a_1, \ldots, a_k \rangle$ and $\vec{b} = \langle b_0, b_1, \ldots, b_m \rangle$ be compressed sequences in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$ and $b_m > 0$. If there does not exist a positive rational α such that $\vec{a} = \alpha \cdot \vec{b}$, then there exist subsets A and B of $\mathbb{N}$ such that $\mathbb{N} = A \cup B$ and there does not exist $\vec{y} \in \mathbb{N}^\omega$ with $MT(\vec{b})\vec{y} \in A^\omega$ and there does not exist $\vec{x} \in \mathbb{N}^\omega$ with $MT(\vec{a})\vec{x} \in B^\omega$.*

Proof. This follows from [8, Corollary 3.9] and the fact that the assumptions that $a_k > 0$ and $b_m > 0$ force the conclusion that there does not exist $\vec{x} \in \mathbb{N}^\omega$ with all entries of $MT(\vec{a})\vec{x}$ less than or equal to 0 or with all entries of $MT(\vec{b})\vec{x}$ less than or equal to 0. $\square$

Definition 2.5. Let α and δ be positive ordinals. An $\alpha \times \delta$ matrix A is *centrally image partition regular* (centrally IPR) if and only if A is admissible and whenever C is a central subset of $\mathbb{N}$, there is some $\vec{x} \in \mathbb{N}^\delta$ such that $A\vec{x} \in C^\alpha$.

We thus have that if $k \in \omega$ and $\vec{a} = \langle a_0, a_1, \ldots, a_k \rangle$ is a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$, then $MT(\vec{a})$ is centrally IPR if and only if $k = 0$.

On the other hand, all finite IPR matrices are centrally IPR as are the matrices W, $\mathbf{F}$, and $\mathbf{D}$.

Definition 2.6. Let A be an $\omega \times \omega$ matrix. Then A is a *restricted triangular* matrix if and only if all entries of A are from $\mathbb{Z}$ and there exist $d \in \mathbb{N}$ and an increasing function $j : \omega \to \omega$ such that for all $i \in \omega$,

(1) $a_{i,j(i)} \in \{1, 2, \ldots, d\}$,

(2) for all $l > j(i)$, $a_{i,l} = 0$, and

(3) for all $k > i$ and all $t \in \{1, 2, \ldots, d\}$, $t | a_{k,j(i)}$.

It was shown in [12] that any restricted triangular matrix is centrally IPR. In particular, if each $a_{i,j(i)} = 1$, then condition (3) is automatically satisfied.

The next class of matrices begins a process of extending given IPR matrices.

Theorem 2.7. *Let A be an $\omega \times \omega$ centrally image partition regular matrix and let $\langle b_n \rangle_{n=0}^{\infty}$ be a sequence in $\mathbb{N}$. Let*

$$B = \begin{pmatrix} b_0 & 0 & 0 & \dots \\ 0 & b_1 & 0 & \dots \\ 0 & 0 & b_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \text{ Then } \begin{pmatrix} \mathbf{O} & B \\ A & \mathbf{O} \\ A & B \end{pmatrix}$$

is centrally image partition regular.

Proof. [12, Theorem 4.7]. □

Theorem 2.8. *Let A be a finite IPR matrix and let B be an infinite IPR matrix. Then $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is IPR.*

Proof. A combinatorially proof attributed to V. Rödl is in [9, Lemma 2.3]. A short proof using the algebra of $\beta\mathbb{N}$ can be found in [16, Theorem 2.4]. □

We have seen that Milliken-Taylor matrices determined by compressed sequences with more than one entry are not centrally IPR. However, by choosing rows of such matrices with fixed nonzero sums, one does get centrally IPR matrices. If $m > 0$ the conclusion of Theorem 2.9 is trivial, since any central set will include a multiple of m and one can let $\vec{x}$ be a constant sequence. (Theorem 3.7 in [9] asserts more than just that M is centrally IPR.)

Theorem 2.9. *Let $k \in \mathbb{N}$, let $\langle a_0, a_1, \dots, a_k \rangle$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$, and let $m \in \mathbb{Z} \setminus \{0\}$. Let M be an admissible matrix such that*

(1) the compressed form of each row is $\langle a_1, a_2, \dots, a_k \rangle$ and

(2) the sum of each row is m.

Then M is centrally IPR.

Proof. [9, Theorem 3.7]. □

In the following theorem, if $\sum_{i=0}^{m} a_i > 0$ and $\sum_{i=0}^{n} b_i > 0$, then the conclusion of the theorem is trivial (although [14, Theorem 2] asserts more than that the matrix is IPR.) But in fact $\sum_{i=0}^{m} a_i$ is allowed to be 0 or negative and $\sum_{i=0}^{n} b_i$ is allowed to be negative.

Theorem 2.10. *Let $m, n \in \omega$ and let $\vec{a} = \langle a_0, a_1, \dots, a_m \rangle$ and $\vec{b} = \langle b_0, b_1, \dots, b_n \rangle$ be sequences in $\mathbb{Z} \setminus \{0\}$. Assume that $a_m > 0$, $b_n > 0$, and $\sum_{i=1}^{n} b_i \neq 0$. Let A be an $\omega \times \omega$ matrix with all rows whose nonzero entries are $a_0, a_1, \dots, a_m$ in order and let B be an $\omega \times \omega$ matrix with all rows whose nonzero entries are $b_0, b_1, \dots, b_m$ in order. Then $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is IPR.*

Proof. [14, Theorem 2]. □

The fact that the matrix in the next theorem is image partition regular established that it is possible to have an IPR matrix which does not have the property that whenever $\mathbb{R}$ is finitely colored, there must exist monochromatic images arbitrarily close to zero.

Theorem 2.11. *The following matrix is IPR.*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 4 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 8 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Proof. [1, Theorem 16]. □

In fact, the matrix of Theorem 2.11 is centrally IPR. The proofs in [1] used the notion of upper asymptotic density. The properties of upper asymptotic density used are shared by *Banach density* and any central subset of $\mathbb{N}$ has positive Banach density. (See [13, Definition 20.1 and Theorems 20.5 and 20.6].)

Theorem 2.12. *Let $k \in \omega$ and let $\vec{a} = \langle a_0, a_1, \dots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$. Let I be the $\omega \times \omega$ identity matrix, let A be an $\omega \times \omega$ matrix with all rows whose nonzero entries are $a_0, a_1, \dots, a_m$ in order, and let $M = \begin{pmatrix} I \\ A \end{pmatrix}$. If*

(1) $\sum_{i=0}^{k} a_i = 0$ *and* $a_k = 1$,

(2) $\sum_{i=0}^{k} a_i = 1$, *or*

(3) $a_0 = a_1 = \dots = a_k = 1$,

then M is IPR.

Proof. [5, Lemma 2.7]. □

It is conjectured in [5] that the conditions of Theorem 2.12 are also necessary, and this conjecture is proved in special cases.

We have seen that Milliken-Taylor matrices determined by compressed sequences of length greater than 1 are not centrally IPR. However, if the final terms are all equal to 1, a shift suffices, as seen in the following theorem.

Theorem 2.13. *Let $m \in \omega$ and for each $i \in \{0,1,\ldots,m\}$, let $k(i) \in \mathbb{N}$, let $\vec{a}_i = \langle a_{i,0}, a_{i,1}, \ldots, a_{i,k(i)} \rangle$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_{i,k(i)} = 1$. Let $\overline{0}$ and $\overline{1}$ be the length ω constant column vectors. Then*

$$\begin{pmatrix} \overline{1} & \overline{0} & \ldots & \overline{0} & MT(\vec{a}_0) \\ \overline{0} & \overline{1} & \ldots & \overline{0} & MT(\vec{a}_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{0} & \overline{0} & \ldots & \overline{1} & MT(\vec{a}_m) \\ \overline{0} & \overline{0} & \ldots & \overline{0} & \mathbf{F} \end{pmatrix}$$

is centrally IPR.

Proof. [10, Corollary 6.4]. □

A generaliztion of Theorem 2.13 (minus the conclusion that the matrix is centrally IPR) is a consequence of Theorem 3.26 below. In that generalization $\mathbf{F}$ is replaced by an arbitrary Milliken-Taylor matrix and the requirement that $a_{i,k(i)} = 1$ is replaced by the requirement that all of the final entries are the same positive number.

The most recent class of infinite matrices known by us to be IPR (before the results of the present paper) is given in the following theorem.

Theorem 2.14. *Let $u, v \in \mathbb{N}$ and let D be a $u \times v$ IPR matrix. Let A be an $\omega \times v$ matrix all of whose rows are rows of D, let B be an $\omega \times \omega$ centrally IPR matrix, and let M be a Milliken-Taylor matrix. Then $\begin{pmatrix} A & B & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & M \end{pmatrix}$ is image partition regular.*

Proof. [16, Theorem 2.19]. □

3. New classes of infinite IPR matrices

Our first result replaces the assumption in Theorem 2.14 that B is centrally IPR by the assumption that B is simply IPR. As did the authors of [16], we utilize the following set introduced in [9].

Definition 3.1. Let α and δ be positive ordinals and let A be an $\alpha \times \delta$ admissible matrix. Then $C(A) = \{p \in \beta\mathbb{N} :$ for every $P \in p$, there exists $\vec{x} \in \mathbb{N}^\delta$ such that $A\vec{x} \in P^\alpha\}$.

Lemma 3.2. *Let A be an admissible matrix.*

(1) The set $C(A)$ is compact and $C(A) \neq \emptyset$ if and only if A is IPR.

(2) If A is a finite IPR matrix, then $C(A)$ is a subsemigroup of $(\beta\mathbb{N}, +)$.

(3) If A is IPR, then $C(A)$ is a left ideal of $(\beta\mathbb{N},\cdot)$.

(4) If A is a finite IPR matrix, then $C(A)$ is a two sided ideal of $(\beta\mathbb{N},\cdot)$.

(5) If A is a finite IPR matrix, then all minimal idempotents of $(\beta\mathbb{N},+)$ are members of $C(A)$.

Proof. Statements (1) and (2) are [9, Lemma 2.5]. Statements (3) and (4) are [16, Lemma 2.3]. Statement (5) holds because finite IPR matrices are centrally IPR. $\square$

Lemma 3.3. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ IPR matrix. Let $r \in \beta\mathbb{Z}$ and let p be a minimal idempotent in $(\beta\mathbb{N},+)$.*

(1) If $r \in C(A)$, then $r + p \in C(A)$.

(2) If $r \in -C(A)$, then $r + p \in C(A)$.

(3) If $r \in C(A)$, then $r + (-p) \in -C(A)$.

(4) If $r \in -C(A)$, then $r + (-p) \in -C(A)$.

Proof. [16, Lemma 2.18]. $\square$

In the next lemmas, the products of the form $\frac{a}{b}p$ are computed in $(\beta\mathbb{Q}_d,\cdot)$, where $\mathbb{Q}_d$ is the set of rationals with the discrete topology.

Lemma 3.4. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ IPR matrix. Let $k \in \omega$ and let $\langle a_0, a_1, \ldots, a_k\rangle$ be a sequence in $\mathbb{Z}\setminus\{0\}$ with $a_k > 0$. Let p be a minimal idempotent in $(\beta\mathbb{N},+)$. Then $a_0p+a_1p+\ldots+a_kp \in C(A)$ and $\frac{a_0}{a_k}p+\frac{a_1}{a_k}p+\ldots+\frac{a_{k-1}}{a_k}p+p \in C(A)$.*

Proof. By [13, Lemma 5.19.2], each $|a_i|\cdot p$ is a minimal idempotent in $(\beta\mathbb{N},+)$ as is each $|\frac{a_i}{a_k}|\cdot p$. Since A is centrally IPR, each a_ip is in $C(A)$ or $-C(A)$ as is each $\frac{a_i}{a_k}p$ depending on whether a_i is positive or negative. The results then follow by repeated applications of Lemma 3.3. $\square$

Lemma 3.5. *Let $k \in \omega$ and let $\vec{a} = \langle a_0, a_1, \ldots, a_k\rangle$ be a compressed sequence in $\mathbb{Z}\setminus\{0\}$ with $a_k > 0$. Let p be an idempotent in $(\beta\mathbb{N},+)$. Then*

(1) $a_0p + a_1p + \ldots + a_kp \in C\big(MT(\vec{a})\big)$ and

(2) $\frac{a_0}{a_k}p + \frac{a_1}{a_k}p + \ldots + \frac{a_{k-1}}{a_k}p + p \in C\big(MT(\vec{a})\big)$.

Proof. Conclusion (1) is an immediate consequence of [9, Corollary 3.6]. To verify conclusion (2), let $q = \frac{a_0}{a_k}p + \frac{a_1}{a_k}p + \ldots + \frac{a_{k-1}}{a_k}p + p$ and let $Q \in q$. By (1), $a_kq \in C\big(MT(\vec{a})\big)$. Since p is an idempotent, $a_k\mathbb{N} \in p$. Also $a_kQ \in a_kq$. Thus by [9, Corollary 3.6], we may pick $\vec{x} \in (a_k\mathbb{N})^\omega$ such that $MT(\vec{a})\vec{x} \in (a_kQ)^\omega$. Then $\frac{\vec{x}}{a_k} \in \mathbb{N}^\omega$ and $MT(\vec{a})\frac{\vec{x}}{a_k} \in Q^\omega$. $\square$

Lemma 3.6. *Let $u, v \in \mathbb{N}$ and let D be a $u \times v$ IPR matrix. Let α and δ be positive ordinals, let A be an $\alpha \times v$ matrix, all of whose rows are rows of D, and let B be an $\alpha \times \delta$ matrix which is IPR. If $r \in C(A)$ and $p \in C(B)$, then $r + p \in C\begin{pmatrix} A & B \end{pmatrix}$.*

Proof. Let $H \in r + p$. Then $X = \{z \in \mathbb{N} : -z + H \in p\} \in r$ so pick $\vec{w} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{w} \in X^\alpha$. Then $\{y_i : i < \alpha\}$ is finite so $P = \bigcap_{i<\alpha}(-y_i + H) \in p$. Pick $\vec{z} \in \mathbb{N}^\delta$ such that $B\vec{z} \in P^\alpha$. Then $\begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} \vec{w} \\ \vec{z} \end{pmatrix} \in H^\alpha$. $\square$

Theorem 3.7. *Let $u, v \in \mathbb{N}$ and let D be a $u \times v$ IPR matrix. Let α and δ be positive ordinals, let A be an $\alpha \times v$ matrix, whose rows are the rows of D, let B be an $\alpha \times \delta$ matrix which is IPR, and let M be a Milliken-Taylor matrix. Then $\begin{pmatrix} A & B & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & M \end{pmatrix}$ is IPR.*

Proof. Note that $C(A) = C(D)$ so $C(A)$ is an ideal of $(\beta\mathbb{N}, \cdot)$ and a compact subsemigroup of $(\beta\mathbb{N}, +)$. Also $C(B)$ is a left ideal of $(\beta\mathbb{N}, \cdot)$ so pick $p \in C(A) \cap C(B)$. Pick a compressed sequence $\langle a_i \rangle_{i=0}^k$ in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$ such that $M = MT(\vec{a})$. We need to show that $C\begin{pmatrix} A & B \end{pmatrix} \cap C(M) \neq \emptyset$.

Now $p \in C(A)$ so $C(A) + p$ is a left ideal of $C(A)$. Pick a minimal left ideal L of $C(A)$ such that $L \subseteq (C(A) + p)$ and pick an idempotent $q \in L$. Now all minimal idempotents of $(\beta\mathbb{N}, +)$ are in $C(A)$ so $K(\beta\mathbb{N}, +) \cap C(A) \neq \emptyset$ and therefore q is minimal in $(\beta\mathbb{N}, +)$ by [13, Theorem 1.65]. Pick $x \in C(A)$ such that $q = x + p$.

Let $r = a_0 q + a_1 q + \ldots + a_k q$. By Lemma 3.5, $r \in C(M)$, so it suffices to show that $r \in C\begin{pmatrix} A & B \end{pmatrix}$. By Lemma 3.4, $r \in C(A)$.

Note that $a_k q = a_k x + a_k p$. Also $a_k q + a_k q = a_k(q + q) = a_k q$ so $r = r + a_k q = r + a_k x + a_k p$. We have that $r + a_k x \in C(A)$ and $a_k p \in C(B)$. By Lemma 3.6, $r + a_k x + a_k p \in C\begin{pmatrix} A & B \end{pmatrix}$. $\square$

Our next new result involves the notion of *thick* sets. A subset A of $\mathbb{N}$ is thick if and only if it contains arbitrarily long integer intervals. Equivalently, by [13, Theorem 4.48], A is thick if and only if there is a left ideal L of $(\beta\mathbb{N}, +)$ with $L \subseteq \overline{A}$. Since any left ideal contains a minimal idempotent, it is immediate that any thick set is central, and thus contains images of any centrally IPR matrix.

Vitaly Bergelson has asked in a personal communication whether every thick set in $\mathbb{N}$ contains solutions to any partition regular system of linear homogeneous equations. By [11, Theorem 2.4] an affirmative answer to Bergelson's question would follow if one could show that any thick set contains images of any infinite IPR matrix. That in turn would follow from the validity of the following conjecture.

Conjecture 3.8. *If α and δ are positive ordinals, A is $\alpha \times \delta$ IPR matrix, B is a thick subset of $\mathbb{N}$, $r \in \mathbb{N}$, and $B = \bigcup_{i=1}^r C_i$, then there exist $\vec{x} \in \mathbb{N}^\delta$ and $i \in \{1, 2, \ldots, r\}$ such that $A\vec{x} \in C_i^\alpha$.*

Notice that the assertion in Conjecture 3.8 that whenever B is finitely colored, there is a monochromatic image of A is stronger than the assertion that B contains an image of A. Consider for example the following $\omega \times \omega$ matrix.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 2 & 0 & 1 & 0 & \dots \\ 3 & 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & 1 & \dots \\ 2 & 0 & 0 & 1 & \dots \\ 3 & 0 & 0 & 1 & \dots \\ 4 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The assertion that a set B is thick is precisely the assertion that there exists $\vec{x} \in \mathbb{N}^\omega$ such that $A\begin{pmatrix} 1 \\ \vec{x} \end{pmatrix} \in B^\omega$. But it is easy to color $\mathbb{N}$ in two colors so that neither color class contains arbitrarily long arithmetic progressions with a fixed increment.

Definition 3.9. $\mathcal{M} = \{A :$ there exist positive ordinals α and δ such that A is an $\alpha \times \delta$ admissible matrix and whenever T is a thick subset of $\mathbb{N}$ and T is finitely colored, there exists $\vec{x} \in \mathbb{N}^\delta$ such that the entries of $A\vec{x}$ are monochromatic$\}$.

Conjecture 3.8 is the assertion that $\mathcal{M}$ is exactly the set of IPR matrices.

Lemma 3.10. *Let α and δ be positive ordinals, and let A be an admissible $\alpha \times \delta$ matrix. The following statements are equivalent.*

(a) $A \in \mathcal{M}$.

(b) For every minimal left ideal L of $(\beta\mathbb{N}, +)$, $L \cap C(A) \neq \emptyset$.

(c) For every left ideal L of $(\beta\mathbb{N}, +)$, $L \cap C(A) \neq \emptyset$.

Proof. To see that (a) implies (b), assume that $A \in \mathcal{M}$, let L be a minimal left ideal of $(\beta\mathbb{N}, +)$, and suppose that $L \cap C(A) = \emptyset$. For each $p \in L$, pick $C_p \in p$ such that there does not exist $\vec{x} \in \mathbb{N}^\delta$ such that $A\vec{x} \in C_p^\alpha$. Then $L \subseteq \bigcup_{p \in L} \overline{C_p}$. Minimal left ideals of $(\beta\mathbb{N}, +)$ are closed so pick $F \in \mathcal{P}_f(L)$ such that $L \subseteq \bigcup_{p \in F} \overline{C_p}$. Let $B = \bigcup_{p \in F} C_p$. Since $L \subseteq \overline{B}$, B is thick. Since $A \in \mathcal{M}$, pick $\vec{x} \in \mathbb{N}^\delta$ and $p \in F$ such that $A\vec{x} \in C_p^\alpha$. This is a contradiction.

Since every left ideal of $(\beta\mathbb{N}, +)$ contains a minimal left ideal, it is trivial that (b) implies (c).

To see that (c) implies (a), let B be a thick subset of $\mathbb{N}$ and pick a left ideal L of $(\beta\mathbb{N}, +)$ such that $L \subseteq \overline{B}$. Pick $p \in L \cap C(A)$. Let $r \in \mathbb{N}$ and let $B = \bigcup_{i=1}^r C_i$.

Pick $i \in \{1, 2, \ldots, r\}$ such that $C_i \in p$. Since $p \in C(A)$, pick $\vec{x} \in \mathbb{N}^\delta$ such that $A\vec{x} \in C_i^\alpha$. □

Lemma 3.11. *Let A be an admissible matrix. If A is either centrally IPR or a Milliken-Taylor matrix, then $A \in \mathcal{M}$.*

Proof. Assume first that A is centrally IPR. Let L be a minimal left ideal of $(\beta\mathbb{N}, +)$ and pick a minimal idempotent $p \in L$. Then $p \in L \cap C(A)$ so Lemma 3.10 applies.

Now assume that we have $k \in \omega$ and a compressed sequence $\vec{a} = \langle a_0, a_1, \ldots, a_k \rangle$ in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$ such that $A = MT(\vec{a})$. Let L be a left ideal of $\beta\mathbb{N}$ and let q be an idempotent in L. By [13, Lemma 5.19.2], there is an idempotent $p \in \beta\mathbb{N}$ such that $a_k p = q$. It follows from Lemma 3.5 that $a_0 p + a_1 p + \ldots + a_k p \in C(A) \cap L$. □

Theorem 3.12. *Let B be an IPR matrix and let $M \in \mathcal{M}$. Then $\begin{pmatrix} \overline{1} & B & \mathbf{O} \\ \overline{0} & \mathbf{O} & M \end{pmatrix}$ is IPR.*

Proof. Assume that α and δ are positive ordinals and that B is an $\alpha \times \delta$ IPR matrix. Pick $p \in C(B)$ and let $L = \beta\mathbb{N} + p$. Pick $q \in L \cap C(M)$ and pick $r \in \beta\mathbb{N}$ such that $q = r + p$. Then $r \in \beta\mathbb{N} = C(\overline{1})$ so by Lemma 3.6, $q \in C\begin{pmatrix} \overline{1} & B \end{pmatrix}$. □

The next theorem has a (possibly) stronger hypothesis and stronger conclusion than Theorem 3.12. (As we have noted, if Conjecture 3.8 holds, then the content of Theorems 3.12 and 3.13 are identical.)

Theorem 3.13. *Let $B, M \in \mathcal{M}$. Then $\begin{pmatrix} \overline{1} & B & \mathbf{O} \\ \overline{0} & \mathbf{O} & M \end{pmatrix} \in \mathcal{M}$.*

Proof. Let L be a minimal left ideal of $(\beta\mathbb{N}, +)$. Pick $q \in L \cap C(M)$. We will show that $q \in C\begin{pmatrix} \overline{1} & B \end{pmatrix}$. Pick $p \in L \cap C(B)$. Then $q \in L = \beta\mathbb{N} + p$ so pick $r \in \beta\mathbb{N}$ such that $q = r + p$. By Lemma 3.6, $q \in C\begin{pmatrix} \overline{1} & B \end{pmatrix}$. □

Since the hypothesis on B and the conclusion of Theorem 3.13 are the same, we can iterate the process.

Corollary 3.14. *Let $m \in \omega$ and for $i \in \{0, 1, \ldots, m+1\}$, let $M_i \in \mathcal{M}$. Then*

$$\begin{pmatrix} \overline{1} & M_0 & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \overline{1} & M_1 & \ldots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & \overline{1} & M_m & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} & M_{m+1} \end{pmatrix} \in \mathcal{M}.$$

Proof. Apply Theorem 3.13 $m + 1$ times. □

Theorem 3.15. *Let $u, v \in \mathbb{N}$, let D be a $u \times v$ IPR matrix, and let A be an $\omega \times v$ matrix, whose rows are the rows of D. Let M_0 and M_1 be Milliken-Taylor matrices. Then* $\begin{pmatrix} A & M_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & M_0 \end{pmatrix} \in \mathcal{M}$.

Proof. Let $k, m \in \omega$ and let $\vec{a} = \langle a_0, a_1, \ldots, a_k \rangle$ and $\vec{b} = \langle b_0, b_1, \ldots, b_m \rangle$ be condensed sequences in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$ and $b_m > 0$ such that $M_0 = MT(\vec{a})$ and $M_1 = MT(\vec{b})$. Let L be a minimal left ideal of $(\beta\mathbb{N}, +)$ and let p be an idempotent in L. Since D is finite and IPR, $p \in C(D)$. Note that by [13, Exercise 4.3.8], $L = \beta\mathbb{N} + p = \beta\mathbb{Z} + p$.

Let $q = \frac{a_0}{a_k}p + \frac{a_1}{a_k}p + \ldots + \frac{a_{k-1}}{a_k}p + p$ and let $q' = \frac{b_0}{b_m}p + \frac{b_1}{b_m}p + \ldots + \frac{b_{m-1}}{b_m}p + p$. Then q and q' are in L. By Lemma 3.4, $q, q' \in C(D) = C(A)$. By Lemma 3.5, $q \in C\big(MT(\vec{a})\big)$ and $q' \in C\big(MT(\vec{b})\big)$.

Let $L' = L \cap C(A)$. Then $q, q' \in L'$. By [13, Theorem 1.65] L' is a minimal left ideal of $C(A)$ so $L' = C(A) + p$. Pick $r \in C(A)$ such that $q = r + q'$. By Lemma 3.6, $q \in C\begin{pmatrix} A & MT(\vec{b}) \end{pmatrix}$ so $L \cap C\begin{pmatrix} A & MT(\vec{b}) \end{pmatrix} \cap C\big(MT(\vec{a})\big) \neq \emptyset$. $\square$

Theorem 3.16. *Let $u, v \in \mathbb{N}$ and let D be a $u \times v$ IPR matrix. Let α and δ be positive ordinals and let A be an $\alpha \times v$ matrix, whose rows are the rows of D. Let B be an $\alpha \times \delta$ centrally IPR matrix, and let M be a Milliken-Taylor matrix. Then* $\begin{pmatrix} A & B & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & M \end{pmatrix} \in \mathcal{M}$.

Proof. Let $k \in \omega$ and let $\vec{a} = \langle a_0, a_1, \ldots, a_k \rangle$ be a condensed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$ such that $M = MT(\vec{a})$. Let L be a minimal left ideal of $(\beta\mathbb{N}, +)$ and let p be an idempotent in L. Since D is finite and IPR, $p \in C(D)$. Note that by [13, Exercise 4.3.8], $L = \beta\mathbb{N} + p = \beta\mathbb{Z} + p$. Let $q = \frac{a_0}{a_k}p + \frac{a_1}{a_k}p + \ldots + \frac{a_{k-1}}{a_k}p + p$. Then $q \in L$. By Lemma 3.4, $q \in C(D) = C(A)$. By Lemma 3.5, $q \in C(M)$. Since p is an idempotent in L and L is minimal, $q = q + p$. Since B is centrally IPR, $p \in C(B)$. By Lemma 3.6, $q \in C\begin{pmatrix} A & B \end{pmatrix}$. So $L \cap C\begin{pmatrix} A & B \end{pmatrix} \cap C(M) \neq \emptyset$. $\square$

We now introduce two more classes of matrices. As far as we know, it is possible that $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{R}$ are all equal to the set of all IPR matrices.

Definition 3.17. $\mathcal{N} = \{M : M$ is an admissible matrix and for any finite IPR matrix D, $C(D) \cap C(M) \cap K(\beta\mathbb{N}, +) \neq \emptyset\}$.

Definition 3.18. $\mathcal{R} = \{M : M$ is an admissible matrix and for any finite IPR matrix D and any left ideal L of $C(D)$, $L \cap C(M) \neq \emptyset\}$.

Lemma 3.19. $\mathcal{R} \subseteq \mathcal{N}$.

Proof. Let $M \in \mathcal{R}$ and let D be a finite IPR matrix. Pick a minimal left ideal L of $C(D)$. All minimal idempotents of $(\beta\mathbb{N}, +)$ are in $C(D)$ so by [13, Theorem

1.65] we may pick a minimal left ideal T of $(\beta\mathbb{N},+)$ such that $T \cap C(D) = L$ and in particular $L \subseteq K(\beta\mathbb{N},+)$. Thus $\emptyset \neq L \cap C(M) = C(D) \cap C(M) \cap T \subseteq C(D) \cap C(M) \cap K(\beta\mathbb{N},+)$. □

Lemma 3.20. *All centrally IPR matrices and all Milliken-Taylor matrices are in $\mathcal{R}$ (and thus in $\mathcal{N}$).*

Proof. Let D be a finite IPR matrix and let L be a left ideal of $C(D)$. We may presume that L is minimal in $C(D)$. As above, pick a minimal left ideal T of $(\beta\mathbb{N},+)$ such that $T \cap C(D) = L$. Pick an idempotent $p \in T$. If M is centrally IPR, then we have that $p \in L \cap C(M)$.

Assume we have $k \in \omega$ and a condensed sequence $\vec{a} = \langle a_0, a_1, \ldots, a_k\rangle$ in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$ such that $M = MT(\vec{a})$. Let $q = \frac{a_0}{a_k}p + \frac{a_1}{a_k}p + \ldots + \frac{a_{k-1}}{a_k}p + p$. Since $T = \beta\mathbb{Z} + p$, $q \in T$ and by Lemma 3.4, $q \in C(D)$. By Lemma 3.5, $q \in C(M)$. Therefore $q \in L \cap C(M)$. □

Theorem 3.21. *let $u, v \in \mathbb{N}$, and let D be a $u \times v$ IPR matrix. Let α and δ be positive ordinals and let A be an $\alpha \times v$ matrix, whose rows are the rows of D. Let B be an $\alpha \times \delta$ member of $\mathcal{R}$ and let $M \in \mathcal{N}$. Then $\begin{pmatrix} A & B & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & M \end{pmatrix} \in \mathcal{N}$.*

Proof. We show first that $C(A) \cap C(M) \cap K(\beta\mathbb{N},+) \subseteq C\begin{pmatrix} A & B & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & M \end{pmatrix}$. So let $q \in C(A) \cap C(M) \cap K(\beta\mathbb{N},+)$. Let $L = C(A) + q$. Since $B \in \mathcal{R}$, pick $r \in L \cap C(B)$. Since $q \in K(\beta\mathbb{N},+) \cap C(A) = K\big(C(A)\big)$ (by [13, Theorem 1.65]), L is a minimal left ideal of $C(A)$, so $L = L + r$. Pick $s \in L$ such that $q = s + r$. By Lemma 3.6, $q \in C\begin{pmatrix} A & B \end{pmatrix}$ so $q \in C\begin{pmatrix} A & B & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & M \end{pmatrix}$.

Now let $P = \begin{pmatrix} A & B & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & M \end{pmatrix}$. To see that $P \in \mathcal{N}$, let E be a finite IPR matrix. We need to show that $C(E) \cap C(P) \cap K(\beta\mathbb{N},+) \neq \emptyset$.

The rows of $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & E \end{pmatrix}$ are the rows of a finite IPR matrix so, since $M \in \mathcal{N}$,

$$C\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & E \end{pmatrix} \cap C(M) \cap K(\beta\mathbb{N},+) \neq \emptyset\,.$$

Also $C\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & E \end{pmatrix} \cap C(M) \cap K(\beta\mathbb{N},+) \subseteq C(A) \cap C(M) \cap K(\beta\mathbb{N},+) \subseteq C(P)$.

Therefore $\emptyset \neq C\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & E \end{pmatrix} \cap C(M) \cap K(\beta\mathbb{N},+) \subseteq C(E) \cap C(P) \cap K(\beta\mathbb{N},+)$. □

As with Corollary 3.14, we immediately get the following corollary.

Corollary 3.22. *Let $m \in \omega$ and for $i \in \{0, 1, \ldots, m\}$, let $M_i \in \mathcal{R}$, let A_i be a matrix with rows indexed by the same positive ordinal as the rows of M_i, and assume that the rows of A_i are all rows of a finite IPR matrix. Let $M_{m+1} \in \mathcal{N}$. Then*

$$\begin{pmatrix} A_0 & M_0 & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & A_1 & M_1 & \ldots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & A_m & M_m & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} & M_{m+1} \end{pmatrix} \in \mathcal{N}\,.$$

In Corollary 3.22 we could let each M_i be a Milliken-Taylor matrix. If each one is determined by a compressed sequence with the same last term, we will get a stronger result in Theorem 3.26 (except that we only conclude that the resulting matrix is IPR rather than a member of $\mathcal{N}$).

If $F, H \in \mathcal{P}_f(\mathbb{N})$ we write $F < H$ to mean that $\max F < \min H$.

Definition 3.23. Let $k \in \omega$ and let $\vec{a} = \langle a_0, a_1, \ldots, a_k\rangle$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$. Let $i \in \omega$ and let $\langle x_n\rangle_{n=i}^\infty$ be a sequence in $\mathbb{N}$. Then $MT(\vec{a}, \langle x_n\rangle_{n=i}^\infty) = \{\sum_{t=0}^k a_t \sum_{n\in F_t} x_n :$ for each $i \in \{0, 1, \ldots, k\}$, $F_i \in \mathcal{P}_f(\mathbb{N})$, $F_0 < F_1 < \ldots < F_k$, and $\min F_0 \geq i\}$.

Note that if $\vec{x} = \langle x_n\rangle_{n=i}^\infty$, then $MT(\vec{a}, \langle x_n\rangle_{n=i}^\infty)$ is the set of entries of $MT(\vec{a})\vec{x}$.

Lemma 3.24. *Let $k \in \mathbb{N}$ and let $\langle a_0, a_1, \ldots a_k\rangle$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$. Let p be an idempotent in $\beta\mathbb{N}$, let $q = a_0 p + a_1 p + \ldots + a_n p$, and let $Q \in q$. There exist $D \in p$ and a function $P : \bigcup_{m=1}^\infty \mathbb{N}^m \to \mathcal{P}(\mathbb{N})$ such that, if $\langle x_t\rangle_{t=0}^\infty$ is a sequence in $\mathbb{N}$ and*

(1) $x_0 \in D$ and

(2) for each $n \in \omega$, $x_{n+1} \in P(x_0, x_1, \ldots, x_n)$,

then $MT(\vec{a}, \langle x_t\rangle_{t=1}^\infty) \subseteq D$.

Proof. For $m \in \{0, 1, \ldots, k\}$ and $n \in \omega$, let

$$\mathcal{F}_{n,m} = \{\langle F_0, F_1, \ldots, F_m\rangle : \begin{array}{l} \text{for } i \in \{0,1,\ldots,m\}\,,\ F_i \in \mathcal{P}_f(\{0,1,\ldots,n\}) \\ \text{and if } i < m, \text{ then } F_i < F_{i+1}\}\,, \end{array}$$

noting that if $m > n$, then $\mathcal{F}_{n,m} = \emptyset$.

Let C_0 and for $m \in \{0, 1, \ldots, k-1\}$ and $x_0, x_1, \ldots, x_m$ in $\mathbb{N}$, let $B(x_0, x_1, \ldots, x_m)$ be as in the proof of [13, Theorem 15.33].

Let $D = C_0^\star$. Given $n < \omega$ and $\langle x_i\rangle_{i=0}^n$ in $\mathbb{N}$, let

$$\begin{aligned} &P(x_0, x_1, \ldots, x_n) = C_0^\star \cap \textstyle\bigcap\{-\sum_{t\in F} x_t + C_0^\star : F \in \mathcal{P}_f(\{0,1,\ldots,n\})\} \\ &\cap \textstyle\bigcap_{m=0}^{k-1} \bigcap\{B(\sum_{t\in F_0} x_t, \ldots, \sum_{t\in F_m} x_t) : \langle F_0, \ldots, F_m\rangle \in \mathcal{F}_{n,m}\} \\ &\cap \textstyle\bigcap_{m=1}^{k} \bigcap\{-\sum_{t\in F_m} x_t + B(\sum_{t\in F_0} x_t, \ldots, \sum_{t\in F_{m-1}} x_t) : \langle F_0, \ldots, F_m\rangle \in \mathcal{F}_{n,m}\}\,. \end{aligned}$$

(Here for notational convenience we are taking $\bigcap \emptyset = \mathbb{N}$.)

The proof of [13, Theorem 15.33] establishes that D and P are as required. $\square$

Lemma 3.25. *For each $i < \omega$, let $k(i) \in \mathbb{N}$ and let $\vec{a}_i = \langle a_{i,j} \rangle_{j=0}^{k(i)}$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_{i,k(i)} > 0$. Let p be an idempotent in $(\beta\mathbb{N}, +)$, for each $i < \omega$, let $q_i = a_{i,0}p + a_{i,1}p + \ldots + a_{i,k(i)}p$, and let $Q_i \in q_i$. There exists an increasing sequence $\langle x_n \rangle_{n=0}^{\infty}$ in $\mathbb{N}$ such that for each $i < \omega$, $MT(\vec{a}_i, \langle x_n \rangle_{n=i}^{\infty}) \subseteq Q_i$.*

Proof. For $i < \omega$ pick $D_i \in p$ and $P_i : \bigcup_{m=1}^{\infty} \mathbb{N}^m \to \mathcal{P}(\mathbb{N})$ as guaranteed by Lemma 3.24. Pick $x_0 \in D_0$, pick $x_1 \in D_1 \cap P_0(x_0)$, pick $x_2 \in D_2 \cap P_0(x_0, x_1) \cap P_1(x_1)$, and in general, pick $x_{n+1} \in D_{n+1} \cap \bigcap_{i=0}^{n} P_i(x_i, x_{i+1}, \ldots, x_n)$. $\square$

Theorem 3.26. *Let $d \in \mathbb{N}$ and let $m \in \omega$. For $i \in \{0, 1, \ldots, m+1\}$, let $k(i) \in \omega$, let $\vec{a}_i = \langle a_{i,j} \rangle_{j=0}^{k(i)}$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_{i,k(i)} = d$, and let $M_i = MT(\vec{a}_i)$. For $i \in \{0, 1, \ldots, m\}$, let $v_i \in \mathbb{N}$ and let A_i be an $\omega \times v_i$ matrix with all rows in some finite IPR matrix. Then*

$$N = \begin{pmatrix} A_0 & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & M_0 \\ \mathbf{O} & A_1 & \mathbf{O} & \ldots & \mathbf{O} & M_1 \\ \mathbf{O} & \mathbf{O} & A_2 & \ldots & \mathbf{O} & M_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & A_m & M_m \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & M_{m+1} \end{pmatrix} \text{ is IPR.}$$

Proof. Pick a minimal idempotent p in $(\beta\mathbb{N}, +)$. By [13, Lemma 5.19.2] dp is a minimal idempotent so by [13, Exercise 4.3.8] $\beta\mathbb{Z} + dp = \beta\mathbb{N} + dp$. For $i \in \{0, 1, \ldots, m+1\}$, let $q_i = a_{i,0}p + a_{i,1}p + \ldots + a_{i,k(i)}p$. By Lemma 3.4, $q_i \in C(A_j)$ for each $j \in \{0, 1, \ldots, m+1\}$. Given $i \in \{0, 1, \ldots, m\}$, q_i and q_{m+1} are in $(\beta\mathbb{N} + dp) \cap C(A_i)$ which is a minimal left ideal of $C(A_i)$ by [13, Theorem 1.65], so pick $r_i \in C(A_i)$ such that $q_{m+1} = r_i + q_i$. We will show that if $B \in q_{m+1}$ there exists $\vec{w}$ with entries in $\mathbb{N}$ such that all entries of $N\vec{w}$ are in B. So let $B \in q_{m+1}$ be given.

Let $i \in \{0, 1, \ldots, m\}$ and let $D_i = \{x \in \mathbb{N} : -x + B \in q_i\}$. Then $D_i \in r_i$ and $r_i \in C(A_i)$ so pick $\vec{z}_i \in \mathbb{N}^{v_i}$ such that $\vec{y}_i = A_i\vec{z}_i \in D_i^{\omega}$. Now $\{y_{i,j} : j < \omega\}$ is finite so $Q_i = \bigcap_{j<\omega}(-y_{i,j} + B) \in q_i$. Let $Q_{m+1} = B$.

By Lemma 3.25, pick a sequence $\langle x_n \rangle_{n=0}^{\infty}$ in $\mathbb{N}$ such that for each $i \in \{0, 1, \ldots, m+1\}$, $MT(\vec{a}_i, \langle x_n \rangle_{n=i}^{\infty}) \subseteq Q_i$. For $n \in \omega$, let $c_n = x_{m+n+1}$.

$$\vec{w} = \begin{pmatrix} \vec{z}_0 \\ \vec{z}_1 \\ \vdots \\ \vec{z}_m \\ \vec{c} \end{pmatrix}.$$

Then, given $i \in \{0,1,\ldots,m\}$, $\begin{pmatrix} \mathbf{O} & \ldots & A_i & \ldots & \mathbf{O} & M_i \end{pmatrix} \vec{w} = A_i \vec{z}_i + M_i \vec{c} \in B^\omega$ and $\begin{pmatrix} \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & M_{m+1} \end{pmatrix} \vec{w} = M_{m+1}\vec{c} \in B^\omega$ so $\vec{w}$ is as required. $\square$

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