Sets and mappings in βS which are not Borel

Neil Hindman and Dona Strauss

Abstract. We extend theorems proved in [4] by showing that, if S is a countably infinite right cancellative and weakly left cancellative discrete semigroup, then the following subsets of βS are not Borel: the set of idempotents, the smallest ideal, any semiprincipal right ideal defined by an element of S^*, and S^*S^*. This has the imediate corollary that, if S is any infinite right cancellative and weakly left cancellative semigroup, the set of idempotents in βS is not Borel. We extend a theorem proved in [1], which states that for any infinite discrete group G and any $p \in G^*$, $\lambda_p : \beta G \to \beta G$ is not Borel, by showing that this theorem holds for all infinite semigroups which are right cancellative and very weakly left cancellative. We show that continuous maps between compact spaces map Baire sets to universally measurable sets, although this is far from being the case for Borel sets.

Contents

1. Introduction

Let $(S, \cdot)$ be a discrete semigroup. We take the Stone-Čech compactification βS of S to be the set of ultrafilters on S with the points of S identified with the principal ultrafilters. Given $A \subseteq S$, we let $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets of βS as well as a basis for the closed sets. The operation on S extends to βS so that the function ρ_p defined by $\rho_p(x) = x \cdot p$ is continuous for each $p \in \beta S$. Furthermore, S is contained in the topological center of βS, meaning that the function λ_y defined by $\lambda_y(x) = y \cdot x$ is continuous for each $y \in S$. Given $p \in \beta S$ and an indexed family $\langle x_s \rangle_{s \in S}$ and a point y in a topological space X, $p\text{–}\lim_{s \in S} x_s = y$

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if and only if for every neighborhood U of y, $\{s \in S : x_s \in U\} \in p$. If X is compact and Hausdorff, then $p\text{–}\lim_{s\in S} x_s$ is guaranteed to exist uniquely. For $p, q \in \beta S$, $pq = p\text{–}\lim_{s\in S} q\text{–}\lim_{t\in S} st$. For $A \subseteq S$, $A \in pq$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$ where $s^{-1}A = \{t \in S : st \in A\}$.

If $A \subseteq S$, A^* will denote $c\ell_{\beta S}(A) \setminus A$. We write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.

Every compact Hausdorff right topological semigroup T has important algebraic properties, including the fact that it has at least one idempotent. If V is a subset of T, $E(V)$ will denote the set of idempotents in V. T has a smallest two sided ideal, $K(T)$, which is the union of all of the minimal right ideals and the union of all of the minimal left ideals of T. Every right ideal of T contains a minimal right ideal, and every left ideal of T contains a minimal left ideal. The intersection of a minimal right ideal and a minimal left ideal is a group; and all the subgroups of T which arise in this way are algebraically isomorphic and are homeomorphic if they lie in the same minimal right ideal. See [3, Part I] for the facts mentioned here, and any other unfamiliar assertions encountered. We remark that the maximal groups in $K(T)$ need not be homeomorphic in general. In fact, if S is an infinite cancellative and commutative semigroup, then by [3, Lemma 6.40 and Theorem 7.42] the maximal groups contained in any minimal left ideal of βS lie in $2^{\mathfrak{c}}$ homeomorphism classes.

We shall use $\mathbb{N}$ to denote the set of positive integers, ω to denote the set of non-negative integers, $\mathbb{Z}$ to denote the set of all integers and $\mathbb{R}$ to denote the set of real numbers. $\mathbb{H}$ will denote $\bigcap_{n\in\mathbb{N}} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$. This is a subsemigroup of $\beta\mathbb{N}$ which contains all the idempotents.

Anyone who has worked with $\beta\mathbb{N}$, will not be surprised to learn that some of the algebraically defined subsets of $\beta\mathbb{N}$ are not topologically simple, even though they are very simple to define algebraically. It was shown in [4] that the following subsets of $\beta\mathbb{N}$ are not Borel: the set of idempotents; any semiprincipal right ideal of $\mathbb{N}^*$; the smallest ideal of $\beta\mathbb{N}$; the set of idempotents in any left ideal of $\beta\mathbb{N}$; $\mathbb{N}^* + \mathbb{N}^*$; and $\mathbb{H} + \mathbb{H}$. These results were extended to infinite countable semigroups which can be algebraically embedded in compact Hausdorff topological groups.

A subset X of a semigroup S is a *left solution set* if and only if there exist $a, b \in S$ such that $X = \{x \in S : ax = b\}$. A semigroup S is *weakly left cancellative* provided that all left solution sets in S are finite. If $|S| = \kappa \geq \omega$, then S is *very weakly left cancellative* provided the union of any set of fewer than κ left solution sets has cardinality less than κ.

In Section 2 in the present paper, we extend some of the results of [4] by showing that, if S is any infinite countable right cancellative and weakly left cancellative semigroup, then the following subsets of βS are not Borel: the set of idempotents; any semiprincipal right ideal of $S*$; the smallest ideal of βS; S^*S^*. As an immediate corollary, we obtain the result that, if S is an arbitrary infinite right cancellative and weakly left cancellative semigroup, then the set of idempotents in βS is not Borel.

In Section 3 we extend a theorem due to E. Glasner [1] by showing that, if S is an arbitrary infinite cancellative semigroup and if $p \in S^*$, then the map $\lambda_p : \beta S \to \beta S$ is not Borel. E. Glasner proved this theorem in the case in which S is a group, and the methods that we use are based on his.

In Section 4 we discuss continuous images of Borel sets. An elegant example, due to D. Fremlin, shows that continuous functions from $\beta\mathbb{N}$ to metric spaces, need not map Borel sets to universally measurable sets. However, any continuous function from a compact Hausdorff space to a compact Hausdorff space, does map Baire sets to universally measurable sets.

2. Subsets of βS which are not Borel

Throughout this section we will let S be a countably infinite right cancellative and weakly left cancellative discrete semigroup. We will prove that the following subsets of βS are not Borel: the set of idempotents; the smallest ideal; any semiprincipal right ideal defined by an element of S^*; and S^*S^*. The proof is based on the following lemma.

Lemma 2.1. *Every Borel subset of βS is the union of a family of compact subsets of βS of cardinality at most $\mathfrak{c}$.*

Proof. The proof is identical to the proof of [4, Lemma 3.1], where it was stated for $\beta\mathbb{N}$. □

Definition 2.2. We enumerate S as a sequence and write $s \prec t$ if s precedes t in this sequence.

Lemma 2.3. *There is a sequence $\langle s_n\rangle_{n=1}^{\infty}$ in S such that for each $n \in \mathbb{N}$,*

1. *$s_n \prec s_{n+1}$;*
2. *if $a, b \preceq s_n$, then $ab \prec s_{n+1}$; and*
3. *if $a \preceq s_n$ and $ab \preceq s_n$, then $b \prec s_{n+1}$.*

Proof. We construct $\langle s_n\rangle_{n=1}^{\infty}$ inductively. One can do this because, given n, $\{s_n\} \cup \{ab : a, b \preceq s_n\}$ is finite and since S is weakly left cancellative, given $a, c \preceq s_n$, $\{b \in S : ab = c\}$ is finite. □

We define $\phi : S \to \mathbb{N}$ by $\phi(t) = \min\{n \in \mathbb{N} : t \preceq s_n\}$. Then ϕ extends to a continuous mapping from βS to $\beta\mathbb{N}$, which we shall also denote by ϕ.

We claim that, for every $a, b \in S$ and every $n > 2 \in \mathbb{N}$, if $a \preceq s_{n-2}$ and $s_{n-1} \prec b \preceq s_n$, then $s_{n-2} \prec ab \prec s_{n+1}$ and hence that $\phi(ab) \in \phi(b) + \{-1, 0, 1\}$. By condition (2) we have directly that $ab \prec s_{n+1}$. Suppose that $ab \preceq s_{n-2}$. Then by condition (3) with n replaced by $n-2$, we have $b \prec s_{n-1}$, a contradiction. So we have for every $a \in S$ and all sufficiently large $b \in S$, $\phi(ab) \in \{\phi(b) - 1, \phi(b), \phi(b) + 1\}$ If $x, y \in S^*$, $\phi(xy) = x\text{-}\lim_{a\in S} y\text{-}\lim_{b\in S} \phi(ab)$. Therefore $\phi(xy) \in \{\phi(y) - 1, \phi(y), \phi(y) + 1\}$.

We put $P = \{s_n : n \in \mathbb{N}\}$. We observe that $\phi(s_n) = n$ for every $n \in \mathbb{N}$, and so $\phi[P^*] = \mathbb{N}^*$ and hence $|\phi[P^*]| = 2^{\mathfrak{c}}$.

Lemma 2.4. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S^* on which ϕ is injective. Then $c\ell\{x_n : n \in \mathbb{N}\}$ meets $S^* \setminus (S^*S^*)$.*

Proof. We may suppose that $\{\phi(x_n) : n \in \mathbb{N}\}$ is discrete, because any infinite subset of a Hausdorff space has an infinite (strongly) discrete subset.

We claim that ϕ is injective on $c\ell\{x_n : n \in \mathbb{N}\}$. To see this, suppose that p and q are distinct elements of $c\ell\{x_n : n \in \mathbb{N}\}$ and $\phi(p) = \phi(q)$. Pick $A \in p$ and $B \in q$ such that $A \cap B = \emptyset$. Then $\phi(p) \in c\ell(\{\phi(x_n) : x_n \in \overline{A}\})$ and $\phi(q) \in c\ell(\{\phi(x_n) : x_n \in \overline{B}\})$. So, by [3, Theorem 3.40], there exists $m \in \mathbb{N}$ such that $\phi(x_m) \in c\ell(\{\phi(x_n) : n \in \mathbb{N} \setminus \{m\}\})$ – contradicting the assumption that $\{\phi(x_n) : n \in \mathbb{N}\}$ is discrete.

Let x be a point of accumulation of $\langle x_n \rangle_{n=1}^{\infty}$. We claim that $x \notin S^*S^*$. To see this suppose, on the contrary, that $x = yz$ for some $y, z \in S^*$. We have observed that ϕ assumes at most three values on $\beta S z$. So, if $M = \{n \in \mathbb{N} : \phi(x_n) \notin \phi[\beta S z]\}$, then $x \in c\ell\{x_n : n \in M\}$. Also, for every $a \in S$, $x \in c\ell\{bz : b \in S, a \prec b\}$. It follows from [3, Theorem 3.40] that $x_n \in \beta S z$ for some $n \in M$, or for each $a \in S$, there exists $v_a \in c\ell\{x_n : n \in \mathbb{N}\}$ and $b_a \in S$ such that $a \prec b$ and $v_a = b_a z$. The first possibility contradicts the definition of M, and so the second possibility must hold for each $a \in S$. This implies that $\phi(v_a) \in \phi(z) + \{-1, 0, 1\}$. Since ϕ is injective on $c\ell\{x_n : n \in \mathbb{N}\}$, it follows that $|\{v_a : a \in S\}| \leq 3$. However, $\{b_a : a \in S\}$ is infinite. This is a contradiction, because the map $s \mapsto sz$, which maps b_a to v_a, is injective by [3, Lemma 6.28(i)] because S is right cancellative. So this establishes our claim. □

Corollary 2.5. *On any Borel subset B of S^*S^*, ϕ assumes at most $\mathfrak{c}$ values.*

Proof. If C is any compact subset of S^*S^*, $\phi[C]$ is finite so Lemma 2.1 applies. □

Theorem 2.6. *The following subsets of βS are not Borel: the set of idempotents; the smallest ideal; S^*S^*; and any principal right ideal defined by an element of S^*.*

Proof. We shall show that ϕ assumes $2^{\mathfrak{c}}$ values on the intersection of each of these sets with S^*S^*. This will be sufficient, because $E(\beta S) \cap S^* = E(\beta S) \cap S^*S^*$, $K(\beta S) \subseteq S^*S^*$ and, for any $q \in S^*$, $q\beta S \setminus S^*S^*$ is countable. So, if any of these sets were Borel, their intersections with S^*S^* would also be Borel. We define an equivalence relation $\equiv$ on βS by stating that $x \equiv y$ if $\phi(x) \in \mathbb{Z} + \phi(y)$. Then the elements of P^* belong to $2^{\mathfrak{c}}$ distinct equivalence classes. For every $p \in P^*$, there is an idempotent e_p in the left ideal $\beta S p$ of βS. Since $\phi(e_p) \in \phi(p) + \{-1, 0, 1\}$, $e_p \equiv p$. So the elements of $E(\beta S)$ belong to $2^{\mathfrak{c}}$ distinct equivalence classes, and hence $|\phi(E(\beta S) \cap S^*S^*)| = 2^{\mathfrak{c}}$. Similarly, each left ideal $\beta S p$ meets $K(\beta S)$. So $K(\beta S)$ is a subset of S^*S^* on which ϕ assumes $2^{\mathfrak{c}}$ values. Finally, let $q \in S^*$. Since $qP^* \subseteq S^*S^*$ and

ϕ assumes $2^{\mathfrak{c}}$ distinct values on qP^*, S^*S^* is not Borel. Similarly, because $qP^* \subseteq q\beta S$, $q\beta S$ is not Borel. □

Corollary 2.7. *Let R be an arbitrary infinite right cancellative and weakly left cancellative semigroup. Then the set of idempotents in βR is not Borel.*

Proof. R contains a countably infinite semigroup T and $c\ell_{\beta R}(T)$ is a compact subsemigroup of βR which is a copy of βT. If $E(\beta R)$ were a Borel subset of βR, $E(\beta R) \cap (\overline{T}) = E(\overline{T})$ would be a Borel subset of βT, contradicting Theorem 2.6. □

Note that the hypothesis of right cancellative and weakly left cancelative in Theorem 2.6 and Corollary 2.7 cannot be weakened to left cancellative or right cancellative, as shown by the examples of right zero and left zero semigroups. Nor can it be weakened to weakly right cancellative and weakly left cancellative as shown by the example $(\mathbb{N}, \vee)$, where $x \vee y = \max\{x, y\}$. If S denotes one of these semigroups, then every element of βS is idempotent.

We remark that the results of Theorem 2.6 are stronger than the statement that the sets considered are not Borel, because they show that they cannot be expressed as the union of $\mathfrak{c}$ or fewer compact subsets. The set of subsets of βS which can be expressed as the union of $\mathfrak{c}$ or fewer compact subsets, is strictly larger than the set of Borel subsets. It contains the analytic subsets of βS, if these are defined as the set of subsets of βS which can be obtained from the Borel sets by applying operation (A). (For a definition of this operation, see, for example, [5, Chapter II, Section 5].)

As shown in the proof of [4, Lemma 3.1], if X is an arbitrary compact Hausdorff space of weight at most $\mathfrak{c}$, the family $\sigma(X)$ of subsets A of X for which A and $X \setminus A$ are unions of $\mathfrak{c}$ or fewer compact subsets, is a σ-algebra which contains the Borel subsets of X. We claim that, if X and Y are compact Hausdorff spaces of weight at most $\mathfrak{c}$ and if $f : X \to Y$ is a continuous open mapping, then $f[\sigma(X)] \subseteq \sigma(Y)$. To see this, let $A \in \sigma(X)$. Clearly, $f[A]$ is the union of $\mathfrak{c}$ or fewer compact subsets of Y. A is also the intersection of a family $\mathcal{U}$ of open subsets of X for which $|\mathcal{U}| \leq \mathfrak{c}$. Let $\mathcal{V} = \{f^{-1}\big[f[U]\big] : U \in \mathcal{U}\}$. Then $Y \setminus f[A] = \bigcup\{Y \setminus f[V] : V \in \mathcal{V}\}$. So $Y \setminus f[A]$ is also the union of $\mathfrak{c}$ or fewer compact subsets of Y. In particular, $\pi_1[\sigma(Y \times X)] \subseteq \sigma(Y)$.

We have therefore shown that the subsets of βS discussed above, are not analytic and are not projective.

We are grateful to D. Saveliev for a very helpful correpondence about these concepts.

3. λ_p is not Borel

Throughout this section S will denote an infinite semigroup of cardinality κ which is right cancellative, very weakly left cancellative, and has a designated left identity e. (S may or may not have other left identities.) We remind the reader that a semigroup S of cardinality κ is very weakly left cancellative if the union of fewer than κ left solution sets has cardinality less than κ.

Ω will denote the set ${}^S\{0,1\}$ of functions from S to $\{0,1\}$ with the product topology. We work in the dynamical system $\langle \Omega, T_s\rangle_{s\in S}$ where $T_s : \Omega \to \Omega$ is defined by $T_s(w) = w \circ \rho_s$. That is, for $t \in S$, $T_s(w)(t) = w(ts)$. For $p \in S^*$, $T_p : \Omega \to \Omega$ is defined by $T_p(w) = p\text{-}\lim_{s\in S} T_s(w)$. Note that, given $t \in S$ and $w \in \Omega$, $T_p(w)(t) = \big(p\text{-}\lim_{s\in S} T_s(w)\big)(t) = p\text{-}\lim_{s\in S} w(ts)$.

Ω can be given the structure of a compact topological group by noting that $\Omega = {}^S\mathbb{Z}_2$. We shall use λ to denote normalised Haar measure on Ω, and shall use $\mathfrak{B}_\lambda$ to denote the σ-algebra of subsets of Ω generated by the Borel sets and the λ-null sets.

The following lemma is well known. We include a proof, however, because the proof is short and simple.

Lemma 3.1. *If $p \in S^*$, then $\{1_A : A \in p\}$ is not $\mathfrak{B}_\lambda$-measurable, where $1_A \in \Omega$ is the characteristic function of A.*

Proof. Suppose that $P = \{1_A : A \in p\}$ is λ-measurable. Then $1_S + P = \{1_{S\setminus A} : A \in p\}$ so P and $1_S + P$ are disjoint subsets of $G = {}^S\mathbb{Z}_2$ whose union is all of G. So $\lambda(P) = \lambda(1_S + P) = \frac{1}{2}$. By [2, Theorem A, Chapter 12, Section 61, and Theorem B, Chapter 12, Section 62], $P + P$ contains a neighborhood of 0 in G. So there exists $F \in \mathcal{P}_f(S)$ such that $\bigcap_{x\in F} \pi_x^{-1}[\{0\}] \subseteq P + P = \{1_{A\Delta B} : A, B \in p\}$. But then $1_{S\setminus F} \in P + P$, so $S \setminus F \notin p$. Consequently p must be a principle ultrafilter, a contradiction. $\square$

Lemma 3.2. *We can choose $w_0 \in \Omega$ such that $\{T_s(w_0) : s \in S\}$ is dense in Ω and the function $\psi : \beta S \to \omega$ defined by $\psi(p) = T_p(w_0)$ is a continuous surjection.*

Proof. We enumerate $\mathcal{P}_f(S)$ as a κ-sequence $\langle F_\alpha\rangle_{\alpha<\kappa}$. For $\alpha < \kappa$ let $\tau_\alpha = |F_\alpha|$ and $\delta_\alpha = 2^{\tau_\alpha}$. We note that if $\emptyset \neq A \subseteq S$, $|A| < \kappa$, and $\alpha < \kappa$, then $\{s \in S : F_\alpha s \cap A \neq \emptyset\} = \bigcup_{a\in F_\alpha} \bigcup_{b\in A} \{s \in S : as = b\}$, so is the union of fewer than κ left solution sets and thus, since S is very weakly left cancellative, $|\{s \in S : F_\alpha s \cap A \neq \emptyset\}| < \kappa$. Consequently we may inductively choose $\{s_{\alpha,t} : \alpha < \kappa \text{ and } t \in \{1, 2, \ldots, \delta_\alpha\}\}$ so that $F_\alpha s_{\alpha,t} \cap F_\sigma s_{\sigma,r} = \emptyset$ whenever $\alpha, \sigma < \kappa$, $t \in \{1, 2, \ldots, \delta_\alpha\}$, $r \in \{1, 2, \ldots, \delta_\sigma\}$, and $(\alpha, t) \neq (\sigma, r)$.

For each $\alpha < \kappa$, enumerate the set of functions from F_α to $\{0,1\}$ as $\langle f_{\alpha,t}\rangle_{t=1}^{\delta_\alpha}$. We define $w_0 \in \omega$ on $\bigcup_{\alpha<\kappa} \bigcup_{t=1}^{\delta_\alpha} F_\alpha s_{\alpha,t}$ by, for $a \in F_\alpha$ and $t \in \{1, 2, \ldots, \delta_\alpha\}$, $w_0(as_{\alpha,t}) = f_{\alpha,t}(a)$. (We are using here the fact that S is right cancellative.) Define $w_0(x)$ at will for $x \in S \setminus \bigcup_{\alpha<\kappa} \bigcup_{t=1}^{\delta_\alpha} F_\alpha s_{\alpha,t}$.

To see that $\{T_s(w_0) : s \in S\}$ is dense in Ω, let U be a nonempty basic open set in Ω. Pick $\alpha < \kappa$ and $t \in \{1, 2, \ldots, \delta_\alpha\}$ such that $U = \bigcap_{a\in F_\alpha} \pi_\alpha^{-1}[\{f_{\alpha,t}(a)\}]$. Then for $a \in F_\alpha$, $T_{s_{\alpha,t}}(w_0)(a) = w_0(as_{\alpha,t}) = f_{\alpha,t}(a)$ so $T_{s_{\alpha,t}}(w_0) \in U$.

It is routine to verify that the function ψ is continuous. To see that ψ is a surjection, let $w \in \Omega$. We claim that for each $F \in \mathcal{P}_f(S)$ we can choose $s(F) \in S$ such that for each $a \in F$, $T_{s(F)}(w_0)(a) = w(a)$. To see this, let $F \in \mathcal{P}_f(S)$ be given and pick $\alpha < \kappa$ such that $F = F_\alpha$. Pick $t \in \{1, 2, \ldots, \delta_\alpha\}$ such that $f_{\alpha,t}(a) = w(a)$ for each $a \in F_\alpha$. Then $T_{s_{\alpha,t}}(w_0)(a) = w_0(as_{\alpha,t}) = w(a)$. Let $s(F) = s_{\alpha,t}$.

Direct $\mathcal{P}_f(S)$ by inclusion and let p be a limit point in βS of the net $\langle s(F)\rangle_{F\in\mathcal{P}_f(S)}$. It is routine to verify that $T_p(w_0) = w$. □

Definition 3.3. We fix $w_0 \in \Omega$ and $\psi : \beta S \to \Omega$ as guaranteed by Lemma 3.2.

Definition 3.4. If μ is a probability measure on a compact space X, $\mathfrak{B}_\mu$ will denote the σ-algebra of subsets of X generated by the Borel subsets and the μ-null subsets. We shall say that a subset of X is *universally measurable* if it is a member of $\mathfrak{B}_\mu$ for every probability mesaure μ defined on X.

We remind the reader that a subset A of X is in $\mathfrak{B}_\mu$ if and only if $\sup(\{\mu(C) : C \text{ is compact and } C \subseteq A\}) = \inf(\{\mu(U) : U \subseteq X \text{ is open and } A \subseteq U\})$.

We are grateful to E. Glasner for sending us a proof of the following lemma.

Lemma 3.5. *Let X and Y be compact Hausdorff spaces, and let $f : X \to Y$ be a continuous surjection. Then $f[B]$ is universally measurable for every universally measurable subset B of X for which $B = f^{-1}\big[f[B]\big]$.*

Proof. Let μ be a probability measure on Y. It follows from the Hahn Banach Theorem and the Riesz Representation Theorem, that there is a probability measure ν on X for which $\nu(g \circ f) = \mu(g)$ for every continuous $g : Y \to \mathbb{R}$. Let $\varepsilon > 0$. We can choose a compact subset C of B for which $\nu(C) + \varepsilon > \nu(B)$ and a compact subset D of $X \setminus B$ for which $\nu(D) + \varepsilon > \nu(X \setminus B)$. We can then choose disjoint open subsets U and V of Y such that $f[C] \subseteq U$ and $f[D] \subseteq V$, and $\mu(U) < \mu(f[C]) + \varepsilon$ and $\mu(V) < \mu(f[D]) + \varepsilon$. Let g and h be continuous functions from Y to [0,1] such that $g = 1$ on $f[C]$, $g = 0$ on $Y \setminus U$, $h = 1$ on $f[D]$, and $h = 0$ on $Y \setminus V$. Then $\nu(C) \leq \nu(g \circ f) = \mu(g) \leq \mu(f[C]) + \varepsilon$ and $\nu(D) \leq \nu(h \circ f) = \mu(h) < \mu(f[D]) + \varepsilon$. Now $\nu(C) + \nu(D) > 1 - 2\varepsilon$. So $\mu(f[C]) + \mu(f[D]) > 1 - 4\varepsilon$ and hence $\mu(Y \setminus f[D]) < \mu(f[C]) + 4\varepsilon$. Since $Y \setminus f[D]$ is an open set containing $f[B]$ and $f[C]$ is a compact set contained in $f[B]$, it follows that $f[B] \in \mathfrak{B}_\mu$. □

Definition 3.6. Let $p \in S^*$. We put $\mathcal{Q}_p = \{q \in \beta S : \psi(pq)(e) = 1\}$.

Lemma 3.7. *Let $p \in S^*$. Then $\psi[\mathcal{Q}_p] = \{1_A : A \in p\}$.*

Proof. Let $D=\{s\in S: w_0(s)=1\}$. Note that for any $q\in S^*$,

$$\begin{array}{rcl}\psi(pq)(e)=1 & \Leftrightarrow & (pq)\text{–}\lim\limits_{s\in S} T_s(w_0)(e)=1\\ & \Leftrightarrow & (pq)\text{–}\lim\limits_{s\in S} w_0(s)=1\\ & \Leftrightarrow & \{s\in S: w_0(s)=1\}\in pq\\ & \Leftrightarrow & \{s\in S: s^{-1}D\in q\}\in p\,.\end{array}$$

and

$$\begin{array}{rcl}\{s\in S:\psi(q)(s)=1\} & = & \{s\in S: T_q(w_0)(s)=1\}\\ & = & \{s\in S: q\text{–}\lim\limits_{t\in S} w_0(st)=1\}\\ & = & \{s\in S:\{t\in S: w_0(st)=1\}\in q\}\\ & = & \{s\in S: s^{-1}D\in q\}\,.\end{array}$$

Consequently, if $q\in\mathcal{Q}_p$ and $A=\{s\in S:\psi(q)(s)=1\}$, then $\psi(q)=1_A$ and $A\in p$.

Now assume $A\in p$ and pick, by Lemma 3.2 $q\in\beta S$ such that $\psi(q)=1_A$. Then $A=\{s\in S: s^{-1}D\in q\}\in p$ so $\psi(pq)(e)=1$. □

Theorem 3.8. *For each $p\in S^*$, the mapping $\lambda_p:\beta S\to\beta S$ is not Borel.*

Proof. By Lemmas 3.1, 3.5, and 3.7, $\mathcal{Q}_p$ is not a Borel set. Since $\mathcal{Q}_p=\lambda_p^{-1}[\{x\in\beta S:\psi(x)(e)=1\}]$ and $\{x\in\beta S:\psi(x)(e)=1\}$ is compact, λ_p is not Borel. □

As in Section 2, we remark that the preceding theorem need not hold if we weaken our hypothesis to left cancellativity, right cancellativity or weak cancellativity. If S is a left zero semigroup, a right zero semigroup or $(\mathbb{N},\vee)$, then $\lambda_p:\beta S\to\beta S$ is Borel for every $p\in\beta S$.

4. Images of Borel Sets

In this section we address the question of which compact spaces X and Y have the property that, whenever $f:X\to Y$ is continuous, $f[B]$ is a universally measurable subset of Y whenever B is a Borel subset of X. We remark that X and Y have this property if they are metric spaces. However, the following elegant result, due to D. Fremlin in personal correspondence, shows that this property fails dramatically in the case in which $X=\mathbb{N}^*$.

Theorem 4.1. *Let $f:\mathbb{N}^*\to Y$ be a continuous surjection onto a compact metric space. Then, for every subset E of Y, there is an open subset U of $\mathbb{N}^*$ such that $f[U]=E$.*

Proof. For every $y\in Y$, $f^{-1}[\{y\}]$ is a non-empty G_δ subset of $\mathbb{N}^*$. It therefore contains a non-empty open subset U_y of $\mathbb{N}^*$ by [3, Theorem 3.36]. If $U=\bigcup\{U_y:y\in E\}$, then U is open in $\mathbb{N}^*$ and $f[U]=E$. □

We shall show that continuous mappings between compact Hausdorff do map Baire sets to universally measurable sets, where we define the Baire subsets of a compact Hausdorff space X to be the sets in the smallest σ-algebra of subsets of X containing the compact G_δ subsets of X. (Other definitions exist in the literature.)

Definition 4.2. A *determining system* in a space X is a family $\mathfrak{U}$ of subsets of X indexed by the set of finite sequences of positive integers. The nucleus $N(\mathfrak{U})$ of $\mathfrak{U}$ is $\bigcup\{A_{n_1} \cap A_{n_1 n_2} \cap A_{n_1 n_2 n_3} \ldots : \langle n_i \rangle_{i=1}^{\infty}$ is a sequence in $\mathbb{N}\}$.

We shall call such a system a *compact determing system* if all the sets in the system are compact and $A_{n_1 n_2 \ldots n_k n_{k+1}} \subseteq A_{n_1 n_2 \ldots n_k}$ for all positive integers $n_1, n_2, \ldots, n_{k+1}$.

Determining systems were first defined by Alexandrov in 1916. In any topological space, every determining system of universally measurable sets has a nucleus which is universally measurable by [5, Theorem 5.5].

Lemma 4.3. *The set of nuclei of compact determining systems in a compact Hausdorff space X is closed under countable unions and countable intersections.*

Proof. Suppose that $\mathfrak{U}(m) = \{A_{n_1 n_2 \ldots n_k}(m) : \langle n_i \rangle_{i=1}^{k}$ is a finite sequence in $\mathbb{N}\}$ is a compact determining system for each $m \in \mathbb{N}$. Let $N(m) = N(\mathfrak{U}(m))$ for each $m \in \mathbb{N}$.

Then $\bigcup_{m=1}^{\infty} N(m)$ is the nucleus of the system $\{B_{n_1 n_2 \ldots n_k} : \langle n_i \rangle_{i=1}^{k}$ is a finite sequence in $\mathbb{N}\}$ defined by putting $B_{n_1 n_2 \ldots n_k} = A_{n_2 n_3 \ldots n_k}(n_1)$ if $k > 1$, and $B_n = X$ for every $n \in \mathbb{N}$.

To see that $\bigcap_{m=1}^{\infty} N(m)$ is the nucleus of a compact determining system

$$\{C_{n_1 n_2 \ldots n_k} : \langle n_i \rangle_{i=1}^{k} \text{ is a finite sequence in } \mathbb{N}\},$$

choose a partition of $\mathbb{N}$ into a sequence $\langle E_n \rangle_{n=1}^{\infty}$ of infinite pairwise disjoint subsets. For each finite sequence $\sigma = \langle n_1 n_2 \ldots n_k \rangle$ of positive integers and each $m \in \mathbb{N}$, let σ_m be the subsequence of σ formed by the integers n_i for which $i \in E_m$. Then put $C_\sigma = \bigcap_{m \in \mathbb{N}} A_{\sigma_m}(m)$, with $A_\emptyset(m)$ defined to be X. $\square$

Lemma 4.4. *Let X be a compact Hausdorff space. If B is a compact G_δ subset of X or a σ-compact subset of X, then B is the nucleus of a compact determining system.*

Proof. Note that any compact set C is the nucleus of a compact determining system defined by $A_{n_1 n_2 \ldots n_k} = C$. A compact G_δ is the intersection of a sequence $\langle C_n \rangle_{n=1}^{\infty}$ of compact sets, so the conclusion follows from Lemma 4.3. $\square$

Lemma 4.5. *Let X be a compact Hausdorff space. Every Baire subset of X is the nucleus of a compact determing system.*

Proof. If B is a compact G_δ subset of X, then both B and $X \setminus B$ are nuclei of compact determining systems, because $X \setminus B$ is σ-compact. Since the set of nuclei of compact determining systems is closed under countable unions and intersections, it contains all the Baire subsets of X. $\square$

Theorem 4.6. *Let X and Y be compact Hausdorff spaces and let $f : X \to Y$ be a continuous surjection. If B is a Baire subset of X, then $f[B]$ is a universally measurable subset of Y.*

Proof. B is the nucleus of a compact determining system

$$\{A_{n_1 n_2 \ldots n_k} : \langle n_i \rangle_{i=1}^{k} \text{ is a finite sequence in } \mathbb{N}\}\,,$$

and so $f[B]$ is the nucleus of the compact determining system

$$\{f[A_{n_1 n_2 \ldots n_k}] : \langle n_i \rangle_{i=1}^{k} \text{ is a finite sequence in } \mathbb{N}\}\,,$$

because for every decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of compact subsets of X,

$$f[\textstyle\bigcap_{n=1}^{\infty} C_n] = \bigcap_{n=1}^{\infty} f[C_n]\,.$$

It follows that $f[B]$ is universally measurable by [5, Theorem 5.5]. □

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Department of Mathematics, Howard University, Washington, DC 20059, USA.
`nhindman@aol.com`

Department of Pure Mathematics, University of Leeds, Leeds LS2 9J2, UK.
`d.strauss@hull.ac.uk`