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Sets Partition Regular for n Equations Need Not Solve n + 1.

by

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ABSTRACT. An $n \times m$ rational matrix A is said to be partition regular if for every finite coloring of $\mathbb{N}$ there is a monochromatic vector $\vec{x} \in \mathbb{N}^m$ with $A\vec{x} = \vec{0}$. A set $D \subseteq \mathbb{N}$ is said to be partition regular for A (or for the system of equations $A\vec{x} = \vec{0}$) if for every finite coloring of D there is a monochromatic $\vec{x} \in D^m$ with $A\vec{x} = \vec{0}$.

In this paper we show that for every n there is a set that is partition regular for every partition regular system of n equations but not for every system of $n+1$ equations. We give several related results and we also prove a "uniform" extension of this result: for each n we give a set D which is uniformly partition regular for n equations in the sense that given any finite coloring of D some one class solves all partition regular systems of n equations, but D is not partition regular for (and in fact contains no solution to) a particular partition regular system of $n + 1$ equations, namely that system describing a length $n + 2$ arithmetic progression with its increment. We give applications to the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of the discrete set $\mathbb{N}$.

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1. **Introduction**. Theorems in Ramsey Theory often state that whenever a sufficiently large structure of some kind is partitioned into k cells (or "k-colored") a smaller structure of the same kind must be contained in one cell (or be "monochrome"). For example, the simplest non-trivial version of Ramsey's Theorem itself says that whenever the edges of a complete graph on 6 vertices (a K_6) are 2-colored, there must be a monochrome triangle. One can naturally ask (so Erdös did) whether any graph with the property that whenever it is 2-colored there will necessarily be a monochrome triangle must contain a K_6. In fact it turned out [3] that there is a graph with this property which contains no K_4.

In [14], Spencer gave a similar result for van der Waerden's Theorem. He provided a simple elegant proof that given any n and k there is a set A such that whenever A is k-colored there is a monochrome length n arithmetic progression, but A contains no length $n+1$ arithmetic progression. See [12] for some extensions of Spencer's result.

In a similar vein, it was shown in [11] that given any n and k in $\mathbb{N}$there is a sequence $C_1, C_2, \ldots, C_n$ of pairwise disjoint sets with $FU(\langle C_i\rangle_{i=1}^n)$ monochrome, but S contains no $FU(\langle C_i\rangle_{i=1}^{n+1})$. (Here $FU(\langle C_i\rangle_{i=1}^n) = \{\bigcup_{t\in F} C_t : \emptyset \neq F \subseteq \{1,2,\ldots,n\}\}$.) As a consequence, one obtains (see [10, Corollary 3.8]) the fact that for any $n \in \mathbb{N}$ there is a subset A of $\mathbb{N}$a sequence $\langle x_t\rangle_{t=1}^n$ with $FS(\langle x_t\rangle_{t=1}^n)$ monochrome, but A contains no $FS(\langle x_t\rangle_{t=1}^{n+1})$. (Where $FS(\langle x_t\rangle_{t=1}^n) = \{\sum_{t\in F} x_t : \emptyset \neq F \subseteq \{1,2,\ldots,n\}\}$.)

We describe this situation (where the number of colors is unrestricted) by saying that A is "partition regular" for $FS(\langle x_t\rangle_{t=1}^n)$.

1.1 Definition. Let $\mathcal{C}$ be any set of sets. A set B is *partition regular* for $\mathcal{C}$ if whenever B is partitioned into finitely many cells, one cell contains some member of $\mathcal{C}$.

We are concerned in this paper with solving systems of equations of the form $A\vec{x} = \vec{0}$ where A is a rational matrix. (It would be equally general, but less convenient, to assume that all entries of A are integers.) Such a system is said to be partition regular if in the sense of Definition 1.1 $\mathbb{N}$ is partition regular for the set $\mathcal{C}$ of all solution sets for the given system. That is, the given system is partition regular if and only if whenever $\mathbb{N}$ is finitely colored there is a monochrome solution to the given system.

The problem of which systems are partition regular was completely solved by Rado [13]. The solution involves a notion known as the "columns condition".

1.2 Definition. Let $n, v \in \mathbb{N}$ and let A be an $n \times v$ matrix with rational entries. Then A satisfies the *columns condition* if the columns $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_v$ of A can be ordered so that there exist $m \in \mathbb{N}$ and $k_1, k_2, \ldots, k_m$ in $\mathbb{N}$ with $1 \leq k_1 < k_2 \ldots < k_m = v$ such that

(1) $\sum_{i=1}^{k_1}$

(2) if $m > 1$ and $t \in \{2, 3, \ldots, m\}$, there exist $\alpha_{1,t}, \alpha_{2,t}, \ldots \alpha_{k_{t-1},t}$ in $\mathbb{Q}$ such that $\sum_{i=k_{t-1}+1}^{k_t}$

1.3 Theorem (Rado [13]). *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with rational entries. Let $\vec{x} = (x_1, x_2, \ldots, x_v)^T$. The system of equations $A\vec{x} = \vec{0}$ is partition regular (over $\mathbb{N}$) if and only if A satisfies the columns condition.* $\Box$

Our aim in this paper is to investigate how "independent" are the notions of partition regularity for various systems of equations. We will show in Section 3 that for each $n \in \mathbb{N}$ there exist a subset D of $\mathbb{N}$ and a partition regular system of $n+1$ linear equations $B\vec{x} = \vec{0}$ such that D contains no solutions to $B\vec{x} = \vec{0}$ but given any finite coloring of D and any partition regular system of n linear equations (with rational coefficients) $A\vec{x} = \vec{0}$, D will contain a monochrome solution to $A\vec{x} = \vec{0}$. (We remark to the interested reader that the partition regular system of $n+1$ equations that we use is the system describing a length $n+2$ arithmetic progression together with its increment.)

Our constructions in Section 3 are somewhat complicated. We will introduce much of the machinery in a simpler setting in Section 2. There we will establish the following fact. Given any $s \in \mathbb{Q}^+ = \{q \in \mathbb{Q} : q > 0\}$ and any finite set $F \subseteq \mathbb{Q}^+ \backslash \{1, s\}$ there is a set D that is partition regular for the equation $x = y + s \cdot z$ but contains no solution to $x = y + t \cdot z$ for any $t \in F$. (We do not know whether one can allow $1 \in F$.) In particular, given $t \notin \{1, s\}$ there is a set partition regular for $x = y + s \cdot z$ containing no solution to $x = y + t \cdot z$. As a consequence, we obtain a set E that is partition regular for each single partition regular equation but contains no solution to the system

$$x_3 = x_1 + 2x_2$$
$$x_4 = x_1 + 3x_2 \ .$$

One might ask for the stronger conclusion that whenever E is finitely colored some one cell will contain solutions to all single partition regular equations. The sets we construct in Section 2 specifically fail to satisfy this requirement – indeed, the reader will see that this failure is absolutely built in to the way we construct these sets.

However, it *is* possible to insist on the far stronger requirement, and in Section 4 we give a proof of this result. This result has consequences for the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. For a survey of numerous applications of the structure of $\beta\mathbb{N}$ to combinatorial number theory see [8].

All of our equations will have rational coefficients and all of the numbers with which we will deal will be rational – we will sometimes not mention this fact in order to improve the flow of the prose.

2. Solving single equations. As a simple consequence of Rado's Theorem (Theorem 1.3), a single equation is partition regular if and only if some subset of its non-zero coefficients sums to 0. (This special case is much simpler than the general result. See [5, Section 3.2]. We remark also that it can be derived using recurrence of sets of positive density in a fashion similar to that used in [1].) The following well known lemma says that the only "interesting" single partition regular equations are $x = y + s \cdot z$ for positive rationals s.

2.1 Lemma. *Let $c_1 \cdot x_1 + c_2 \cdot x_2 + ... + c_v \cdot x_v = 0$ be a partition regular equation. Then there exists a positive rational s such that any set containing a solution to $x = y + s \cdot z$ also contains a solution to $c_1 \cdot x_1 + c_2 \cdot x_2 + ... + c_v \cdot x_v = 0$.*

Proof. If $\sum_{i=1}^{v} c_i = 0$ then any $x_1 = x_2 = ... = x_v$ solves the equation, so we may assume we have $\sum_{i=1}^{v} c_i \neq 0$. Since $c_1 \cdot x_1 + c_2 \cdot x_2 + ... + c_v \cdot x_v = 0$ is partition regular we may presume we have $k \geq 2$ such that c_1 is nonzero and $c_1 + c_2 + ... + c_k = 0$. Let $s = |(\sum_{i=1}^{v} c_i)/c_1|$. Note that also $s = |(\sum_{i=k+1}^{v} c_i)/c_1|$. Assume we have x, y, and z such that $x = y + s \cdot z$. If $(\sum_{i=1}^{v} c_i)/c_1 < 0$, let $x_1 = x$, $x_2 = x_3 = ... = x_k = y$, and $x_{k+1} = x_{k+2} = ... = x_v = z$. If $(\sum_{i=1}^{v} c_i)/c_1 > 0$, let $x_1 = y$, $x_2 = x_3 = ... = x_k = x$, and $x_{k+1} = x_{k+2} = ... = x_v = z$. $\Box$

As a consequence of Lemma 2.1, we are interested in solutions to equations of the form $x = z + s \cdot w$. Note that x, z, and w are the three entries of the matrix product $\begin{pmatrix} 1 & s \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$. The matrix $\begin{pmatrix} 1 & s \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ is of a special kind which we will use throughout this paper.

2.2 Definition. Let $u, v \in \mathbb{N}$. A $u \times v$ matrix C is a *monic first entries* matrix if all entries of C are rational, no row of C is $\vec{0}$ and the first (leftmost) nonzero entry in each row is 1.

Now we start to describe some sets that will be useful as building blocks for us.

2.3 Definition. Let C be a $u \times v$ monic first entries matrix. A sequence $\langle Q_m \rangle_{m=1}^{\infty}$ is *C-useful* if

(1) for each $m \in \mathbb{N}$, Q_m is a finite set of positive rationals,

(2) for each $m \in \mathbb{N}$, $Q_m \subseteq Q_{m+1}$, and

(3) for each $m \in \mathbb{N}$, each $q \in Q_m$, and each $i \in \{1, 2, \ldots, v\}$ there exists $\vec{x} \in (Q_{m+1} \cup \{0\})^v$ such that $x_i = q$ and $x_{i+1} = x_{i+2} = ... = x_v = 0$ and all entries of $C\vec{x}$ are in $Q_{m+1} \cup \{0\}$.

Given any equation $x = z + s \cdot w$ we will construct a special C-useful sequence where $C = \begin{pmatrix} 1 & s \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$. The set D defined below will assist in this construction.

2.4 Definition. Let C be a $u \times v$ monic first entries matrix, let $\langle Q_m \rangle_{m=1}^{\infty}$ be a C-useful sequence, and let $\langle y_m \rangle_{m=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each $m \in \mathbb{N}$ and each $q \in Q_m$, $q \cdot y_m \in \mathbb{N}$. Then $D(\langle Q_m \rangle_{m=1}^{\infty}, \langle y_m \rangle_{m=1}^{\infty}) =$

$$\{\sum_{m \in F} q_m \cdot y_m : F \text{ is a finite nonempty subset of } \mathbb{N} \text{ and } q_m \in Q_m \text{ for each } m \in F\}.$$

Our next result is the most difficult result in the paper. It is the "workhorse" of this paper. In a sense, having proved it, a lot of the rest of the work is some (admittedly rather delicate) fine tuning. To be more precise, it is Theorem 2.5 that we shall use when proving the "positive" parts of statements – that certain sets are partition regular for certain equations.

Because the proof is rather complicated, we postpone it until the end of this section. After stating Theorem 2.5 we shall immediately give an example of how to apply it.

The statement deals with the notion of "image partition regularity". (What we have been calling "partition regular", namely finding monochrome $\vec{x}$ with $A\vec{x} = \vec{0}$, is "kernel partition regular".) A matrix is *image partition regular* if whenever $\mathbb{N}$ is finitely colored there is some $\vec{x}$ with the entries of $A\vec{x}$ monochrome.

2.5 Theorem. *Let C be a $u \times v$ monic first entries matrix, let $\langle Q_m \rangle_{m=1}^{\infty}$ be a C-useful sequence, and let $\langle y_m \rangle_{m=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each $m \in \mathbb{N}$ and each $q \in Q_m$, $q \cdot y_m \in \mathbb{N}$. Then $D = D(\langle Q_m \rangle_{m=1}^{\infty}, \langle y_m \rangle_{m=1}^{\infty})$ is image partition regular for C. That is, whenever D is finitely colored, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $C\vec{x}$ belong to D and are monochrome.*

As an application of Theorem 2.5, let us find one of the sets claimed in the Introduction. Given $s \in \mathbb{Q}^+$ and a finite $F \subseteq \mathbb{Q}^+\setminus\{1, s\}$ we wish to produce a set that is partition regular for the equation $x = y+s\cdot z$ but contains no solution to $x = z+t\cdot w$ for any $t \in F$. Given such s and F our strategy is simple. We let $C = \begin{pmatrix} 1 & s \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and produce a C-useful sequence $\langle Q_m\rangle_{m=1}^{\infty}$ and a sequence $\langle y_m\rangle_{m=1}^{\infty}$ such that $D(\langle Q_m\rangle_{m=1}^{\infty}, \langle y_m\rangle_{m=1}^{\infty})$ contains no solution to $x = z + t \cdot w$ for any $t \in F$. We see now that by choosing the sequence $\langle y_m\rangle_{m=1}^{\infty}$ to be thin enough, we are able to concentrate entirely on the individual Q_m's.

2.6 Lemma. *Let C be a monic first entries matrix and let $\langle Q_m\rangle_{m=1}^{\infty}$ be a C-useful sequence. Let F be a finite set of positive rationals and let $c = \max F$. For each $m \in \mathbb{N}$, let $d_m = \max Q_m$ and let*

$$b_m = \min\{|x - (z + t \cdot w)| : t \in F \text{ and } x, z, w \in Q_m \cup \{0\} \text{ and } x \neq z + t \cdot w\}.$$

Assume that for each $m \in \mathbb{N}$ we have $y_m \cdot q \in \mathbb{N}$ for all $q \in Q_m$ and if $m > 1$ then $y_m \cdot b_m > (c + 1) \cdot \sum_{k=1}^{m-1} y_k \cdot d_k$. Let $\{t_1, t_2, \ldots, t_\ell\} \subseteq F$ and consider the system of equations

$$\begin{aligned} x_3 &= x_1 + t_1 \cdot x_2 \\ x_4 &= x_1 + t_2 \cdot x_2 \\ &\cdot \\ &\cdot \\ x_{\ell+2} &= x_1 + t_\ell \cdot x_2 \end{aligned}$$

If there is a solution to this system in $D(\langle Q_m\rangle_{m=1}^{\infty}, \langle y_m\rangle_{m=1}^{\infty})$, then for some m there is a solution in $Q_m \cup \{0\}$ with $x_2 \neq 0$.

Proof. For each $x \in D = D(\langle Q_m\rangle_{m=1}^{\infty}, \langle y_m\rangle_{m=1}^{\infty})$, pick $q(x) \in \times_{m=1}^{\infty}(Q_m\cup\{0\})$ such that $x = \sum_{m=1}^{\infty} q(x)_m \cdot y_m$. Assume we have a solution to the specified system in D. It suffices to show that $q(x_{i+2})_m = q(x_1)_m+t_i\cdot q(x_2)_m$ for each $m \in \mathbb{N}$ and $i \in \{1, 2, \ldots, \ell\}$ – for then one simply chooses any $m \in \mathbb{N}$ with $q(x_2)_m \neq 0$. So let $x = x_{i+2}$, $z = x_1$, $w = x_2$, and $t = t_i$ and suppose one has some $m \in \mathbb{N}$ such that $q(x)_m \neq q(z)_m+t\cdot q(w)_m$, and pick the largest such m. Note that $|q(x)_m - q(z)_m - t \cdot q(w)_m| \geq b_m$.

Case 1. $q(x)_m > q(z)_m + t \cdot q(w)_m$. Then we have

$$\begin{aligned}
0 &= x - z - t \cdot w \\
&= (q(x)_m - q(z)_m - t \cdot q(w)_m) \cdot y_m + \sum_{k=1}^{m-1} (q(x)_k - q(z)_k - t \cdot q(w)_k) \cdot y_k \\
&\geq b_m \cdot y_m - \sum_{k=1}^{m-1} (d_k + t \cdot d_k) \cdot y_k \\
&\geq b_m \cdot y_m - (1 + c) \cdot \sum_{k=1}^{m-1} d_k \cdot y_k \\
&> 0,
\end{aligned}$$

a contradiction.

Case 2. $q(x)_m < q(z)_m + t \cdot q(w)_m$. Then

$$\begin{aligned}
0 &= z + t \cdot w - x \\
&= (q(z)_m + t \cdot q(w)_m - q(x)_m) \cdot y_m + \sum_{k=1}^{m-1} (q(z)_k + t \cdot q(w)_k - q(x)_k) \cdot y_k \\
&\geq b_m \cdot y_m - \sum_{k=1}^{m-1} d_k \cdot y_k \\
&> 0,
\end{aligned}$$

again a contradiction. $\square$

We are now ready to construct a set that was promised in the Introduction.

2.7 Theorem. *Let s be a positive rational and let F be a finite subset of $\mathbb{Q}^+ \setminus \{1, s\}$. Then there is a set $D \subseteq \mathbb{N}$ that is partition regular for the equation $x = z + s \cdot w$ but contains no solutions to $x = z + t \cdot w$ for any $t \in F$.*

Proof. Let C be the matrix $\begin{pmatrix} 1 & s \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$. By Lemma 2.6 it suffices to produce a C-useful sequence $\langle Q_m \rangle_{m=1}^{\infty}$ such that for all $t \in F$ and all $m \in \mathbb{N}$, $Q_m \cup \{0\}$ does not contain a solution to $x = z + t \cdot w$ with $w \neq 0$. Indeed, assume we have done this and choose a sequence $\langle y_m \rangle_{m=1}^{\infty}$ as required by Lemma 2.6 and let $D = D(\langle Q_m \rangle_{m=1}^{\infty}, \langle y_m \rangle_{m=1}^{\infty})$. By Theorem 2.5 D is partition regular for the equation $x = z + s \cdot w$. By Lemma 2.6 D contains no solution to $x = z + t \cdot w$ for any $t \in F$.

So we set out to construct $\langle Q_m \rangle_{m=1}^{\infty}$. Let

$$c = \max(F \cup \{1/t : t \in F\} \cup \{2/(1-t) : t \in F\} \cup \{t/s : t \in F\}) \cup \{s/t : t \in F\})$$

and let

$$\begin{aligned} d = \min(&\{t \in F : t > 1\} \cup \{1/t : t \in F \text{ and } t < 1\} \cup \{t/s : t \in F \text{ and } t > s\} \\ &\cup \{s/t : t \in F \text{ and } t < s\} \cup \{2/(1+t) : t \in F \text{ and } t < 1\}). \end{aligned}$$

Note that $d > 1$.

We show next that it suffices to produce the sequence $\langle Q_m \rangle_{m=1}^{\infty}$ so that

(*) for each $m \in \mathbb{N}$ and each a and b in Q_m with $a < b$, one has that $b/a < d$ or $b/a > 1 + c \cdot d$, and moreover if $b/a < d$ then there is some $q \in Q_m$ such that $b - a = q \cdot s$.

Indeed, assume that we have constructed $\langle Q_m \rangle_{m=1}^{\infty}$ satisfying (*), and suppose that we have some $m \in \mathbb{N}$ and some x, z, w in $Q_m \cup \{0\}$ and some $t \in F$ with $x = z + t \cdot w$ and $w \neq 0$. Note that $z \neq 0$ since if it were we would have $x/w = t$. This is forbidden by (*) because if $t > 1$ one has $d \leq t \leq c$ and if $t < 1$ one has $d \leq 1/t \leq c$.

Next we claim that $w/z \leq d$. Suppose instead $w/z > d$. Then $w/z > 1 + c \cdot d > 2$. If $t > 1$ we have $x/w = z/w + t \geq d$ and $x/w = z/w + t < 1/2 + t < 1 + c \cdot d$, a contradiction. If $t < 1$ we have $w/z > c \geq 2/(1-t)$ so $z/w < (1-t)/2$ so $x/w = z/w + t < (1+t)/2$ and hence $w/x > 2/(1+t) \geq d$; but $w/x = 1/t - z/(x \cdot t) < 1/t \leq c$, a contradiction.

Thus we have $w/z \leq d$. Consequently $x/z = 1 + t \cdot (w/z) \leq 1 + c \cdot d$. Hence we may pick $q \in Q_m$ such that $x - z = s \cdot q$, that is $t \cdot w = s \cdot q$. But then $q \neq w$ since $t \neq s$, while the ratios t/s and s/t are forbidden for two members of Q_m.

It thus suffices to produce a C-useful sequence $\langle Q_m \rangle_{m=1}^{\infty}$ satisfying (*). But this is essentially a triviality. We let $Q_1 = \{1\}$. (Or insert any other positive rational.) Given Q_m, which is finite, write $Q_m = \{q_1, q_2, \ldots, q_\ell\}$. We need to choose for each q_j some r_j and add r_j and $r_j + s \cdot q_j$ to Q_{m+1}. (In the definition of C-useful, the $i = 1$ requirement has no content so we are making sure all entries of $\begin{pmatrix} 1 & s \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_j \\ q_j \end{pmatrix}$ are in $Q_{m+1} \cup \{0\}$.) To start out we may assume $q_\ell = \max Q_m$, we choose r_1 such that $r_1 > q_\ell \cdot (1 + c \cdot d)$ and $(r_1 + s \cdot q_1)/r_1 < d$. Given that we have chosen r_{j-1}, we pick r_j such that $r_j > (r_{j-1} + s \cdot q_{j-1}) \cdot (1 + c \cdot d)$ and $(r_j + s \cdot q_j)/r_j < d$. $\Box$

Observe that the proof does not work if one allows $1 \in F$. There is a good reason for this. Sets of the form $D = D(\langle Q_m \rangle_{m=1}^{\infty}, \langle y_m \rangle_{m=1}^{\infty})$ are always partition regular for the equation $x = y + z$. One way to see this is to pick any $q_m \in Q_m$ and note that $FS(\langle q_m \cdot y_m \rangle_{m=1}^{\infty}) \subseteq D$, where $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$. Consequently, any finite partition of D will have one cell containing $FS(\langle x_n \rangle_{n=1}^{\infty})$ for some sequence $\langle x_n \rangle_{n=1}^{\infty}$. (See for example [7, Lemma 3.8].)

However, we believe that this fault is due merely to our inability to construct "better" sets than D.

2.8 Conjecture. *For each positive rational $s \neq 1$ there is a set D that is partition regular for $x = y + s \cdot z$ but contains no solution to $x = y + z$.*

In fact, we believe that a much stronger statement is true. Let A and B be partition regular matrices. Say that A *Rado-dominates* B if every set which is partition regular for A is partition regular for B. Say that A *solution-dominates* B if every solution to $A\vec{x} = \vec{0}$ contains a solution to $B\vec{y} = \vec{0}$. To be more precise, if A is an $n \times m$ matrix and B is an $r \times k$ matrix, A *solution-dominates* B if there is a function $f : \{1, 2, \ldots, k\} \longrightarrow \{1, 2, \ldots, m\}$ such that whenever $A\vec{x} = \vec{0}$ we have

$$B \begin{pmatrix} x_{f(1)} \\ x_{f(2)} \\ \cdot \\ \cdot \\ x_{f(k)} \end{pmatrix} = \vec{0}.$$

Thus we trivially have that if A solution-dominates B then A Rado-dominates B.

2.9 Conjecture. *Let A and B be partition regular matrices. Then A Rado-dominates B if and only if A solution-dominates B.*

Let us remark that to ask, as we did in Conjecture 2.8, for a set that is partition regular for one matrix and contains no solution to the second is the same as asking for a set that is partition regular for one but not for the other. (To see this, assume D is partition regular for A but not for B. Pick a finite partition $\mathcal{F}$ of D, no cell of which contains a solution to B. Some one cell must be partition regular for A.)

We now turn our attention to constructing a set that is partition regular for every single partition regular equation but not for a certain partition regular system of 2 equations. Before we start, however, let us point out that the situation just described is not so clear when one deals with a set which is partition regular for several, perhaps infinitely many matrices but is not partition regular for some other specified matrix.

However, the specified matrix which we will deal with subsequently will in all cases be the matrix for the terms of an arithmetic progression together with its increment. One can derive in this case (as a consequence of Lemma 2.10) using a construction like that in the proof of Theorem 2.11, that for such a matrix it is equivalent to ask for a set not partition regular for the matrix or to ask for a set containing no solution for the matrix. However, we will not spell out this argument, because the sets we produce always satisfy the stronger conclusion anyway.

We now start to work towards a set, as mentioned above, that is partition regular for every single partition regular equation but not for a certain partition regular system of 2 equations. The next lemma will be used again in Section 3. It will provide us with a way to piece together partition regular sets is a useful fashion.

2.10 Lemma. *Let $\ell \in \mathbb{N}$, let $a \in \{1,2\}$, and let $\langle D_n\rangle_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathbb{N}$. Assume that no D_n contains a solution to the system:*

$$\begin{aligned} x_{a+2} &= x_1 + a\cdot x_2 \\ x_{a+3} &= x_1 + (a+1)\cdot x_2 \\ &\;\;\cdot \\ &\;\;\cdot \\ x_{\ell+1} &= x_1 + (\ell-1)\cdot x_2\ . \end{aligned}$$

Then there is a sequence $\langle r_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\bigcup_{n=1}^{\infty} r_n\cdot D_n$ contains no solution to the same system.

Proof. Let $r_1 = 1$ and inductively given $n > 1$, let $k_n = \max\bigcup_{j=1}^{n-1} r_j\cdot D_j$ and let $r_n = k_n\cdot\ell+1$. Now we claim that if $t\in\{1,2,\ldots,\ell-1\}$ and $\{x,z,w\}\subseteq\bigcup_{n=1}^{\infty} r_n\cdot D_n$ with $x = z+t\cdot w$, then there exists n with $\{x,z,w\}\subseteq r_n\cdot D_n$. To see this pick the largest n such that $\{x,z,w\}\cap r_n\cdot D_n\neq\emptyset$ and suppose $\{x,z,w\}\backslash r_n\cdot D_n\neq\emptyset$. Since $x > z$ and $x > w$ there are three possibilities for $\{x,z,w\}\cap r_n\cdot D_n$, namely $\{x\}$, $\{x,w\}$, and $\{x,z\}$. Suppose for example that $\{x,z,w\}\cap r_n\cdot D_n = \{x\}$. Then $0 < x = z+t\cdot w \le (1+t)\cdot k_n \le \ell\cdot k_n < r_n$ while r_n divides x, a contradiction. The other two cases are handled similarly.

Now suppose we have $\{x_1, x_2, x_3, \ldots, x_{\ell+1}\} \subseteq \bigcup_{n=1}^{\infty} r_n \cdot D_n$ satisfying the system:

$$\begin{aligned} x_{a+2} &= x_1 + a \cdot x_2 \\ x_{a+3} &= x_1 + (a+1) \cdot x_2 \\ &\cdot \\ &\cdot \\ x_{\ell+1} &= x_1 + (\ell - 1) \cdot x_2 \ . \end{aligned}$$

For $t \in \{a, a+1, \ldots, \ell-1\}$, pick $n(t) \in \mathbb{N}$ such that $\{x_{t+2}, x_1, x_2\} \subseteq r_{n(t)} \cdot D_{n(t)}$. Since $x_1 \in r_{n(t)} \cdot D_{n(t)}$ for all t we have $n(t) = n(a)$. Let $n = n(a)$. Then we have $\{x_1, x_2, \ldots, x_{\ell+1}\} \subseteq r_n \cdot D_r$ so $\{x_1/r_n,\ x_2/r_n, \ldots, x_{\ell+1}/r_n\} \subseteq D_n$, a contradiction. $\square$

Note that by Rado's Theorem the system of Lemma 2.10 with $a = 1$ is partition regular. (This is the strengthening of van der Waerden's Theorem which requires that the increment belong to the chosen color class as well.)

2.11 Theorem. *There is a subset E of $\mathbb{N}$ which is partition regular for every single partition regular homogeneous linear equation with rational coefficients but contains no solution to the system*

$$\begin{aligned} x_3 &= x_1 + 2 \cdot x_2 \\ x_4 &= x_1 + 3 \cdot x_2 \ . \end{aligned}$$

Proof. By Lemma 2.1 it suffices to produce a set E which is partition regular for every equation of the form $x = z + s \cdot w$ for $s \in \mathbb{Q}^+$ but contains no solution to

$$\begin{aligned} x_3 &= x_1 + 2 \cdot x_2 \\ x_4 &= x_1 + 3 \cdot x_2 \ . \end{aligned}$$

Given $s \in \mathbb{Q}^+ \backslash \{2\}$, pick by Theorem 2.7 a set $H_s \subseteq \mathbb{N}$ which is partition regular for $x = z + s \cdot w$ but contains no solution to $x = z + 2 \cdot w$. Also pick a set $H_2 \subseteq \mathbb{N}$ which is partition regular for $x = z + 2 \cdot w$ but contains no solution to $x = z + 3 \cdot w$.

Given $s \in \mathbb{Q}^+$ and $k \in \mathbb{N}$, pick by compactness (see [5, Section 1.5]) a finite subset $H_{s,k}$ of H_s such that whenever $H_{s,k}$ is k-colored there is a monochrome solution to $x = z + s \cdot w$. (Strictly speaking, we don't need to appeal to compactness, since we have built up our set in Theorem 2.5 using finite sets.)

Let $\langle D_n\rangle_{n=1}^{\infty}$ enumerate $\{H_{s,k} : s \in \mathbb{Q}^+ \text{ and } k \in \mathbb{N}\}$ and choose a sequence $\langle r_n\rangle_{n=1}^{\infty}$ as guaranteed by Lemma 2.10 with $a = 2$ and $\ell = 4$. Let $E = \bigcup_{n=1}^{\infty} r_n \cdot D_n$. Then by Lemma 2.10 E contains no solution to

$$x_3 = x_1 + 2 \cdot x_2$$
$$x_4 = x_1 + 3 \cdot x_2 \ .$$

It is clear that E is partition regular for every single partition regular equation. Indeed, let E be finitely colored, say with k colors, and let $s \in \mathbb{Q}^+$ and pick n such that $D_n = H_{s,k}$. Color x in $H_{s,k}$ by the color given to $r_n \cdot x$ in E. Pick monochrome x, z, w in $H_{s,k}$ with $x = z + s \cdot w$. Then $r_n \cdot x$, $r_n \cdot z$, and $r_n \cdot w$ are monochrome in E and $r_n \cdot x = r_n \cdot z + s \cdot r_n \cdot w$. $\Box$

We complete this section by providing the postponed proof of Theorem 2.5. For those who are familiar with such arguments, we point out that in the special case where, in the definition of C-useful, we actually always have $x_1 = ... = x_{i-1} = 0$, one could view what we are doing as in some sense reproving Ramsey's Theorem, but replacing the appeals to the pigeon-hole principle by appeals to the Hales-Jewett Theorem. We first state the form of the Hales-Jewett Theorem which we will be using.

By an *n-variable word* over an alphabet A we mean a word over the alphabet $A \cup \{v_1, v_2, \ldots, v_n\}$ in which each v_i actually occurs, where $v_1, v_2, \ldots, v_n$ are symbols not in A. Given an n-variable word $w(v_1, v_2, \ldots, v_n)$ and $(a_1, a_2, \ldots, a_n) \in A^n$, $w(a_1, a_2, \ldots, a_n)$ has its obvious meaning – it is the word obtained from $w(v_1, v_2, \ldots, v_n)$ by replacing each occurrence of v_i by a_i.

2.12 Lemma. (Hales-Jewett). *Let A be a finite alphabet and let $k, n \in \mathbb{N}$. Then there exists $p \in \mathbb{N}$ such that whenever the length p words over A are k-colored, there is an n-variable word $w(v_1, v_2, \ldots, v_n)$ of length p, all of whose instances $w(a_1, a_2, \ldots, a_n)$ for $(a_1, a_2, \ldots, a_n) \in A^n$ have the same color.*

Proof. The usually stated version of the Hales-Jewett Theorem ([6], or see [5]) is for a single variable. Apply this version to the alphabet A^n. $\Box$

2.13 Definition. Given finite subsets Q and $S_1, S_2, \ldots, S_p$ of $\mathbb{N}$ and a sequence $\langle y_m\rangle_{m=1}^{\infty}$ in $\mathbb{N}$, the *Q-span* of $(S_1, S_2, \ldots, S_p)$ with respect to $\langle y_m\rangle_{m=1}^{\infty}$ is $\{\sum_{t=1}^{p} \ell_t \cdot \sum_{m\in S_t} y_m : \ell_t \in (Q \cup \{0\})$ for each $t \in \{1, 2, \ldots, p\}$ and $\ell \neq \vec{0}\}$.

2.14 Lemma. *Let C, $\langle Q_m\rangle_{m=1}^{\infty}$ and $\langle y_m\rangle_{m=1}^{\infty}$ be as in Theorem 2.5 and let $\alpha, k, n \in \mathbb{N}$. Then there exists $p \in \mathbb{N}$ such that, given any pairwise disjoint nonempty* *subsets $S_1, S_2, \ldots, S_p$ of $\mathbb{N}$ with $\min \bigcup_{t=1}^{p} S_p \geq \alpha$ and any k-coloring φ of the set $D = D(\langle Q_m \rangle_{m=1}^{\infty}, \langle y_m \rangle_{m=1}^{\infty})$ there exist pairwise disjoint nonempty subsets $H_0, H_1, \ldots, H_n$ of $\{1, 2, \ldots, p\}$ and pairwise disjoint nonempty subsets $T_1, T_2, \ldots, T_n$ of $\mathbb{N}$ and $x \in D$ such that*

(1) for all $j \in \{1, 2, \ldots, n\}$, $T_j = \bigcup_{t \in H_j} S_t$,

(2) there exists $\ell \in \bigtimes_{t \in H_0} Q_\alpha$ such that $x = \sum_{t \in H_0} \ell_t \cdot \sum_{m \in S_t} y_m$ (so x is in the Q_α-span of $(S_1, S_2, \ldots, S_p)$ over $\langle y_m \rangle_{m=1}^{\infty}$), and

(3) for any z in the Q_α-span of $(T_1, T_2, \ldots, T_n)$ over $\langle y_m \rangle_{m=1}^{\infty}$, $\varphi(x) = \varphi(x + z)$.

Proof. Let $A = Q_\alpha \cup \{0\}$. Pick p as guaranteed by Lemma 2.11 for A, n, and $k+1$. Let W be the set of length p words over A. Define $\psi : W \backslash \{\vec{0}\} \longrightarrow D$ by

$$\psi(w) = \sum_{t=1}^{p} \ell_t \cdot \sum_{m \in S_t} y_m,$$

where $w = \ell_1 \ell_2 \ldots \ell_p$ and each $\ell_t \in A$. Color $W \backslash \{\vec{0}\}$ using k colors by $\varphi \circ \psi$, and give $\vec{0}$ its own color. Pick $w(v_1, v_2, \ldots, v_n)$ such that all occurrences are monochrome. Note that not all occurrences are $\vec{0}$ so $w(0, 0, \ldots, 0) \neq \vec{0}$. Let $w = \ell_1 \ell_2 \ldots \ell_p$, where each $\ell_t \in A \cup \{v_1, v_2, \ldots, v_n\}$. Let $H_0 = \{t \in \{1, 2, \ldots, p\} : \ell_t \in Q_\alpha\}$. By the observation above, $H_0 \neq \emptyset$. For $j \in \{1, 2, ..., n\}$, let $H_j = \{t \in \{1, 2, \ldots, p\} : \ell_t = v_j\}$ and let $T_j = \bigcup_{t \in H_j} S_t$. Let $x = \sum_{t \in H_0} \ell_t \cdot \sum_{m \in S_t} y_m$. Since $\min \bigcup_{t=1}^{p} S_t \geq \alpha$ we have $x \in D$. Conclusions (1) and (2) hold directly.

To verify (3) let $q = s_1 s_2 \ldots s_n$ be a length n word over A with $q \neq \vec{0}$ and let $z = \sum_{t=1}^{n} s_t \cdot \sum_{m \in T_t} y_m$. Then $x = \psi(w(0, 0, \ldots, 0))$ and $x + z = \psi(w(s_1, s_2, \ldots, s_n))$ so $\varphi(x) = \varphi(x + z)$. ▯

Finally we have:

Proof of Theorem 2.5. Let φ be a finite coloring of D, say with k colors. Let $\gamma = k \cdot (v - 1) + 1$ (so γ is the pigeon-hole number to obtain v objects of one color). Define numbers $n(1), n(2), \ldots, n(\gamma)$ by reverse induction as follows. Given $n, \alpha \in \mathbb{N}$ let $p(n, \alpha)$ be the number guaranteed by Lemma 2.14 for α, k, and n. Let $n(\gamma) = 1$ and for $i \in \{1, 2, \ldots, \gamma - 1\}$, let $n(\gamma - i) = p(n(\gamma - i + 1), i + 1)$. For $i \in \{1, 2, \ldots, n(1)\}$ let $S_{0,i} = \{\gamma + i\}$. Pick pairwise disjoint nonempty subsets $H_{1,0}$, $H_{1,1}, \ldots, H_{1,n(2)}$ of $\{1, 2, \ldots, n(1)\}$ and $\ell(1) \in \bigtimes_{t \in H_{1,0}} Q_\gamma$ as guaranteed by Lemma 2.14. Define $x(1) = \sum_{t \in H_{1,0}} \ell(1)_t \cdot \sum_{m \in S_{0,t}} y_m$ and for $j \in \{1, 2, \ldots, n(2)\}$, let $S_{1,j} = \bigcup_{t \in H_{1,j}} S_{0,t}$. Then for any z in the Q_γ-span of $(S_{1,1}, S_{1,2}, \ldots, S_{1,n(2)})$ over $\langle y_m \rangle_{m=1}^\infty$ we have that $\varphi(x(1)) = \varphi(x(1) + z)$.

Inductively, given $r \in \{2, 3, \ldots, \gamma - 1\}$ and pairwise disjoint $S_{r-1,1}$, $S_{r-1,2}, \ldots$, $S_{r-1,n(r)}$, recall that $n(r) = p(n(r+1), \gamma - r + 1)$ and so we may pick pairwise disjoint nonempty subsets $H_{r,0}$, $H_{r,1}, \ldots, H_{r,n(r+1)}$ and $\ell(r) \in \times_{t \in H_{r,0}} Q_{\gamma-r+1}$ as guaranteed by Lemma 2.14. Let $x(r) = \sum_{t \in H_{r,0}} \ell(r)_t \cdot \sum_{m \in S_{r-1,t}} y_m$ and for $j \in \{1, 2, \ldots, n(r+1)\}$ let $S_{r,j} = \bigcup_{t \in H_{r,i}} S_{r-1,t}$. Then for any z in the $Q_{\gamma-r+1}$-span of $(S_{r,1}, S_{r,2}, \ldots, S_{r,n(r+1)})$ over $\langle y_m \rangle_{m=1}^\infty$ one has $\varphi(x(r)) = \varphi(x(r) + z)$.

Finally we have $S_{\gamma-1,1}$. Let $H_{\gamma,0} = \{1\}$. Pick $\ell(\gamma) \in Q_1$, and let

$$x(\gamma) = \ell(\gamma) \cdot \sum_{m \in S_{\gamma-1,1}} y_m = \sum_{t \in H_{\gamma,0}} \ell(\gamma) \cdot \sum_{m \in S_{\gamma-1,t}} y_m.$$

Pick by the pigeon-hole principle some $\delta(1) < \delta(2) < \ldots < \delta(v)$ such that $\varphi(x(\delta(1))) = \varphi(x(\delta(2))) = \ldots = \varphi(x(\delta(v)))$ and let $c = \varphi(x(\delta(1)))$.

For $r \in \{1, 2, \ldots, v\}$, let $\alpha(r) = \gamma - \delta(r) + 1$, so that $\ell(\delta(r)) \in \times_{t \in H_{\delta(r),0}} Q_{\alpha(r)}$. Then if $r < v$ and z is in the $Q_{\alpha(r)}$-span of $(S_{\delta(r),1}, S_{\delta(r),2}, \ldots, S_{\delta(r),n(\delta(r)+1)})$ over $\langle y_m \rangle_{m=1}^\infty$ one has $\varphi(x(\delta(r))) = \varphi(x(\delta(r)) + z) = c$.

Now given $r \in \{2, 3, \ldots, v\}$ and $t \in H_{\delta(r),0}$ we have $\ell(\delta(r))_t \in Q_{\alpha(r)}$. Since $\langle Q_m \rangle_{m=1}^\infty$ is a C-useful sequence, we may pick some $\vec{w}_{r,t} \in (Q_{\alpha(r)+1} \cup \{0\})^v$ such that $w_{r,t}(r) = \ell(\delta(r))_t$ and $w_{r,t}(r+1) = w_{r,t}(r+2) = \ldots = w_{r,t}(v) = 0$ and all entries of $C\vec{w}_{r,t}$ are in $Q_{\alpha(r)+1} \cup \{0\}$.

Now let $z_v = x(\delta(v))$, and for $j \in \{1, 2, \ldots, v-1\}$ let

$$z_j = x(\delta(j)) + \sum_{r=j+1}^{v} \sum_{t \in H_{\delta(r),0}} w_{r,t}(j) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m.$$

To complete the proof we show that for each entry μ of $C\vec{z}$ one has $\varphi(\mu) = c$. To this end let $(a_1, a_2, \ldots, a_v)$ be a row of C and let $\mu = \sum_{j=1}^{v} a_j \cdot z_j$. Let i be the coordinate of the first nonzero entry of $(a_1, a_2, \ldots, a_v)$, so that $a_i = 1$. Thus $\mu = z_i + \sum_{j=i+1}^{v} a_j \cdot z_j$.

If $i = v$, then $\mu = z_v = x(\delta(v))$ so $\delta(\mu) = c$ as required. Thus we assume that $i < v$. Then

$$\mu = x(\delta(i)) + \sum_{r=i+1}^{v} \sum_{t \in H_{\delta(r),0}} w_{t,t}(i) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m + \sum_{j=i+1}^{v} a_j \cdot z_j.$$

So, putting

$$\tau = \sum_{r=i+1}^{v} \sum_{t \in H_{\delta(r),0}} w_{r,t}(i) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m + \sum_{j=i+1}^{v} a_j \cdot z_j,$$

it suffices to show that either τ belongs to the $Q_{\alpha(i)}$-span of $(S_{\delta(i),1}, S_{\delta(i),2}, \ldots, S_{\delta(i),n(\delta(i)+1)})$ over $\langle y_m \rangle_{m=1}^{\infty}$ or else $\tau = 0$. Now, we have

$$\sum_{j=i+1}^{v} a_j \cdot z_j$$

$$= a_v \cdot x(\delta(v)) + \textstyle\sum_{j=i+1}^{v-1} a_j \cdot (x(\delta(j)) + \sum_{r=j+1}^{v} \sum_{t \in H_{\delta(r),0}} w_{r,t}(j) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m)$$

$$= \textstyle\sum_{r=i+1}^{v} a_r \cdot x(\delta(r)) + \sum_{j=i+1}^{v} \sum_{r=j+1}^{v} \sum_{t \in H_{\delta(r),0}} a_j \cdot w_{r,t}(j) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m$$

$$\begin{aligned} &= \textstyle\sum_{r=i+1}^{v} \sum_{t \in H_{\delta(r),0}} a_r \cdot \ell(\delta(r))_t \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m + \\ &\quad \textstyle\sum_{r=i+2}^{v} \sum_{j=i+1}^{r-1} \sum_{t \in H_{\delta(r),0}} a_j \cdot w_{r,t}(j) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m \\ &= \textstyle\sum_{r=i+1}^{v} \sum_{t \in H_{\delta(r),0}} a_r \cdot w_{r,t}(r) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m + \\ &\quad \textstyle\sum_{r=i+2}^{v} \sum_{j=i+1}^{r-1} \sum_{t \in H_{\delta(r),0}} a_j \cdot w_{r,t}(j) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m \\ &= \textstyle\sum_{t \in H_{\delta(i+1),0}} a_{i+1} \cdot w_{i+1,t}(i+1) \cdot \sum_{m \in S_{\delta(i+1)-1,t}} y_m + \\ &\quad \textstyle\sum_{r=i+2}^{v} \sum_{t \in H_{\delta(r),0}} \sum_{j=i+1}^{r} a_j \cdot w_{r,t}(j) \cdot \sum_{m + S_{\delta(r)-1,t}} y_m. \end{aligned}$$

Therefore

$$\begin{aligned} \tau &= \textstyle\sum_{r=i+1}^{v} \sum_{t \in H_{\delta(r),0}} w_{r,t}(i) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m + \\ &\quad \textstyle\sum_{t \in H_{\delta(i+1),0}} a_{i+1} \cdot w_{i+1,t}(i+1) \cdot \sum_{m \in S_{\delta(i+1)-1,t}} y_m + \\ &\quad \textstyle\sum_{r=i+2}^{v} \sum_{t \in H_{\delta(r),0}} \sum_{j=i+1}^{r} a_j \cdot w_{r,t}(j) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m \\ &= \textstyle\sum_{r=i+2}^{v} \sum_{t \in H_{\delta(r),0}} \sum_{j=i}^{r} a_j \cdot w_{r,t}(j) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m + \\ &\quad \textstyle\sum_{t \in H_{\delta(i+1),0}} (w_{i+1,t}(i) + a_{i+1} \cdot w_{i+1,t}(i+1)) \cdot \sum_{m \in S_{\delta(i+1)-1,t}} y_m. \end{aligned}$$

Now observe that if $j > r$ then $w_{r,t}(j) = 0$ and if $j < i$ then $a_j = 0$ so

$$w_{i+1,t}(i) + a_{i+1} \cdot w_{i+1,t}(i+1) = \sum_{j=1}^{v} a_j \cdot w_{i+1,t}(i)$$

and for $r \geq i+2$ we have

$$\sum_{j=i}^{r} a_j \cdot w_{r,t}(j) = \sum_{j=1}^{v} a_j \cdot w_{r,t}(j).$$

Consequently, we have

$$\tau = \sum_{r=i+1}^{v} \sum_{t \in H_{\delta(r),0}} (\sum_{j=1}^{v} a_j \cdot w_{r,t}(j)) \cdot \sum_{m \in S_{\delta(r)-1,t}} y_m.$$

Now we are given that for $r \geq i+1$ we have

$$\sum_{j=1}^{v} a_j \cdot w_{r,t}(j) \in Q_{\alpha(r)+1} \cup \{0\} \subseteq Q_{\alpha(i)} \cup \{0\},$$

so we have $\tau = 0$ or τ is in $Q_{\alpha(i)}$-span of $(S_{\delta(i),1}, S_{\delta(i),2}, ..., S_{\delta(i),n(\delta(i)+1)})$, provided we can show that given (r_1, t_1) and (r_2, t_2) with $r_1, r_2 \in \{i+1, i+2, \ldots, v\}$ and $t \in H_{\delta(r_1),0}$ and $t_2 \in H_{\delta(r_2),0}$, if $(r_1, t_1) \neq (r_2, t_2)$ then $S_{\delta(r_i)-1,t_1} \cap S_{\delta(r_2)-1,t_2} = \emptyset$. If $r_1 = r_2$ this is immediate so assume $r_1 < r_2$. Then $S_{\delta(r_2)-1,t_2}$ is a union of sets of the form $S_{\delta(r_1)-1,s}$ with $s \in H_{r_1,q}$ for some $q > 0$ while $t_1 \in H_{r_1,0}$. $\square$

3. Solving systems of n equations for n $\geq$ 2. We show in this section that given $n \geq 2$, there is a set $E \subseteq \mathbb{N}$ which is partition regular for every partition regular system of n equations, but fails to have any solution to a specified system of $n+1$ equations, namely the equations describing a length $n+2$ arithmetic progression and its increment.

We show first that we can reduce the problem to one involving monic first entries matrices. The proof is a minor variation of the standard method of converting kernel partition regular matrices to image partition regular ones. (Recall that a matrix is image partition regular if whenever $\mathbb{N}$ is finitely colored there is some $\vec{x}$ with the entries of $A\vec{x}$ monochrome.)

3.1. Lemma. *Let $n \in \mathbb{N}$. Let a partition regular system of n homogeneous linear equations with rational coefficients be given. Then there is a monic first entries matrix C with at most n rows having more than one nonzero entry, such that, for any $y \in \mathbb{N}^m$, the entries of $C\vec{y}$ contain a solution to the given system.*

Proof. Let A be the $n \times u$ coefficient matrix of the given system. Then A satisfies the columns condition so assume the columns $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_u$ of A have been ordered as required by the columns condition. We assume no $\vec{c}_i = \vec{0}$. Pick $v \in \mathbb{N}$ and $k_1, k_2, \ldots, k_v$ with $1 < k_1 < \ldots < k_v = u$ so that $\sum_{i=1}^{k_1} \vec{c}_i = \vec{0}$ and for $t \in \{2, 3, \ldots, v\}$, $\sum_{i=k_{t-1}+1}^{k_t} \vec{c}_i$ is a linear combination of $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{k_{t-1}}\}$ over $\mathbb{Q}$.

Let $I = \{1\} \cup \{i \in \{2, 3, \ldots, u\} : \vec{c}_i \notin \mathrm{Span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{i-1}\}\}$ and note that since rank $A \leq n$ we have $|I| \leq n$. Given $t \in \{2, 3, \ldots, v\}$, pick $\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{k_{t-1},t}$ in $\mathbb{Q}$ such that $\sum_{i=k_{t-1}+1}^{k_t} \vec{c}_i = \sum_{i=1}^{k_{t-1}} \alpha_{i,t} \cdot \vec{c}_i$. By elementary linear algebra we may assume that if $i \notin I$, then $\alpha_{i,t} = 0$. Let $k_0 = 0$. Now define the $u \times v$ matrix C by, for $i \in \{1, 2, \ldots, u\}$ and $t \in \{1, 2, \ldots, v\}$,

$$c_{i,t} = \begin{cases} \alpha_{i,t} & \text{if } t > 1 \text{ and } i \leq k_{t-1} \\ 1 & \text{if } k_{t-1} < i \leq k_t \\ 0 & \text{if } i > k_t \end{cases} .$$

Then C is a monic first entries matrix whose only rows with more than one nonzero entries are labelled by members of I.

Further, given any $y_1, y_2, \ldots, y_v$ in $\mathbb{N}$ and $i \in \{1, 2, \ldots, u\}$, let $x_i = \sum_{t=1}^{v} c_{i,t} \cdot y_t$. (So $\vec{x} = C\vec{y}$.) Then one can routinely verify that $A\vec{x} = \vec{0}$. (See [5, pp. 79 - 80].) $\square$

Now we use a strategy similar to that used in Section 2. In outline, here is what we will do. Given any monic first entries matrix C with at most n rows with more than one nonzero entry, we produce a C-useful sequence $\langle Q_m \rangle_{m=1}^{\infty}$ such that for no m does $Q_m \cup \{0\}$ contain a solution to the system

$$\begin{aligned} x_3 &= x_1 + x_2 \\ x_4 &= x_1 + 2 \cdot x_2 \\ &\;\;\vdots \\ x_{n+3} &= x_1 + (n+1) \cdot x_2 \end{aligned}$$

with $x_2 \neq 0$. Then using Lemma 2.6 we produce a sequence $\langle y_m \rangle_{m=1}^{\infty}$ so that the existence of a solution to this system in $D = D(\langle Q_m \rangle_{m=1}^{\infty}, \langle y_m \rangle_{m=1}^{\infty})$ would imply the existence of a solution in some $Q_m \cup \{0\}$ with $x_2 \neq 0$. Consequently D is partition regular for C (by Theorem 2.5) but has no solution to the specified system of $n+1$ equations. Finally we piece the solutions together as in the proof of Theorem 2.11.

The construction of the sequence $\langle Q_m \rangle_{m=1}^{\infty}$ is similar to the corresponding construction in Section 2.

3.2 Definition. Let C be a $u \times v$ monic first entries matrix.

(a) Given $\ell \in \{1, 2, \ldots, u\}$, $\mu(\ell) = \min\{j \in \{1, 2, \ldots, v\} : c_{\ell,j} \neq 0\}$.

(b) Given $n \in \mathbb{N}$, C is n-sparse provided that for each $i \in \{1, 2, \ldots, v\}$ we have

$$|\{\ell \in \{1, 2, \ldots, u\} : \mu(\ell) = i \text{ and row } \ell \text{ of } C \text{ has more than one nonzero entry }\}| \leq n.$$

The reader might wonder why we define n-sparse the way we do, since the matrices we are considering in this section all have the stronger property that at most n rows have more than one non-zero entry. It is true that, for this section, we do not need the generality of n-sparse matrices as defined above. However, this generality will yield an enormous payoff when we get to Section 4.

3.3. Lemma. *Let $n \in \mathbb{N}$ and let C be a $u \times v$ n-sparse monic first entries matrix. Then there is a C-useful sequence $\langle Q_m \rangle_{m=1}^{\infty}$ such that, for each $m \in \mathbb{N}$ and each $r \in Q_m$, we have $|\{s \in Q_m : 1 \leq s/r \leq 2n+2\}| \leq n+1$, and, given $r, s \in Q_m$, if $1 \leq s/r \leq 2n+2$, then $s/r < 2$.*

Proof. We construct $\langle Q_m \rangle_{m=1}^{\infty}$ analogously to the proof of Theorem 2.7 adding new elements as required by the notion of C-useful, at each stage adding things much bigger than we have added already.

We show that given any finite set Q of positive rationals and any $q \in Q$ there is a finite set $F = F(q, Q)$ of positive rationals such that

(a) Given $s \in F$ and $r \in Q$, $s/r > 2n + 2$.

(b) Given $r, s \in F$ with $r < s$, either $s/r > 2n + 2$ or $s/r < 2$.

(c) Given $r \in F$, $|\{s \in F : 1 \leq s/r \leq 2n + 2\}| \leq n + 1$.

(d) For all $i \in \{1, 2, \ldots, v\}$, there exists $\vec{x} \in (Q \cup F \cup \{0\})^v$ such that $x_i = q$ and $x_{i+1} = x_{i+2} = ... = x_v = 0$ and each entry of $C\vec{x}$ is in $Q \cup F \cup \{0\}$.

(When we have shown we can do this we will proceed to the construction of the sequence $\langle Q_m \rangle_{m=1}^{\infty}$.)

We construct an upper triangular $u \times v$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & . & . & . & a_{1,v} \\ 0 & a_{2,2} & . & . & . & a_{2,v} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & a_{v,v} \end{pmatrix}$$

where for each j, $a_{j,j} = q$. The off diagonal assignments proceed from left to right and, within columns, from bottom to top: $a_{1,2}, a_{2,3}, a_{1,3}, a_{3,4}, a_{2,4}, ...$ When we have chosen $a_{i,j}$, we let

$$\begin{aligned} R_{i,j} &= G_{i,j} \cup \{a_{i,j}\} \cup \{\sum_{t=1}^{j} c_{\ell,t} \cdot a_{t,j} : \ell \in \{1, 2, \ldots, u\} \text{ and } \mu(\ell) = i\} \\ &= G_{i,j} \cup \{a_{i,j}\} \cup \{\alpha_{i,j} + \sum_{t=i+1}^{j} c_{\ell,t} \cdot a_{t,j} : \ell \in \{1, 2, \ldots, u\} \text{ and } \mu(\ell) = i\}, \end{aligned}$$

where $G_{i,j}$ is the set of values which have been chosen before step (i,j). (So $G_{1,2}=Q$, $G_{i,j}=R_{1,j-1}$ if $1<i=j-1$, and $G_{i,j}=R_{i+1,j}$ otherwise.) We then let $F=R_{1,v}$.

Note that for any (i,j) one has $|R_{i,j}\backslash G_{i,j}|\leq n+1$. Indeed by assumption there are at most n rows ℓ of C with $\mu(\ell)=i$ having more than one nonzero entry. If there is a row ℓ of C with $\mu(\ell)=i$ having exactly one nonzero entry, $a_{i,j}+\sum_{t=i+1}^{j}c_{\ell,t}\cdot a_{t,j}=a_{i,j}$.

Note also that the manner of the construction guarantees that (d) holds. To see this, let $j\in\{1,2,\ldots,v\}$ and let

$$\vec{x}=\begin{pmatrix} a_{1,j}\\ a_{2,j}\\ \cdot\\ \cdot\\ a_{j,j}\\ 0\\ 0\\ \cdot\\ \cdot\\ 0\end{pmatrix}.$$

Then let a row ℓ be given. If $\mu(\ell)>j$, then $\sum_{t=1}^{v}c_{\ell,t}\cdot x_t=0$. If $\mu(\ell)=j$, then $\sum_{t=1}^{v}c_{\ell,t}\cdot x_t=a_{j,j}=q\in Q$. If $\mu(\ell)=i<j$, then we have

$$\sum_{t=1}^{v}c_{\ell,t}\cdot x_t=a_{i,j}+\sum_{t=i+1}^{j}c_{\ell,t}\cdot a_{t,j}\in R_{i,j}.$$

It thus suffices to show that at step (i,j) we can choose $a_{i,j}\in\mathbb{Q}^+$ so that:
(1) Given $s\in R_{i,j}\backslash G_{i,j}$ and $r\in G_{i,j}$, $s/r>2n+2$; and
(2) Given $r,s\in R_{i,j}\backslash G_{i,j}$ with $r<s$ one has $s/r<2$.
To this end let $b=\max G_{i,j}$ and let

$$d=\max\{|\sum_{t=i+1}^{j}c_{\ell,t}\cdot a_{t,j}|:\ell\in\{1,2,\ldots,u\}\text{ and }\mu(\ell)=i\}.$$

Pick $a_{i,j}\geq\max\{b\cdot(2n+2)+d,\,4d\}$. Then given $s\in R_{i,j}\backslash G_{i,j}$ and $r\in G_{i,j}$ one has $n\leq b$ and $s\geq a_{i,j}-d\geq b\cdot(2n+2)$ so (1) holds. Given $r,s\in R_{i,j}\backslash G_{i,j}$ with $r<s$ one has $r\geq a_{i,j}-d$ and $s\leq a_{i,j}+d$ so (2) holds.

We are now ready to choose our C-useful sequence. Let $Q_1=\{1\}$. Inductively given $Q_m=\{q_1,q_2,\ldots,q_k\}$, let $Q_{m+1,0}=Q_m$ and let $Q_{m+1,1}=Q_m\cup F(q_1,Q_{m+1,0})$ and $Q_{m+1,t+1}=Q_{m+1,t}\cup F(q_{t+1},Q_{m+1,t})$. Let $Q_{m+1}=Q_{m+1,k}$.

By condition (d) the sequence $\langle Q_m\rangle_{m=1}^{\infty}$ is C-useful. Also, given $m\in\mathbb{N}$ and $r,s\in Q_{m+1}$ with $r\leq s$, one has some $t,p\in\{1,2,\ldots,k\}$ with $r\in Q_{m+1,t}$ and $s\in Q_{m+1,p}$.

Immediately $t \leq p$. If $t < p$ then by (a) $s/r > 2n + 2$. Thus $s \in F(q_t, Q_{m+1,t-1})$. So (c) guarantees that $|\{s \in Q_{m+1} : 1 \leq s/r \leq 2n + 2\}| \leq n + 1$. $\Box$

The conditions on ratios given in Lemma 3.3 ensure that the sets $Q_m \cup \{0\}$ do not contain arithmetic progressions of $n + 2$ terms with their common difference.

3.4 Lemma. *Let $n \in \mathbb{N}$ and let C be an n-sparse monic first entries matrix. Then there is a C-useful sequence $\langle Q_m \rangle_{m=1}^{\infty}$ such that for each m, $Q_m \cup \{0\}$ does not contain a solution to the system*

$$\begin{aligned} x_3 &= x_1 + x_2 \\ x_4 &= x_1 + 2 \cdot x_2 \\ &\cdot \\ &\cdot \\ x_{n+3} &= x_1 + (n + 1) \cdot x_2 \end{aligned}$$

with $x_2 \neq 0$.

Proof. Let $\langle Q_m \rangle_{m=1}^{\infty}$ be as guaranteed by Lemma 3.3. Let m be given and suppose $Q_m \cup \{0\}$ contains a solution to the specified system with $x_2 \neq 0$. If we had $x_1 = 0$ we would have $x_4 = 2 \cdot x_2$ so $x_4/x_2 = 2$ contradicting Lemma 3.3. Thus $x_1 > 0$. Now suppose $x_1 > (n + 1) \cdot x_2$. Then given $j \in \{3, 4, \ldots, n + 3\}$ we have

$$x_j = x_1 + (j - 2) \cdot x_2 \leq x_1 + (n + 1) \cdot x_2 < 2 \cdot x_1,$$

so

$$\{x_1, x_3, x_4, \ldots, x_{n+3}\} \subseteq \{s \in Q_m : 1 \leq s/x_1 \leq 2n + 2\},$$

a contradiction to Lemma 3.3. Thus we must have $0 < x_1 \leq (n+1) \cdot x_2$. But then given $j \in \{3, 4, \ldots, n + 3\}$ we have

$$x_j = x_1 + (j - 2) \cdot x_2 \leq 2 \cdot (n + 1) \cdot x_2,$$

so

$$\{x_2, x_3, x_4, \ldots, x_{n+3}\} \subseteq \{s \in Q_m : 1 \leq s/x_2 \leq 2n + 2\},$$

again a contradiction to Lemma 3.3. $\Box$

Analogously to the case of kernel partition regular matrices, we say that a set D is *image partition regular* for a $u \times v$ image partition regular matrix C provided that for any finite coloring of D there is some $\vec{x} \in \mathbb{N}^v$ with the entries of $C\vec{x}$ belonging to D and monochrome.

3.5 Theorem. *Let $n \in \mathbb{N}$ and let C be an n-sparse first entries matrix. Then there is a set D which is image partition regular for C but contains no solution to the system*

$$\begin{aligned} x_3 &= x_1 + x_2 \\ x_4 &= x_1 + 2 \cdot x_2 \\ &\;\cdot \\ &\;\cdot \\ x_{n+3} &= x_1 + (n+1) \cdot x_2 \ . \end{aligned}$$

Proof. Pick $\langle Q_m \rangle_{m=1}^{\infty}$ as guaranteed by Lemma 3.4. Choose a sequence $\langle y_m \rangle_{m=1}^{\infty}$ as in Lemma 2.6 (where $F = \{1, 2, \ldots, n+1\}$). Let $D = D(\langle Q_m \rangle_{m=1}^{\infty}, \langle y_m \rangle_{m=1}^{\infty})$. Then by Theorem 2.5, D is partition regular for C. By Lemmas 2.6 and 3.4, D contains no solution to the specified system. $\square$

We now mimic the proof of Theorem 2.11.

3.6 Theorem. *Let $n \in \mathbb{N}$. There is a set E which is image partition regular for every n-sparse monic first entries matrix but contains no solution to the system*

$$\begin{aligned} x_3 &= x_1 + x_2 \\ x_4 &= x_1 + 2 \cdot x_2 \\ &\;\cdot \\ &\;\cdot \\ x_{n+3} &= x_1 + (n+1) \cdot x_2 \ . \end{aligned}$$

Proof. For each n-sparse monic first entries matrix C there is, by Theorem 3.5, a set which is partition regular for C and contains no solution to the specified system so by compactness there is, for each $k \in \mathbb{N}$, a finite set $H_{C,k}$ which contains no solution to the specified system and is such that whenever $H_{C,k}$ is k-colored, there is some $\vec{x}$ with the entries of $C\vec{x}$ monochrome. (See the discussion of the use of compactness in the proof of Theorem 2.11.)

Let $\langle D_s\rangle_{s=1}^{\infty}$ enumerate $\{H_{C,k} : k \in \mathbb{N}$ and C is an n-sparse monic first entries matrix$\}$. Choose a sequence $\langle r_s\rangle_{s=1}^{\infty}$ as guaranteed by Lemma 2.10 with $a = 1$ and $\ell = n + 2$. Let $E = \bigcup_{s=1}^{\infty} r_s \cdot D_s$. Then as in the proof of Theorem 2.11 we see that E is as required. $\Box$

3.7 Corollary. *Let $n \in \mathbb{N}$. There is a set E which is partition regular for every partition regular system of n linear homogeneous equations but contains no solution to the system*

$$\begin{aligned} x_3 &= x_1 + x_2 \\ x_4 &= x_1 + 2 \cdot x_2 \\ &\cdot \\ &\cdot \\ x_{n+3} &= x_1 + (n+1) \cdot x_2 \ . \end{aligned}$$

Proof. By Lemma 3.1, the set E produced in Theorem 3.6 is as required. $\Box$

We shall see in Theorem 4.2 that in fact Theorem 3.6 implies a rather stronger conclusion.

4. Uniform partition regularity and some subsemigroups of $\beta\mathbb{N}$. The Stone-Čech compactification $\beta\mathbb{N}$ of the discrete set $\mathbb{N}$ supports operations $+$ and $\cdot$ extending ordinary addition and multiplication and making $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$ compact left topological semigroups. These structures, and the interactions of various ideals and subsemigroups of $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$, have had several combinatorial consequences. We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$. (For a general background about $\beta\mathbb{N}$ and its applications, see the survey [8].)

We will see here that our results, specifically Theorem 3.6, have consequences for some special subsemigroups of $\beta\mathbb{N}$.

4.1 Definition. For $n \in \mathbb{N}$, let $G_n = \{p \in \beta\mathbb{N}$: for every $B \in p$ and for every $m \in \mathbb{N}$ and every $n \times m$ kernel partition regular matrix A, there exists $\vec{x} \in B^m$ with $A\vec{x} = \vec{0}\}$

Thus G_n is the set of ultrafilters, every member of which contains solutions to every partition regular system of n equations. Each G_n is a subsemigroup of $(\beta\mathbb{N}, +)$ and a two-sided ideal of $(\beta\mathbb{N}, \cdot)$, and each $G_{n+1} \subseteq G_n$. Our aim in this section is to show that for every n, this inclusion is strict. (We remark that it is not at all obvious that for each n there exists an $m > n$ with $G_m \neq G_n$. Indeed, it is not even obvious that we do not have $G_n = G_1$ for all n.)

We need to strengthen Corollary 3.7 in a "uniform" way, by insisting that, in any finite coloring of E, some class should actually contain solutions to all partition regular systems of n equations. Now it would seem that our methods so far, involving "piecing together" finite sets, are absolutely useless for this, since by their very nature we may need different color classes for different partition regular matrices. However, it is now that our rather general definition of "n-sparse" pays off.

4.2 Theorem. *Let $n \in \mathbb{N}$. There is a set E which contains no solution to the system*

$$\begin{aligned} x_3 &= x_1 + x_2 \\ x_4 &= x_1 + 2 \cdot x_2 \\ &\cdot \\ &\cdot \\ x_{n+3} &= x_1 + (n+1) \cdot x_2, \end{aligned}$$

and whenever E is partitioned into finitely many cells, one of these cells contains a solution to every partition regular system of n homogeneous linear equations.

Proof. Pick E as guaranteed by Theorem 3.6. Suppose the conclusion fails and let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a partition of E no cell of which contains solutions to every partition regular system of n equations. For each $i \in \{1, 2, \ldots, k\}$ pick A_i, a $n \times m_i$ kernel partition regular matrix such that for no $\vec{x} \in {F_i}^{m_i}$ is $A_i \vec{x} = \vec{0}$. Pick by Lemma 3.1 an n-sparse monic $u_i \times v_i$ first entries matrix C_i such that for any $\vec{y} \in \mathbb{N}^{v_i}$, the entries of $C_i \vec{y}$ contain a solution to $A_i \vec{x} = \vec{0}$. Let C be the diagonal sum of $C_1, C_2, \ldots, C_k$. That is,

$$C = \begin{pmatrix} C_1 & \mathbf{O} & . & . & . & \mathbf{O} \\ \mathbf{O} & C_2 & . & . & . & \mathbf{O} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mathbf{O} & \mathbf{O} & . & . & . & C_k \end{pmatrix}.$$

Let $u = \sum_{i=1}^{k} u_i$ and $v = \sum_{i=1}^{k} v_i$. Then C is a $u \times v$ monic first entries matrix. Further, a brief consideration shows that C is n-sparse. Pick some i and some $\vec{z} \in \mathbb{N}^{v_i}$ such that all entries of $C\vec{z}$ are in F_i. Then there is some $\vec{y} \in \mathbb{N}^{v_i}$ such that all entries of $C_i \vec{y}$ are in F_i and hence F_i contains a solution to $A_i \vec{x} = \vec{0}$, a contradiction. $[]$

We wish to remark that it is very fortunate that the diagonal sums constructed above are indeed n-sparse.

4.3 Corollary. *For each $n \in \mathbb{N}$, the set G_{n+1} is strictly contained in G_n.*

Proof. Let E be as guaranteed by Theorem 4.2, and let $C = \{A \subseteq E : A$ contains a solution to every partition regular system of n homogeneous linear equations$\}$. Now, by Theorem 4.2, whenever E is partitioned into finitely many cells, one of these contains a member of C. So by [5, Theorem 6.2.3], there is an ultrafilter q on E, every member of which contains a member of C. Let $p = \{A \subseteq \mathbb{N} : A \cap E \in q\}$. Then p is an ultrafilter on $\mathbb{N}$, and $E \in p$. Thus $p \in G_n \backslash G_{n+1}$. $\Box$

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