This paper was published in *Combinatorics, Probability, and Computing*, **12** (2003), 571-583. To the best of my knowledge this is the final version as it was submitted to the publisher. – NH.

OPEN PROBLEMS IN PARTITION REGULARITY

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Abstract

A finite or infinite matrix A with rational entries is called partition regular if whenever the natural numbers are finitely coloured there is a monochromatic vector x with $Ax = 0$. Many of the classical theorems of Ramsey Theory may naturally be interpreted as assertions that particular matrices are partition regular.

While in the finite case partition regularity is well understood, very little is known in the infinite case. Our aim in this paper is to present some of the natural and appealing open problems in the area.

§0. Introduction

Let A be an $m \times n$ matrix with rational entries. We say that A is *kernel partition regular*, or simply *partition regular*, if for every finite colouring of the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ there is a monochromatic vector $x \in \mathbb{N}^n$ with $Ax = 0$. In other words, A is partition regular if for every positive integer k, and every function $c : \mathbb{N} \to \{1, \ldots, k\}$, there is a vector $x = (x_1, \ldots, x_n) \in \mathbb{N}^n$ with $c(x_1) = \ldots = c(x_n)$ such that $Ax = 0$. We may also speak of the 'system of equations $Ax = 0$' being partition regular.

Many of the classical results of Ramsey Theory may naturally be considered as statements about partition regularity. For example, Schur's Theorem [29], that in any finite colouring of the natural numbers we may solve $x + y = z$ in one colour class, is precisely the assertion that the 1×3 matrix $(1, 1, -1)$ is partition regular. As another example,

† This author acknowledges support received from the National Science Foundation (USA) via grant DMS-0070593.

the theorem of van der Waerden [32] that, for any m, every finite colouring of the natural numbers contains a monochromatic arithmetic progression with m terms, is (with the strengthening that we may also choose the common difference of the sequence to have the same colour) exactly the statement that the $(m-1) \times (m+1)$ matrix

$$\begin{pmatrix} 1 & 1 & -1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & -1 & \dots & 0 & 0 \\ 1 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

is partition regular.

Note that not all matrices are partition regular. For example, the 1×2 matrix $(2, -1)$ is not partition regular. Indeed, if it were then in any finite colouring of the natural numbers there would exist an x such that x and $2x$ had the same colour, and this is certainly not the case (for example, colour x by the parity of the largest n such that 2^n divides x).

The partition regular matrices were characterised by Rado [28]. To state Rado's result, we need a small amount of notation. Let A be an $m \times n$ rational matrix, with columns $a^{(1)}, \dots, a^{(n)}$. We say that A has the *columns property* if there is a partition of $[n] = \{1, \dots, n\}$, say $[n] = D_0 \cup D_1 \cup \dots \cup D_k$, such that

$$\sum_{i \in D_0} a^{(i)} = 0$$

and for every $r = 1, \dots, k$ we have

$$\sum_{i \in D_r} a^{(i)} \in \left\langle a^{(i)} : i \in D_0 \cup \dots \cup D_{r-1} \right\rangle,$$

where $\langle \cdot \rangle$ denotes rational linear span. For example, a $1 \times n$ matrix has the columns property if and only if some of its (non-zero) entries sum to zero, so that the above matrix $(1, 1, -1)$ certainly has the columns property. It is also easy to see that the $(m-1) \times (m+1)$ van der Waerden matrix above has the columns property.

Rado [28] proved that a matrix is partition regular if and only if it has the columns property. This reduces partition regularity to a property that is very tangible and, moreover, can be checked in finite time for any particular matrix. One of the remarkable features of Rado's Theorem is that neither direction is obvious. For general background about Rado's Theorem, see Deuber [7] or Graham, Rothschild and Spencer [11]. However, we stress that this paper can be read without any background knowledge of this sort.

This seminal work of Rado in the 1930s initiated the study of partition regularity. Another viewpoint, and another characterisation, were introduced by Deuber in the 1970s, and used by him to clarify and answer some questions left open by Rado. Roughly speaking, Deuber proved that the only partition regular systems consist of iterated versions of arithmetic progressions. More precisely, for positive integers m, p, c, and positive integers $u_1, \ldots, u_m$, the (m, p, c)*-set* generated by $u_1, \ldots, u_m$ consists of all sums of the form

$$\sum_{i=k}^{m} \lambda_i u_i,$$

where $k \in \{1, 2, \ldots, m\}$, $\lambda_k = c$, and $\lambda_i \in \{1, 2, \ldots, p\}$ for all $i > k$. Thus for example a $(2, p, 1)$-set is just an arithmetic progression of p terms, together with its common difference. What Deuber showed [6] is that a matrix A is partition regular if and only if there exist m, p, c such that every (m, p, c)-set contains a solution of $Ax = 0$. Again, see [7] or [11] for a full discussion of this.

Thanks to this work of Rado and Deuber, the situation for finite matrices is now in many ways well understood, although there are a few outstanding open questions. However, in the infinite case far less is known. If A is an infinite matrix, with rational entries and only finitely many non-zero entries in each row, we say as before that A is *partition regular* if whenever the natural numbers are finitely coloured there is a monochromatic vector x with $Ax = 0$.

The first example of a (non-trivial) infinite partition regular system of equations was constructed in 1974 [12], proving a conjecture of Graham and Rothschild: in any finite colouring of the natural numbers there is a sequence $x_1, x_2, \ldots$ of natural numbers such that the set

$$FS(x_1, x_2, \ldots) = \left\{ \sum_{i \in I} x_i : \ I \subset \mathbb{N}, \, I \text{ finite and non-empty} \right\}$$

is monochromatic. This is also known as the Finite Sums Theorem. (It is worth remarking that the finite analogue of this, known as Folkman's Theorem, stating that, for any m, in any finite colouring of the natural numbers we may find $x_1, \ldots, x_m$ with $FS(x_1, \ldots, x_m)$ monochromatic, follows easily from Rado's Theorem). While some other infinite partition regular systems are now known, the general problem of deciding which systems are partition regular looks hopelessly out of reach at present.

We believe that there are a number of open questions about partition regularity that are extremely attractive. In many cases, these are questions for which it seems absurd that they could still be unanswered. Our aim in this paper is to present some of these questions.

We should mention that this is not supposed to be a survey paper. We have made no attempt to give a comprehensive list of results. For a wide-ranging survey, the reader may consult Deuber [7]. For general background on all aspects of Ramsey theory, see [11]. Let us also point out that a large part of the modern approach to partition regularity is based on ultrafilters, or more precisely the structure of the space $\beta\mathbb{N}$, the Stone-Čech compactification of the natural numbers. For example, the Finite Sums Theorem follows immediately from the existence of an idempotent for addition on $\beta\mathbb{N}$. To keep this paper self-contained, however, we have refrained from giving questions that directly involve ultrafilters. We refer the interested reader to [18] for a comprehensive introduction to ultrafilters and their uses.

The plan of the paper is as follows. In Section 1 we consider the finite case, pointing out the beautiful questions that still remain open despite the work of Rado and Deuber. In Section 2 we turn to the infinite case. We look at the general question of which systems of equations are partition regular. Most particular instances of this question have not been answered. In Section 3 we look at the relationship between partition regularity over different spaces: the natural numbers, the rationals and the reals. Here again, while in the finite case everything is known, very little is known for infinite systems. In Section 4 we examine another notion of partition regularity, called image partition regularity. This is in many ways an even more fundamental and natural notion than that of partition regularity itself. The connections between it and the usual (kernel) partition regularity are still not fully understood. Finally, in Section 5 we turn our attention to 'sparse' results. This is concerned with just how much structure a subset of the natural numbers needs to have in order to support a partition theorem that is true for the set of natural numbers itself.

§1. Finite systems

There are two important questions thrown up by Rado's Theorem that have still not been answered. In Rado's Theorem itself, neither direction of the implication is obvious, but which way do we expect to be harder? One would imagine that going from partition regular to columns property ought to be easier, as it should just be a matter of choosing some clever colourings and extracting information from them, whereas going from columns property to partition regularity one should actually need to prove some Ramsey theorems.

This is correct, but with some reservations. There are by now several ways known to prove partition regularity from the columns property, often based on van der Waerden's Theorem or the Hales-Jewett Theorem (the abstract analogue of van der Waerden's

Theorem – see [11]). But there is still essentially only one way known to go in the other direction. For each prime p, define a colouring c_p of the natural numbers as follows: for a natural number x, we set $c_p(x)$ to be the least significant non-zero digit in the base p expansion of x. By considering a monochromatic solution for a colouring c_p, with p large, it turns out that one can read off the columns property.

But this raises an intriguing question. Having proved Rado's Theorem, we know that if a matrix A is partition regular for each of the colourings c_p (meaning the obvious thing: for each c_p there is a monochromatic vector x with $Ax = 0$) then A is partition regular for *all* colourings. So the c_p colourings in some sense 'generate' all colourings. But is there a direct (as opposed to via Rado's Theorem) proof of this?

Question 1. *Is there a direct proof that if a matrix is partition regular for every c_p colouring then it is partition regular for every finite colouring?*

A positive answer to this might open the door to a similar attack on colourings for infinite matrices – this would be very important, in view of the fact that we do not have the luxury of a classification theorem for infinite partition regular matrices.

Another obvious question is the following. Suppose we have an $m \times n$ matrix A that is *not* partition regular. Then we know that there is a bad k-colouring, for some k. How large need k be? Is there a bound on k (for fixed m and n)? This is Rado's Boundedness Conjecture [28].

Question 2. *Given m and n, is there a $k = k(m, n)$ such that any $m \times n$ matrix that is partition regular for all k-colourings must be partition regular for all finite colourings?*

Remarkably, the answer to this is not known for *any* non-trivial values of m and n. It is not even known for $m = 1$ and $n = 3$. In other words, it is not known if there exists a constant k such that whenever the matrix (a, b, c) is not partition regular there is a bad colouring with k colours. The best that is known is merely that k must be at least 4. It is amusing to note that, if one tries to tackle this by joining together some of the colour classes in the c_p colourings, then one finds that the result one would need to make this approach work is precisely Rado's Boundedness Conjecture itself.

There is another major open question on finite systems, concerning mixing addition and multiplication. Results where we just replace sums by products are straightforward:

for example, to show that whenever the natural numbers are finitely coloured there must exist x and y with x, y, xy all of the same colour we just consider the set $\{2^n : n \in \mathbb{N}\}$ and apply Schur's Theorem.

Things are very different if we wish to involve both addition and multiplication. There is a positive result in this direction [13]: whenever the natural numbers are finitely coloured, there is a colour class containing all finite sums from some sequence $x_1, x_2, \ldots$ and also all finite products from some sequence $y_1, y_2, \ldots$. What one would like, though, is an analogue of Folkman's Theorem in which addition and multiplication may be iterated. But even the most trivial case of this is unknown.

Question 3. *If the natural numbers are finitely coloured, must there exist x, y with $x, y, x+y, xy$ monochromatic?*

It is rather extraordinary that this is not even known if we do not care about the colour of x and y. In other words, it is no known if in any finite colouring of the natural numbers there exist x and y such that $x + y$ and xy have the same colour.

§2. Infinite systems

Apart from the Finite Sums Theorem, which infinite systems of equations are partition regular? The first extension was the Milliken-Taylor Theorem [20,31]. For a k-tuple $a = (a_1, \ldots, a_k)$ of natural numbers, and a sequence $x_1, x_2, \ldots$ of natural numbers, we write $FS_a(x_1, x_2, \ldots)$ for the set of all sums of the form

$$\sum_{i \in I_1} a_1 x_i + \sum_{i \in I_2} a_2 x_i + \ldots + \sum_{i \in I_k} a_k x_i \ ,$$

where $I_1, \ldots, I_k$ are non-empty finite subsets of $\mathbb{N}$ such that $\max I_t < \min I_{t+1}$ for all t. Then the Milliken-Taylor Theorem asserts that, for any a, whenever the natural numbers are finitely coloured there exists a sequence $x_1, x_2, \ldots$ such that $FS_a(x_1, x_2, \ldots)$ is monochromatic.

The careful reader will notice that, as it stands, this system is not quite given as the kernel of some linear equations, because the variables x_i are not required to be the same colour. However, this is not a problem, as we may rewrite the equations in terms of some different variables (see the discussion of $B(A)$ in Section 4, for example). Alternatively, we may think of the theorem as asserting that a certain matrix has a monochromatic vector in its *image* instead of in its kernel – this point will be explored more fully in Section 4.

Let us briefly remark that, roughly speaking, the Milliken-Taylor Theorem may be proved by copying down the proof of Ramsey's Theorem, replacing the appeals to the pigeonhole principle with appeals to the Finite Sums Theorem. We also remark that one cannot allow any of the I_t to be empty, or allow the I_t to interleave in any way: this may be shown by some simple bad colourings. For example, if we allowed sums of the form x_i+2x_j and also $2x_i+x_j$ $(i<j)$, then it is easy to see that we would have a monochromatic x and y with y close to $2x$, and for this we may find a bad colouring similar to the bad colouring for $y=2x$.

Remarkably, these are close to being the only infinite partition regular systems known. Let us describe some of the other known systems. Firstly, one can extend the Finite Sums Theorem in the following way: rather than having a sequence $x_1, x_2, \ldots$ of natural numbers, which we combine by summing together a finite number, we may allow for example a sequence $S_1, S_2, \ldots$, with each S_i an arithmetic progression of length i, and we may take any sum $\sum_{i \in I} x_i$, where I is a finite non-empty set and $x_i \in S_i$ for all i. More generally, given any sequence $A_1, A_2, \ldots$ of finite partition regular matrices, then whenever the natural numbers are finitely coloured there exist sets $S_1, S_2, \ldots$, with S_i a solution set for A_i for all i, such that all sums of the form $\sum_{i \in I} x_i$, where I is a finite non-empty set and $x_i \in S_i$ for all i, are the same colour. This was proved in [8], and might be viewed as the Finite Sums Theorem 'extended' by Rado's Theorem. Similarly, the Milliken-Taylor Theorem may be extended in the same way.

One of the key properties of finite partition regular matrices is what is called 'consistency': if A and B are partition regular then so is the diagonal sum of A and B. Putting it another way, if we can guarantee to solve $Ax=0$ in a colour class and we can guarantee to solve $By=0$ in a colour class then we can guarantee to solve them both in the *same* colour class. (Note that this is trivial by the columns property.) This turns out to have many consequences (see [7]). Unfortunately, it does not hold in the infinite case. Indeed, it was shown in [9] that the diagonal sum of two Milliken-Taylor systems is never partition regular, except in trivial cases. This implies that there cannot be a 'universal' partition regular system, that all partition regular systems are subsystems of.

Nevertheless, one family of systems that might be viewed as coming from consistency *is* known. We illustrate with an example. It is an immediate consequence of Ramsey's Theorem that whenever the natural numbers are finitely coloured there is a sequence $x_1, x_2, \ldots$ such that all sums $x_i + x_j$ $(i \neq j)$ are the same colour (one just induces a colouring of the pairs of natural numbers by giving the pair $\{x, y\}$ the colour of $x+y$, and applies Ramsey's Theorem for pairs). It follows in the same way that whenever the natural numbers are finitely coloured there is a sequence $y_1, y_2, \ldots$ such that all sums $y_i + 2y_j$ $(i < j)$ are the same colour. In [16] it was proved that these two systems are consistent: in other words, whenever the natural numbers are finitely coloured there exist sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ such that the sets $\{x_i + x_j : \ i < j\}$ and $\{y_i + 2y_j : \ i < j\}$ are not only monochromatic but also contained in the same colour class. Other similar results are also proved in [16], with the methods involving structural results for the space $\beta\mathbb{N}$. No direct (non-ultrafilter) proofs of these results are known.

But this (with the other results proved in [16]) is the entire known list of partition regular systems. They all seem to come from tinkering with or combining Milliken-Taylor systems, which themselves could be viewed as minor modifications of the Finite Sums system. Is there some underlying principle here?

Question 4. *Which infinite systems of equations are partition regular?*

Could it be that they are all in some way built up from Milliken-Taylor-type systems? It seems on the face of it rather unlikely, but all of the known examples do fit into this framework. As a 'concrete' instance of Question 4, one intriguing fact is that, in all the partition regular systems known, any one variable can occur with coefficients only from a bounded set. The set of all coefficients (for all the variables) can certainly be unbounded, as may be seen for example by taking the sets S_i above to be (m, p, c)-sets, with p increasing, but what about for one variable?

Question 5. *Is there an infinite partition regular system in which there exists a variable appearing with an unbounded set of coefficients?*

Note that such a system would definitely *not* come from a Milliken-Taylor system by some 'local' modification. Note also that, formally, of course one requires that the system cannot be rewritten in such a way that each variable occurs with only bounded coefficients. (Thus for example one would not be allowed to answer Question 5 by writing the Schur system $x + y = z$ as the infinite set of equations $x + y = z, 2x + 2y = 2z, 3x + 3y = 3z, \ldots$.) We remark also that it is very easy to think up plausible systems for Question 5: the trouble is that they always seem to fail to some unpleasant colouring.

§3. Partition regularity over different spaces

What is special about $\mathbb{N}$ in the definition of partition regularity? What happens if we move to different spaces?

In the integers, nothing changes. Let us say that a finite or infinite matrix A with rational entries (and, as always, only finitely many non-zero entries in each row) is *partition regular over* $\mathbb{Z}$ if whenever $\mathbb{Z} \setminus \{0\}$ is finitely coloured there is a monochromatic vector x with $Ax = 0$. Then if a matrix is partition regular (over $\mathbb{N}$) then it is certainly partition regular over $\mathbb{Z}$. Conversely, if a matrix has a bad k-colouring over $\mathbb{N}$ then it has a bad $2k$-colouring over $\mathbb{Z}$: we just copy the colouring from the positive to the negative integers, but using a new set of k colours.

What about the rationals? A matrix A with rational entries is *partition regular over* $\mathbb{Q}$ if whenever $\mathbb{Q} \setminus \{0\}$ is finitely coloured there is a monochromatic vector x with $Ax = 0$. In the finite case, nothing changes: a simple compactness argument shows that if a matrix is partition regular over $\mathbb{Q}$ then it is partition regular over $\mathbb{Z}$. (See for example [7]).

But for infinite systems we do not know what happens. It is easy to write down systems that are not partition regular over $\mathbb{N}$ for the simple reason that they do not even have solution sets in $\mathbb{N}$ – for example, any system that forces an infinite decreasing sequence, such as the system of equations $x_1 - x_2 = y_1$, $x_2 - x_3 = y_2$, $x_3 - x_4 = y_3, \ldots$. However, in practice it usually seems possible to find a bad colouring of $\mathbb{Q}$, often by extending the c_p colourings by writing rationals in a terminating decimal way (such as 'factorial base') and looking at how the expansion ends. For example, to find a bad colouring for the above system, write the fractional part of a (non-integer) rational x as $\sum_{n=2}^{t} x_n/n!$, where $x_n \in \{0, 1, \ldots, n-1\}$ for all n and $x_t \neq 0$. Colouring by the parity of the digit x_t and the two digits before it, a moment's thought shows that there cannot be a monochromatic system of the above form. (To be more precise, this gives an 8-colouring of all x with $t \geq 3$, and we use a ninth colour for the set of all x that are integer or have $t \leq 2$.)

Question 6. *If a system is partition regular over* $\mathbb{Q}$*, must it be partition regular over* $\mathbb{N}$*?*

Finally, let us consider what happens when we move up to the reals. A matrix A with rational entries is *partition regular over* $\mathbb{R}$ if whenever $\mathbb{R} \setminus \{0\}$ is finitely coloured there is a monochromatic vector x with $Ax = 0$. In the finite case is not too hard to extend from $\mathbb{Q}$ to $\mathbb{R}$. Indeed, Rado himself had shown [28] that a finite matrix A, with *real* entries, is partition regular over the reals if and only if it satisfies the columns property over the reals.

However, in the infinite case the reals really are richer than the rationals. Although it is very often possible to extend in some way a bad colouring of $\mathbb{Q}$ to a bad colouring of $\mathbb{R}$ (usually by considering the coefficients of a real with respect to a fixed Hamel basis – see [17]), this cannot always be done.

For example, consider the system mentioned above that is not partition regular over $\mathbb{Q}$. We claim, however, that it is partition regular over $\mathbb{R}$. Indeed, suppose we are given a finite colouring of the (positive) reals. We may induce a colouring of the pairs of reals by giving the pair $\{x, y\}$, where $x < y$, the colour of $y - x$. Now, by the Baumgartner-Hajnal theorem [1], there is a monochromatic set of order type $\omega + 1$ for this colouring of the pairs: say it is $z_0 < z_1 < z_2 < \ldots < z_\omega$. We may now take $x_i = z_\omega - z_i$ and $y_i = z_{i+1} - z_i$.

To end this section, let us mention that there *are* many examples of positive results for colourings of the reals with additional properties, such as each colour class being measurable or having the property of Baire (see [27, 24, 4]). There are even such results mixing addition and multiplication (see [3]). However, these results tend to work by showing that the desired result would be trivial for open or closed sets and then showing that the given sets are 'close' enough to open sets that the results go through. Currently, there seems no hope of removing the measurability conditions in these statements.

§4. Image partition regularity

There is another very natural way to cast van der Waerden's Theorem and Schur's Theorem in terms of matrices. We say that a matrix A, with rational entries, is *image partition regular* if whenever the natural numbers are finitely coloured there is a vector x, with entries in the natural numbers, such that Ax is monochromatic. For example, Schur's theorem is the assertion that the 3×2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ is image partition regular.

We mention in passing that in fact most theorems asserting that a given system is partition regular actually go via the notion of image partition regularity. For example, the Milliken-Taylor Theorem is naturally a statement about image, not kernel, partition regularity. Rather more seriously, van der Waerden's Theorem (without the strengthening to include the common difference) *cannot* be written as the assertion that a particular matrix is kernel partition regular. So it seems that perhaps image partition regularity is a more fundamental notion than the usual kernel partition regularity.

Similarly, we say that A is *image partition regular over* $\mathbb{Z}$ or $\mathbb{Q}$ if whenever $\mathbb{Z} \setminus \{0\}$ or $\mathbb{Q} \setminus \{0\}$ respectively is finitely coloured there is a vector x, with entries from $\mathbb{Z}$ or $\mathbb{Q}$ respectively, such that Ax is monochromatic. As with kernel partition regularity, for finite matrices it is quite easy to see that the notions of image partition regularity over $\mathbb{Z}$ and $\mathbb{Q}$ coincide.

In the finite case, the image partition matrices over $\mathbb{N}$ and $\mathbb{Z}$ were characterised in [15]: it turns out that there is a close connection with the columns property. For example, a finite matrix A is kernel partition regular if and only if there exists an image partition regular matrix B with $AB = 0$ (see [18, Lemma 15.15]). Let us mention that the situation for $\mathbb{N}$ is rather more subtle and difficult than the situation for $\mathbb{Z}$ – see [15]. Also, there are several other notions of image partition regularity (for example, not allowing any x_i to be zero), but they are all closely related to the definition we have given above.

There is a natural way to pass from a matrix A to a matrix $B(A)$ that gives information about the linear dependences among the rows of A. Given a matrix A, choose a maximal linearly independent (over $\mathbb{Q}$) set of rows of A: say $\{r_i : i \in I\}$. Write the other rows, say $\{s_j : j \in J\}$, as linear combinations of the r_i. This gives a $J \times (I \cup J)$ matrix B (whose right-hand side is -1 times the $J \times J$ identity matrix).

It is clear that A is image partition regular over $\mathbb{Q}$ if and only if $B(A)$ is partition regular over $\mathbb{Q}$. Indeed, the forward implication is obvious. For the reverse implication, given a finite colouring of $\mathbb{Q}$, find a monochromatic vector in the kernel of $B(A)$, in other words some suitable values for the r_i and s_j, and then solve for the original variables to give the correct values of the r_i – this is always possible, because the r_i are linearly independent.

What happens over the integers? (This is the natural place to work for this kind of problem, instead of the natural numbers, because clearly some of the variables may turn out to be negative when we solve for them.) Certainly if A is image partition regular over $\mathbb{Z}$ then $B(A)$ is partition regular. The converse is easily seen to hold when A is finite. But for infinite A it fails for simple reasons if A is allowed to have general rational entries. For example, we may take any A that has independent rows (so that $B(A)$, being empty, is trivially partition regular) but is not image partition regular over $\mathbb{Z}$, such as the system of images $x_n + \frac{1}{n}y$, $n = 1, 2, \ldots$. However, it is not known what happens for integer matrices.

Question 7. *Let A be a matrix, with integer entries, with $B(A)$ partition regular. Must A be image partition regular over $\mathbb{Z}$?*

There is a very appealing way in which this might be true. When we solve for the original variables, we might obtain non-integer values, but this would not be the case if we could insist that the various intermediate variables (the r_i) were multiples of given integers.

Question 8. *If a matrix A is partition regular, with variables x_i, $i \in I$ and y_j, $j \in J$ and equations of the form $y_j = \sum c_{ij} x_i$ (one equation for each j), is it true that in any finite colouring of $\mathbb{N}$ there is a monochromatic vector x with $Ax = 0$ satisfying given congruence constraints $x_i \equiv 0 \pmod{d_i}$?*

It is hard to see how this could fail to be true, but it remains unproved. Finally, let us point out that affirmative answers to both Question 7 and Question 6 would answer the following analogue of Question 6.

Question 9. *Let A be a matrix with integer entries. If A is image partition regular over $\mathbb{Q}$, must it be image partition regular over $\mathbb{Z}$?*

Again, easy examples show that this is false if we are allowed non-integer entries.

§5. Sparse partition regularity

Let A be a (finite or infinite) partition regular matrix. We say that a subset D of $\mathbb{N}$ is *partition regular* for A if whenever D is finitely coloured there is a monochromatic vector x (meaning that the entries of x all belong to D and have the same colour) with $Ax = 0$. If a set is partition regular for one matrix A, when must it be partition regular for another matrix B? Thus one is asking about how the notions of partition regular for different matrices affect each other. For example, the set of all integers whose base k expansion does not contain the digit $k-1$ does not contain an arithmetic progression with k terms, but it is easily seen (via the Hales-Jewett Theorem) to be partition regular for arithmetic progressions of $k-1$ terms.

A much deeper result is that, for any k, there is a set that is partition regular for $FS(x_1, \ldots, x_{k-1})$ but is not partition regular for (and indeed contains no copy of) $FS(x_1, \ldots, x_k)$. This is due to Nešetřil and Rödl [21], and is based on their amalgamation technique. This is a technique that has been used to prove many other 'sparse' results. For example, Frankl, Graham and Rödl [10] proved an 'induced restricted' Hales-Jewett Theorem, and Prömel and Voigt [26] proved a sparse van der Waerden Theorem – see [22] for some of the most powerful results in this direction.

A related result, in [2], is that, for any m, there exists a set that is partition regular for every partition regular matrix with m rows but not for every partition regular matrix with $m+1$ rows.

But the situation between two general finite matrices is very unclear. If A and B are partition regular matrices, let us say that A *Rado-dominates* B if every set that is partition regular for A is also partition regular for B. One rather trivial way in which this can occur is if every solution set for A actually contains a solution set for B: if this is the case we say that A *solution-dominates* B.

Is this the only way for one matrix to Rado-dominate another?

Question 10. *Let A and B be finite partition regular matrices. Is it true that A Rado-dominates B if and only if A solution-dominates B?*

The trouble is that it is very hard to construct sets that are rich enough in structure to be partition regular for a given system and yet manage to avoid having other additional structure. Question 10 is known to be true for single equations (matrices with one row) [19]. This is based on a sparse version of Rado's Theorem.

In the infinite case, absolutely nothing is known. The very first question one could possibly ask would be as follows, where we consider a considerable weakening of the Finite

Sums theorem. We write $FS_{\leq k}(x_1, x_2, \ldots)$ for the set of all sums $\sum_{i \in I} x_i$, where I is a non-empty set of size at most k.

Question 11. *For each k, does there exist a set D such that D is partition regular for the system $FS_{\leq k}(x_1, x_2, \ldots)$ but not for the system $FS_{\leq k+1}(x_1, x_2, \ldots)$?*

This question is introduced in [23], where the answer is conjectured to be yes. Amusingly, the same question also appears in [14], where the answer is conjectured to be no. The answer is not even known in the case $k = 2$. And weakening the condition that D is not partition regular for $FS_{\leq k+1}(x_1, x_2, \ldots)$ to the condition that D should not be partition regular for (or, equivalently, contain a copy of) $FS(x_1, x_2, \ldots)$ does not seem to help. In other words, it is not known whether or not there is a set D such that whenever D is finitely coloured there is a sequence $x_1, x_2, \ldots$ with all x_i and all $x_i + x_j$ $(i \neq j)$ of the same colour, and yet D does not contain any set of the form $FS(x_1, x_2, \ldots)$.

Part of the problem seems to be that it is very hard to prove results about sums of given sizes from an infinite sequence without invoking the whole Finite Sums Theorem. Even the following is unknown.

Question 12. *Is there a proof that whenever $\mathbb{N}$ is finitely coloured there is a sequence $x_1, x_2, \ldots$ such that all x_i and all $x_i + x_j$ $(i \neq j)$ have the same colour, that does not also prove the Finite Sums Theorem?*

It seems truly remarkable that this can be unknown. In a sense, all the evidence in the finite case points towards the answer to Question 11 being yes, but the fact that Question 12 has not been solved may suggest that things are genuinely different in the infinite case.

One more important open problem concerns finding structures that fall only partly inside a given set. This is sometimes called an 'induced' Ramsey problem.

Question 13. *Is there a set $D \subset \mathbb{N}$ such that, whenever D is finitely coloured, there is a sequence $x_1, x_2, \ldots$, with no x_i belonging to D, such that $FS(x_1, x_2, \ldots) \setminus \{x_1, x_2, \ldots\}$ is contained in D and monochromatic?*

We remark that in all finite versions of the problem such sets do exist: this (as well as related results) is due to Prömel [25] (see also [5]). Curiously, no known proof of the Finite Sums Theorem seems to shed any light on this problem.

Question 13 is important because it turns out that a negative answer to it would imply a negative answer to the Continuous Homomorphism Problem. This asks if there is a non-trivial continuous homomorphism from $\beta\mathbb{N}$ to itself. By difficult work of Strauss [30] and Zelenyuk [33], this is equivalent to the question of whether one ultrafilter p can 'absorb' another ultrafilter q, in the sense that all of $p+p, p+q, q+p, q+q$ are equal to p. If such p and q existed then it is not hard to see that any set belonging to p but not q would give a positive answer to Question 13.

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