# Image partition regular matrices and concepts of largeness

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Abstract. We show that for several notions of largeness in a semigroup, if $u, v \in \mathbb{N}$, $A$ is a $u \times v$ matrix satisfying restrictions that vary with the notion of largeness, and if $C$ is a large subset of $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is large in $\mathbb{N}^v$. We show that in most cases the restrictions on $A$ are necessary. Several other results, including some generalizations, are also obtained. Included is a simple proof that if $u > 1$, then $\beta(\mathbb{N}^v)$ is not isomorphic to $(\beta\mathbb{N})^u$.

Contents



## 1. Introduction

The starting point of this investigation is the notion of image partition regularity of matrices over the set $\mathbb{N}$ of positive integers.

**Definition 1.1.** Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with rational entries. The matrix $A$ is *image partition regular over* $\mathbb{N}$ (denoted IPR/$\mathbb{N}$) if and only if whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are monochromatic.

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Some of the major old results in Ramsey Theory are naturally represented by image partition regular matrices. For example, van der Waerden's Theorem is the assertion that for any $k \in \mathbb{N}$, the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & k \end{pmatrix}$$

is image partition regular and Schur's Theorem is the assertion that the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is image partition regular.

The first characterizations of matrices that are IPR/$\mathbb{N}$ were obtained in 1993 in [9]. Other characterizations have been obtained over the years. The one of these that most concerns us involves the notion of *central*. Central subsets of $\mathbb{N}$ were introduced by Furstenberg in [5], defined in terms of notions from topological dynamics. Furstenberg proved the original version of the Central Sets Theorem [5, Proposition 8.21] and showed that any central subset of $\mathbb{N}$ contains a kernel of every kernel partition regular matrix. That is, if the $u \times v$ matrix $A$ has the property that whenever $\mathbb{N}$ is finitely colored, there exists a monochromatic $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = \vec{0}$, then every central subset of $\mathbb{N}$ contains all of the entries of such an $\vec{x}$.

We use a different, but equivalent, definition of central set, which we will present in the next section. (The equivalence was established in [17] by H. Shi and H. Yang.)

**Theorem 1.2.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) *$A$ is image partition regular.*
(b) *For every central set $C$ in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$.*
(c) *For every central set $C$ in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$.*

**Proof.** These are the first three statements of [10, Theorem 2.10]. Unfortunately, as published there were some gaps in the proof. See [14, Theorem 15.24] or the version of [10] posted on `nhindman.us/reprint.html` for a complete proof. $\square$

Statement (c) of Theorem 1.2 is an example of a common phenomenon in Ramsey Theory. One wants to know that a set is nonempty, and one shows that in some sense it is large. For example, Furstenberg [4] proved Szemerédi's Theorem, namely that any subset of $\mathbb{N}$ with positive upper density contains arbitrarily long arithmetic progressions, by showing that the set of starting points of a length $k$ arithmetic progression in such a set is large.

Of particular interest to us for this phenomenon is the notion of C-set. We write ${}^{\mathbb{N}}S$ for the set of sequences in $S$ and $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of a set $X$.

**Definition 1.3.** Let $(S,+)$ be a commutative semigroup and let $A \subseteq S$. Then $A$ is a *C-set* in $S$ if and only if there exist functions $\alpha : \mathcal{P}_f({}^{\mathbb{N}}S) \to S$ and $H : \mathcal{P}_f({}^{\mathbb{N}}S) \to \mathcal{P}_f(\mathbb{N})$ such that

(1) if $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and
(2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in A$.

The currently strongest version of the Central Sets Theorem for a commutative semigroup $(S,+)$ is the assertion that any central subset of $S$ is a C-set. Many of the strong properties of central sets are derivable from the fact that they satisfy the Central Sets Theorem. It is natural to ask whether Theorem 1.2 remains true if "central set" is replaced by "C-set".

In fact, in [3, Theorem 1.10], the following theorem was stated without proof.

**Theorem 1.4.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) *$A$ is image partition regular.*
(b) *For every C-set $C$ in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$.*
(c) *For every C-set $C$ in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a C-set in $\mathbb{N}^v$.*

We will provide a proof of Theorem 1.4 in Section 4.

In addition to central sets and C-sets, there are several other notions of size that make sense in any commutative semigroup. We shall define these notions in Section 2, and describe the relationships that hold among them. In Section 3, we will establish several preliminary results. In Section 4 we will prove, if $\Psi$ is any one of seventeen of the notions, a theorem of the following form, where $X$ is $\mathbb{Q}$, $\mathbb{Z}$, $\{x \in \mathbb{Q} : x \geq 0\}$, or $\omega = \mathbb{N} \cup \{0\}$ and $Y$ is "$A$ is IPR/$\mathbb{N}$", "$A$ has no row equal to $\vec{0}$", or "$A$ is IPR/$\mathbb{N}$ and $\operatorname{rank}(A) = u$".

**Theorem 1.5.** *Let $u, v \in \mathbb{N}$, let $A$ be a $u \times v$ matrix with entries from $X$, and assume that $A$ satisfies $Y$. Let $C$ be a subset of $\mathbb{N}$ which is a $\Psi$-set in $\mathbb{N}$. Then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a $\Psi$-set in $\mathbb{N}^v$.*

We shall also show that in each case, the restriction on entries of $A$ is necessary.

Section 5 consists of some more general results, as well as some other observations.

We conclude the introduction with the background expected of the reader. The proofs of results in Sections 2, 3, and 4 rely heavily on results in the book [14], and all of the results needed in these sections can be found in that book. In Section 5 we assume a knowledge of some of the concepts of functional analysis and we use some of the well known theorems of functional analysis, such as the Riesz Representation Theorem and Day's Fixed Point Theorem.

## 2. Definitions

For all but two of the notions that we are studying, we will utilize a characterization in terms of the algebraic structure of the Stone-Čech compactification of a discrete commutative semigroup $(S,+)$. For this paper except for Theorem 5.2, $S$ will always be $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N}^v$, or $\mathbb{Z}^v$ for some $v \in \mathbb{N}$. We give a very brief introduction to this structure now. For a detailed introduction see [14, Part I].

We let $\beta S = \{p : p$ is an ultrafilter on $S\}$, identifying the principal ultrafilters on $S$ with the points of $S$ so that we may assume that $S \subseteq \beta S$. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. While $\overline{A}$ is the closure of $A$ in $\beta S$, more importantly, $\{\overline{A} : A \subseteq S\}$ is a basis for the topology of $\beta S$. The operation $+$ on $S$ extends to an operation, also denoted $+$, on $\beta S$ so that $(\beta S,+)$ is a right topological semigroup with $S$ contained in the topological center of $\beta S$. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q + p$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x+q$ is continuous. Despite the fact that it is denoted by $+$, the operation on $\beta S$ is not commutative. In fact, if $S = \mathbb{N}^v$ or $S = \mathbb{Z}^v$, the topological center of $\beta S$ is equal to $S$; that is, if $p \in S^* = \beta S \setminus S$, then $\lambda_p$ is not continuous. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$.

As does any compact Hausdorff right topological semigroup, $\beta S$ has idempotents and a smallest two sided ideal, denoted $K(\beta S)$, which is the union of all of the minimal left ideals of $\beta S$ and also the union of all of the minimal right ideals of $\beta S$. An idempotent in $\beta S$ is an element of $K(\beta S)$ if and only if it is mimimal with respect to the ordering of idempotents wherein $p \leq q$ if and only if $p + q = q + p = p$. Such idempotents are simply said to be minimal. Minimal left ideals of $\beta S$ are closed. The intersection of any minimal left ideal with any minimal right ideal is a group, and any two such groups are isomorphic. Given a subset $X$ of $\beta S$, we let $E(X) = \{p \in X : p+p = p\}$. We will use the fact that if $L$ is a minimal left ideal of $\beta\mathbb{N}$, then it is also a minimal left ideal of $\beta\mathbb{Z}$.

Given a property $\Psi$ of a subset of $S$, there is a dual property $\Psi^*$ defined as follows. If $A \subseteq S$, then $A$ has property $\Psi^*$ if and only if $A$ has nonempty intersection with any subset $B$ of $S$ which has property $\Psi$. All of the notions we will consider are closed under supersets. In that situation, $A$ has property $\Psi^*$ if and only if $S \setminus A$ does not have property $\Psi$. Further, under the same assumption, property $\Psi$ implies property $\theta$ if and only if property $\theta^*$ implies property $\Psi^*$ and property $\Psi^{**}$ is the same as property $\Psi$.

Two of our basic notions involve the property of positive *Banach density* introduced by Bergelson in [1].

**Definition 2.1.** Let $v \in \mathbb{N}$, let $S = \mathbb{N}^v$ or $S = \mathbb{Z}^v$, and let $A \subseteq S$. Then the *Banach density* of $A$,

$$d(A) = \sup\{\alpha \in [0,1] : (\forall n \in \mathbb{N})(\exists k_1, k_2, \ldots, k_v \in \{m \in \mathbb{N} : m > n\}) \\ (\exists \vec{a} \in S)(|A \cap (\vec{a} + \times_{i=1}^{v}\{0,1,\ldots,k_i - 1\})| \geq \alpha \cdot \textstyle\prod_{i=1}^{v} k_i)\}\,.$$

Note that if $v = 1$, the Banach density of $A$ is commonly denoted $d^*(A)$, reserving the notation $d(A)$ for the asymptotic density of $A$.

In some papers such as [13] we have used the more general notion of *Følner density*, which is also more complicated. It is a recent result of Bergelson and Glasscock [2, Theorem 3.5 and Corollary 3.6] that for subsets of $\mathbb{Z}^v$ or $\mathbb{N}^v$, the Banach density and Følner density are equal.

**Definition 2.2.** Let $v \in \mathbb{N}$ and let $S = \mathbb{N}^v$ or $S = \mathbb{Z}^v$. Then $\Delta(S) = \{p \in \beta S : (\forall A \in p)(d(A) > 0)\}$.

We shall show in Theorem 3.1 that $\Delta(S)$ is a closed two sided ideal of $\beta S$.

As we define the notions, we will frequently give equivalent characterizations. For the proofs (or references to the proofs) see [8].

**Definition 2.3.** Let $v \in \mathbb{N}$, let $S = \mathbb{N}^v$ or $S = \mathbb{Z}^v$, and let $A \subseteq S$.

(1) $A$ is a *Q-set* if and only if there exists a sequence $\langle \vec{x}_n \rangle_{n=1}^{\infty}$ in $S$ such that whenever $m < n$, $\vec{x}_n \in \vec{x}_m + A$.
(2) $A$ is a *P-set* if and only if for each $k \in \mathbb{N}$, there exist $\vec{a}, \vec{d} \in S$ such that $\{\vec{a}, \vec{a} + \vec{d}, \ldots, \vec{a} + k\vec{d}\} \subseteq A$.
(3) $A$ is an *IP-set* if and only if there exists a sequence $\langle \vec{x}_n \rangle_{n=1}^{\infty}$ in $S$ such that $FS(\langle \vec{x}_n \rangle_{n=1}^{\infty}) \subseteq A$, where $FS(\langle \vec{x}_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} \vec{x}_n : F \in \mathcal{P}_f(\mathbb{N})\}$. Equivalently, $A$ is an IP-set if and only if there is an idempotent $p \in \beta S$ such that $A \in p$.
(4) $A$ is a *J-set* if and only if for every $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $\vec{a} \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $\vec{a} + \sum_{n \in H} f(n) \in A$.
(5) $J(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a J-set})\}$.

It is shown in [14, Section 14.4] that $J(S)$ is a two sided ideal of $\beta S$ and that a subset $A$ of $S$ is a C-set if and only if there is an idempotent in $\overline{A} \cap J(S)$. (The proof of Theorem 14.14.4 should be moved to after Lemma 14.14.6, since one needs to know $J(S) \neq \emptyset$.)

**Lemma 2.4.** *Let $v \in \mathbb{N}$, let $S = \mathbb{N}^v$ or $S = \mathbb{Z}^v$, and let $A \subseteq S$. If $d(A) > 0$, then $\overline{A} \cap \Delta(S) \neq \emptyset$.*

**Proof.** It is a routine exercise to establish that if $B$ and $C$ are subsets of $S$, then $d(B \cup C) \leq d(B) + d(C)$. The conclusion is then an immediate consequence of [14, Theorem 3.11]. □

**Definition 2.5.** Let $v \in \mathbb{N}$, let $S = \mathbb{N}^v$ or $S = \mathbb{Z}^v$, and let $A \subseteq S$.

(1) $A$ is a *B-set* if and only if $d(A) > 0$. Equivalently $A$ is a B-set if and only if $\overline{A} \cap \Delta(S) \neq \emptyset$.
(2) $A$ is a D-set if and only if there is an idempotent in $\overline{A} \cap \Delta(S)$.
(3) $A$ is a *PS-set* if and only if there exists $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $\vec{x} \in S$ such that $F + \vec{x} \subseteq \bigcup_{\vec{t} \in G}(-\vec{t} + A)$. Equivalently $A$ is a PS-set if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.
(4) $A$ is a *QC-set* if and only if there is an idempotent in $\overline{A} \cap c\ell K(\beta S)$.
(5) $A$ is *central* if and only if there is an idempotent in $\overline{A} \cap K(\beta S)$.
(6) $A$ is *syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{\vec{t} \in G}(-\vec{t} + A)$. Equivalently $A$ is syndetic if and only if for every left ideal $L$ of $\beta S$, $\overline{A} \cap L \neq \emptyset$.
(7) $A$ is an *SC-set* if and only if for every left ideal $L$ of $\beta S$, there is an idempotent in $\overline{A} \cap L$.
(8) $A$ is *thick* if and only if for every $F \in \mathcal{P}_f(S)$, there exists $\vec{x} \in S$ such that $F + \vec{x} \subseteq A$. Equivalently $A$ is thick if and only if there exists a left ideal $L$ of $\beta S$ such that $L \subseteq \overline{A}$.

The names Q, P, IP, PS, QC, and SC come from quotient, progression, infinite dimensional parallelepiped, piecewise syndetic, quasi central, and strongly central respectively. (If quotient seems confusing, consider that when written multiplicatively, $x_n \in x_m \cdot A$ says that $\frac{x_n}{x_m} \in A$.) The names C, J, B, and D, have no particular significance.

The implications in Figure 1 are established in [8] and examples are given in $S = \mathbb{N}$ showing that the only implications that hold in general are those that follow from the diagram and transitivity.

Recall that a property of subsets of a set $X$ is *partition regular over* $X$ if and only if whenever the union of two subsets of $X$ has that property, at least one of them does.

**Theorem 2.6.** *A property listed in Figure 1 is partition regular over* $\mathbb{N}$ *if and only if it is one of the properties implied by central.*

**Proof.** Each of central, QC, PS, D, C, and IP is determined by membership in an ultrafilter, so is partition regular. As remarked in the proof of Lemma 2.4, it is a routine exercise to establish the partition regularity of B. The partition regularity of J is [14, Lemma 14.14.6]. The partition regularity of P and Q are easy consequences of van der Waerden's Theorem and Ramsey's Theorem for pairs respectively.

If $B = \bigcup_{n=0}^{\infty}\{2^{2n}, 2^{2n}+1, \ldots, 2^{2n+1}-1\}$, then neither $B$ nor $\mathbb{N} \setminus B$ is syndetic, so no property that implies syndetic is partition regular.

Neither $2\mathbb{N}$ nor $2\mathbb{N}-1$ is thick, so no property that implies thick is partition regular. (This fact will also follow from the fact that SC* is not partition regular, but is much simpler.)

Let $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}} 2^n\mathbb{N}$. For $x \in \mathbb{N}$, let $\text{supp}(x) = F$ where $x = \sum_{t \in F} 2^t$. And let $B = \{x \in \mathbb{N} : \min \text{supp}(x) \text{ is even}\}$. By [14, Lemma 6.8] $\overline{B} \cap \mathbb{H}$ and

$\mathbb{H}\setminus\overline{B}$ are right ideals of $\mathbb{H}$. We show now that $B$ and $\mathbb{N}\setminus B$ are both SC-sets, so neither is SC*. So let $L$ be a left ideal of $\beta\mathbb{N}$. Then $L\cap\mathbb{H}$ is a left ideal of $\mathbb{H}$ so $L\cap\mathbb{H}\cap\overline{B}$ contains a subgroup of $\mathbb{H}$ hence has an idempotent. Also $L\cap\mathbb{H}\setminus\overline{B}$ contains a subgroup of $\mathbb{H}$ hence has an idempotent. □

## 3. Preliminary results

The following theorem is known, but the proof used the notion of Følner density. Since it has a simple proof using Banach density, we present it. (The notions are equivalent, but the proof of that fact in [2] is not particularly easy.)

**Theorem 3.1.** *Let $v\in\mathbb{N}$ and let $S=\mathbb{N}^v$ or $S=\mathbb{Z}^v$. Then $\Delta(S)$ is a closed two sided ideal of $\beta S$.*

**Proof.** By Lemma 2.4, $\Delta(S)\neq\emptyset$. From the definition it is immediate that $\Delta(S)$ is closed. Let $p\in\Delta(S)$ and let $q\in\beta S$. To see that $\Delta(S)$ is a left ideal, let $A\in q+p$. Then $\{\vec{x}\in S:-\vec{x}+A\in p\}\in q$ so pick $\vec{x}\in S$ such that $-\vec{x}+A\in p$. Let $0<\alpha<d(-\vec{x}+A)$. To see that $d(A)\geq\alpha$, let $n\in\mathbb{N}$ and pick $k_1,k_2,\ldots,k_v\in\{m\in\mathbb{N}:m>n\}$ and $\vec{a}\in S$ such that $|(-\vec{x}+A)\cap(\vec{a}+\times_{i=1}^{v}\{0,1,\ldots,k_i-1\})|\geq\alpha\cdot\prod_{i=1}^{v}k_i$. Then $|A\cap(\vec{x}+\vec{a}+\times_{i=1}^{v}\{0,1,\ldots,k_i-1\})|\geq\alpha\cdot\prod_{i=1}^{v}k_i$.

To see that $\Delta(S)$ is a right ideal, let $A\in p+q$ and let $B=\{\vec{x}\in S:-\vec{x}+A\in q\}$. Let $0<\alpha<d(B)$. To see that $d(A)\geq\alpha$, let $n\in\mathbb{N}$ and pick $k_1,k_2,\ldots,k_v\in\{m\in\mathbb{N}:m>n\}$ and $\vec{a}\in S$ such that $|B\cap(\vec{a}+\times_{i=1}^{v}\{0,1,\ldots,k_i-1\})|\geq\alpha\cdot\prod_{i=1}^{v}k_i$. Pick $\vec{y}\in\bigcap\{-\vec{x}+A:\vec{x}\in B\cap(\vec{a}+\times_{i=1}^{v}\{0,1,\ldots,k_i-1\})$ Then $|A\cap(\vec{y}+\vec{a}+\times_{i=1}^{v}\{0,1,\ldots,k_i-1\})|\geq\alpha\cdot\prod_{i=1}^{v}k_i$. □

**Lemma 3.2.** *Let $v\in\mathbb{N}$ and for $j\in\{1,2,\ldots,v\}$ let $\widetilde{\pi}_j:\beta(\mathbb{Z}^v)\to\beta\mathbb{Z}$ be the continuous extension of the projection onto the $j^{th}$ coordinate. Let $\Theta=\{p\in\beta(\mathbb{N}^v):(\forall j\in\{1,2,\ldots,v\})(\widetilde{\pi}_j(p)\in\mathbb{N}^*)\}$. Then $\Theta$ is a left ideal of $\beta(\mathbb{Z}^v)$ and a right ideal of $\beta(\mathbb{N}^v)$.*

**Proof.** First let $p\in\Theta$, let $q\in\beta(\mathbb{Z}^v)$, and let $j\in\{1,2,\ldots,v\}$. It is a routine exercise to show that for each $k\in\mathbb{N}$, $\mathbb{N}\setminus\{1,2,\ldots,k\}\in\widetilde{\pi}_j(q+p)$.

Now assume that $p\in\Theta$, $q\in\beta(\mathbb{N}^v)$, and $j\in\{1,2,\ldots,v\}$. Suppose that $\widetilde{\pi}_j(p+q)\notin\mathbb{N}^*$ and pick $k\in\mathbb{N}$ such that $\widetilde{\pi}_j(p+q)=k$. The fact that $\mathbb{N}\setminus\{1,2,\ldots,k\}\in\widetilde{\pi}_j(p)$ leads quickly to a contradiction. □

**Lemma 3.3.** *Let $F\in\mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z})$ and let $k\in\mathbb{N}$. There exists a sequence $\langle K_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n\in\mathbb{N}$, $\max K_n<\min K_{n+1}$ and for each $f\in F$ and each $n\in\mathbb{N}$, $\sum_{t\in K_n}f(t)\in k\mathbb{Z}$.*

**Proof.** This is a routine proof by induction on $|F|$, using the fact that if $f\in{}^{\mathbb{N}}\mathbb{Z}$ and $K$ is a set of $k$ elements of $\mathbb{N}$ such that for $i,j\in K$, $f(i)\equiv f(j)$ mod $k$, then $k$ divides $\sum_{t\in K}f(t)$. □

$$
\begin{tikzcd}
\text{Q*} \arrow[d] & \text{P*} \arrow[d] \\
\text{IP*} \arrow[d] & \text{J*} \arrow[d] \arrow[dl] \\
\text{C*} \arrow[d] & \text{B*} \arrow[d] \arrow[dl] \\
\text{D*} \arrow[d] & \text{PS*} \arrow[dl] \arrow[dd] \\
\text{QC*} \arrow[d] & \\
\text{central*} \arrow[d] \arrow[dr] & \text{thick=syndetic*} \arrow[d] \\
\text{SC} \arrow[d] \arrow[dr] & \text{SC*} \arrow[d] \\
\text{syndetic} \arrow[dd] & \text{central} \arrow[d] \\
 & \text{QC} \arrow[dl] \arrow[d] \\
\text{PS} \arrow[d] & \text{D} \arrow[dl] \arrow[d] \\
\text{B} \arrow[d] & \text{C} \arrow[dl] \arrow[d] \\
\text{J} \arrow[d] & \text{IP} \arrow[d] \\
\text{P} & \text{Q}
\end{tikzcd}
$$

Figure 1: Implications for $S = \mathbb{N}^v$ or $\mathbb{Z}^v$.

**Lemma 3.4.** *Let $v \in \mathbb{N}$, let $\emptyset \neq H \subseteq \{1,2,\ldots,v\}$ and for $j \in H$, let $b_j \in \mathbb{N}$. Let $B \subseteq \mathbb{N}^v$ and assume that $d(B) > \alpha > 0$. Let $\gamma = \dfrac{\alpha}{|H| \cdot \sum_{j\in H} b_j}$. Then $d(\{\sum_{j\in H} b_j x_j : \vec{x} \in B\}) \geq \gamma$.*

**Proof.** Suppose that $d(\{\sum_{j\in H} b_j x_j : \vec{x} \in B\}) < \gamma$ and pick $n \in \mathbb{N}$ such that for all $k > n$ and all $a \in \mathbb{N}$,

$$|\{\textstyle\sum_{j\in H} b_j x_j : \vec{x} \in B\} \cap (a + \{0,1,\ldots,k-1\})| < \gamma \cdot k\,.$$

Pick $k_1, k_2, \ldots, k_v \in \{k \in \mathbb{N} : k > n\}$ and $\vec{a} \in \mathbb{N}^v$ such that

$$|B \cap (\vec{a} + \times_{j=1}^{v} \{0,1,\ldots,k_j - 1\})| \geq \alpha \cdot \textstyle\prod_{j=1}^{v} k_j\,.$$

Since $\sum_{j\in H} b_j a_j \in \mathbb{N}$ and $\sum_{j\in H} b_j k_j > n$, we have that

$$\begin{aligned} &|\{\textstyle\sum_{j\in H} b_j x_j : \vec{x} \in B\} \cap (\sum_{j\in H} b_j a_j + \{0,\ldots,\sum_{j\in H} b_j k_j - 1\})| \\ &< \gamma \cdot \textstyle\sum_{j\in H} b_j k_j\,. \end{aligned}$$

Let $D = B \cap (\vec{a} + \times_{j=1}^{v} \{0,1,\ldots,k_j - 1\})$. If $\vec{x} \in D$, then $\sum_{j\in H} b_j a_j \leq \sum_{j\in H} b_j x_j < \sum_{j\in H} b_j a_j + \sum_{j\in H} b_j k_j$ so

$$\begin{aligned} &|\{\textstyle\sum_{j\in H} b_j x_j : \vec{x} \in D\}| \\ &= |\{\textstyle\sum_{j\in H} b_j x_j : \vec{x} \in D\} \cap (\sum_{j\in H} b_j a_j + \{0,1,\ldots,\sum_{j\in H} b_j k_j - 1\})| \\ &\leq |\{\textstyle\sum_{j\in H} b_j x_j : \vec{x} \in B\} \cap (\sum_{j\in H} b_j a_j + \{0,1,\ldots,\sum_{j\in H} b_j k_j - 1\})| \\ &< \gamma \cdot \textstyle\sum_{j\in H} b_j k_j\,. \end{aligned}$$

Pick $r \in H$ such that $k_r = \max\{k_t : t \in H\}$. Let $m = \dfrac{\prod_{t\in H} k_t}{k_r}$. If $u \in \{\sum_{j\in H} b_j x_j : \vec{x} \in D\}$, then $|\{\vec{x} \in D : \sum_{j\in H} b_j x_j = u\}| \leq m$ because the value of $x_r$ is determined once other values of $x_j$ have been determined. Therefore $|\{\sum_{j\in H} b_j x_j : \vec{x} \in D\}| \geq \dfrac{|D|}{m}$. And $|D| \geq \alpha \cdot \prod_{j=1}^{v} k_j \geq \alpha \cdot \prod_{j\in H} k_j = \alpha \cdot m \cdot k_r$ so $|\{\sum_{j\in H} b_j x_j : \vec{x} \in D\}| \geq \alpha \cdot k_r = \gamma \cdot |H| \cdot k_r \cdot \sum_{j\in H} b_j \geq \gamma \cdot (\sum_{j\in H} k_j) \cdot (\sum_{j\in H} b_j) \geq \gamma \cdot \sum_{j\in H} k_j b_j$. This contradiction completes the proof. □

**Lemma 3.5.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\omega$ and no row equal to $\vec{0}$. Define $T : \mathbb{N}^v \to \mathbb{N}^u$ by $T(\vec{x}) = A\vec{x}$. Let $B$ be a subset of $\mathbb{N}^v$ such that $d(B) > 0$. Then for each $i \in \{1,2,\ldots,u\}$, $d(\pi_i \circ T[B]) > 0$.*

**Proof.** Let $i \in \{1,2,\ldots,u\}$. Let $H = \{j \in \{1,2,\ldots,v\} : a_{i,j} > 0\}$ and for $j \in H$, let $b_j = a_{i,j}$. By Lemma 3.4, $d(\{\sum_{j\in H} b_j x_j : \vec{x} \in B\}) > 0$ and for $\vec{x} \in B$, $\sum_{j\in H} b_j x_j = \sum_{j=1}^{v} a_{i,j} x_j = \pi_i\big(T(\vec{x})\big)$. □

**Lemma 3.6.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\omega$ and no row equal to $\vec{0}$. Define $T : \mathbb{N}^v \to \mathbb{N}^u$ by $T(\vec{x}) = A\vec{x}$ and let*

*$\widetilde{T}:\beta(\mathbb{N}^v)\to(\beta\mathbb{N})^u$ be its continous extension. Let $q\in J(\mathbb{N}^v)$ and let $\widetilde{T}(q)=\begin{pmatrix}p_1\\ \vdots\\ p_u\end{pmatrix}$. For each $i\in\{1,2,\ldots,u\}$, $p_i\in J(\mathbb{N})$.*

**Proof.** Let $i\in\{1,2,\ldots,u\}$ and let $B\in p_i$. We need to show that $B$ is a J-set in $\mathbb{N}$. Let $F\in\mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ be given. Let $k=\sum_{j=1}^v a_{i,j}$. Then $k\in\mathbb{N}$. Pick a sequence $\langle K_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ as guaranteed by Lemma 3.3. For $f\in F$ and $n\in\mathbb{N}$, define $h_f(n)=\frac{1}{k}\sum_{t\in K_n}f(t)$ and define $\vec{g}_f(n)=\begin{pmatrix}h_f(n)\\ \vdots\\ h_f(n)\end{pmatrix}$. Now $\pi_i^{-1}[\,\overline{B}\,]$ is a neighborhood of $\widetilde{T}(q)$ so pick $D\in q$ such that $\widetilde{T}[\,\overline{D}\,]\subseteq\pi_i^{-1}[\,\overline{B}\,]$. Then $D$ is a J-set in $\mathbb{N}^v$ so pick $\vec{b}\in\mathbb{N}^v$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that for each $f\in F$, $\vec{b}+\sum_{n\in H}\vec{g}_f(n)\in D$. Then $\sum_{j=1}^v a_{i,j}\big(b_j+\sum_{n\in H}h_f(n)\big)=\pi_i\circ T\big(\vec{b}+\sum_{n\in H}\vec{g}_f(n)\big)\in B$. Let $c=\sum_{j=1}^v a_{i,j}b_j$ and let $G=\bigcup_{n\in H}K_n$. Then for $f\in F$, $c+\sum_{t\in G}f(t)=c+\sum_{n\in H}\sum_{t\in K_n}f(t)\cdot\frac{1}{k}\sum_{j=1}^v a_{i,j}=\sum_{j=1}^v a_{i,j}\big(b_j+\sum_{n\in H}h_f(n)\big)\in B$. $\square$

Recall that we are interested in proving theorems of the form of Theorem 1.5.

**Lemma 3.7.** *Let $u,v\in\mathbb{N}$. Let $\Psi$ be a property of subsets of $\mathbb{N}$ and of $\mathbb{N}^v$ which is closed under passage to supersets. Assume that whenever $C$ is a $\Psi$-set in $\mathbb{N}$ and $k\in\mathbb{N}$, then $kC$ is a $\Psi$-set in $\mathbb{N}$. Let $Y$ be one of the statements "$A$ is IPR/$\mathbb{N}$", "$A$ has no row equal to $\vec{0}$", or "$A$ is IPR/$\mathbb{N}$ and $\mathrm{rank}(A)=u$". Let $X=\mathbb{Z}$ or $X=\omega$. Then statement (a) implies statement (b).*

(a) *Let $A$ be a $u\times v$ matrix with entries in $X$ and assume $Y$. Then whenever $C$ is a $\Psi$-set in $\mathbb{N}$, $\{\vec{x}\in\mathbb{N}^v:A\vec{x}\in C^u\}$ is a $\Psi$-set in $\mathbb{N}^v$.*

(b) *Let $A$ be a $u\times v$ matrix with entries in $\{\frac{x}{n}:x\in X$ and $n\in\mathbb{N}\}$ and assume $Y$. Then whenever $C$ is a $\Psi$-set in $\mathbb{N}$, $\{\vec{x}\in\mathbb{N}^v:A\vec{x}\in C^u\}$ is a $\Psi$-set in $\mathbb{N}^v$.*

**Proof.** Assume that (a) holds, let $A$ be a $u\times v$ matrix with entries in $\{\frac{x}{n}:x\in X$ and $n\in\mathbb{N}\}$, and let $C$ be a $\Psi$-set in $\mathbb{N}$. Pick $k\in\mathbb{N}$ such that the entries of $kA$ are in $X$. Then $kC$ is a $\Psi$-set in $\mathbb{N}$ so $\{\vec{x}\in\mathbb{N}^v:(kA)\vec{x}\in(kC)^u\}$ is a $\Psi$-set in $\mathbb{N}^v$. Since $\{\vec{x}\in\mathbb{N}^v:(kA)\vec{x}\in(kC)^u\}=\{\vec{x}\in\mathbb{N}^v:A\vec{x}\in C^u\}$, we are done. $\square$

**Lemma 3.8.** *Let $\Psi$ be any of C, D, QC, central, SC*, SC, central*, QC*, D*, C*, IP*, or Q* and let $k\in\mathbb{N}$. Whenever $C$ is a $\Psi$-set in $\mathbb{N}$, one has $kC$ is a $\Psi$-set in $\mathbb{N}$.*

**Proof.** If $p$ is $\beta\mathbb{N}$ and $k\mathbb{N}\in p$, then $\frac{1}{k}p\in\beta\mathbb{N}$, where $\frac{1}{k}p$ is computed in $(\beta\mathbb{Q}_d,\cdot)$. Further, by [14, Lemma 5.19.2], if $p$ is an idempotent, so are $kp$

and $\frac{1}{k}p$ and if $p$ is a minimal idempotent, so are $kp$ and $\frac{1}{k}p$. Given $D \subseteq \mathbb{N}$ and $p \in \beta\mathbb{N}$, $D \in kp$ if and only if $(\frac{1}{k}D) \cap \mathbb{N} \in p$ and $D \in \frac{1}{k}p$ if and only if $kD \in p$.

Case $\Psi = \mathrm{C}$. Assume that $C$ is a C-set. Pick an idempotent $p \in J(\mathbb{N}) \cap \overline{C}$. We claim that $kp \in J(\mathbb{N})$, so let $D \in kp$. We will show that $D$ is a J-set. So let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$. Let $\langle K_n \rangle_{n=1}^{\infty}$ be as guaranteed by Lemma 3.3. For each $f \in F$ and $n \in \mathbb{N}$, let $g_f(n) = \frac{1}{k}\sum_{t \in K_n} f(t)$. Now $k^{-1}D \in p$ so pick $a \in \mathbb{N}$ and $G \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + \sum_{n \in G} g_f(n) \in k^{-1}D$. Let $H = \bigcup_{n \in G} K_n$. Then for $f \in F$, $ka + \sum_{t \in H} f(t) \in D$.

Thus $kp$ is an idempotent in $J(\mathbb{N})$ and $kC \in kp$.

Case $\Psi = \mathrm{D}$. Assume that $C$ is a D-set. We first establish that if $p \in \Delta(\mathbb{N})$, then $kp \in \Delta(\mathbb{N})$. So assume that $p \in \Delta(\mathbb{N})$ and let $D \in kp$. Then $(\frac{1}{k}D) \cap \mathbb{N} \in p$ so $d\big((\frac{1}{k}D) \cap \mathbb{N}\big) > 0$. A simple computation establishes that $d(D) \geq \frac{1}{k}d\big((\frac{1}{k}D) \cap \mathbb{N}\big)$.

Now pick an idempotent $p \in \Delta(\mathbb{N}) \cap \overline{C}$. Then $kp$ is an idempotent in $\Delta(\mathbb{N}) \cap \overline{k}C$,

Case $\Psi = \mathrm{QC}$. Assume that $C$ is a QC-set. We claim that if $p \in c\ell K(\beta\mathbb{N})$, then $kp \in c\ell K(\beta\mathbb{N})$. To see this, let $D \in kp$. Then $(\frac{1}{k}D) \cap \mathbb{N} \in p$ so $(\frac{1}{k}D) \cap \mathbb{N}$ is piecewise syndetic so $k\big((\frac{1}{k}D) \cap \mathbb{N}\big)$ is piecewise syndetic and $k\big((\frac{1}{k}D) \cap \mathbb{N}\big) \subseteq D$.

Pick an idempotent $p \in \overline{C} \cap c\ell K(\beta\mathbb{N})$. Then $kp$ is an idempotent in $\overline{kC} \cap c\ell K(\beta\mathbb{N})$.

Case $\Psi = \mathrm{central}$. This is [14, Lemma 15.23.2].

Case $\Psi = \mathrm{SC}^*$. Assume that $C$ is an SC*-set. Pick a left ideal $L$ of $\beta\mathbb{N}$ such that $E(L) \subseteq \overline{C}$. We may assume that $L$ is a minimal left ideal. Pick an idempotent $q \in L$, so that $L = \beta\mathbb{N} + q$. Then $kq$ is a minimal idempotent so $\beta\mathbb{N} + kq$ is a minimal left ideal of $\beta\mathbb{N}$. We claim that $E(\beta\mathbb{N} + kq) \subseteq \overline{kC}$. So let $p \in E(\beta\mathbb{N} + kq)$. Since $kq$ is a right identity for $\beta\mathbb{N} + kq$, we have that $p + kq = p$ and thus $\frac{1}{k}p + q = \frac{1}{k}p$ so that $\frac{1}{k}p \in L$. Therefore $\frac{1}{k}p \in \overline{C}$ so that $p \in \overline{kC}$.

Case $\Psi = \mathrm{SC}$. Assume that $C$ is an SC-set. Let $L$ be a minimal left ideal of $\beta\mathbb{N}$. We want to show that $E(L) \cap \overline{kC} \neq \emptyset$. Pick an idempotent $q \in L$. Then $\frac{1}{k}q$ is a minimal idempotent in $\beta\mathbb{N}$ so $\beta\mathbb{N} + \frac{1}{k}q$ is a minimal left ideal of $\beta\mathbb{N}$. Pick $p \in E(\beta\mathbb{N} + \frac{1}{k}q) \cap \overline{C}$. Then $p = p + \frac{1}{k}q$, so $kp = kp + q \in E(L) \cap \overline{kC}$.

Case $\Psi = \mathrm{central}^*$. Assume that $C$ is a central* set. Let $p$ be a minimal idempotent in $\beta\mathbb{N}$. Then $\frac{1}{k}p$ is a minimal idempotent so $C \in \frac{1}{k}p$ and thus $kC \in p$.

Case $\Psi = \mathrm{QC}^*$. Assume that $C$ is a QC*-set. Let $p$ be an idempotent in $c\ell K(\beta\mathbb{N})$. Then $\frac{1}{k}p$ is an idempotent. Further, if $D \in \frac{1}{k}p$, then $kD \in p$ so $kD$ is piecewise syndetic and thus $D$ is piecewise syndetic so $\frac{1}{k}p \in c\ell K(\beta\mathbb{N})$. Thus $C \in \frac{1}{k}p$ and so $kC \in p$.

Case $\Psi = \text{D*}$. Assume that $C$ is a D*-set. Observe that if $p \in \Delta(\mathbb{N}) \cap \overline{k\mathbb{N}}$, then $\frac{1}{k}p \in \Delta(\mathbb{N})$. To see this, let $D \subseteq \mathbb{N}$ with $D \in \frac{1}{k}p$, then $kD \in p$ so $d(kD) > 0$ and thus $d(D) > 0$.

Let $p$ be an idempotent in $\Delta(\mathbb{N})$. Then $\frac{1}{k}p$ is an idempotent in $\Delta(\mathbb{N})$ so $C \in \frac{1}{k}p$ and thus $kC \in p$.

Case $\Psi = \text{C*}$. Assume that $C$ is a C*-set. We claim that if $p \in J(\mathbb{N}) \cap \overline{k\mathbb{N}}$, then $\frac{1}{k}p \in J(\mathbb{N})$. To see this, let $D \in \frac{1}{k}p$ and let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$. Then $\{kf : f \in F\} \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ and $kD \in p$ so pick $a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + \sum_{t \in H} kf(t) \in kD$. Note that $a = kb$ for some $b \in \mathbb{N}$. Then for each $f \in F$, $b + \sum_{t \in H} f(t) \in D$.

Now, if $p \in E\big(J(\mathbb{N})\big)$, then $\frac{1}{k}p \in E\big(J(\mathbb{N})\big)$ so $C \in \frac{1}{k}p$ so $kC \in p$.

Case $\Psi = \text{IP*}$. Assume that $C$ is an IP*-set. Given $p \in E(\beta\mathbb{N})$, $\frac{1}{k}p \in E(\beta\mathbb{N})$ so $C \in \frac{1}{k}p$ and thus $kC \in p$.

Case $\Psi = \text{Q*}$. Assume that $C$ is a Q*-set. We need to show that $\mathbb{N} \setminus kC$ is not a Q-set so that for any increasing sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$, there exist $m < n$ such that $x_n - x_m \in kC$. So let such a sequence be given. Choose $i \in \{0, 1, \ldots, k-1\}$ and a subsequence $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$ such that $y_1 > i$ and for each $n \in \mathbb{N}$, $y_n \equiv i \pmod{k}$. For $n \in \mathbb{N}$, pick $z_n \in \mathbb{N}$ such that $y_n = i + kz_n$. Since $C$ is a Q*-set, pick $m < n$ such that $z_n - z_m \in C$. Then $y_n - y_m = kz_n - kz_m \in kC$. $\square$

The following lemma tells us that if $\Psi$ is any of the properties that imply "thick", and Theorem 1.5 holds for $\Psi$, then $A$ cannot have any entries that are not in $\mathbb{Z}$.

**Lemma 3.9.** *Let $u, v \in \mathbb{N}$, let $A$ be a $u \times v$ matrix with entries from $\mathbb{Q}$. If $A$ has some entry which is not in $\mathbb{Z}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in \mathbb{N}^u\}$ is not thick in $\mathbb{N}^v$.*

**Proof.** Assume that $A$ has an entry which is not in $\mathbb{Z}$. Pick $i \in \{1, 2, \ldots, u\}$ and $j \in \{1, 2, \ldots, v\}$ such that $a_{i,j} \in \mathbb{Q} \setminus \mathbb{Z}$. Suppose that $B = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in \mathbb{N}^u\}$ is thick in $\mathbb{N}^v$. Define $\vec{x} \in \mathbb{N}^v$ by $x_t = 1$ if $t \neq j$ and $x_j = 2$. Let $\overline{1}$ be the vector with all entries equal to 1. Pick $\vec{w} \in \mathbb{N}^v$ such that $\{\vec{x}, \overline{1}\} + \vec{w} \subseteq B$. Let $\vec{y} = A(\overline{1} + \vec{w})$ and let $\vec{z} = A(\vec{x} + \vec{w})$. Then $y_i = \sum_{t=1}^{v} a_{i,t}(1 + w_t)$ and $z_i = \sum_{t=1}^{v} a_{i,t}(x_t + w_t) = y_i + a_{i,j}$ so one can't have both $y_i \in \mathbb{N}$ and $z_i \in \mathbb{N}$. $\square$

In a fashion similar to Lemma 3.9 we see that if $\Psi$ is any of the properties that imply "syndetic", and Theorem 1.5 holds for $\Psi$, then $A$ cannot have any negative entries.

**Lemma 3.10.** *Let $u, v \in \mathbb{N}$, let $A$ be a $u \times v$ matrix with entries from $\mathbb{Q}$. If $A$ has some entry which is negative, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in \mathbb{N}^u\}$ is not syndetic in $\mathbb{N}^v$.*

**Proof.** Assume that $A$ has a negative entry, and pick $i \in \{1, 2, \ldots, u\}$ and $j \in \{1, 2, \ldots, v\}$ such that $a_{i,j} < 0$. For $n \in \mathbb{N}$, define $\vec{x}^{(n)} \in \mathbb{N}^v$ by, for

$t \in \{1,2,\ldots,v\}$, $x_t^{(n)} = 1$ if $t \neq j$ and $x_j^{(n)} = n$. For $m \in \mathbb{N}$, let $C_m = \{\vec{x}^{(n)} : n \in \mathbb{N}$ and $n > m\}$. Pick $q \in \beta(\mathbb{N}^v)$ such that $\{C_m : m \in \mathbb{N}\} \subseteq q$. Then $\beta(\mathbb{N}^v) + q$ is a left ideal of $\beta(\mathbb{N}^v)$. Let $B = \{\vec{w} \in \mathbb{N}^v : A\vec{w} \in \mathbb{N}^u\}$ We claim that $(\beta(\mathbb{N}^v) + q) \cap \overline{B} = \emptyset$, showing that $B$ is not syndetic. Suppose that we have $z \in (\beta(\mathbb{N}^v) + q) \cap \overline{B}$ and pick $r \in \beta(\mathbb{N}^v)$ such that $z = r + q$.

Define $T : \mathbb{N}^v \to \mathbb{Z}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(\mathbb{N}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. Now $z \in \overline{B} = c\ell_{\beta(\mathbb{N}^v)} T^{-1}[\mathbb{N}^u]$ and it is a routine exercise to show that $c\ell_{\beta(\mathbb{N}^v)} T^{-1}[\mathbb{N}^u] = \widetilde{T}^{-1}[(\beta\mathbb{N})^u]$ so $\widetilde{T}(z) \in (\beta\mathbb{N})^u$ so that $\pi_i\big(\widetilde{T}(z)\big) \in \beta\mathbb{N}$. Pick $D \in z$ such that $\pi_i \circ \widetilde{T}[\,\overline{D}\,] \subseteq \beta\mathbb{N}$. Since $D \in r + q$, pick $\vec{y} \in \mathbb{N}^v$ such that $-\vec{y} + D \in q$. Pick $m > \sum_{t \in \{1,2,\ldots,v\} \setminus \{j\}} a_{i,t}(y_t + 1)$ and pick $n > m$ such that $\vec{x}^{(n)} \in C_m \cap (-\vec{y} + D)$. Then $\pi_i\big(T(\vec{y} + \vec{x}^{(n)})\big) \in \mathbb{N}$. But $\pi_i\big(T(\vec{y} + \vec{x}^{(n)})\big) = \sum_{t \in \{1,2,\ldots,v\} \setminus \{j\}} a_{i,t}(y_t + 1) + a_{i,j} x_j^{(n)} \leq m - x_j^{(n)} = m - n < 0$, a contradiction. □

The main point of the following lemma is that for $f \in F$ and $n \in \mathbb{N}$, $f(n)$ is allowed to be negative.

**Lemma 3.11.** *Assume that $C$ is a $C$-set in $\mathbb{N}$ and $k \in \mathbb{N}$. For each $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z})$, there exist $a \in k\mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + \sum_{t \in H} f(t) \in C$.*

**Proof.** Let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z})$ and pick $\langle K_n \rangle_{n=1}^{\infty}$ as guaranteed by Lemma 3.3. For $f \in F$ define $g_f \in {}^{\mathbb{N}}\mathbb{Z}$ by $g_f(n) = \sum_{t \in K_n} f(t)$. For each $n \in \mathbb{N}$ pick $b(n) \in \mathbb{N}$ such that for each $f \in F$, $b(n) + g_f(n) \in \mathbb{N}$. For each $f \in F$, define $h_f \in {}^{\mathbb{N}}\mathbb{N}$ by $h_f(n) = b(n) + g_f(n)$. Pick an idempotent $p \in J(\mathbb{N}) \cap \overline{C}$. Let $C' = C \cap k\mathbb{N}$. Then $C' \in p$ so pick $c \in \mathbb{N}$ and $G \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $c + \sum_{n \in G} h_f(n) \in C'$. Let $a = c + \sum_{n \in G} b(n)$, let $H = \bigcup_{n \in G} K_n$ and let $f \in F$. Then $a + \sum_{t \in H} f(t) = c + \sum_{n \in G} b(n) + \sum_{n \in G} \sum_{t \in K_n} f(t) = c + \sum_{n \in G} b(n) + \sum_{n \in G} g_f(n) = c + \sum_{n \in G} h_f(n) \in C'$. Since $a + \sum_{t \in H} f(t) \in k\mathbb{N}$ and $\sum_{t \in H} f(t) \in k\mathbb{Z}$, we have $a \in k\mathbb{N}$. □

**Lemma 3.12.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}$. Given $C \subseteq \mathbb{N}$, let $D(C) = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$. Let $p$ be an idempotent in $J(\mathbb{N})$. If for every $C \in p$, $D(C)$ is a J-set in $\mathbb{N}^v$, then for every $C \in p$, $D(C)$ is a C-set in $\mathbb{N}^v$.*

**Proof.** Define $T : \mathbb{N}^v \to \mathbb{Z}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(\mathbb{N}^v) \to (\beta\mathbb{Z})^u$ be the continuous extension of $T$. Let $\overline{p} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \in (\beta\mathbb{N})^u$. We claim that $\widetilde{T}^{-1}[\{\overline{p}\}] \cap J(\mathbb{N}^v) \neq \emptyset$. Given $C \in p$ we have that $T^{-1}[C^u] = D(C)$ is a J-set in $\mathbb{N}^v$ so $\overline{T^{-1}[C^u]} \cap J(\mathbb{N}^v) \neq \emptyset$. Consequently $\bigcap_{C \in p} \big(\overline{T^{-1}[C^u]} \cap J(\mathbb{N}^v)\big) \neq \emptyset$. Further, it is routine to verify that $\bigcap_{C \in p} \overline{T^{-1}[C^u]} \subseteq \widetilde{T}^{-1}[\{\overline{p}\}]$ so $\widetilde{T}^{-1}[\{\overline{p}\}] \cap J(\mathbb{N}^v) \neq \emptyset$ as claimed. Then $\widetilde{T}^{-1}[\{\overline{p}\}] \cap J(\mathbb{N}^v)$ is a compact subsemigroup of $\beta(\mathbb{N}^v)$ so pick an idempotent $q \in \widetilde{T}^{-1}[\{\overline{p}\}] \cap J(\mathbb{N}^v)$. Given $C \in p$, we have $\overline{C}^u$ is a neighborhood of $\overline{p}$. Pick $B \in q$ such that $\widetilde{T}[\,\overline{B}\,] \subseteq \overline{C}^u$. Then $B \subseteq T^{-1}[C^u] = D(C)$. □

**Definition 3.13.** Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then $A$ is a *first entries matrix* if and only if

(1) $A$ has no row consisting of all zeroes;
(2) for $i \in \{1, 2, \ldots, u\}$ and $j \in \{1, 2, \ldots, v\}$, if $a_{i,j}$ is the first nonzero entry in row $i$, then $a_{i,j} > 0$; and
(3) if $i, k \in \{1, 2, \ldots, u\}$, $j \in \{1, 2, \ldots, v\}$, $a_{i,j}$ is the first nonzero entry in row $i$, and $a_{k,j}$ is the first nonzero entry in row $k$, then $a_{i,j} = a_{k,j}$.

**Lemma 3.14.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}$ such that* $\mathrm{rank}(A) = u$. *There exists $k \in \mathbb{N}$ such that $k\mathbb{Z}^u \subseteq \{A\vec{x} : \vec{x} \in \mathbb{Z}^v\}$.*

**Proof.** We may presume that the first $u$ columns of $A$ are linearly independent and let $B$ consist of those columns. Let $k$ be the determinant of $B$. We may presume that $k > 0$. Let $\vec{y} \in k\mathbb{Z}^u$. Pick $\vec{x} \in \mathbb{Q}^u$ such that $B\vec{x} = \vec{y}$. By Cramer's rule, $\vec{x} \in \mathbb{Z}^u$. Define $\vec{z} \in \mathbb{Z}^v$ by $z_i = x_i$ if $i \in \{1, 2, \ldots, u\}$ and $z_i = 0$ if $i > u$. Then $A\vec{z} = \vec{y}$. □

**Lemma 3.15.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}$. Let $X = \mathbb{N}$ or $X = \mathbb{Z}$. Define $T : X^v \to \mathbb{Z}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(X^v) \to (\beta\mathbb{Z})^u$ be its continuos extension. Let $p$ be an idempotent in $\beta\mathbb{N}$. If for every $P \in p$ there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in P^u$, then there is an idempotent $q \in \beta(\mathbb{N}^v)$ such that $\widetilde{T}(q) = \overline{p} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \in (\beta\mathbb{N})^u$. If $p \in K(\beta\mathbb{N})$, then $\widetilde{T}\big[K\big(\beta(\mathbb{N}^v)\big)\big] = \widetilde{T}[\beta(\mathbb{N}^v)] \cap \big(K(\beta\mathbb{N})\big)^u$.*

**Proof.** We have that $\overline{p} \in c\ell_{(\beta\mathbb{Z})^u} T[\mathbb{N}^v] = \widetilde{T}[\beta(\mathbb{N}^v)]$, so $\widetilde{T}^{-1}[\{\overline{p}\}] \cap \beta(\mathbb{N}^v) \neq \emptyset$. Since $\widetilde{T}$ is a homomorphism, $\widetilde{T}^{-1}[\{\overline{p}\}] \cap \beta(\mathbb{N}^v)$ is a compact subsemigroup of $\beta(\mathbb{N}^v)$ and thus has an idempotent $q$.

Now assume that $p \in K(\beta\mathbb{N})$. Then $\widetilde{T}[\beta(\mathbb{N}^v)] \cap \big(K(\beta\mathbb{N})\big)^u \neq \emptyset$. By [14, Theorem 2.23], $\big(K(\beta\mathbb{N})\big)^u = K\big((\beta\mathbb{N})^u\big)$ so by [14, Theorem 1.65], $K(\widetilde{T}[\beta(\mathbb{N}^v)]) = \widetilde{T}[\beta(\mathbb{N}^v)] \cap K\big((\beta\mathbb{N})^u\big)$. By [14, Exercise 1.7.3], $\widetilde{T}\big[K\big(\beta(\mathbb{N}^v)\big)\big] = K(\widetilde{T}[\beta(\mathbb{N}^v)]) = \widetilde{T}[\beta(\mathbb{N}^v)] \cap \big(K(\beta\mathbb{N})\big)^u$. □

**Lemma 3.16.** *Let $u, v, d \in \mathbb{N}$, let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}$, let $B$ be a $u \times d$ matrix with entries from $\mathbb{Z}$, and assume that whenever $C$ is a $C$-set in $\mathbb{N}$ and $\langle \vec{b}(n) \rangle_{n=1}^\infty$ is a sequence in $\mathbb{Z}^d$, there exist $\vec{x} \in \mathbb{N}^v$ and $\vec{y} \in FS(\langle \vec{b}(n) \rangle_{n=1}^\infty)$ such that all entries of $A\vec{x} + B\vec{y}$ are in $C$. Then whenever $C$ is a $C$-set in $\mathbb{N}$ and $\langle \vec{b}(n) \rangle_{n=1}^\infty$ is a sequence in $\mathbb{Z}^d$, there exist a sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ and a sequence $\langle \vec{x}(n) \rangle_{n=1}^\infty$ in $\mathbb{N}^v$ such that* $\max H_n < \min H_{n+1}$ *for each $n \in \mathbb{N}$ and $FS(\langle A\vec{x}(n) + B \cdot \sum_{t \in H_n} \vec{b}(t) \rangle_{n=1}^\infty) \subseteq C^u$.*

**Proof.** Let a C-set $C$ in $\mathbb{N}$ and a sequence $\langle \vec{b}(n)\rangle_{n=1}^{\infty}$ in $\mathbb{Z}^d$ be given. Pick an idempotent $p \in J(\mathbb{N})$ such that $C \in p$ and let $C^\star = \{x \in C : -x + C \in p\}$. By [14, Lemma 4.14], if $x \in C^\star$, then $-x + C^\star \in p$. Then $C^\star$ is a C-set in $\mathbb{N}$, so pick $\vec{x}(1) \in \mathbb{N}^v$ and $\vec{y}(1) \in FS(\langle \vec{b}(n)\rangle_{n=1}^{\infty})$ such that all entries of $A\vec{x}(1) + B\vec{y}(1)$ are in $C^\star$. Pick $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $\vec{y}(1) = \sum_{t\in H_1} \vec{b}(t)$.

Let $n \in \mathbb{N}$ and assume we have chosen $\langle \vec{x}(t)\rangle_{t=1}^{n}$ in $\mathbb{N}^v$ and $\langle H_t\rangle_{t=1}^{n}$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_t < \min H_{t+1}$ for all $t \in \{1, 2, \ldots, n-1\}$ and for each $F \in \mathcal{P}_f(\{1, 2, \ldots, n\})$, all entries of $\sum_{t\in F} \big(A\vec{x}(t) + B \cdot \sum_{k\in H_t} \vec{b}(k)\big)$ are in $C^\star$.

For $F \in \mathcal{P}_f(\{1, 2, \ldots, n\})$, let $E_F$ be the set of entries of $\sum_{t\in F} \big(A\vec{x}(t) + B \cdot \sum_{k\in H_t} \vec{b}(k)\big)$, and let $D = \bigcup\{E_F : F \in \mathcal{P}_f(\{1, 2, \ldots, n\})\}$. Let $G = C^\star \cap \bigcap_{y\in D}(-y + C^\star)$ and let $m = \max H_n$. Then $G$ is a C-set so pick $\vec{x}(n+1) \in \mathbb{N}^v$ and $\vec{y}(n+1) \in FS(\langle \vec{b}(t)\rangle_{t=m+1}^{\infty})$ such that all entries of $A\vec{x}(n+1)+B\vec{y}(n+1)$ are in $G$. Pick $H_{n+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{n+1} > m$ and $\vec{y}(n+1) = \sum_{t\in H_{n+1}} \vec{b}(t)$. The induction hypothesis is satisfied. □

If $C$ is a central subset of $\mathbb{N}$, then the following theorem follows from [15, Theorem 4.4].

**Theorem 3.17.** *Let $u, v, d \in \mathbb{N}$, let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}$ which is IPR/$\mathbb{N}$, and let $B$ be a $u \times d$ matrix with entries from $\mathbb{Z}$. Let $C$ be a C-set in $\mathbb{N}$ and let $U$ be an IP-set in $\mathbb{Z}^d$. There exist $\vec{x} \in \mathbb{N}^v$ and $\vec{y} \in U$ such that all the entries of $A\vec{x} + B\vec{y}$ are in $C$.*

**Proof.** It will be sufficient to prove that, given any sequence $\langle \vec{b}(n)\rangle_{n=1}^{\infty}$ in $\mathbb{Z}^d$, there exist $\vec{x} \in \mathbb{N}^v$ and $\vec{y} \in FS(\langle \vec{b}(n)\rangle_{n=1}^{\infty})$ such that all the entries of $A\vec{x} + B\vec{y}$ are in $C$ so let $\langle \vec{b}(n)\rangle_{n+1}^{\infty}$ be given.

We shall first prove our theorem for the case in which $A$ is a first entries matrix.

(1) Suppose that the first column of $A$ is a constant vector whose entries are all equal to $c \in \mathbb{N}$. Let $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_u$ denote the rows of $B$. Assume first that $v = 1$. For $i \in \{1, 2, \ldots, u\}$, define $f_i \in {}^{\mathbb{N}}\mathbb{Z}$ by $f_i(n) = \vec{s}_i \cdot \vec{b}(n)$ and pick by Lemma 3.11, $m \in c\mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $i \in \{1, 2, \ldots, u\}$, $m + \sum_{t\in H} f_i(t) \in C$. The conclusion holds with $\vec{x} = (m/c)$ and $\vec{y} = \sum_{t\in H} \vec{b}(t)$.

Now assume that $v > 1$, let $M$ consist of the last $v - 1$ columns of $A$, let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_u$ denote the rows of $M$, and pick any vector $\vec{a} \in \mathbb{N}^{v-1}$. For $i \in \{1, 2, \ldots, u\}$, define $f_i \in {}^{\mathbb{N}}\mathbb{Z}$ by $f_i(n) = \vec{r}_i \cdot \vec{a} + \vec{s}_i \cdot \vec{b}(n)$ and pick by Lemma 3.11, $m \in c\mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $i \in \{1, 2, \ldots, u\}$, $m + \sum_{t\in H} f_i(t) \in C$. The conclusion holds with $\vec{x} = \begin{pmatrix} m/c \\ |H|\vec{a} \end{pmatrix}$ and $\vec{y} = \sum_{t\in H} \vec{b}(t)$.

(2) It follows from (1) that our claim holds if $u = 1$. So we shall assume that $u > 1$ and that our claim holds for all smaller values of $u$, with $\vec{y} \in$

$FS\langle\vec{b}(n)\rangle$. We may suppose that $v > 1$, because a first entries matrix with a single column consists of a constant column, in which case (1) applies. We may also suppose that the first column of $A$ is not identically zero, because we can ensure this, if necessary, by interchanging the first column of $A$ and the first column which is not identically zero. We may also suppose that the first column of $A$ is not a constant vector, because otherwise our claim follows from (1).

So by rearranging rows, we can write $A$ in block form as $A = \begin{pmatrix} \overline{0} & D \\ \overline{c} & E \end{pmatrix}$, where $D$ is a first entries $w \times (v-1)$ matrix over $\mathbb{Z}$ for a positive integer $w < u$, $E$ is a $(u-w) \times (v-1)$ matrix over $\mathbb{Z}$, $\overline{0}$ is a column vector with $w$ entries all equal to 0 and $\overline{c}$ is a column vector with $u-w$ entries all equal to a positive integer $c$.

Let $B_1$ denote the matrix formed by the first $w$ rows of $B$, and let $B_2$ denote the matrix formed by the last $u-w$ rows of $B$. It follows from our inductive assumption and Lemma 3.16 that we can select a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ with $\max H_n < \min H_{n+1}$ for every $n \in \mathbb{N}$, and a sequence $\langle\vec{v}(n)\rangle_{n=1}^\infty$ of vectors in $\mathbb{N}^{v-1}$, such that $FS(\langle D\vec{v}(n) + B_1 \cdot \sum_{t\in H_n} \vec{b}(t)\rangle_{n=1}^\infty) \subseteq C^w$.

Let $\vec{e}_{w+1}, \vec{e}_{w+2}, \ldots, \vec{e}_u$ denote the rows of $E$ and let $\vec{s}_{w+1}, \vec{s}_{w+2}, \ldots, \vec{s}_u$ denote the rows of $B_2$. For each $i \in \{w+1, w+2, \ldots, u\}$, we define $g_i : \mathbb{N} \to \mathbb{Z}$ by $g_i(n) = \vec{e}_i \cdot \vec{v}(n) + \vec{s}_i \cdot \sum_{t\in H_n} \vec{b}(t)$. Pick by Lemma 3.11, $m \in c\mathbb{N}$ and $K \in \mathcal{P}_f(\mathbb{N})$ such that $m + \sum_{t\in K} g_i(t) \in C$ for every $i \in \{w+1, w+2, \ldots, u\}$. It follows that our theorem holds with $\vec{x} = \begin{pmatrix} m/c \\ \sum_{n\in K} \vec{v}(n) \end{pmatrix}$ and $\vec{y} = \sum\{\vec{b}(t) : t \in \bigcup_{n\in K} H_n\}$.

(3) Finally, assume that $A$ is any $u \times v$ matrix with entries in $\mathbb{Z}$, which is IPR/$\mathbb{N}$. By [14, Theorem 15.24(g)], there exists $m \in \mathbb{N}$ and a $v \times m$ matrix $G$ with entries in $\omega$ and no row equal to 0, such that $AG$ is a first entries matrix. The fact that our theorem holds for $AG$, implies that it holds for $A$. $\square$

**Corollary 3.18.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Q}$ which is image partition regular over $\mathbb{N}$. Then for any C-set $C$ in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a C-set in $\mathbb{N}^v$.*

**Proof.** By Lemmas 3.7 and 3.8 it suffices to assume that the entries of $A$ are in $\mathbb{Z}$. By Lemma 3.12, it suffices to show that for any C-set $C$ in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a J-set in $\mathbb{N}^v$. So let $C$ be a C-set in $\mathbb{N}$, let $m \in \mathbb{N}$, and let $f_1, f_2, \ldots, f_m$ be functions from $\mathbb{N}$ to $\mathbb{N}^v$. We show that there exist $\vec{x} \in \mathbb{N}^v$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $i \in \{1, 2, \ldots, m\}$, $A\big(\vec{x} + \sum_{n\in H} f_i(n)\big) \in C^u$.

Define $mu \times mv$ matrices $M$ and $B$ by

$$M = \begin{pmatrix} A & A & \dots & A \\ A & A & \dots & A \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \dots & A \end{pmatrix} \text{ and } B = \begin{pmatrix} A & O & \dots & O \\ O & A & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A \end{pmatrix},$$

where $O$ denotes the zero $u \times v$ matrix. Trivially the matrix $\begin{pmatrix} A \\ A \\ \vdots \\ A \end{pmatrix}$ is IPR/$\mathbb{N}$, so by [14, Theorem 15.24(k)], $M$ is IPR/$\mathbb{N}$. Define $g : \mathbb{N} \to \mathbb{N}^{mv}$ by $g(n) = \begin{pmatrix} f_1(n) \\ f_2(n) \\ \vdots \\ f_m(n) \end{pmatrix}$.

Since $FS(\langle g(n) \rangle_{n+1}^{\infty})$ is an IP set in $\mathbb{Z}^{mv}$, it follows from Theorem 3.17 that there exist $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m \in \mathbb{N}^v$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that all the entries of $M \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_m \end{pmatrix} + B \cdot \sum_{n \in H} g(n)$ are in $C$. This implies that all the entries of $A\big(\vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_m + \sum_{n \in H} f_i(n)\big)$ are in $C$ for every $i \in \{1, 2, \dots, m\}$ as required. □

We now see that Theorem 3.17 characterizes $C$-subsets of $\mathbb{N}$.

**Theorem 3.19.** *Let $C$ be a subset of $\mathbb{N}$ which satisfies the conclusion of Theorem 3.17. Then $C$ is a J-set. Hence, if $p$ is an idempotent in $\beta\mathbb{N}$, every member of $p$ satisfies the conclusion of Theorem 3.17 if and only if every member of $p$ is a C-set.*

**Proof.** Let $u \in \mathbb{N}$ and let $f_1, f_2, \dots, f_u$ be functions from $\mathbb{N}$ to $\mathbb{N}$. Define $g : \mathbb{N} \to \mathbb{N}^u$ by $g(n) = \begin{pmatrix} f_1(n) \\ f_2(n) \\ \vdots \\ f_m(n) \end{pmatrix}$ for every $n \in \mathbb{N}$. Let $A$ denote the $u \times 1$ matrix whose entries are all 1, and let $B$ denote the identity $u \times u$ matrix. Since $FS(\langle g(n) \rangle_{n=1}^{\infty})$ is an IP set in $\mathbb{Z}^u$, it follows that there exists $x \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $Ax + B \cdot \sum_{n \in H} g(n) \in C^u$. I.e. $x + \sum_{n \in H} f_i(n) \in C$ for every $i \in \{1, 2, \dots, u\}$. So $C$ is a J-set in $\mathbb{N}$. □

**Lemma 3.20.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}$ which is IPR/$\mathbb{N}$. Define $T : \mathbb{Z}^v \to \mathbb{Z}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(\mathbb{Z}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. Assume that $k \in \mathbb{N}$ and whenever $P \subseteq k\mathbb{N}$*

*with* $d(P) > 0$, $d(T^{-1}[P^u]) > 0$, *where the latter density is computed in* $\mathbb{Z}^v$. *Then for every D-set* $C \subseteq \mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ *is a D-set in* $\mathbb{N}^v$.

**Proof.** Let $C$ be a D-set in $\mathbb{N}$ and pick an idempotent $p \in \Delta(\mathbb{N}) \cap \overline{C}$. Since $p$ is an idempotent, $k\mathbb{N} \in p$. Thus for every $P \in p$, $d(P \cap k\mathbb{N}) > 0$ so by assumption $d(T^{-1}[(P \cap k\mathbb{N})^u]) > 0$ and thus by Lemma 2.4, $\overline{T^{-1}[(P \cap k\mathbb{N})^u]} \cap \Delta(\mathbb{Z}^v) \neq \emptyset$. Pick $r \in \Delta(\mathbb{Z}^v) \cap \bigcap_{P \in p} \overline{T^{-1}[(P \cap k\mathbb{N})^u]}$ and note that $\widetilde{T}(r) = \overline{p} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \in (\beta\mathbb{N})^u$.

Given any $P \in p$, $P$ is a D-set, hence a C-set, in $\mathbb{N}$ so by Corollary 3.18, $\mathbb{N}^v \cap T^{-1}[P^u]$ is a C-set in $\mathbb{N}^v$ and consequently $J(\mathbb{N}^v) \cap \overline{T^{-1}[P^u]} \neq \emptyset$. If $s \in \bigcap_{P \in p} \overline{T^{-1}[P^u]}$, then $\widetilde{T}(s) = \overline{p}$, so $\{s \in J(\mathbb{N}^v) : \widetilde{T}(s) = \overline{p}\}$ is a compact semigroup, so has an idempotent $q$.

Let $\Theta = \{p \in \beta(\mathbb{N}^v) : (\forall j \in \{1, 2, \ldots, v\})(\widetilde{\pi}_j(p) \in \mathbb{N}^*)\}$. Note that $q \in \Theta$ since if we had $j \in \{1, 2, \ldots, v\}$ such that $\widetilde{\pi}_j(q) = k \in \mathbb{N}$, we would have $k = \widetilde{\pi}_j(q) = \widetilde{\pi}_j(q + q) = k + k$. Thus by Lemma 3.2, $r + q \in \Theta \subseteq \beta(\mathbb{N}^v)$. By Theorem 3.1, $r + q \in \Delta(\mathbb{Z}^v)$ so $r + q \in \Delta(\mathbb{Z}^v) \cap \beta(\mathbb{N}^v)$. It is routine to verify that if $B \subseteq \mathbb{N}^v$, then its Banach density is the same whether it is computed in $\mathbb{N}^v$ or in $\mathbb{Z}^v$, so $r + q \in \Delta(\mathbb{N}^v)$.

Now $\widetilde{T}(r + q) = \overline{p} + \overline{p} = \overline{p}$, so $\{s \in \Delta(\mathbb{N}^v) : \widetilde{T}(s) = \overline{p}\} \neq \emptyset$ so there is an idemmpotent $s \in \Delta(\mathbb{N}^v)$ such that $\widetilde{T}(s) = \overline{p}$. Then $T^{-1}[C^u] \in s$ so $T^{-1}[C^u]$ is a D-set in $\mathbb{N}^v$. $\square$

## 4. Large preimages of matrices

We begin by proving Theorem 1.4. As we noted in the introduction, this theorem was stated in [3]. However, for the proof, the reader was referred indirectly to a proof of Theorem 1.2. The known proofs of that theorem all utilize strongly the fact that a central set is a member of an idempotent which is minimal with respect to the usual ordering of idempotents, so they cannot be simply adapted to prove Theorem 1.4.

**Theorem 1.4.** *Let* $u, v \in \mathbb{N}$ *and let* $A$ *be a* $u \times v$ *matrix with entries from* $\mathbb{Q}$. *The following statements are equivalent.*

(a) *$A$ is image partition regular.*
(b) *For every C-set $C$ in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$.*
(c) *For every C-set $C$ in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a C-set in $\mathbb{N}^v$.*

**Proof.** It is trivial that (b) implies (a) and that (c) implies (b). The fact that (a) implies (c) is Corollary 3.18. $\square$

We now set out to prove theorems of the form of Theorem 1.5 for each of our notions for which such a theorem is possible. In each case, assuming that the entries of $A$ are rational, the restrictions on the entries of $A$ are necessary by Lemmas 3.9 and 3.10.

**Theorem 4.1.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Q}$ which is IPR/$\mathbb{N}$. Let $\Psi$ be any of C, central, or SC*. If $C$ is a $\Psi$-set in $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a $\Psi$-set in $\mathbb{N}^v$.*

**Proof.** By Lemmas 3.7 and 3.8 we may assume that the entries of $A$ are integers.

The case $\Psi = \mathrm{C}$ is Corollary 3.18 and the case $\Psi =$ central is [14, Theorem 15.24(i)] so assume that $\Psi = \mathrm{SC}^*$ and let $C$ be an SC*-set in $\mathbb{N}$.

Define $T : \mathbb{N}^v \to \mathbb{Z}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(\mathbb{N}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. Pick a minimal left ideal $L$ of $\beta\mathbb{N}$ such that $E(L) \subseteq \overline{C}$ and note that $L$ is also a left ideal of $\beta\mathbb{Z}$. Pick an idempotent $p \in L$ and let $\overline{p} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \in (\beta\mathbb{N})^u$. Since every member of $p$ is central, we have by [14, Theorem 15.24(h)] that $(\forall P \in p)(\exists \vec{x} \in \mathbb{N}^v)(A\vec{x} \in P^u)$. By Lemma 3.15, pick an idempotent $q \in \beta(\mathbb{N}^v)$ such that $\widetilde{T}(q) = \overline{p}$. Let $M = \beta(\mathbb{N}^v) + q$. We claim that $E(M) \subseteq c\ell_{\beta(\mathbb{N}^v)} T^{-1}[C^u]$ so that $T^{-1}[C^u]$ is an SC* set in $\mathbb{N}^v$.

To this end, let $r \in E(M)$ and note that $r = r + q$. Let $\widetilde{T}(r) = \begin{pmatrix} s_1 \\ \vdots \\ s_u \end{pmatrix}$. Since $\widetilde{T}$ is a homomorphism, we have that each $s_i$ is an idempotent. At this point we only know that it is an idempotent in $\beta\mathbb{Z}$. But $\widetilde{T}(r) = \widetilde{T}(r) + \widetilde{T}(q) = \widetilde{T}(r) + \overline{p}$, so each $s_i + p \in L$. So $\widetilde{T}(r) \in \big(E(L)\big)^u$ so $r \in \widetilde{T}^{-1}[\big(E(L)\big)^u] \subseteq \widetilde{T}^{-1}[\overline{C}^u] = c\ell_{\beta(\mathbb{N}^v)} T^{-1}[C^u]$. □

It is interesting to note that by Theorem 2.6 all of the properties for which the entries of $A$ are allowed to be arbitrary rationals are partition regular except SC*.

**Theorem 4.2.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Q}$ which is IPR/$\mathbb{N}$ such that* $\mathrm{rank}(A) = u$. *Let $\Psi$ be D or QC. If $C$ is a $\Psi$-set in $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a $\Psi$-set in $\mathbb{N}^v$.*

**Proof.** By Lemmas 3.7 and 3.8 we may assume that the entries of $A$ are integers.

Case $\Psi = \mathrm{D}$. Assume that $C$ is a D-set in $\mathbb{N}$. We will assume that $v > u$; if $v = u$, let $\delta = 0$ in the argument that follows. We assume that the first $u$ columns of $A$ are linearly independent, let $B$ consist of these first $u$ columns, let $D$ consist of the last $v-u$ columns of $A$, let $k = \det(B)$, which we assume is positive, and let $F = B^{-1}$. Let $P \subseteq k\mathbb{N}$ with $d(P) > \alpha > 0$. By Lemma 3.20, it suffices to show that $d(T^{-1}[P^u]) > 0$.

Let $\delta = \max\big\{|a_{i,j}| : i \in \{1, 2, \ldots, u\}$ and $j \in \{u+1, u+2, \ldots, v\}\big\}$. For $i \in \{1, 2, \ldots, u\}$, let $S_i = \{j \in \{1, 2, \ldots, u\} : f_{i,j} > 0\}$, let $M_i = \{j \in \{1, 2, \ldots, u\} : f_{i,j} < 0\}$, let $s_i = \sum_{j \in S_i} f_{i,j}$, let $m_i = \sum_{j \in M_i} f_{i,j}$, and let

$r_i = (s_i - m_i)(1 + 2\delta k)$. (Note that $S_i$ or $M_i$ could be empty. We take $\sum_{t\in\emptyset} x_t = 0$.)

Let $\gamma = \dfrac{\alpha}{k^{v-u}\prod_{i=1}^{u}(r_i+1)}$. We shall show that $d(T^{-1}[P^u]) \geq \gamma$. To this end, let $n \in \mathbb{N}$. Pick $l \in k\mathbb{N}$ and $a \in \mathbb{N}$ such that $l > n$ and $|P \cap \{a, a+1, \ldots, a+l-1\}| > \alpha \cdot l$. Since $P \subseteq k\mathbb{N}$, we may presume that $a \in k\mathbb{N}$. Let $R = P \cap \{a, a+1, \ldots, a+l-1\}$.

Define $\psi : R^u \times \{k, 2k, \ldots, lk\}^{v-u} \to \mathbb{Z}^v$ by, for $\vec{y} \in R^u$ and $\vec{z} \in \{k, 2k, \ldots, lk\}^{v-u}$, $\psi\begin{pmatrix}\vec{y}\\ \vec{z}\end{pmatrix} = \begin{pmatrix}\vec{x}\\ \vec{z}\end{pmatrix}$, where $\vec{x} = F \cdot (\vec{y} - D\vec{z})$. Then $T\begin{pmatrix}\vec{x}\\ \vec{z}\end{pmatrix} = \vec{y} \in P^u$. Note that $\vec{x} \in \mathbb{Z}^u$ since all entries of $\vec{y}$ and all entries of $\vec{z}$ are divisible by $k$. Note also that $\psi$ is injective.

It now suffices to show that

$$\begin{aligned}&\psi[R^u \times \{k, 2k, \ldots, lk\}^{v-u}]\\ &\subseteq \vec{b} + (\times_{i=1}^{u}\{0, 1, \ldots, (r_i+1)l - 1\} \times \{1, 2, \ldots, lk\}^{v-u})\end{aligned}$$

where $b_i = (s_i + m_i)a + m_i l - (s_i - m_i)\delta lk$ if $i \in \{1, 2, \ldots, u\}$ and $b_i = 0$ if $i \in \{u+1, u+2, \ldots, v\}$ for then $|T^{-1}[P^u] \cap (\vec{b} + \times_{i=1}^{u}\{0, 1, \ldots, (r_i+1)l-1\} \times \{1, 2, \ldots, lk\}^{v-u})| \geq l^{v-u}(\alpha l)^u = \gamma \cdot (lk)^{v-u}\prod_{i=1}^{u}\big((r_i+1)l\big)$. For this, it in turn suffices to let $\vec{y} \in R^u$ and $\vec{z} \in \{k, 2k, \ldots, lk\}^{v-u}$, let $\vec{x} = F \cdot (\vec{y} - D\vec{z})$, and show that for $i \in \{1, 2, \ldots, u\}$, $b_i \leq x_i < b_i + (r_i+1)l$.

Let $\vec{h} = D\vec{z}$ and note that for $j \in \{1, 2, \ldots, u\}$, $|h_j| \leq \delta lk$ and so $a - \delta lk \leq y_j - h_j < a + l + \delta lk$. Let $i \in \{1, 2, \ldots, u\}$. Then $x_i = \sum_{j\in S_i} f_{i,j}(y_j - h_j) + \sum_{j\in M_i} f_{i,j}(y_j - h_j)$ so $x_i \geq s_i(a - \delta lk) + m_i(a + l + \delta lk) = b_i$ and $x_i \leq s_i(a + l + \delta lk) + m_i(a - \delta lk) = b_i + r_i l < b_i + (r_i+1)l$.

<u>Case $\Psi = \mathrm{QC}$</u>. Assume that $C$ is a QC-set in $\mathbb{N}$ and pick an idempotent $p \in \overline{C} \cap c\ell_{\beta\mathbb{N}} K(\beta\mathbb{N})$. Define $T : \mathbb{Z}^v \to \mathbb{Z}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(\mathbb{Z}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. We claim that it suffices to show that for each $P \in p$, there exists $r \in K\big(\beta(\mathbb{N}^v)\big)$ such that $\widetilde{T}(r) \in \overline{P}^u$. Suppose we have done this. For $P \in p$, let $D_P = \{r \in c\ell K\big(\beta(\mathbb{N}^v)\big) : \widetilde{T}(r) \in \overline{P}^u\}$. Then $\{D_P : P \in p\}$ is a collection of closed subsets of $\beta(\mathbb{N}^v)$ with the finite intersection property so, lettin $R = \bigcap\{D_P : P \in p\}$, we have that $R \neq \emptyset$. If $q \in R$, then $\widetilde{T}(q) = \overline{p} = \begin{pmatrix}p\\ \vdots\\ p\end{pmatrix}$. Thus $R$ is a compact subsemigroup of $c\ell K\big(\beta(\mathbb{N}^v)\big)$, so pick an idempotent $q \in R$. Then $\widetilde{T}(q) \in \overline{C}^u = \overline{C^u}$ so that $T^{-1}[C^u]$ is a QC-set.

We set out to show to that for each $P \in p$, there exists $r \in K\big(\beta(\mathbb{N}^v)\big)$ such that $\widetilde{T}(r) \in \overline{P}^u$, so let $P \in p$ be given. By Lemma 3.14, pick $k \in \mathbb{N}$ such that $k\mathbb{Z}^u \subseteq T[\mathbb{Z}^v]$ so that $(c\ell_{\beta\mathbb{Z}} k\mathbb{Z})^u = c\ell_{(\beta\mathbb{Z})^u}(k\mathbb{Z}^u) \subseteq \widetilde{T}[\beta(\mathbb{Z}^v)]$. For each $P \in p$, by Corollary 3.18, $\{\vec{x} \in \mathbb{N}^v : T(\vec{x}) \in P^u\}$ is a C-set in $\mathbb{N}^v$ and in particular is nonempty. Pick by Lemma 3.15 an idempotent $q \in \beta(\mathbb{N}^v)$ such that $\widetilde{T}(q) = \overline{p}$. Let $P^\star = \{x \in P : -x + P \in p\}$. Then $P^\star \cap k\mathbb{N} \in p$ so pick $r \in K(\beta\mathbb{N})$ such that $P^\star \cap k\mathbb{N} \in r$.

Now $\overline{r} = \begin{pmatrix} r \\ \vdots \\ r \end{pmatrix} \in (c\ell_{\beta\mathbb{Z}} k\mathbb{Z})^u \subseteq \widetilde{T}[\beta(\mathbb{Z}^v)]$ and by [14, Theorem 2.23], $\overline{r} \in K\big((\beta\mathbb{N})^u\big) \subseteq K\big((\beta\mathbb{Z})^u\big)$ so $\widetilde{T}[\beta(\mathbb{Z}^v)] \cap K\big((\beta\mathbb{Z})^u\big) \neq \emptyset$ and thus by [14, Theorem 1.65], $K(\widetilde{T}[\beta(\mathbb{Z}^v)]) = \widetilde{T}[\beta(\mathbb{Z}^v)] \cap K\big((\beta\mathbb{Z})^u\big)$. By [14, Exercise 1.7.3], $\widetilde{T}\big[K\big(\beta(\mathbb{Z}^v)\big)\big] = K(\widetilde{T}[\beta(\mathbb{Z}^v)])$. Thus $\overline{r} \in \widetilde{T}\big[K\big(\beta(\mathbb{Z}^v)\big)\big]$ so pick $q' \in K\big(\beta(\mathbb{Z}^v)\big)$ such that $\widetilde{T}(q') = \overline{r}$.

Now $P^\star \in r$ so $P \in r + p$. Then $\widetilde{T}(q' + q) = \widetilde{T}(q') + \widetilde{T}(q) = \overline{r} + \overline{p} = \overline{r + p} \in \overline{P}^u$. So it finally suffices to show that $q' + q \in K\big(\beta(\mathbb{N}^v)\big)$. We have $q' \in K\big(\beta(\mathbb{Z}^v)\big)$, so $q' + q \in K\big(\beta(\mathbb{Z}^v)\big)$. Note that since $q$ is an idempotent in $\beta(\mathbb{N})^v$ and $\mathbb{N}$ has no idempotents, for each $j \in \{1, 2, \ldots, v\}$, $\widetilde{\pi}_j(q) \in \mathbb{N}^*$ and thus $q \in \Theta = \{s \in \beta(\mathbb{N}^v) : (\forall j \in \{1, 2, \ldots, v\})(\widetilde{\pi}_j(s) \in \mathbb{N}^*)\}$. By Lemma 3.2, $\Theta$ is a left ideal of $\beta(\mathbb{Z}^v)$ so $q' + q \in \Theta \cap K\big(\beta(\mathbb{Z}^v)\big) \subseteq \beta\mathbb{N}^v \cap K\big(\beta(\mathbb{Z}^v)\big) = K\big(\beta(\mathbb{N}^v)\big)$. □

**Theorem 4.3.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\{x \in \mathbb{Q} : x \geq 0\}$ which is IPR/$\mathbb{N}$ such that* $\mathrm{rank}(A) = u$. *If $C$ is an SC-set in $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is an SC-set in $\mathbb{N}^v$.*

**Proof.** By Lemmas 3.7 and 3.8 we may assume that the entries of $A$ are in $\omega$. Assume that $C$ is an SC-set in $\mathbb{N}$. Let $M$ be a minimal left ideal of $\beta(\mathbb{N}^v)$ and pick an idempotent $q \in M$. Define $T : \mathbb{Z}^v \to \mathbb{Z}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(\mathbb{Z}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. Note that, since the entries of $A$ come from $\omega$, $\widetilde{T}[\beta(\mathbb{N}^v)] \subseteq (\beta\mathbb{N})^u$. Let $\widetilde{T}(q) = \begin{pmatrix} p_1 \\ \vdots \\ p_u \end{pmatrix} \in (\beta\mathbb{N})^u$. For each $i \in \{1, 2, \ldots, u\}$, pick an idempotent $p_i' \in (\beta\mathbb{N} + p_i) \cap \overline{C}$. By Lemma 3.14, pick $k \in \mathbb{N}$ such that $k\mathbb{Z}^u \subseteq T[\mathbb{Z}^v]$. Then $(c\ell_{\beta\mathbb{Z}} k\mathbb{Z})^u = c\ell_{(\beta\mathbb{Z})^u}(k\mathbb{Z}^u) \subseteq \widetilde{T}[\beta(\mathbb{Z}^v)]$ so we have some $q' \in \beta(\mathbb{Z}^v)$ such that $\widetilde{T}(q') = \vec{p}\,' = \begin{pmatrix} p_1' \\ \vdots \\ p_u' \end{pmatrix}$.

Let $\Theta = \{s \in \beta(\mathbb{N}^v) : (\forall j \in \{1, 2, \ldots, v\})(\widetilde{\pi}_j(s) \in \mathbb{N}^*)\}$. By Lemma 3.2, $\Theta$ is a left ideal of $\beta(\mathbb{Z}^v)$ and a two sided ideal of $\beta(\mathbb{N}^v)$ so $M \subseteq \Theta$ so $q \in \Theta$ and $q' + q \in \Theta \subseteq \beta(\mathbb{N}^v)$. Also $q' + q = q' + q + q \in M$. For each $i \in \{1, 2, \ldots, u\}$, $p_i' \in \beta\mathbb{N} + p_i$ so $p_i' + p_i = p_i'$ and therefore $\widetilde{T}(q' + q) = \vec{p}\,'$. Thus, $\{r \in M : \widetilde{T}(r) = \vec{p}\,'\}$ is a compact subsemigroup of $M$, which therefore has an idemptoent $r$ such that $\widetilde{T}(r) \in \overline{C}^u$. □

**Theorem 4.4.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}$ which is IPR/$\mathbb{N}$. If $C$ is a thick set in $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a thick set in $\mathbb{N}^v$.*

**Proof.** Define $T:\mathbb{N}^v\to\mathbb{Z}^u$ by $T(\vec{x})=A\vec{x}$ and let $\widetilde{T}:\beta(\mathbb{N}^v)\to(\beta\mathbb{Z})^u$ be its continuous extension. Assume that $C$ is a thick set in $\mathbb{N}$, pick a minimal left ideal $L$ of $\beta\mathbb{N}$ such that $L\subseteq\overline{C}$, and pick an idempotent $p\in L$. Given any $P\in p$, $P$ is central so by [14, Theorem 15.24(h)], there exists $\vec{x}\in\mathbb{N}^v$ such that $A\vec{x}\in P^u$. By Lemma 3.15, pick an idempotent $q\in\beta(\mathbb{N}^v)$ such that $\widetilde{T}(q)=\overline{p}=\begin{pmatrix}p\\ \vdots\\ p\end{pmatrix}\in(\beta\mathbb{N})^u$.

We claim that $\mathbb{N}^v+q\subseteq c\ell_{\beta(\mathbb{N}^v)}T^{-1}[C^u]$. To see this, let $\vec{x}\in\mathbb{N}^v$. Then $\widetilde{T}(\vec{x}+q)=T(\vec{x})+\overline{p}\in L^u$ so $\vec{x}+q\in\widetilde{T}^{-1}[L^u]\subseteq\widetilde{T}^{-1}[\overline{C}^u]=c\ell_{\beta(\mathbb{N}^v)}T^{-1}[C^u]$. Thus $\beta(\mathbb{N}^v)+q=c\ell_{\beta(\mathbb{N}^v)}(\mathbb{N}^v+q)\subseteq c\ell_{\beta(\mathbb{N}^v)}T^{-1}[C^u]$ so $C^u$ is thick in $\mathbb{N}^v$. □

**Theorem 4.5.** *Let $u,v\in\mathbb{N}$ and let $A$ be a $u\times v$ matrix with entries from $\omega$ which is IPR/$\mathbb{N}$. If $C$ is a PS*-set in $\mathbb{N}$, then $\{\vec{x}\in\mathbb{N}^v:A\vec{x}\in C^u\}$ is a PS*-set in $\mathbb{N}^v$.*

**Proof.** Define $T:\mathbb{N}^v\to\mathbb{N}^u$ by $T(\vec{x})=A\vec{x}$ and let $\widetilde{T}:\beta(\mathbb{N}^v)\to(\beta\mathbb{N})^u$ be its continuous extension. Assume that $C$ is a PS*-set in $\mathbb{N}$. Pick an idempotent $p\in K(\beta\mathbb{N})$. Given any $P\in p$, $P$ is central so by [14, Theorem 15.24(h)], there exists $\vec{x}\in\mathbb{N}^v$ such that $A\vec{x}\in P^u$. By Lemma 3.15, $\widetilde{T}\big[K\big(\beta(\mathbb{N}^v)\big)\big]=\widetilde{T}[\beta(\mathbb{N}^v)]\cap\big(K(\beta\mathbb{N})\big)^u\subseteq\overline{C}^u$. Consequently $K\big(\beta(\mathbb{N}^v)\big)\subseteq c\ell_{\beta(\mathbb{N}^v)}T^{-1}[C^u]$ so that $T^{-1}[C^u]$ is a PS*-set in $\mathbb{N}^v$. □

**Theorem 4.6.** *Let $u,v\in\mathbb{N}$ and let $A$ be a $u\times v$ matrix with entries from $\{x\in\mathbb{Q}:x\geq 0\}$ which is IPR/$\mathbb{N}$. Let $\Psi$ be central* or QC*. If $C$ is a $\Psi$-set in $\mathbb{N}$, then $\{\vec{x}\in\mathbb{N}^v:A\vec{x}\in C^u\}$ is a $\Psi$-set in $\mathbb{N}^v$.*

**Proof.** By Lemmas 3.7 and 3.8 we may assume that the entries of $A$ are in $\omega$. Define $T:\mathbb{N}^v\to\mathbb{N}^u$ by $T(\vec{x})=A\vec{x}$ and let $\widetilde{T}:\beta(\mathbb{N}^v)\to(\beta\mathbb{N})^u$ be its continuous extension.

<u>Case $\Psi=$ central*</u>. Assume that $C$ is central* in $\mathbb{N}$ so that $E\big(K(\beta\mathbb{N})\big)\subseteq\overline{C}$. Pick an idempotent $p\in K(\beta\mathbb{N})$. By [14, Theorem 15.24(h)], for every $P\in p$ there exists $\vec{x}\in\mathbb{N}^v$ such that $A\vec{x}\in P^u$. By Lemma 3.15, we have that $\widetilde{T}\big[K\big(\beta(\mathbb{N}^v)\big)\big]=\widetilde{T}[\beta(\mathbb{N}^v)]\cap\big(K(\beta\mathbb{N})\big)^u$. We claim that $E\big(K\big(\beta(\mathbb{N}^v)\big)\big)\subseteq c\ell_{\beta(\mathbb{N}^v)}T^{-1}[C^u]$ so that $T^{-1}[C^u]$ is central* in $\mathbb{N}^v$. So let $q$ be an idempotent in $K\big(\beta(\mathbb{N}^v)\big)$. Then $\widetilde{T}(q)$ is an idempotent in $\big(K(\beta\mathbb{N})\big)^u$ so $\widetilde{T}(q)\in\overline{C}^u$ as required.

<u>Case $\Psi=$ QC*</u>. Assume that $C$ is a QC*-set in $\mathbb{N}$. Pick an idempotent $p\in c\ell_{\beta\mathbb{N}}K(\beta\mathbb{N})$. Then for each $P\in p$, $P$ is a C-set so by Corollary 3.18, there exists $\vec{x}\in\mathbb{N}^v$ such that $A\vec{x}\in P^u$. By Lemma 3.15, we have that

$\widetilde{T}\big[K\big(\beta(\mathbb{N}^v)\big)\big] = \widetilde{T}[\beta(\mathbb{N}^v)] \cap \big(K(\beta\mathbb{N})\big)^u\big)$. Thus

$$\begin{aligned}
&\widetilde{T}\big[c\ell_{\beta(\mathbb{N}^v)}K\big(\beta(\mathbb{N}^v)\big)\big] \\
&= c\ell_{(\beta\mathbb{N})^u}\widetilde{T}\big[K\big(\beta(\mathbb{N}^v)\big)\big] \\
&= c\ell_{(\beta\mathbb{N})^u}\big(\widetilde{T}[\beta(\mathbb{N}^v)] \cap \big(K(\beta\mathbb{N})\big)^u\big) \\
&\subseteq \widetilde{T}[\beta(\mathbb{N}^v)] \cap c\ell_{(\beta\mathbb{N})^u}\big(K(\beta\mathbb{N})\big)^u \\
&= \widetilde{T}[\beta(\mathbb{N}^v)] \cap \big(c\ell_{\beta\mathbb{N}}K(\beta\mathbb{N})\big)^u .
\end{aligned}$$

So, if $q$ is an idempotent in $c\ell_{\beta(\mathbb{N}^v)}K\big(\beta(\mathbb{N}^v)\big)$. then $\widetilde{T}(q)$ is an idempotent in $\big(c\ell_{\beta\mathbb{N}}K(\beta\mathbb{N})\big)^u$ and is thus in $\overline{C}^u$. □

**Theorem 4.7.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\{x \in \mathbb{Q} : x \geq 0\}$ which has no row equal to $\vec{0}$. Let $\Psi$ be any of $D^*$, $C^*$, $IP^*$, or $Q^*$. If $C$ is a $\Psi$-set in $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a $\Psi$-set in $\mathbb{N}^v$.*

**Proof.** By Lemmas 3.7 and 3.8 we may assume that the entries of $A$ are in $\omega$. Define $T : \mathbb{N}^v \to \mathbb{N}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(\mathbb{N}^v) \to (\beta\mathbb{N})^u$ be its continuous extension.

Case $\Psi = \mathrm{D}^*$. Assume that $C$ is a D*-set in $\mathbb{N}$. Let $q$ be an idempotent in $\Delta(\mathbb{N}^v)$. We need to show that $\{\vec{x} \in \mathbb{N}^v : T(\vec{x}) \in C^u\} \in q$. Let $\begin{pmatrix} p_1 \\ \vdots \\ p_u \end{pmatrix} = \widetilde{T}(q)$. Let $i \in \{1, 2, \ldots, u\}$. Then $p_i$ is an idempotent. It suffices to show that $p_i \in \Delta(\mathbb{N})$, for then $p_i \in \overline{C}$. So let $D \in p_i$ and pick $B \in q$ such that $\pi_i \circ \widetilde{T}[\,\overline{B}\,] \subseteq \overline{D}$. Then $d(B) > 0$ so by Lemma 3.5, $d(\pi_i \circ T[B]) > 0$ and so $d(D) > 0$.

Case $\Psi = \mathrm{C}^*$ Assume that $C$ is a C*-set in $\mathbb{N}$. Let $q$ be an idempotent in $J(\mathbb{N}^v)$ and let $\begin{pmatrix} p_1 \\ \vdots \\ p_u \end{pmatrix} = \widetilde{T}(q)$. Let $i \in \{1, 2, \ldots, u\}$. Then $p_i$ is an idempotent and by Lemma 3.6, $p_i \in J(\mathbb{N})$ so $C \in p_i$.

Case $\Psi = \mathrm{IP}^*$. Asume that $C$ is an IP*-set in $\mathbb{N}$. Let $q$ be an idempotent in $\beta(\mathbb{N}^v)$. Let $\begin{pmatrix} p_1 \\ \vdots \\ p_u \end{pmatrix} = \widetilde{T}(q)$. Let $i \in \{1, 2, \ldots, u\}$. Then $p_i$ is an idempotent so $C \in p_i$.

Case $\Psi = \mathrm{Q}^*$. Asume that $C$ is a Q*-set in $\mathbb{N}$. Let $B = \{x \in \mathbb{N}^v : A\vec{x} \in C^u\}$ and suppose that $B$ is not a Q*-set in $\mathbb{N}^v$. Then $\mathbb{N}^v \setminus B$ is a Q-set in $\mathbb{N}^v$ so pick a sequence $\langle \vec{s}_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}^v$ such that whenever $m < n$, $\vec{s}_n \in \vec{s}_m + (\mathbb{N}^v \setminus B)$. Write $\vec{s}_n = \begin{pmatrix} s_{n,1} \\ \vdots \\ s_{n,v} \end{pmatrix}$. For each $n \in \mathbb{N}$ and each $j \in \{1, 2, \ldots, v\}$, $s_{n+1,j} > s_{n,j}$. Given $m < n$ in $\mathbb{N}$, we have that $\vec{s}_n - \vec{s}_m \notin B$

so we may pick $i_{m,n} \in \{1,2,\ldots,u\}$ such that $\pi_i \circ T(\vec{s}_n - \vec{s}_m) \notin C$. By Ramsey's Theorem for pairs, pick $i \in \{1,2,\ldots,u\}$ and infinite $M \subseteq \mathbb{N}$ such that whenever $m < n$ in $M$, $i_{m,n} = i$. Enumerate $M$ in order as $\langle k(n)\rangle_{n=1}^\infty$. For $n \in \mathbb{N}$, let $x_n = \sum_{j=1}^v a_{i,j} s_{k(n),j}$. Then whenever $m < n$ in $\mathbb{N}$, $x_n - x_m \in \mathbb{N} \setminus C$, a contradiction. $\square$

**Theorem 4.8.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\omega$ which has no row equal to $\vec{0}$. Let $\Psi$ be any of $B^*$, $J^*$, or $P^*$. If $C$ is a $\Psi$-set in $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a $\Psi$-set in $\mathbb{N}^v$.*

**Proof.** Define $T : \mathbb{N}^v \to \mathbb{N}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(\mathbb{N}^v) \to (\beta\mathbb{N})^u$ be its continuous extension.

Case $\Psi = \mathrm{B}^*$. Assume that $C$ is a B*-set in $\mathbb{N}$. Let $q \in \Delta(\mathbb{N}^v)$. We need to show that $\{\vec{x} \in \mathbb{N}^v : T(\vec{x}) \in C^u\} \in q$. Let $\begin{pmatrix} p_1 \\ \vdots \\ p_u \end{pmatrix} = \widetilde{T}(q)$. Let $i \in \{1,2,\ldots,u\}$. It suffices to show that $p_i \in \Delta(\mathbb{N})$, for then $p_i \in \overline{C}$. So let $D \in p$ and pick $B \in q$ such that $\pi_i \circ \widetilde{T}[\,\overline{B}\,] \subseteq \overline{D}$. Then $d(B) > 0$ so by Lemma 3.5, $d(\pi_i \circ T[B]) > 0$ and so $d(D) > 0$.

Case $\Psi = \mathrm{J}^*$. Assume that $C$ is a J*-set in $\mathbb{N}$. Let $q \in J(\mathbb{N}^v)$ and let $\begin{pmatrix} p_1 \\ \vdots \\ p_u \end{pmatrix} = \widetilde{T}(q)$. Let $i \in \{1,2,\ldots,u\}$. By Lemma 3.6, $p_i \in J(\mathbb{N})$ so $C \in p_i$.

Case $\Psi = \mathrm{P}^*$. Assume that $C$ is P*-set in $\mathbb{N}$. Pick $k \in \mathbb{N}$ such that there do not exist $a, d \in \mathbb{N}$ such that $\{a, a+d, \ldots, a+(k-1)d\} \subseteq \mathbb{N} \setminus C$. By van der Waerden's Theorem, pick $m \in \mathbb{N}$ such that whenever $\{1,2,\ldots,m\}$ is $u$-colored, there is a monochromatic length $k$ arithmetic progression.

Let $B = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ and suppose that $B$ is not a P*-set in $\mathbb{N}^v$. Then $\mathbb{N}^v \setminus B$ is a P-set so pick $\vec{s}$ and $\vec{d}$ in $\mathbb{N}^v$ such that $\{\vec{s}+\vec{d}, \vec{s}+2\vec{d}, \ldots, \vec{s}+m\vec{d}\} \subseteq \mathbb{N}^v \setminus B$. Then for $t \in \{1,2,\ldots,m\}$, $A(\vec{s}+t\vec{d}) \notin C^u$ so pick $i(t) \in \{1,2,\ldots,u\}$ such that $\pi_{i(t)}\big(A(\vec{s}+t\vec{d})\big) \notin C$. Pick $i \in \{1,2,\ldots,u\}$, and $t, c \in \{1,2,\ldots,m\}$ such that $t + kc \leq m$ and $i(t+c) = i(t+2c) = \ldots = i(t+kc) = i$. Let $b = \sum_{j=1}^v a_{i,j}s_j + t \cdot \sum_{j=1}^v a_{i,j}d_j$ and $e = c \cdot \sum_{j=1}^v a_{i,j}d_j$. Then $\{b+e, b+2e, \ldots, b+ke\} \subseteq \mathbb{N} \setminus C$, a contradiction. $\square$

We have no results of the form of Theorem 1.5 for the properties that are implied by syndetic or IP. For those properties that are implied by syndetic, none such are possible because $2\mathbb{N}+1$ is syndetic in $\mathbb{N}$ and if $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, then $\{\vec{x} \in \mathbb{N}^2 : A\vec{x} \in (2\mathbb{N}+1)^2\} = \emptyset$.

In the cases of IP and Q, if $C = FS(\langle 2^{2n}\rangle_{n=1}^\infty)$, then $C$ contains no length 3 arithmetic progression, so if $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$, then $\{\vec{x} \in \mathbb{N}^2 : A\vec{x} \in C^3\} = \emptyset$.

This leaves open the possibility that there may be a positive result if one adds the assumption that $\mathrm{rank}(A) = u$. We form the weakest version as a question.

**Question 4.9.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\omega$ which is IPR/$\mathbb{N}$ such that $\mathrm{rank}(A) = u$.*

1. *If $C$ is an IP-set in $\mathbb{N}$, must $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ be an IP-set in $\mathbb{N}^v$?*
2. *If $C$ is a Q-set in $\mathbb{N}$, must $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ be a Q-set in $\mathbb{N}^v$?*

By Lemmas 3.9 and 3.10 we see that the restrictions on the entries of $A$ in all of the results of this section are needed.

For our results about C, D, QC, central, SC, SC*, thick, PS*, central*, and QC* we assume that the matrix $A$ is IPR/$\mathbb{N}$. That assumption is necessary for C, D, QC, and central because these are partition regular notions. We do not know whether that assumption (rather than the weaker assumption that no row is $\vec{0}$) is needed for the other listed notions. However, the following result tells us that, if $\mathrm{rank}(A) = u$, the IPR/$\mathbb{N}$ assumption is needed in Theorem 4.4 for thick and in Theorem 4.1 for SC*.

**Theorem 4.10.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}$ such that $\mathrm{rank}(A) = u$. If for every thick set $C$ in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\} \neq \emptyset$, then $A$ is IPR/$\mathbb{N}$.*

**Proof.** Define $T : \mathbb{Z}^v \to \mathbb{Z}^u$ by $T(\vec{x}) = A\vec{x}$ and let $\widetilde{T} : \beta(\mathbb{Z}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension.

Pick a minimal left ideal $L$ of $\beta\mathbb{N}$ and let $\mathcal{D} = \{P \subseteq \mathbb{N} : L \subseteq \overline{P}\}$ and direct $\mathcal{D}$ by reverse inclusion. Given $P \in \mathcal{D}$, $P$ is a thick set so we may pick $\vec{x}_P$ such that $A\vec{x}_P \in P^u$. Let $q$ be a cluster point of the net $\langle \vec{x}_P \rangle_{P \in \mathcal{D}}$ in $\beta\mathbb{N}^v$. We claim that $\widetilde{T}(q) \in L^u$. To see this, let $i \in \{1, 2, \ldots, u\}$ and suppose $\pi_i \circ \widetilde{T}(q) \notin L$. Pick $R \in \pi_i \circ \widetilde{T}(q)$ such that $\overline{R} \cap L = \emptyset$ and pick $B \in q$ such that $\pi_i \circ \widetilde{T}[\,\overline{B}\,] \subseteq \overline{R}$. Pick $P \in \mathcal{D}$ such that $P \subseteq \mathbb{N} \setminus R$ and $\vec{x}_P \in B$. Since $\vec{x}_P \in B$, $\pi_i\big(T(\vec{x}_P)\big) \in R$ while $T(\vec{x}_P) \in P^u$, a contradiction.

Since $q \in \widetilde{T}^{-1}[L^u] \cap \beta(\mathbb{N}^v)$, we have that $\widetilde{T}^{-1}[L^u] \cap \beta(\mathbb{N}^v)$ is a left ideal of $\beta(\mathbb{N}^v)$ so we may pick a minimal left ideal $L'$ of $\beta(\mathbb{N}^v)$ such that $L' \subseteq \widetilde{T}^{-1}[L^u] \cap \beta(\mathbb{N}^v)$. Pick an idempotent $w \in L'$ and let $\widetilde{T}(w) = \begin{pmatrix} s_1 \\ \vdots \\ s_u \end{pmatrix}$. Then each $s_i$ is an idempotent in $L$.

Pick an idempotent $p \in L$. By Lemma 3.14, pick $k \in \mathbb{N}$ such that $k\mathbb{Z}^u \subseteq T[\mathbb{Z}^v]$, so that $c\ell_{\beta(\mathbb{Z}^u)}(k\mathbb{Z}^u) \subseteq \widetilde{T}[\beta(\mathbb{Z}^v)]$. Pick $r \in \beta(\mathbb{Z}^v)$ such that $\widetilde{T}(r) = \overline{p} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix}$.

Let $\Theta = \{s \in \beta(\mathbb{N}^v) : (\forall j \in \{1,2,\ldots,v\})(\widetilde{\pi}_j(s) \in \mathbb{N}^*)\}$. Then $w \in \Theta$ so by Lemma 3.2, $r+w \in \Theta \subseteq \beta(\mathbb{N}^v)$. Since $p+s_i = p$ for each $i \in \{1,2,\ldots,u\}$ we have that $\widetilde{T}(r+w) = \overline{p}$.

To see that $A$ is IPR/$\mathbb{N}$, let a finite coloring of $\mathbb{N}$ be given and pick a color class $B$ which is a member of $p$. Then $T^{-1}[B^u] \cap \mathbb{N}^v \in r+w$ so is nonempty. □

We note now that the assumption that no row of $A$ is $\vec{0}$ is indeed weaker than the assumption that $A$ is IPR/$\mathbb{N}$, even if the assumption that $\mathrm{rank}(A) = u$ is added. The matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ has nonnegative entries, no row equal to 0, and $\mathrm{rank}(A) = 2$ and $A$ is not IPR/$\mathbb{N}$. If it were, by [14, Theorem 15.24(b)] there would exist positive rationals $t_1$ and $t_2$ such that $B = \begin{pmatrix} t_1 & t_2 & -1 & 0 \\ 2t_1 & 3t_2 & 0 & -1 \end{pmatrix}$ satisfies the columns condition. The only values of $t_1$ and $t_2$ making $B$ satisfy the columns condition are $t_1 = 2$ and $t_2 = -1$.

For our results about D, QC, and SC we assume that $\mathrm{rank}(A) = u$, the number of rows of $A$. We do not know whether that assumption is needed for any of these notions. In the case of SC we suspect it may be needed since we cannot answer the following question.

**Question 4.11.** *Let* $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ *and let $D$ be the set defined in* [12, *Definition* 4.1] *(which was shown to be QC but not central). Is* $\{\vec{x} \in \mathbb{N}^2 : A\vec{x} \in D^3\}$ *a QC-set in* $\mathbb{N}^2$*?*

We remark that similar results are easily obtainable if the restrictions on negative entries of $A$ are deleted, the assumption that $A$ is IPR/$\mathbb{N}$ is replaced by the assumption that $A$ is IPR/$\mathbb{Z}$, and the conclusion that $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a $\Psi$-set in $\mathbb{N}^v$ is replaced by the conclusion that $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in C^u\}$ is a $\Psi$-set in $\mathbb{Z}^v$. From a Ramsey Theoretic point of view, these results are much less interesting because some of the entries of $\vec{x}$ are allowed to be zero. (Consider $A\vec{x}$ where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.)

## 5. Related results and generalizations

In the preceding sections, we studied the properties of a continuous homomorphism $\widetilde{T} : \beta(\mathbb{Z}^v) \to (\beta\mathbb{Z})^u$, where $u$ and $v$ are positive integers. It seems worth pointing out that no continuous function from $\beta(\mathbb{Z}^v)$ to $(\beta\mathbb{Z})^u$ can be bijective if $u > 1$. This was shown by Glicksberg [7] in 1959. An easy way to see that $\beta(\mathbb{Z}^v)$ and $(\beta\mathbb{Z})^u$ are not homeomorphic is to note that by [6, Theorem 14.25], $\beta(\mathbb{Z}^v)$ is an F-space while by [6, Exercise 14Q(1)], $(\beta\mathbb{Z})^u$ is not.

We also realized that $\beta(\mathbb{Z}^v)$ and $(\beta\mathbb{Z})^u$ are not algebraically isomorphic if $u > 1$, after we received an email from Aninda Chakraborty asking whether they might be. So no homomorphism from $\beta\mathbb{Z}^v$ to $(\beta\mathbb{Z})^u$ can be bijective. We checked with some experts and the following result appears to be new.

**Theorem 5.1.** *Let $u, v \in \mathbb{N}$ with $u > 1$. Then $\beta(\mathbb{Z}^v)$ and $(\beta\mathbb{Z})^u$ are not isomorphic and $\beta(\mathbb{N}^v)$ and $(\beta\mathbb{N})^u$ are not isomorphic .*

**Proof.** By [14, Corollary 6.23], if $e$ and $f$ are idempotents in $\beta(\mathbb{Z}^v)$ and $(\beta(\mathbb{Z}^v)+e)\cap(\beta(\mathbb{Z}^v)+f)\neq\emptyset$, then $e+f=e$ or $f+e=f$; in particular if $e$ and $f$ are idempotents in $\beta(\mathbb{N}^v)$ and $(\beta(\mathbb{N}^v)+e)\cap(\beta(\mathbb{N}^v)+f)\neq\emptyset$, then $e+f=e$ or $f+e=f$. We show now that the corresponding statement is not valid in $(\beta\mathbb{N})^u$ and thus not in $(\beta\mathbb{Z})^u$. To see this pick an idempotent $p$ in $\beta\mathbb{N}\setminus K(\beta\mathbb{N})$. By [14, Theorem 1.60] pick an idempotent $q$ in $K(\beta\mathbb{N})$ such that $q < p$. Let $e = (p, p, \ldots, p, q) \in (\beta\mathbb{Z})^u$ and $f = (q, q, \ldots, q, p) \in (\beta\mathbb{Z})^u$. Then $e + f = f + e$ so $\big((\beta\mathbb{N})^u + e\big) \cap \big((\beta\mathbb{N})^u + f\big) \neq \emptyset$. However, $e + f \neq e$ and $f + e \neq f$, because $q = p + q = q + p \neq p$, since $p \notin K(\beta\mathbb{N})$. $\square$

The case $I = K(\beta\mathbb{N})$ of the following theorem was proved in [11, Theorem 2.12]. We remark that it follows from Theorem 5.2, that every member of any idempotent in $J(\mathbb{N})$ contains $2^{\mathfrak{c}}$ idempotents in $J(\mathbb{N})$, and every member of any idempotent in $\Delta(\mathbb{N})$ contains $2^{\mathfrak{c}}$ idempotents in $\Delta(\mathbb{N})$. We write the operation additively in this theorem since we are mostly concerned with additive semigroups in this paper, but we are not assuming that $S$ is commutative.

**Theorem 5.2.** *Let $(S, +)$ be a countable discrete left cancellative semigroup, let $I$ be an ideal of $\beta S$, let $p$ be an idempotent in $I \cap S^*$, and let $P \in p$. Then $\overline{P} \cap I$ contains $2^{\mathfrak{c}}$ idempotents.*

**Proof.** Let $P^\star = \{x \in P : -x + P \in p\}$. Then by [14, Lemma 4.14], $P^\star \in p$ and for each $x \in P^\star$, $-x + P^\star \in p$. For $F \in \mathcal{P}_f(P^\star)$, let $R_F = P^\star \cap \bigcap_{s\in F}(-s + P^\star)$. Let $V = \bigcap\{\overline{R_F} : F \in \mathcal{P}_f(P^\star)\}$ and note that $p \in V$. We claim that $V$ is a subsemigroup of $\beta S$. To see this, we use [14, Theorem 4.20]. Let $F \in \mathcal{P}_f(P^\star)$ and let $x \in R_F$. Then $x \in P^\star$ and $F + x \subseteq P^\star$. Let $G = \{x\} \cup (F + x)$. We show that $x + R_G \subseteq R_F$, so let $y \in R_G$. Then $y \in -x + P^\star$ so $x + y \in P^\star$. Given $z \in F$, $z + x \in G$ so $y \in -(z + x) + P^\star$ and thus $x + y \in -z + P^\star$ as required. By [14, Corollary 4.29], $S^*$ is a subsemigroup of $\beta S$ and $p \in V \cap S^*$, so $V \cap S^*$ is a subsemigroup of $\beta S$. Let $V' = V \cap S^*$.

Since $S$ is countable, $S^*$ is a $G_\delta$ set in $\beta S$. So we have that $V'$ is a $G_\delta$ subset of $S^*$, so by [14, 6.32], $V'$ contains a topological and algebraic copy of $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$. By [14, Lemma 6.8], $\mathbb{H}$ contains all of the idempotents of $(\beta\mathbb{N}, +)$. By [14, Theorem 6.9], $\beta\mathbb{N}$ contains $2^{\mathfrak{c}}$ minimal left ideals. Choosing one idempotent from each minimal left ideal of $\beta\mathbb{N}$, one has a set $W$ of $2^{\mathfrak{c}}$ idempotents with the property that if $q$ and $r$ are distinct members of $W$, then $q + r \neq q$ and $r + q \neq r$. Therefore, there is a set $T$ of $2^{\mathfrak{c}}$ idempotents in $V'$ with the same property. We claim that $(\beta S+q)\cap(\beta S+r)=\emptyset$ if $q$ and $r$ are distinct elements of $T$. To see this, suppose that $(\beta S+q)\cap(\beta S+r)\neq\emptyset$. By [14, Theorem 6.19] we may suppose that there exists $s\in S$ and $x\in\beta S$ such that $s+q=x+r$. Then $s+q+r=x+r=s+q$. It follows [14, Lemma 8.1] that $q+r=q$, contradicting the choice of $T$.

Since $I$ is an ideal of $\beta S$ and $p\in I\cap V'$, we have that $I\cap V'$ is an ideal of $V'$ so that $K(V')\subseteq I$. Given $q\in T$, pick a minimal left ideal $L_q$ of $V'$ with $L\subseteq V'+q$. Then if $q$ and $r$ are distinct members of $T$, $L_q\cap L_r=\emptyset$ and each $L_q$ contains an idempotent which is a member of $I\cap\overline{P}$. □

By a dynamical system we mean a pair $(X,\langle T_s\rangle_{s\in S})$, where $X$ is a compact Hausdorff space, $S$ is a discrete semigroup, $T_s:X\to X$ is a continuous function for every $s\in S$ and $T_s\circ T_t=T_{st}$ for every $s,t\in S$. If $X$ is a compact space, $C(X)$ will denote the space of continuous real-valued functions defined on $X$, with the uniform norm, and $C(X)'$ will denote its dual space. Given a probability measure $\mu$ defined on the Borel subsets of $X$, we shall also view $\mu$ as a linear functional defined on $C(X)$ by $\mu(f)=\int f\,d\mu$. If $(X,\langle T_s\rangle_{s\in S})$ is a dynamical system, we shall say that a Borel measure $\mu$ defined on $X$ is $S$-invariant if $\mu(f)=\mu(f\circ T_s)$ for every $f\in C(X)$ and every $s\in S$. It is well known, and easy to prove, that this is equivalent to the condition that $\mu(T_s^{-1}[B])=\mu(B)$ for every $s\in S$ and every Borel subset $B$ of $X$.

**Theorem 5.3.** *Let $(X,\langle Q_r\rangle_{r\in R})$ and $(Y,\langle T_s\rangle_{s\in S})$ be dynamical systems, and assume that $S$ is left amenable. Let $\phi:Y\to X$ be a continuous surjection and let $\mu$ denote a probability measure on $X$ which is $R$ invariant. Assume that, for each $s\in S$, there exists $r\in R$ such that $\phi\circ T_s=Q_r\circ\phi$. Then there is a probability measure $\nu$ on $Y$ which is $S$-invariant, such that $\nu(f\circ\phi)=\mu(f)$ for every $f\in C(X)$.*

**Proof.** Let $L$ denote the linear subspace $\{f\circ\phi : f\in C(X)\}$ of $C(Y)$. Observe that the map $f\mapsto f\circ\phi$ is an isometry mapping $C(X)$ into $C(Y)$, and that $L$ contains the constant function $\bar{1}$ equal to 1 at every point of $Y$. Define $\rho$ on $L$ by putting $\rho(f\circ\phi)=\mu(f)$ for every $f\in C(X)$. Then $\rho$ is a linear functional of norm 1 on $L$, and $\nu(\bar{1})=1$. By the Hahn Banach Theorem, $\rho$ can be extended to a linear functional of norm 1 on $C(Y)$.

Let $K=\{\nu\in C(Y)' : \nu(\bar{1})=1\,,\ \|\nu\|=1$ and $\nu(f\circ\phi)=\mu(f)\big(\forall f\in C(X)\big)\}$. We have seen that $K$ is nonempty. For every $s\in S$ and every $\nu\in C(Y)'$, we define $\theta_s(\nu)\in C(Y)'$ by putting $\big(\theta_s(\nu)\big)(g)=\nu(g\circ T_s)$ for every $g\in C(Y)$. We claim that, for every $s\in S$, $\theta_s(\nu)\in K$ if $\nu\in K$. To see this, let $f\in C(X)$, let $s\in S$ and let $\nu\in K$. We are assuming that $\phi\circ T_s=Q_r\circ\phi$ for some $r\in R$. So $\nu(f\circ\phi\circ T_s)=\nu(f\circ Q_r\circ\phi)=\mu(f\circ Q_r)=\mu(f)$, and so $\theta_s(\nu)\in K$. Since $K$ is a convex weak* compact subset of $C(Y)'$, it follows from Day's fixed point theorem ([16, Theorem 1.14]) that we can choose a member $\nu$ of $L$ with the property that $\nu=\theta_s(\nu)$ for every $s\in S$. So $\nu(g)=\nu(g\circ T_s)$ for every $g\in C(X)$ and every $s\in S$. By the Riesz

Representation Theorem, $\nu$ can de regarded as a probability measure on $Y$. □

In the following corollary, we use the relationship between the concept of density for subsets of a discrete semigroup $S$ and probability measures on $\beta S$ which are invariant under translation by elements of $S$. This is a powerful tool in analyzing the Ramsey theoretic properties of subsets of $S$ of positive density. If $S$ is left cancellative and left amenable, it is well-known that a subset $A$ of $S$ has positive Følner density if and only if there is a probability measure $\mu$ on $\beta S$, invariant under translation by elements of $S$, for which $\mu(\,\overline{A}\,) > 0$. See [13, Theorem 2.14].

We conclude by proving the case $\Psi = \mathrm{D}$ of Theorem 4.2 as a corollary of Theorem 5.3. Our motive for doing so is that the proof is fairly short and applies to semigroups other than $\mathbb{N}$. For example, it is clear that the proof applies to $\mathbb{Z}$ and to any commutative group in which multiplication by rationals is defined.

**Corollary 5.4.** *Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Q}$ which is IPR/$\mathbb{N}$ such that* $\mathrm{rank}(A) = u$. *If $C$ is a D-set in $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is a D-set in $\mathbb{N}^v$.*

**Proof.** By Lemmas 3.7 and 3.8 we may assume that the entries of $A$ are integers. Let $T$ denote the mapping from $\mathbb{Z}^v$ to $\mathbb{Z}^u$ defined by $A$, and let $\widetilde{T} : \beta\mathbb{Z}^v \to (\beta\mathbb{Z})^u$ denote its continuous extension. Assume that $C$ is a D-set in $\mathbb{N}$ and pick an idempotent $p \in \Delta(\mathbb{N})$ such that $C \in p$. We note from Figure 1 that every member of $p$ is a C-set. So, for every $P \in p$, there exists $\vec{x}_P \in \mathbb{N}^v$ such that $A\vec{x}_P \subseteq P^u$. Direct $p$ by reverse inclusion and let $q$ be a limit point of the net $\langle \vec{x}_P \rangle_{P \in p}$ in $\beta(\mathbb{N}^v)$. Then $\widetilde{T}(q) = \overline{p} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \in (\beta\mathbb{N})^u$.

By Lemma 3.14, there exists $k \in \mathbb{N}$ such that $k\mathbb{Z}^u \subseteq T(\mathbb{Z}^v)$. By [14, Lemma 6.6], $k\mathbb{N} \in p$. Let $P \in p$, with $P \subseteq k\mathbb{N}$. Since $P$ has positive density in $\mathbb{N}$, there is a probability measure $\mu$ on $\beta\mathbb{N}$, invariant under translations by elements of $\mathbb{Z}$, such that $\mu(\overline{P}) > 0$. Let $\mu^u$ denote the product measure $\mu \otimes \mu \otimes \ldots \otimes \mu$ defined on $(\beta\mathbb{N})^u$. Observe that $\mu^u$ is a probability measure invariant under translations by elements of $\mathbb{Z}^u$. We shall apply Theorem 5.3 with $Y = \beta(\mathbb{Z}^v)$, $X = \widetilde{T}[Y]$, $\phi = \widetilde{T}$, $S = \mathbb{Z}^v$, $R = \mathbb{Z}^u$, $T_{\vec{z}} = \lambda_{\vec{z}}$ for every $\vec{z} \in \mathbb{Z}^v$, and $Q_{\vec{z}} = \lambda_{\vec{z}}$ for every $\vec{z} \in \mathbb{Z}^u$. The hypotheses of Theorem 5.3 are satisfied because, for every $\vec{y}$ and $\vec{z}$ in $\mathbb{Z}^v$, $T(\vec{z} + \vec{y}) = T(\vec{z}) + T(\vec{y})$ and so $\phi \circ \lambda_{\vec{z}} = \lambda_{T(\vec{z})} \circ \phi$.

So, by Theorem 5.3, there is an $S$-invariant probability measure $\nu$ on $\beta\mathbb{Z}^v$ for which $\nu(\phi^{-1}[\overline{P}]^u) > 0$. It follows that $T^{-1}[P^u]$ has positive density in $\beta(\mathbb{Z}^v)$ and hence that $\overline{T^{-1}[P^u]} \cap \Delta(\mathbb{Z}^v) \neq \emptyset$ by Lemma 2.4. Let $\mathcal{P}$ denote the set of all members of $p$ contained in $k\beta\mathbb{N}$. Then $\widetilde{T}^{-1}[\{\overline{p}\}] \cap \Delta(\mathbb{Z}^v) = \bigcap_{P \in \mathcal{P}} \overline{T^{-1}[P^u]} \cap \Delta(\mathbb{Z}^v) \neq \emptyset$. Since $\Delta(\mathbb{Z}^v)$ is an ideal in $\beta\mathbb{Z}^v$ by Theorem 3.1,

$K(\widetilde{T}^{-1}[\{\overline{p}\}]) \subseteq \Delta(\mathbb{Z}^v)$. We can choose a minimal idempotent $q'$ of $\widetilde{T}^{-1}[\{\overline{p}\}]$ in the left ideal $\widetilde{T}^{-1}[\{\overline{p}\}] + q$ of $\widetilde{T}^{-1}[\{\overline{p}\}]$. Let

$$\Theta = \{s \in \beta(\mathbb{N}^v) : (\forall j \in \{1, 2, \ldots, v\})(\widetilde{\pi}_j(s) \in \mathbb{N}^*)\}\,.$$

Then $q \in \Theta$ so by Lemma 3.2, $q' \in \Theta \subseteq \beta(\mathbb{N}^v)$. We have shown that $\widetilde{T}^{-1}(C^u)$ is a D-set in $\mathbb{N}^v$. $\square$

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