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Nonconstant Monochromatic Solutions to Systems of Linear Equations

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Abstract. The systems of linear equations (homogeneous or inhomogeneous) that are partition regular, over $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{Q}$, were characterized by Rado. Our aim here is to characterize those systems for which we can guarantee a nonconstant, or injective, solution. It turns out that we thereby recover an equivalence between $\mathbb{N}$ and $\mathbb{Z}$ that is normally lost when one passes from homogeneous to inhomogeneous systems.

1. Introduction

We say that a $u \times v$ matrix A, with entries from $\mathbb{Q}$, is *partition regular* (or *kernel partition regular*) if whenever the positive integers $\mathbb{N}$ are finitely colored there is a vector $x \in \mathbb{N}^v$ that is monochromatic (meaning that all its entries are from the same color class) such that $Ax = 0$. We may also speak of the 'system of equations $Ax = 0$' being partition regular. Many of the classical results of Ramsey Theory may be interpreted as statements that particular matrices are partition regular. For example, Schur's Theorem [9], that whenever $\mathbb{N}$ is finitely colored there exist x, y, z of the same color with $x+y=z$, is precisely the assertion that the 1×3 matrix $(\,1 \quad 1 \quad -1\,)$ is partition regular.

The partition regular matrices were characterized by Rado in the 1930s [7]. To give the characterization, we need to introduce another definition. Let the matrix A have columns $c_1, c_2, \ldots, c_v$. Then we say that A has the *columns property* if there is a partition of $\{1, 2, \ldots, v\}$ as $I_1 \cup I_2 \cup \ldots \cup I_m$ (some $m \geq 1$) such that

(1) $\sum_{i \in I_1} c_i = 0$; and

(2) for each $t > 1$, $\sum_{i \in I_t} c_i$ is a (rational) linear combination of $\{c_i : i \in I_1 \cup \ldots \cup I_{t-1}\}$.

Note that the columns property can be checked in finite time. Rado showed that a matrix is partition regular if and only if it has the columns property. (Although this

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paper is self-contained, the reader who wishes for background information may see [3] or [4].)

What happens over different spaces? We say that A is *partition regular over* $\mathbb{Z}$ (respectively $\mathbb{Q}$) if whenever $\mathbb{Z} \setminus \{0\}$ (respectively $\mathbb{Q} \setminus \{0\}$) is finitely colored there is a monochromatic vector x with $Ax = 0$. Then trivially if A is partition regular over $\mathbb{N}$, it is partition regular over $\mathbb{Z}$, and in fact the converse holds as well: indeed, if we have a bad k-coloring of $\mathbb{N}$ (meaning a coloring with k colors such that there is no monochromatic x with $Ax = 0$), then we may extend this to $\mathbb{Z}$ by coloring $-\mathbb{N}$ the same way as $\mathbb{N}$, but with k new colors – it is easy to check that this is a bad $2k$-coloring of $\mathbb{Z}$. It also turns out that partition regularity over $\mathbb{Z}$ and $\mathbb{Q}$ coincide, by a simple compactness argument (see [3] for details).

Rado went on to consider *inhomogeneous* linear equations. Let A be a $u \times v$ matrix, and let $b \in \mathbb{Q}^u$. Then we say that the system of equations $Ax = b$ is *partition regular over* S (where S is one of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$) if whenever $S \setminus \{0\}$ is finitely colored there is a monochromatic vector x with $Ax = b$. In this inhomogenous setup, partition regularity over $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are definitely not the same. For example, the system $x + y + z = -6$ is partition regular over $\mathbb{Z}$ (just take $x = y = z = -2$) but not partition regular over $\mathbb{N}$.

Rado's characterization of partition regularity in the inhomogeneous case is as follows. If S is $\mathbb{Z}$ or $\mathbb{Q}$, then the system $Ax = b$ (with $b \neq 0$) is partition regular over S if and only if there is a constant solution. More precisely, writing $\overline{c}$ for the vector of the appropriate size all of whose coordinates are c, $Ax = b$ (with $b \neq 0$) is partition regular over S if and only if there exists $d \in S \setminus \{0\}$ such that $A\overline{d} = b$. In a sense, this is saying that if an inhomogeneous system is partition regular over $\mathbb{Z}$ or $\mathbb{Q}$ then it is partition regular for a trivial reason. Rado also showed that, over $\mathbb{N}$, the situation is 'halfway in between': the system $Ax = b$ ($b \neq 0$) is partition regular if and only if *either* there is a $d \in \mathbb{N}$ with $A\overline{d} = b$ *or* A has the columns property and there is a $d \in \mathbb{Z}$ with $A\overline{d} = b$. (These results are in [7] and [8]. To be precise, the cases of $\mathbb{N}$ and $\mathbb{Z}$ are in [7], while the case of $\mathbb{Q}$, although not appearing explicitly, may easily be obtained from results in [8].)

Our main aim in this paper is to consider what happens when we restrict our attention to *nonconstant* solutions ($Ax = b$ with x not a constant vector). There are two natural reasons for wanting to consider this question. Our first reason is that in fact some statements only appear artificially as partition regularity statements. Consider for example van der Waerden's Theorem [10], which says that whenever $\mathbb{N}$ is finitely colored there exist arbitrarily long monochromatic arithmetic progressions. A natural statement of the length 5 instance of this theorem is that the equations

$$x_3 - x_2 = x_2 - x_1$$
$$x_4 - x_3 = x_3 - x_2$$
$$x_5 - x_4 = x_4 - x_3$$

have a monochromatic solution *which is not constant*. The matrix corresponding to this system of equations is

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

which satisfies the columns condition with $m = 1$, so Rado's Theorem only guarantees a constant solution to this system. (It can be made to guarantee a nonconstant solution by adding the equation $x_5 - x_4 = x_6$, that is by requiring that the increment also be the same color as the terms of the progression.) So, apart from the strengthening to insist that the increment is the same color, the natural way to have van der Waerden's Theorem as a partition regularity statement would be to introduce 'nonconstant' as an extra condition.

Our second reason concerns the inhomogeneous results over $\mathbb{Z}$ and $\mathbb{Q}$. The fact that the only way for a system $Ax = b$ to be partition regular is for there to be trivial (constant) solutions suggests that one is not asking the right question. Removing the constant solutions stops this particular phenomenon (and, as we shall see, gives a much richer structure to the characterization).

One rather unexpected consequence of restricting to nonconstant solutions is that it turns out that, even in the inhomogeneous case, partition regularity over $\mathbb{N}$ and $\mathbb{Z}$ now coincide. This will follow from our characterizations. Curiously, this rather pleasant feature seems not to have a direct proof. It seems remarkable that the equivalence could not be 'trivially obvious', given that it is true, but we have been unable to find a direct argument.

Instead of asking for nonconstant monochromatic solutions, one could also ask whether there are injective monochromatic solutions. (In a sense, this is what one really wants in the case of van der Waerden's Theorem. However, any nonconstant solution of those equations is automatically injective.) More generally, one can ask that certain specific coordinates in a monochromatic solution be distinct. We give characterizations here as well.

The plan of the paper is as follows. In Section 2 we present some preliminary results, including a proof of the case $S = \mathbb{Q}$ of the result of Rado on partition regularity of $Ax = b$ over $\mathbb{Q}$ – we include this for completeness, and to make the paper more readable. Then Section 3 contains our main results characterizing the existence of monchromatic nonconstant or injective solutions to $Ax = 0$ and $Ax = b$ in $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$.

2. Preliminaries

We begin by presenting a proof of the characterization of partition regularity of the system $Ax = b$ (where $b \neq 0$) over $\mathbb{Q}$. The proof is based on [4, Lemma 22 and Corollary 24, pp. 87-88] and [8, Lemma 4].

2.1 Lemma. *Let $v \in \mathbb{N}$. There is a coloring χ of $\mathbb{R}$ in $2v$ colors such that there do not exist $\langle x_j \rangle_{j=1}^v$ and $\langle y_j \rangle_{j=1}^v$ in $\mathbb{R}$ with $\chi(x_j) = \chi(y_j)$ for each $j \in \{1, 2, \ldots, v\}$ and $\sum_{j=1}^v (x_j - y_j) = 1$.*

Proof. Define $\chi : \mathbb{R} \to \{0, 1, \ldots, 2v-1\}$ by $\chi(x) = i$ if and only if there is some $m \in \mathbb{Z}$ such that $2m + \dfrac{i}{v} \le x < 2m + \dfrac{i+1}{v}$. Suppose one has $\langle x_j \rangle_{j=1}^v$ and $\langle y_j \rangle_{j=1}^v$ in $\mathbb{R}$ with $\chi(x_j) = \chi(y_j)$ for each $j \in \{1, 2, \ldots, v\}$ and $\sum_{j=1}^v (x_j - y_j) = 1$. Then given j one has some $m_j \in \mathbb{Z}$ such that $2m_j - \frac{1}{v} < x_j - y_j < 2m_j + \frac{1}{v}$. Let $n = \sum_{j=1}^v m_j$. Then $2n - 1 < 1 < 2n + 1$, a contradiction. $\square$

2.2 Lemma. *Assume that the equation $\sum_{j=1}^v c_j x_j = b$ is partition regular over $\mathbb{Q}$ where $b \in \mathbb{Q}$ and each $c_j \in \mathbb{Q}$. Then there is some $d \in \mathbb{Q}$ such that $d \sum_{j=1}^v c_j = b$. In particular, if $\sum_{j=1}^v c_j = 0$, then $b = 0$.*

Proof. If $\sum_{j=1}^v c_j \neq 0$, let $d = \dfrac{b}{\sum_{j=1}^v c_j}$. So assume that $\sum_{j=1}^v c_j = 0$ and suppose that $b \neq 0$. Define a coloring χ^* of $\mathbb{Q}$ by $\chi^*(x) = \chi^*(y)$ if and only if for each $j \in \{1, 2, \ldots, v\}$, $\chi(\frac{c_j x}{b}) = \chi(\frac{c_j y}{b})$, where χ is as guaranteed by Lemma 2.1 for $v - 1$. Pick monochrome $x_1, x_2, \ldots, x_v$ such that $\sum_{j=1}^v c_j x_j = b$. Then $\sum_{j=2}^v \left(\dfrac{c_j x_j}{b} - \dfrac{c_j x_1}{b} \right) = 1$, contradicting Lemma 2.1. $\square$

2.3 Lemma. *Assume that A is a $2 \times v$ matrix with entries from $\mathbb{Q}$, $b \in \mathbb{Q}^2$, and the equation $Ax = b$ is partition regular over $\mathbb{Q}$. Then for any choice of $t_1, t_2 \in \mathbb{Q}$, there is some $d \in \mathbb{Q}$ such that $d(t_1 \sum_{j=1}^v a_{1,j} + t_2 \sum_{j=1}^v a_{2,j}) = t_1 b_1 + t_2 b_2$.*

Proof. Let $c_j = t_1 a_{1,j} + t_2 a_{2,j}$ and let $b = t_1 b_1 + t_2 b_2$. We claim that the equation $\sum_{j=1}^v c_j x_j = b$ is partition regular over $\mathbb{Q}$. So let $\mathbb{Q}$ be finitely colored and pick monochrome x such that $Ax = b$. Then $\sum_{j=1}^v c_j x_j = t_1 \sum_{j=1}^v a_{1,j} x_j + t_2 \sum_{j=1}^v a_{2,j} x_j = t_1 b_1 + t_2 b_2 = b$. Pick d as guaranteed by Lemma 2.2. $\square$

2.4 Lemma. *Assume that A is a $2 \times v$ matrix with entries from $\mathbb{Q}$, $b \in \mathbb{Q}^2$, the equation $Ax = b$ is partition regular over $\mathbb{Q}$, $s_1 = \sum_{j=1}^{v} a_{1,j} \neq 0$, and $s_2 = \sum_{j=1}^{v} a_{2,j} \neq 0$. Then $\dfrac{b_1}{s_1} = \dfrac{b_2}{s_2}$.*

Proof. By Lemma 2.3, if $s_1 t_1 + s_2 t_2 = 0$, then $b_1 t_1 + b_2 t_2 = 0$ so the system

$$s_1 t_1 + s_2 t_2 = 0$$

$$b_1 t_1 + b_2 t_2 = 1$$

is not solvable so $\begin{vmatrix} s_1 & s_2 \\ b_1 & b_2 \end{vmatrix} = 0$. Thus $\dfrac{b_1}{s_1} = \dfrac{b_2}{s_2}$ as required. $\square$

2.5 Theorem (Rado). *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, let $b \in \mathbb{Q}^u \setminus \{0\}$. The system $Ax = b$ is partition regular over $\mathbb{Q}$ if and only if there exists $d \in \mathbb{Q} \setminus \{0\}$ such that $A\overline{d} = b$.*

Proof. Given any $i \in \{1, 2, \ldots, u\}$ if $\sum_{j=1}^{v} a_{i,j} = 0$, then by Lemma 2, $b_i = 0$ so any choice of d will work for that row. If for all $i \in \{1, 2, \ldots, u\}$, $\sum_{j=1}^{v} a_{i,j} = 0$, then we are done. So assume we have some $i \in \{1, 2, \ldots, u\}$ such that $\sum_{j=1}^{v} a_{i,j} \neq 0$ and let $d = \dfrac{b_i}{\sum_{j=1}^{v} a_{i,j}}$. By Lemma 4, $A\overline{d} = b$. $\square$

We shall use the notion of a *first entries matrix*, a notion based on the *mpc-sets* introduced by Deuber in [2]. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is a *first entries matrix* if and only if no row of A is 0 and whenever $i, j \in \{1, 2, \ldots, u\}$ and $k = \min\{t \in \{1, 2, \ldots, v\} : a_{i,t} \neq 0\} = \min\{t \in \{1, 2, \ldots, v\} : a_{j,t} \neq 0\}$, then $a_{i,k} = a_{j,k} > 0$.

2.6 Lemma (Deuber). *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ first entries matrix. Whenever $\mathbb{N}$ is finitely colored, there exists $x \in \mathbb{N}^v$ such that all entries of Ax are monochrome.*

Proof. This is essentially in [2]. The proof may also be found in [6, Theorem 15.24]. $\square$

A matrix satisfying the conclusion of Lemma 2.6 is said to be *image partition regular.*

2.7 Lemma. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ first entries matrix. Whenever $\mathbb{N}$ is finitely colored, there exists $x \in \mathbb{N}^v$ such that all entries of Ax are monochrome and entries of Ax corresponding to unequal rows of A are distinct.*

Proof. By Lemma 2.6 A is image partition regular, so by [5, Theorem 2.10] the conclusion holds. (Statement (n) of [5, Theorem 2.10] refers to finding the image in a given *central* set. One only needs to know that given any finite partition of $\mathbb{N}$, one cell must be central.) $\square$

3. Nonconstant Monochromatic Solutions

In this section we determine precisely those systems of homogeneous and those systems of inhomogeneous solutions which always have nonconstant or injective solutions whenever $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ are finitely colored. Our characterizations typically state that a condition that is clearly necessary is in fact also sufficient. For example, for a matrix A to be nonconstant partition regular (over $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$) we certainly require that A has the columns property and also that there is some nonconstant linear dependence among the columns of A, and statement (e) of Theorem 3.2 asserts that this condition is also sufficient.

A key idea in the proofs will be the general Ramsey philosophy of 'if something can be forced, then it can be forced in a monochromatic way'. Thus for example if we wish to find solutions in which two particular variables x and y are distinct, we do not find such solutions directly, but rather we introduce a new variable z and a new equation $x + z = y$, so that in any solution of the new system we must have $x \neq y$.

3.1 Theorem. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $F \subseteq \{1, 2, \ldots, v\}$. The following statements are equivalent.*

(a) Whenever $\mathbb{N}$ is finitely colored there exists monochromatic $x \in \mathbb{N}^v$ such that $Ax = 0$ and $x_i \neq x_j$ whenever i and j are distinct members of F.

(b) Whenever $\mathbb{Z} \setminus \{0\}$ is finitely colored there exists monochromatic $x \in \mathbb{Z}^v$ such that $Ax = 0$ and $x_i \neq x_j$ whenever i and j are distinct members of F.

(c) Whenever $\mathbb{Q} \setminus \{0\}$ is finitely colored there exists monochromatic $x \in \mathbb{Q}^v$ such that $Ax = 0$ and $x_i \neq x_j$ whenever i and j are distinct members of F.

(d) The matrix A satisfies the columns condition and there exists $x \in \mathbb{Q}^v$ such that $Ax = 0$ and $x_i \neq x_j$ whenever i and j are distinct members of F.

Proof. That (a) implies (b) and (b) implies (c) is trivial. That (c) implies (d) follows immediately from Rado's Theorem.

To see that (d) implies (a) pick m, $\langle I_j \rangle_{j=1}^m$ and $\langle \delta_{i,t} \rangle_{i \in J_t}$ for $t \in \{2, 3, \ldots, m\}$ such that (as guaranteed by the columns condition) we have

(1) $\{I_1, I_2, \ldots, I_m\}$ is a partition of $\{1, 2, \ldots, v\}$;

(2) $\sum_{i \in I_1} c_i = 0$; and

(3) if $m > 1$ and $t \in \{2, 3, \ldots, m\}$, then $\sum_{i \in I_t} c_i = \sum_{i \in J_t} \delta_{i,t} \cdot c_i$, where $J_t = \bigcup_{j=1}^{t-1} I_j$.

Define a $v \times m$ matrix B by, for $i \in \{1, 2, \ldots, v\}$ and $j \in \{1, 2, \ldots, m\}$,

$$b_{i,j} = \begin{cases} 1 & \text{if } i \in I_j \\ -\delta_{i,j} & \text{if } i \in J_j \\ 0 & \text{otherwise.} \end{cases}$$

Then B is a first entries matrix and $AB = 0$. Pick $y \in \mathbb{Q}^v$ such that $Ay = 0$ and $y_i \neq y_j$ whenever i and j are distinct members of F. Let C be the $v \times (m+1)$ matrix whose first m columns are the columns of B and whose final column is y. Then C is a first entries matrix and the rows of C corresponding to members of F are distinct. Let $\mathbb{N}$ be finitely colored and pick by Lemma 2.7 some $x \in \mathbb{N}^{m+1}$ such that all entries of Cx are monochrome and entries of Cx corresponding to unequal rows of C are distinct. Let $z = Cx$. Then $Az = ACx = Ox = 0$. □

Notice that statement (d) of Theorem 3.1 is a computable condition. Notice also that by taking $F = \{1, 2, \ldots, v\}$ in Theorem 3.1 one has a characterization of matrices that are injectively kernel partition regular over $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$.

Observe that the single equation $a_1x_1 + a_2x_2 = 0$ is partition regular if and only if $a_1 = -a_2$, in which case there are no nonconstant solutions (unless $a_1 = a_2 = 0$). On the other hand Theorem 3.1 tells us that if $n > 3$, $a_1, a_2, \ldots, a_n \in \mathbb{Q} \setminus \{0\}$, and the equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$ is partition regular, then it is injectively partition regular.

We have the following characterization of nonconstant partition regularity which includes a second (much more easily) computable condition.

3.2 Theorem. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) Whenever $\mathbb{N}$ is finitely colored there exists monochromatic nonconstant $x \in \mathbb{N}^v$ such that $Ax = 0$.

(b) Whenever $\mathbb{Z}$ is finitely colored there exists monochromatic nonconstant $x \in \mathbb{Z}^v$ such that $Ax = 0$.

(c) Whenever $\mathbb{Q}$ is finitely colored there exists monochromatic nonconstant $x \in \mathbb{Q}^v$ such that $Ax = 0$.

(d) The matrix A satisfies the columns condition and there exists nonconstant $x \in \mathbb{Q}^v$ such that $Ax = 0$.

(e) The matrix A satisfies the columns condition and if the sum of the columns of A is 0, then there exists nonempty $D \subsetneq \{1, 2, \ldots, v\}$ and for each $j \in D$ there exists

$\alpha_j \in \mathbb{Q} \setminus \{0\}$ such that $\sum_{j\in D} \alpha_j c_j = 0$, where c_j is column j of A.

Proof. As in the proof of Theorem 3.1 we have that (a) implies (b), (b) implies (c), and (c) implies (d).

To see that (d) implies (e), assume that $\sum_{j=1}^{v} c_j = 0$ and pick nonconstant $x \in \mathbb{Q}^v$ such that $\sum_{j=1}^{v} x_j c_j = 0$. Then $\sum_{j=2}^{v}(x_j - x_1)c_j = 0$. Let $D = \{j \in \{2, 3, \ldots, v\} : x_j \neq x_1\}$ and for $j \in D$ let $\alpha_j = x_j - x_1$.

To see that (e) implies (a) let m, $\langle I_j \rangle_{j=1}^{m}$, and B be as in the proof that (d) implies (a) in Theorem 3.1. If $m > 1$, pick $i \in I_1$ and $t \in I_2$, note that rows i and t of B are unequal, and let $C = B$. If $m = 1$, then pick nonempty $D \subsetneq \{1, 2, \ldots, v\}$ and for each $j \in D$ pick $\alpha_j \in \mathbb{Q} \setminus \{0\}$ such that $\sum_{j\in D} \alpha_j c_j = 0$. Define $y \in \mathbb{Q}^v$ by

$$y_j = \begin{cases} \alpha_j & \text{if } j \in D \\ 0 & \text{if } j \in \{1, 2, \ldots, v\} \setminus D \end{cases}$$

and let C be the single column of B followed by y. Given $i \in D$ and $t \in \{1, 2, \ldots, v\} \setminus D$, rows i and t of C are unequal.

In either case C is a first entries matrix with two unequal rows such that $AC = 0$. Let $\mathbb{N}$ be finitely colored and pick by Lemma 2.7 some $x \in \mathbb{N}^{m+1}$ such that all entries of Cx are monochrome and entries of Cx corresponding to unequal rows of C are distinct. Let $z = Cx$. Then $Az = ACx = 0x = 0$. $\square$

We now turn our attention to nonconstant monochromatic solutions to inhomogeneous systems of linear equations.

3.3 Theorem. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, let $b \in \mathbb{Q}^u \setminus \{ 0 \}$, and let $F \subseteq \{1, 2, \ldots, v\}$ with $|F| \geq 2$. If $S = \mathbb{Z}$ or $S = \mathbb{Q}$, then the following statements are equivalent.*

(a) Whenever $S \setminus \{0\}$ is finitely colored there exists monochromatic $x \in S^v$ such that $Ax = b$ and $x_i \neq x_j$ whenever i and j are distinct members of F.

(b) There exists $d \in S \setminus \{0\}$ such that $A\overline{d} = b$, A satisfies the columns condition, and there exists $x \in \mathbb{Q}^v$ such that $Ax = b$ and $x_i \neq x_j$ whenever i and j are distinct members of F.

Proof. To see that (a) implies (b), note that we may pick $d \in S \setminus \{0\}$ such that $A\overline{d} = b$ by Theorem 1.4 and trivially $x \in \mathbb{Q}^v$ exists as required. So it suffices to show that the system $Ax = 0$ is partition regular over S. So let $r \in \mathbb{N}$ and let $\varphi : S \setminus \{0\} \to \{1, 2, \ldots, r\}$.

Define $\psi : S \setminus \{0\} \to \{1, 2, \ldots, r+1\}$ by

$$\psi(x) = \begin{cases} \varphi(x-d) & \text{if } x \neq d \\ r+1 & \text{if } x = d\,. \end{cases}$$

Pick $x \in S^v$ such that x is monochromatic with respect to ψ, $Ax = b$, and $x_i \neq x_j$ whenever i and j are distinct members of F. Since $|F| \geq 2$, the constant value of $\psi(x_i)$ cannot be $r+1$. Let $y = x - \overline{d}$. Then y is monochromatic with respect to φ, $y_i \neq y_j$ whenever i and j are distinct members of F, and $Ay = Ax - A\overline{d} = 0$.

To see that (b) implies (a), pick $d \in S \setminus \{0\}$ such that $A\overline{d} = b$ and pick $x \in \mathbb{Q}^v$ such that $Ax = b$ and $x_i \neq x_j$ whenever i and j are distinct members of F. Let $r \in \mathbb{N}$ and let $\varphi : S \setminus \{0\} \to \{1, 2, \ldots, r\}$. Define $\psi : S \setminus \{0\} \to \{1, 2, \ldots, r+1\}$ by

$$\psi(y) = \begin{cases} \varphi(y+d) & \text{if } y \neq -d \\ r+1 & \text{if } y = -d\,. \end{cases}$$

Now $x - \overline{d} \in \mathbb{Q}^v$ and $A(x - \overline{d}) = 0$ so by Theorem 3.1 we may pick $z \in S^v$ such that z is monochromatic with respect to ψ, $Az = 0$, and $z_i \neq z_j$ when i and j are distinct members of F. Since $|F| \geq 2$, the constant value of $\psi(z_i)$ cannot be $r+1$. Let $y = z + \overline{d}$. Then y is monochromatic with respect to φ, $y_i \neq y_j$ whenever i and j are distinct members of F, and $Ay = Az + A\overline{d} = b$. $\square$

Notice that Theorem 3.3 tells us that the single equation $2x_1 - 2x_2 + 2x_3 = 1$ is nonconstantly partition regular over $\mathbb{Q}$ but not over $\mathbb{Z}$. On the other hand the next theorem tells us that nonconstant partition regularity over $\mathbb{Z}$ is equivalent to nonconstant partition regularity over $\mathbb{N}$. As we stated earlier, we have been unable to find a trivial proof of this equivalence.

3.4 Theorem. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, let $b \in \mathbb{Q}^u \setminus \{0\}$, and let $F \subseteq \{1, 2, \ldots, v\}$ with $|F| \geq 2$. The following statements are equivalent.*

(a) Whenever $\mathbb{N}$ is finitely colored there exists monochromatic $x \in \mathbb{N}^v$ such that $Ax = b$ and $x_i \neq x_j$ whenever i and j are distinct members of F.

(b) Whenever $\mathbb{Z} \setminus \{0\}$ is finitely colored there exists monochromatic $x \in \mathbb{Z}^v$ such that $Ax = b$ and $x_i \neq x_j$ whenever i and j are distinct members of F.

(c) There exists $d \in \mathbb{Z} \setminus \{0\}$ such that $A\overline{d} = b$, A satisfies the columns condition, and there exists $x \in \mathbb{Q}^v$ such that $Ax = b$ and $x_i \neq x_j$ whenever i and j are distinct members of F.

Proof. Trivially (a) implies (b) and (b) implies (c) by Theorem 3.3. To see that (c) implies (a) pick $d \in \mathbb{Z} \setminus \{0\}$ such that $A\overline{d} = b$ and pick $x \in \mathbb{Q}^v$ such that $Ax = b$ and

$x_i \neq x_j$ whenever i and j are distinct members of F. Let $r \in \mathbb{N}$ and let $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$. If $d > 0$, define $\psi : \mathbb{N} \to \{1, 2, \ldots, r\}$ by $\psi(y) = \varphi(y + d)$. If $d < 0$, define $\psi : \mathbb{N} \to \{1, 2, \ldots, r - d\}$ by

$$\psi(y) = \begin{cases} \varphi(y + d) & \text{if } y > -d \\ r + y & \text{if } y \leq -d\,. \end{cases}$$

Now $x - \overline{d} \in \mathbb{Q}^v$ and $A(x - \overline{d}) = 0$ so by Theorem 3.1 we may pick $z \in \mathbb{N}^v$ such that z is monochromatic with respect to ψ, $Az = 0$, and $z_i \neq z_j$ when i and j are distinct members of F. Since $|F| \geq 2$, the constant value of $\psi(z_i)$ cannot be $r+t$ for any $t \leq -d$. Let $y = z + \overline{d}$. Then y is monochromatic with respect to φ, $y_i \neq y_j$ whenever i and j are distinct members of F, and $Ay = Az + A\overline{d} = b$. $\square$

We close by remarking that it would be very nice to find a direct short proof for the fact proved above that the notions of nonconstant partition regularity for $Ax = b$ over $\mathbb{N}$ and $\mathbb{Z}$ are the same.

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