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Orderings of the Stone-Čech Remainder of a Discrete Semigroup

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Abstract. The Rudin-Keisler (and in the case the space S is countable, the Rudin-Frolík) order of the Stone-Čech remainder $\beta S\backslash S$ of the discrete space S has often been studied, yielding much useful information about βS. More recently, the Comfort order has been introduced. If $(S,\cdot)$ is a semigroup, then the operation $\cdot$ extends naturally to βS, and the study of the semigroup $(\beta S,\cdot)$ is both fascinating in its own right and useful in terms of applications to Ramsey Theory.

In this paper, we study the Rudin-Keisler and Comfort orders on $\beta S\backslash S$ when S is a semigroup. We show, for example, that the set of Comfort predecessors of a given point $p\in\beta S\backslash S$ is always a subsemigroup of βS, while if S is cancellative, the set of Rudin-Keisler predecessors of a point p is never a subsemigroup.

1. Introduction.

Given a discrete space S, we take the points of βS, the Stone-Čech compactification of S, to be the ultrafilters on S, with the points of S identified with the principal ultrafilters. The topology of βS can be defined by stating that the sets of the form $\{p\in\beta S : A\in p\}$, where A is a subset of S, are a base for the open sets. We note that the sets of this form are clopen and that, for any $p\in\beta S$ and any $A\subseteq S$, $A\in p$ if and only if $p\in\overline{A}$, where $\overline{A}$ denotes $\mathrm{cl}_{\beta S}A$. If A is a subset of S, we shall use A^* to denote $\overline{A}\setminus A$.

If X is any compact Hausdorff space, then any function $f:S\to X$ has a continuous extension $\overline{f}:\beta S\to X$.

The *Rudin-Keisler* order $\leq_{RK}$ on βS is defined by agreeing that $p\leq_{RK} q$ if and only if there is a function $f:S\longrightarrow S$ such that $\overline{f}(q)=p$, where $\overline{f}:\beta S\longrightarrow\beta S$ is the continuous extension of f. A great deal has been learned about this order, especially in the case of countable discrete spaces. (See [6],[7] and [20] for much of what is known.)

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This information has been a powerful tool in studying the structure of βS, showing in a dramatic fashion that it is not true that all ultrafilters were created equal.

We shall use $p <_{RK} q$ to denote that $p \leq_{RK} q$ and $q \not\leq_{RK} p$, and $p \approx_{RK} q$ to denote that $p \leq_{RK} q$ and $q \leq_{RK} p$. We may simply use $\leq$ and $<$ respectively, instead of $\leq_{RK}$ and $<_{RK}$.

Any binary operation $*$ defined on S can be extended in a natural way to a binary operation defined on βS. This can be done by using the using the notion of p-limit introduced in [9]. Given a point $p \in \beta S$ and a function f taking S to a Hausdorff topological space X, $p\text{-}\lim_{x \in S} f(x) = y$ if and only if for every neighborhood U of y, $f^{-1}[U] \in p$. This is equivalent to stating that $\lim_{x \to p} f(x) = y$. It is also equivalent to stating that $\bar{f}(p) = y$ in the case in which X is compact.

Then, given $p, q \in \beta S$, we define $p * q = p\text{-}\lim_{s \in S} q\text{-}\lim_{t \in S} s * t = \lim_{s \to p} \lim_{t \to q} s * t$, where s and t denote elements of S. For any $A \subseteq S$, $A \in p * q$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$, where $s^{-1}A = \{t \in S : s * t \in A\}$. If $*$ is associative on S, its extension to βS is also associative and so $(\beta S, *)$ is a semigroup. It has the property that that, for every $p \in \beta S$, the map $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q * p$ is continuous and thus $(\beta S, *)$ is a compact right topological semigroup. Furthermore, for every $s \in S$, the map $\lambda_s : \beta S \to \beta S$ defined by $\lambda_s(q) = s * q$ is continuous as well. (The reader should be warned that the extension of $*$ is sometimes carried out in the opposite order, making $(\beta S, *)$ a left topological semigroup. In fact this is the case in some of the references to this paper.)

In the case in which S is a semigroup, some relationships between the order $\leq_{RK}$ and the semigroup operation on βS are known, primarily the fact that the points p of $\beta\mathbb{N}$ at which right cancellation holds in the semigroup $(\beta\mathbb{N}, +)$ are characterised by the property that $p <_{RK} q + p$ for all $q \in \beta\mathbb{N}$ [5]. However, one would not expect an intimate relationship because permutations of a semigroup do not normally respect the semigroup operation.

Recently, at the suggestion of W. Comfort, one of us initiated a study of a different order relation on elements of βS. (See [11] and [12].)

In [2], a space Hausdorff X is called *p-compact* provided that whenever $f : S \longrightarrow X$, $p\text{-}\lim_{x \in S} f(x)$ exists in X. In the Comfort order, one says that $p \leq_C q$ if and only if every q-compact space is p-compact. It is easy to check that $p \leq_{RK} q$ implies that $p \leq_C q$. We shall write $p <_C q$ if $p \leq_C q$ and $q \not\leq_C p$, and $p \approx_C$ if $p \leq_C q$ and $q \leq_C p$.

In this paper, we invesigate some of the connections between the relations $\leq_{RK}$ and $\leq_C$ on βS and the semigroup structure of βS.

In Section 1 we consider the tensor product $p \otimes q$ of two elements p and q and show that $p * q \leq_{RK} p \otimes q$ for every binary relation $*$ defined on S. We also show that every $\leq_{RK}$ minimal ultrafilter in $\beta\omega$ is $\leq_C$ minimal.

In Section 2 we establish a strong relationship between the Comfort order and the semigroup structure on βS. That is, we show that for any infinite discrete semigroup $(S, \cdot)$ and any point $p \in \beta S$, the set of Comfort predecessors of p is a subsemigroup of $(\beta S, \cdot)$. We also show that if $(S, \cdot)$ is cancellative, then the corresponding statement about the Rudin-Keisler order fails dramatically: the set of Rudin-Keisler predecessors of any element $p \in S^*$ is not a semigroup. (The restriction that $p \notin S$ is necessary, because, if $p \in S$, the set of Rudin-Keisler predecessors of p is just S.) We also show that there are no ultrafilters which are maximal for the Comfort order. We prove that the right cancellable elements of ω^* preserve order properties in the following sense: for any right cancellable element p of $\beta\omega$ and for any $x, y \in \omega^*$, $x \leq_{RK} y$ if and only if $x + p \leq_{RK} y + p$, and $x \leq_C y$ implies that $x + p \leq_C y + p$.

In Section 3 we present some other results connecting order relations with the semigroup operation of βS. We show that there is a rich set of elements p in βS with the property that $p <_{RK} p + q$ and $q <_{RK} p + q$ for every $q \in S^*$. We prove that for any subset C of $\beta\mathbb{N}$ with at most $\mathfrak{c}$ elements, there is a left ideal L of $\beta\mathbb{N}$ and a right ideal R of $\beta\mathbb{N}$ such that $x <_{RK} y$ for every $x \in C$ and every $y \in L \cup R$. This implies that the $\leq_C$ successors of a given ultrafilter in $\beta\mathbb{N}$ do not normally form a subsemigroup of $\beta\mathbb{N}$. We finally observe that, if p is a P-point in ω^* and if $x \in \omega^*$, then $x \leq_{RK} p$ implies that x is a P-point in ω^* and $x \leq_C p$ imples that x is right cancellable in ω^*. It follows that the set of elements of ω^* which are Comfort equivalent to p is a subsemigroup of ω^*.

We shall use some of the basic algebraic properties of compact right topological semigroups, whose proofs can be found in [1]. Any such semigroup T has a smallest two-sided ideal $K(T)$, which is the union of all the minimal left ideals, as well as being the union of all the minimal right ideals of T. Any minimal left (right) ideal of T is of the form Te (eT) for some idempotent e.

We conclude this introduction with some well known facts whose proofs have not been previously published in the generality in which we shall use them.

1.1 Definition. Let S be a discrete space and let $p, q \in \beta(S)$. The *tensor product* of p and q is

$$p \otimes q = \{A \subseteq S \times S : \{s : \{t : (s,t) \in A\} \in q\} \in p\}.$$

Then $p \otimes q$ is an ultrafilter on $S \times S$ which can be considered as an ultrafilter on S via any fixed bijection. Notice that, if τ is a bijection from $S \times S$ to S, and for $s, t \in S$

one defines $s * t = \tau(s,t)$, then for any p and q in βS, one has $\{\tau[A] : A \in p \otimes q\} = p * q$ and thus results obtained here about the extensions of arbitrary binary operations on S apply to $\otimes$. For other properties not included here and some historical notes concerning $\otimes$ see [6].

1.2 Lemma. *Let S be a discrete space and let $p, q \in S^*$. Then $p < p \otimes q$ and $q < p \otimes q$.*

Proof. If π_1 and π_2 are the projection maps from $S \times S$ onto S, it is easy to see that $\bar{\pi}_1(p \otimes q) = p$ and $\bar{\pi}_2(p \otimes q) = q$. Since there is no member of $p \otimes q$ on which π_1 or π_2 is injective, it follows from [6, Theorem 9.2] that $p \not\approx_{RK} p \otimes q$ and $q \not\approx_{RK} p \otimes q$. $\square$

1.3 Lemma. *Let S be a discrete space and let $p, q \in \beta S$. Then*

$$p \otimes q = \lim_{s \to p} \lim_{t \to q} (s,t)$$

where s and t denote elements of S and the limits are taken to be in $\beta(S \times S)$.

Proof. For each $s \in S$ and $q \in \beta S$, let $f_s : S \to \beta(S \times S)$ and $g_q : S \to \beta(S \times S)$ be defined by $f_s(t) = (s,t)$ and $g_q(s) = \bar{f}_s(q)$. Now $\bar{f}(q) = \lim_{t \to q} f_s(t)$ and $\lim_{t \to q} f_s(t) = s \otimes q$, because, for every $U \in s \otimes q$, we have $f_s^{-1}[U] \in q$. So $g_q(s) = s \otimes q$. Also, $\bar{g}_q(p) = \lim_{s \to p} g_q(s) = p \otimes q$ because, for every $V \in p \otimes q$, we have $g_q^{-1}[V] \in p$. $\square$

1.4 Lemma. *Let S be a discrete space and let $*$ be a binary operation defined on S. Let $p, q, x, y \in \beta S$. If $p \leq q$ and $x \leq y$, then $p * x \leq q \otimes y$.*

Proof. Let $f, g : S \to S$ be functions for which $\bar{f}(q) = p$ and $\bar{g}(y) = x$. We define $h : S \times S \to S$ by $h(s,t) = f(s) * g(t)$. Then

$$\bar{h}(q \otimes y) = \lim_{s \to q} \lim_{t \to y} h(s,t) = \lim_{s \to q} \lim_{t \to y} f(s) * g(t) = \bar{f}(q) * \bar{g}(y) = p * x.$$

$\square$

1.5 Corollary. *Let S be a discrete space and let $*$ be a binary operation defined on S. For every $p, q \in \beta S$, $p * q \leq p \otimes q$. Furthermore, if $h : S \times S \to S$ is defined by $h(s,t) = s * t$, we have $\bar{h}(p \otimes q) = p * q$.*

Proof. The proof is the same as that of Lemma 1.4, with f and g taken to be the identity maps. $\square$

Corollary 1.5 shows that $p \otimes q$ is an RK-upper bound of the set of elements of the form $p * q$, where $*$ denotes any binary operation on S and $p, q \in S^*$. We now show that we frequently have $p \otimes q \approx_{RK} p * q$.

We remind the reader that a subset D of a topological space X is said to be discrete if no point x of D is in $c\ell_X(D \setminus \{x\})$. It is said to be strongly discrete if each point x of D has a neighbourhood U_x in X for which the family $\langle U_x \rangle_{x \in D}$ is pairwise disjoint. If X is regular and D is countable, these two concepts are equivalent.

1.6 Theorem. *Let S be a discrete space and let $*$ be a binary operation defined on S with the property that, for each $s \in S$, the map $t \mapsto s * t$ is injective. Then, for every $p, q \in \beta S$, the following are equivalent.*
(1) $p * q \approx_{RK} p \otimes q$.
(2) There exists $D \in p$ such that $D * q$ is strongly discrete.

Proof. (1) $\Rightarrow$ (2). If $p * q \approx_{RK} p \otimes q$, there is a set $A \in p \otimes q$ on which the mapping $(s,t) \mapsto s * t$ from $S \times S$ to S is injective (by Corollary 1.5 and [6, Theorem 9.2]). We may suppose that A has the form $\bigcup_{s \in D}(\{s\} \times E_s)$, where $D \in p$ and $E_s \in q$ for every $s \in D$. Then, for each $s \in D$, $s * E_s \in s * q$ and $(s * E_s) \cap (s' * E_{s'}) = \emptyset$ if s and s' are distinct elements of D.

(2) $\Rightarrow$ (1). Let $D \in p$ be such that $D * q$ is strongly discrete. Then, for each $s \in D$, there exists $U_s \in s * q$ such that $U_s \cap U_{s'} = \emptyset$ whenever s and s' are distinct elements of D. For each $s \in D$, there exists $E_s \in q$ such that $s * E_s \subseteq U_s$. Then $\{(s,t) \in D \times S : t \in E_s\} \in p \otimes q$ and the mapping $(s,t) \mapsto s * t$ is injective on this set. So $p * q \approx_{RK} p \otimes q$. □

We shall need the following result, which is a consequence of [16, Corollary 2.6].

1.7 Lemma. *If $(S, \cdot)$ is a cancellative discrete semigroup, then, for every $s, t \in S$ and every $p \in \beta S$, $s \cdot p = t \cdot p$ implies that $s = t$.*

1.8 Corollary. *Let $(S, \cdot)$ be a countable cancellative semigroup. If $q \in S^*$ is right cancellable in βS, then $p \cdot q \approx_{RK} p \otimes q$ for every $p \in \beta S$.*

Proof. This follows from Theorem 1.6 and the fact that $S \cdot q$ is discrete in S^* (by [16, Theorem 2.2]). □

The following corollary generalizes a portion of [5, Theorem 2.1], where it was established for $(\mathbb{N}, +)$.

1.9 Corollary. *Let $(S, \cdot)$ be a countable cancellative semigroup. If q is a right cancellable element of S^*, then $p < p \cdot q$ and $q < p \cdot q$ for every $p \in S^*$.*

Proof. This follows from Lemma 1.2 and Corollary 1.8. □

The following example contrasts with the fact that, for any $p, q \in \beta S$, we have $p \otimes q \approx_C q \otimes p$ (Cf. Corollary 2.3 below). We shall need to use a lemma, due to Frolík, which is valid in any F-space. A proof can be found in [16], where it occurs as Lemma 1.1.

1.10 Lemma. *Let S be a discrete space and let A and B be σ-compact subsets of βS. If $\overline{A} \cap \overline{B} \neq \emptyset$, then $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.*

1.11 Theorem. *Let S be a countable discrete space and let $*$ be a binary operation defined on S with the property that, for every $a \in S$, the mapping $b \mapsto a * b$ from S to itself is injective. Suppose that $p, q \in S^*$ and that there is a member A of p for which $a * q \neq a' * q$ whenever a and a' are distinct elements of A and for which $A * q$ is discrete in βS. Then $p * q \leq q * p$ implies that p and q are RK-comparable.*

Proof. For each $a \in A$ we can choose $U_a \in a * q$ with the the property that $U_a \cap U_{a'} = \emptyset$ whenever $a \neq a'$. We can then choose $B_a \in q$ satisfying $a * B_a \subseteq U_a$. We put $V = \bigcup_{a \in A} a * B_a$ and note that $V \in p * q$. Each $v \in V$ has a unique expression of the form $v = a * b$ with $a \in A$ and $b \in B_a$. We can define $\phi_1, \phi_2 : S \to S$ by stating that $\phi_1(v) = a$ and $\phi_2(v) = b$ if $v \in V$ is expressed in this form and then extending these functions arbitrarily to $S \setminus V$. We observe that, for any $x \in \overline{A}$, we have

$$\bar{\phi}_1(x * q) = \lim_{a \to x} \lim_{b \to q} \phi_1(a * b) = \lim_{a \to x} a = x$$

and

$$\bar{\phi}_2(x * q) = \lim_{a \to x} \lim_{b \to q} \phi_2(a * b) = \lim_{a \to x} \lim_{b \to q} b = q.$$

(In these expressions, a denotes an element of A and b an element of B_a.)

Let $f : S \to S$ be a function for which $\bar{f}(q * p) = p * q$. Let $P \in p$ and $Q \in q$, with $P \subseteq A$. Then $p * q$ belongs to each of the sets $\overline{P * q}$ and $\overline{\bar{f}[Q * p]}$. It follows (from Lemma 1.10) that one of the two following alternatives must hold:

i) $\bar{f}(b * p) = x * q$ for some $b \in Q$ and some $x \in \overline{P}$;

ii) $a * q \in \overline{\bar{f}(Q * p)}$ for some $a \in P$.

Now i) implies that the mapping $s \mapsto \phi_2 f(b * s)$ from S to itself has a continuous extension to βS which maps p to q. Thus i) implies that $q \leq p$, and we shall therefore assume that ii) holds for every $P \in p$ and every $Q \in q$.

Statement ii) implies that $a \in \overline{\bar{\phi}_1 \bar{f}(Q * p)}$. Since a is isolated in βS, this implies that $a = \bar{\phi}_1 \bar{f}(b * p)$ for some $b \in Q$. Let $B = \{b \in S : \bar{\phi}_1 \bar{f}(b * p) \in S\}$. Then $B \in q$ because B meets every member of q. We can define $\theta : S \to S$ by stating that $\theta(b) = \bar{\phi}_1 \bar{f}(b * p)$

if $b \in B$ and then extending θ arbitrarily to $S \setminus B$. We have seen that, for every $P \in p$ and $Q \in q$, $\theta[Q] \cap P \neq \emptyset$. Thus $\theta[Q] \in p$ and so $\bar{\theta}(q) = p$ and $p \leq q$. □

1.12 Corollary. *Let S be a countably infinite discrete space and let $p, q \in S^*$. If $p \otimes q \leq q \otimes p$, then p and q are RK-comparable.*

Proof. Let τ be a bijection from $S \times S$ to S and define an operation $*$ on S by $s * t = \tau(s, t)$. Then as we have observed $p * q = \{\tau[A] : A \in p \otimes q\}$ and the hypotheses of Theorem 1.11 are clearly satisfied. □

1.13 Corollary. *Let $(S, \cdot)$ be a countable cancellative semigroup and let q be a right cancellable element of S^*. If $p \in S^*$ and $p \cdot q \leq q \cdot p$, then p and q are RK-comparable.*

Proof. This follows from Theorem 1.11 and the fact that $S \cdot q$ is discrete in S^* (by [16, Theorem 2.2]). □

1.14 Lemma. *Let $(S, \cdot)$ be a countable cancellative semigroup. Then for every $p \in S^*$ there exists $q \in S^*$ such that $q \approx_{RK} p$ and $q \cdot p \approx_{RK} p \otimes p$.*

Proof. We can choose an infinite subset D of S for which $D \cdot p$ is discrete and can choose $q \in D^*$ such that $q \approx_{RK} p$. Using Theorem 1.6, we have $q \cdot p \approx_{RK} q \otimes p \approx_{RK} p \otimes p$. □

1.15 Lemma. *Let $(S, \cdot)$ be a countable cancellative semigroup. For every $p \in S^*$, there exists $q \in S^*$ such that $q \approx_{RK} p$ and $r \cdot q \approx_{RK} r \otimes p$ for every $r \in S^*$.*

Proof. By [14, Corollary 4.4] there is a dense open subset of βS all of whose elements are right cancellable in βS. We can therefore choose an infinite subset D of S with the property that every element of $\overline{D}$ is right cancellable in βS. We can then choose $q \in \overline{D}$ such that $q \approx_{RK} p$. Using Corollary 1.8, we then have $r \cdot q \approx_{RK} r \otimes q \approx_{RK} r \otimes p$ for every $r \in S^*$. □

1.16 Lemma. *Let $(S, \cdot)$ be a cancellative countable semigroup. Then, for every $p, q \in S^*$ there exists $r \in S^*$ such that $r \leq_{RK} p \cdot q$ and $r \leq_{RK} p$.*

Proof. We note that by Lemma 1.7 there is at most one element $s \in S$ for which $s \cdot q = p \cdot q$. Hence, if $P = \{t \in S : t \cdot q \neq p \cdot q\}$, we have $P \in p$.

Suppose that P is arranged as a sequence $\langle s_n \rangle_{n=1}^{\infty}$. For each $s \in P$, we shall define a set $A_s \subseteq P$ so that the following statements hold:

$A_s \in s \cdot q$;

$A_s \notin p \cdot q$;

For every $s, t \in P$, either $A_s = A_t$ or $A_s \cap A_t = \emptyset$.

We define these sets inductively, first choosing A_{s_1} to be any member of $s_1 \cdot q$ which is not a member of $p \cdot q$. We then suppose that A_{s_i} has been defined for every $i \in \{1, 2, \cdots, n\}$ so that the required conditions hold. If $s_{n+1} \cdot q \in \overline{\bigcup_{i=1}^n A_{s_i}}$, we put $A_{s_{n+1}} = A_{s_i}$ where $i \in \{1, 2, \cdots n\}$ and $s_{n+1} \cdot q \in \overline{A_{s_i}}$. Otherwise, we choose $A_{s_{n+1}}$ satisfying $A_{s_{n+1}} \in s_{n+1} \cdot q$, $A_{s_{n+1}} \cap A_{s_i} = \emptyset$ for every $i \in \{1, 2, \cdots, n\}$ and $A_{s_{n+1}} \notin p \cdot q$.

Having defined the sets A_{s_n}, we put $A = \bigcup_{i=1}^{\infty} A_{s_i}$ and define a mapping $f : A \to S$ by stating that $f(a) = s_i$ if i the first integer for which $a \in A_{s_i}$. We put $r = \bar{f}(p \cdot q)$. It is then immediate that $r \leq p \cdot q$. We observe that $r \in S^*$, because $r \in S$ would imply the existence of an integer i with the property that $\{t \in P : t \cdot q \in A_{s_i}\} \in p$. This would imply that $A_{s_i} \in p \cdot q$ – contradicting our choice of the sets A_{s_n}.

For each $s_n \in P$, we can choose i to be the first integer for which $s_n \cdot q \in \overline{A_{s_i}}$. We then have $\bar{f}(s_n \cdot q) = s_i \in S$. So the map $s \mapsto \bar{f}(s \cdot q)$ from P to S has an extension to $\bar{P}$ which maps p to r, and thus $r \leq p$. □

1.17 Definition. Let X be a completely regular Hausdorff space, let S be an infinite discrete space, and let $p \in \beta S$. Then

$$\beta_p(X) = \bigcap \{Y : X \subseteq Y \subseteq \beta X \text{ and } Y \text{ is } p\text{-compact}\} .$$

Notice that trivially $\beta_p(X)$ is p-compact. We remark that $\beta_p(X)$ has the following universal property: If Y is any completely regular Hausdorff p-compact space, then any continuous function from X to Y extends to a continuous function from $\beta_p(X)$ to Y.

We now see how to construct $\beta_p(X)$ from the inside out.

1.18 Lemma. *Let X be a completely regular Hausdorff space, let S be an infinite discrete space, let $\alpha = |S|$, and let $p \in \beta S$. Let $A_0(p, X) = X$. Inductively, let $\sigma < \alpha^+$ be given. If σ is a nonzero limit ordinal, let*

$$A_\sigma(p, X) = \bigcup_{\tau < \sigma} A_\tau(p, X) .$$

If $\sigma = \tau + 1$, let

$$A_\sigma(p, X) = \{p\text{-}\lim_{x \in S} f(x) : \ f : S \longrightarrow A_\tau(p, X) \subseteq \beta S\} .$$

Then $\beta_p(X) = \bigcup_{\sigma < \alpha^+} A_\sigma(p, X)$.

Proof. Let $Z = \bigcup_{\sigma < \alpha^+} A_\sigma(p, X)$. To see that $Z \subseteq \beta_p(X)$, suppose instead that this inclusion fails and pick the first $\sigma < \alpha^+$ such that $A_\sigma(p, X) \backslash \beta_p(X) \neq \emptyset$ and pick

$x \in A_\sigma(p,X)\backslash\beta_p(X)$. Since $X \subseteq \beta_p(X)$, $\sigma > 0$, and trivially $\sigma = \tau + 1$ for some τ. Pick $f : S \longrightarrow A_\tau(p,X)$ such that $x = p\text{-}\lim_{t\in S} f(t)$. Since $A_\tau(p,X) \subseteq \beta_p(X)$ and $\beta_p(X)$ is p-compact, it follows that $x \in \beta_p(X)$, a contradiction.

To show that $\beta_p(X) \subseteq Z$, it suffices to show that Z is p-compact. Let $f : S \longrightarrow Z$ and for each $s \in S$, pick $\sigma(s) < \alpha^+$ such that $f(s) \in A_{\sigma(s)}(p,X)$. Let $\delta = \sup\{\sigma(s) : s \in S\}$. Then $\delta < \alpha^+$ and $f : S \longrightarrow A_\delta(p,X)$ so $p\text{-}\lim_{s\in S} f(s) \in A_{\delta+1}(p,X) \subseteq Z$. □

As a consequence of Lemma 1.18 we see that $\beta_p(S)$ is always relatively small. (Recall [13, Theorem 9.2] that if $|S| = \alpha$, then $|\beta S| = 2^{2^\alpha}$.)

We remark that $A_1(p,S)$ as defined in Lemma 1.18, is equal to $\{x \in \beta S : x \leq_{RK} p\}$.

1.19 Theorem. *Let S be an infinite discrete space and let $|S| = \alpha$. Then for all $p \in \beta S$, $|\beta_p(S)| \leq 2^\alpha$.*

Proof. We show by induction on $\sigma < \alpha^+$ that $|A_\sigma(p,S)| \leq 2^\alpha$ and hence by Lemma 1.18 that $|\beta_p(S)| \leq 2^\alpha \cdot \alpha^+ = 2^\alpha$. We have $|A_0(p,S)| = \alpha$. Given $\sigma < \alpha^+$, such that $|A_\sigma(p,S)| \leq 2^\alpha$, note that $|\{f : \ f : S \longrightarrow A_\tau(p,S)\}| \leq (2^\alpha)^\alpha = 2^\alpha$ and hence $|A_{\sigma+1}(p,S)| \leq 2^\alpha$. Given a limit ordinal τ with $0 < \tau < \alpha^+$ we have that $|A_\tau(p,S)| \leq 2^\alpha \cdot |\tau| = 2^\alpha$. □

We omit the routine proof of the following lemma.

1.20 Lemma. *Let S be an infinite discrete space, let $p \in \beta S$, let X be a p-compact space, let Y be a Hausdorff space, let Z be a p-compact subspace of Y and let $f : X \longrightarrow Y$ be continuous. Then $f^{-1}[Z]$ is p-compact.*

The following theorem provides several convenient characterizations of the Comfort order. It was stated without proof in [12].

1.21 Theorem. *Let S be an infinite discrete space and let $p, q \in \beta S\backslash S$. The following statements are equivalent.*

(1) $p \leq_C q$.

(2) $\beta_p(S) \subseteq \beta_q(S)$.

(3) $p \in \beta_q(S)$.

(4) There is a function $f : S \longrightarrow \beta_q(S)$ such that $\overline{f}(q) = p \notin f[S]$.

(5) $\beta_q(S)$ is p-compact.

(6) $\beta_q(S)\backslash S$ is p-compact.

Proof. (1) implies (2). $\beta_q(S)$ is q-compact, hence p-compact.

(2) implies (3). $p = p\text{-}\lim_{s\in S} s \in A_1(p,S) \subseteq \beta_p(S) \subseteq \beta_q(S)$.

(3) implies (4). Let $\alpha = |S|$. Pick the first $\sigma < \alpha^+$ such that $p \in A_{\sigma+1}(q,S)$. Then $p = q\text{-}\lim_{s\in S} f(s) = \overline{f}(q)$ for some function $f : S \longrightarrow A_\sigma(q,S)$.

(4) implies (3). One has $p = q\text{-}\lim_{s\in S} f(s) \in \beta_q(S)$.

(3) implies (1). Let X be a q-compact space and let $f : S \longrightarrow X$ and denote the continuous extension from βS to βX by $\overline{f}$. By Lemma 1.20 $\overline{f}^{-1}[X]$ is q-compact so that $p \in \beta_q(S) \subseteq \overline{f}^{-1}[X]$ so $\overline{f}(p) \in X$. Thus $p\text{-}\lim_{s\in S} f(s) = \overline{f}(p\text{-}\lim_{s\in S} s) = \overline{f}(p) \in X$.

The assertions that (1) implies (5), that (5) implies (2), and that (5) implies (6) are trivial.

(6) implies (1). Let $\alpha = |S|$ and enumerate S as $\langle s_\sigma\rangle_{\sigma<\alpha}$. Let $\langle S_\sigma\rangle_{\sigma<\alpha}$ be a sequence of pairwise disjoint subsets of S, each of cardinality α such that $S = \bigcup_{\sigma<\alpha} S_\sigma$. For each $\sigma < \alpha$, pick $r_\sigma \in \beta S$ such that $S_\sigma \in r_\sigma$ and $r_\sigma \approx_{RK} q$, that is there is a permutation of S whose extension from βS to βS takes q to r_σ. Notice that each $r_\sigma \in \beta_q(S)\backslash S$ since $\beta_q(S)\backslash S$ is q-compact. Define $f : S \longrightarrow \beta_q(S)\backslash S$ by $f(s_\sigma) = r_\sigma$ and define $g : S \longrightarrow S$ by agreeing that $g(x) = s_\sigma$ if and only if $x \in S_\sigma$. Now, since each $S_\sigma \in r_\sigma$, we have that $\overline{g}\big(f(s_\sigma)\big) = \overline{g}(r_\sigma) = s_\sigma$ so that $\overline{g}\circ f$ is the identity on S and hence $\overline{g}\circ\overline{f}$ is the identity on βS. In particular, $\overline{g}\big(\overline{f}(p)\big) = p$ and hence $p \leq_{RK} \overline{f}(p)$. Also, $\overline{f}(p) \in \beta_q(S)\backslash S \subseteq \beta_q(S)$ so, since (3) implies (1), $\overline{f}(p) \leq_C q$ and thus $p \leq_C q$. □

We see as a consequence of Theorem 1.21 that Lemma 1.4 remains valid if the Rudin-Keisler order is replaced by the Comfort order.

1.22 Corollary. *Let S be a discrete space and let $*$ be a binary operation defined on S. Let $p, q, x, y \in \beta S$. If $p \leq_C q$ and $x \leq_C y$, then $p * x \leq_C q \otimes y$.*

Proof. Let $\alpha = |S|$. We show by induction on $\tau < \alpha^+$ that if $p \in A_\tau(q,S)$ and $x \in A_\tau(y,S)$, then $p * x \leq_C q \otimes y$. If $\tau = 0$, then $p * x \in S$ so the conclusion is trivial so assume that $\tau > 0$ and the conclusion is true for smaller ordinals. If τ is a limit ordinal, then for some $\sigma < \tau$, $p \in A_\sigma(q,S)$ and $x \in A_\sigma(y,S)$ so the conclusion is immediate. Thus we may assume that $\tau = \sigma + 1$ for some σ. Pick $f : S \longrightarrow A_\sigma(q,S)$ and $g : S \longrightarrow A_\sigma(y,S)$ such that $p = q\text{-}\lim_{z\in S} f(z)$ and $x = y\text{-}\lim_{w\in S} g(w)$. Then

$$p * x = q\text{-}\lim_{z\in S} y\text{-}\lim_{w\in S} f(z) * g(w)$$

and for all $z, w \in S$, we have by the induction hypothesis that $f(z) * g(w) \leq q \otimes y$. Thus, by Theorem 1.21 for all $z, w \in S$, $f(z) * g(w) \in \beta_{q\otimes y}$. By Lemma 1.2 $q \leq_C q \otimes y$

and $y \leq_C q \otimes y$ so, again by Theorem 1.21, we have that $\beta_{q\otimes y}$ is both q-compact and y-compact and hence $p * x = q\text{-}\lim_{z\in S} y\text{-}\lim_{w\in S} f(z) * g(w) \in \beta_{q\otimes y}$. □

1.23 Theorem. *Every $\leq_{RK}$ minimal ultrafilter in $\mathbb{N}^*$ is also $\leq_C$ minimal.*

Proof. Let p be a $\leq_{RK}$ minimal ultrafilter in $\mathbb{N}^*$. Throughout this proof, we shall simply use A_σ to denote the set $A_\sigma(p,\mathbb{N})$ defined in Lemma 1.18, and β_p to denote $\beta_p(\mathbb{N})$.

For each $x \in \mathbb{N}^* \cap \beta_p$, we define $\phi(x)$ to be the first ordinal $\sigma < \omega_1$ for which $x \in A_\sigma$. We note that $\phi(x)$ is neither 0 nor a limit ordinal.

Suppose that $x \in \mathbb{N}^* \cap \beta_p$ and that $\phi(x) = \sigma$. Then, by the definition of A_σ, there is a function $f : \mathbb{N} \to A_{\sigma-1}$ such that $\overline{f}(p) = x$. We shall show that there is a set $A \in p$ such that $f_{|A}$ is injective and $f[A]$ is discrete in $\beta\mathbb{N}$.

We shall inductively define a sequence $\langle U_i\rangle_{i=1}^{\infty}$ of clopen subsets of $\beta\mathbb{N}$ with the following properties:

$U_i \cap U_j = \emptyset$ if $i \neq j$;

$f[\{1, 2, \cdots, n\}] \subseteq \bigcup_{i=1}^{n} U_i$;

$x \notin \bigcup_{i=1}^{\infty} U_i$.

We first choose U_1 to be any clopen subset of $\beta\mathbb{N}$ such that $f(1) \in U_1$ and $x \notin U_1$. We then suppose that we have defined U_i for each $i \in \{1, 2, \cdots, n\}$ so that these properties hold. Let r denote the first positive integer for which $f(r) \notin \bigcup_{i=1}^{n} U_i$. We choose U_{n+1} to be a clopen subset of $\beta\mathbb{N}\setminus \bigcup_{i=1}^{n} U_i$ such that $f(r) \in U_{n+1}$ and $x \notin U_{n+1}$. Thus we can define a sequence $\langle U_i\rangle_{i=1}^{\infty}$ as claimed.

We note that, for each $i \in \mathbb{N}$, $x \notin U_i$ and hence $p \notin f^{-1}[U_i]$. Since $\mathbb{N} \subseteq \bigcup_{i=1}^{\infty} f^{-1}[U_i]$, it follows from [6,Theorem 9.6] that there is a set $A \in p$ such that $|A \cap f^{-1}[U_i]| \leq 1$ for every $i \in \mathbb{N}$. So $f_{|A}$ is injective and $f[A]$ is discrete in $\beta\mathbb{N}$.

For each $a \in A$, we can choose $B_a \in f(a)$ such that $B_a \cap B_{a'} = \emptyset$ whenever $a \neq a'$. We can define a function $h : \mathbb{N} \to \mathbb{N}$ by stating that $h(b) = a$ if $b \in B_a$, defining h arbitrarily on $\mathbb{N}\setminus \bigcup_{a\in A} B_a$. For each $a \in A$, $f(a) \in \overline{B_a}$ and so $\overline{h}f(a) = a$. Allowing a to converge to p, shows that $\overline{h}(x) = p$. So $x \geq_{RK} p$ and hence $x \geq_C p$.

We have thus shown that $x \leq_C p$ implies that $x \approx_C p$. So p is $\leq_C$ minimal. □

We remark that, for any weak P-point p in $\mathbb{N}^*$ and any $q \in \mathbb{N}^*$, an easy inductive argument shows that $p \in \beta_q(\mathbb{N})$ if and only if $p \in A_1(q,\mathbb{N})$. So $p \leq_C q$ if and only if $p \leq_{RK} q$. It follows that a weak P-point p in $\mathbb{N}^*$ is $\leq_C$ minimal if and only if it is $\leq_{RK}$ minimal. To see this, suppose that p is $\leq_C$ minimal. Then, for any $q \in \mathbb{N}^*$,

$$\begin{aligned} q \leq_{RK} p \quad &\Rightarrow \quad q \leq_C p \\ &\Rightarrow \quad q \geq_C p \\ &\Rightarrow \quad q \geq_{RK} p \ . \end{aligned}$$

So p is $\leq_{RK}$ minimal.

If we assume CH, there are clearly $\leq_C$ minimal ultrafilters which are not weak P-points of $\mathbb{N}^*$. If p is any $\leq_{RK}$ minimal ultrafilter, then any ultrafilter in $\beta_p(\mathbb{N})\backslash A_1(p, \mathbb{N})$ is an ultrafilter of this kind. We do not know whether every $\leq_C$ minimal ultrafilter is $\leq_C$ equivalent to a $\leq_{RK}$ minimal ultrafilter; nor do we know whether the existence of $\leq_C$ minimal ultrafilters can be demonstrated without CH.

2. Sets of Predecessors as Semigroups.

In this section we establish that for any infinite semigroup $(S, \cdot)$ and any point $p \in \beta S$, the set of Comfort predecessors of p is a subsemigroup of $(\beta S, \cdot)$ and that, if S is cancellative and $p \in \beta S\backslash S$, then the set of Rudin-Keisler predecessors of p is not a semigroup. We begin by establishing the first of these assertions. Notice that by the equivalence of (1) and (3) in Theorem 1.21, the set of Comfort predecessors of p is precisely $\beta_p(S)$.

2.1 Theorem. *Let S be a discrete space and let $*$ be a binary operation on S. For every $p \in \beta S$, the set $\beta_p(S)$ is closed under $*$.*

Proof. Let $q, r \in \beta_p(S)$. Then $q * r = q\text{-}\lim_{s\in S} r\text{-}\lim_{t\in S} s * t$. Since $\beta_p(S)$ is r-compact by Theorem 1.21, for each $s \in S$ one has that $r\text{-}\lim_{t\in S} s * t \in \beta_p(S)$ and hence, since $\beta_p(S)$ is q-compact, $q * r \in \beta_p(S)$. □

2.2 Corollary. *Let S be a discrete space and let $p \in \beta S$. Then $p \approx_C p \otimes p$.*

Proof. By Lemma 1.2 we have $p < p \otimes p$ so $p \leq_C p \otimes p$. By Theorem 2.1, $p \otimes p \leq_C p$.□

2.3 Corollary. *Let S be a discrete space. For every $p, q \in \beta S$, we have $q \otimes p \approx_C p \otimes q$.*

Proof. By Lemma 1.2 we have $q < p \otimes q$ and $p < p \otimes q$. It follows from Theorem 2.1, that $q \otimes p \leq_C p \otimes q$. □

2.4 Theorem. *Let $(S, \cdot)$ be an infinite, discrete, left cancellative semigroup. Let $D \subseteq S$ and let $q \in S^*$. Suppose that $s \cdot q \neq t \cdot q$ whenever s and t are distinct members of D, and that $D \cdot q$ is strongly discrete. Then, for every $x, y \in S^*$* and every $p \in S^* \cap \overline{D}$, *$x \leq_{RK} p$ and $y \leq_{RK} q$ imply that $x \cdot y \leq_{RK} p \cdot q$. Furthermore, $x \leq_C p$ and $y \leq_C q$ imply that $x \cdot y \leq_C p \cdot q$.*

Proof. By Theorem 1.6, we have $p \cdot q \approx_{RK} p \otimes q$. If $x \leq p$ and $y \leq q$, then $x \cdot y \leq p \otimes q$ by Lemma 1.4.

If $x \leq_C p$ and $y \leq_C q$, then $x \leq_C p \otimes q$ and $y \leq_C p \otimes q$, because $p \leq p \otimes q$ and $q \leq p \otimes q$ (by Lemma 1.2). Hence, by Theorem 2.1, $x \cdot y \leq_C p \otimes q$. $\square$

2.5 Corollary. *Let $(S, \cdot)$ be a countable cancellative semigroup. and let p be a right cancellable element of βS. Then, for every $x, y \in S^*$, $x \leq_{RK} y$ implies that $x \cdot p \leq_{RK} y \cdot p$ and $x \leq_C y$ implies that $x \cdot p \leq_C y \cdot p$.*

Proof. Since p is right cancellable, $S \cdot p$ is discrete and therefore, being countable, it is strongly discrete. So Theorem 2.4 applies. $\square$

The following theorem is a converse of Corollary 2.5.

2.6 Theorem. *Let $(S, \cdot)$ be a countable cancellative semigroup. Let q be a right cancellable element of βS and let $p \in \beta S$ satisfy $p \leq q$. Then, for every $x, y \in \beta S$, $x \cdot q \leq y \cdot p$ implies that $x \leq y$.*

Proof. Suppose that $f : S \to S$ is a function for which $\bar{f}(y \cdot p) = x \cdot q$.

Let $B = \{b \in S : \bar{f}(b \cdot p) \in S \cdot q\}$. For each $b \in B$, there is a unique $c \in S$ for which $\bar{f}(b \cdot p) = c \cdot q$ (by Lemma 1.7). We define $g : B \to S$ by putting $g(b) = c$ and define g arbitrarily on the rest of S. We may suppose that $\bar{g}(y) \neq x$ (otherwise $x \leq y$, as we wish to prove). So there is a set $V \in y$ for which $g[V] \notin x$. We choose $V \subseteq B$ in the case in which $B \in y$.

Let $X \in x$ and $Y \in y$, with $Y \subseteq S \setminus B$ if $B \notin y$ and with $Y \subseteq V$ and $X \subseteq S \setminus g[V]$ if $B \in y$. Now $x \cdot q$ is in both $\overline{X \cdot q}$ and $\overline{\bar{f}[Y \cdot p]}$. It follows from Lemma 1.10 that

i) $a \cdot q = \bar{f}(z \cdot p)$ for some $a \in X$ and some $z \in \overline{Y}$, or else

ii) $w \cdot q = \bar{f}(b \cdot p)$ for some $w \in \overline{X}$ and some $b \in Y$.

We first show that ii) can be ruled out. Assuming ii), we have $w \cdot q \leq b \cdot p \leq p$. However, if $w \in S^*$, $p \leq q < w \cdot q$ (by Corollary 1.9). Hence $w \in S$ and therefore $b \in B$. This implies that $B \cap Y \neq \emptyset$ and thus that $B \in y$. So $b \in V$ and $w = g(b) \in X \cap g[V]$ contradicting the assumption that $X \cap g[V] = \emptyset$.

We may now suppose that i) holds for every $X \in x$ and $Y \in y$ satisfying the description above. Let $A = \{a \in S : a \cdot q \in \bar{f}[\beta S \cdot p]\}$. Then $A \in x$, because the assumption that i) holds for every choice of X and Y implies that $A \cap X \neq \emptyset$. For each $a \in A$, put

$$C_a = \{z \in \beta S : a \cdot q = \bar{f}(z \cdot p)\} .$$

We observe that $C_a \cap \overline{\bigcup_{a' \in A \setminus \{a\}} C_{a'}} = \emptyset$, because otherwise we should have $a \cdot q \in \overline{\{a' \cdot q : a' \in A \setminus \{a\}\}}$. This is impossible, because the assumption that q is right cancellable implies that $S \cdot q$ is discrete. So, for each $a \in A$, we can choose a clopen subset U_a of βS such that $C_a \subseteq U_a$ and $U_a \cap U_{a'} = \emptyset$ whenever $a \neq a'$. We define $h : S \cap \bigcup_{a \in A} U_a \to S$ by stating that $h(s) = a$ if $s \in U_a$. So $\bar{h}[U_a] = \{a\}$. Now i) implies that, for each $X \in x$ and $Y \in y$, there exists $a \in A \cap X$ for which $U_a \cap \overline{Y} \neq \emptyset$. So $h[Y] \cap X \neq \emptyset$ and hence $\bar{h}(y) = x$. So $x \leq y$. □

2.7 Corollary. *Let S be a countable cancellative semigroup and let $p \in \beta S$. The following statements are equivalent.*

(1) p is right cancellable in βS.

(2) For every $x, y \in \beta S$, $x \cdot p \leq y \cdot p$ implies that $x \leq y$.

(3) For every $x, y \in \beta S$, $x \cdot p \approx_{RK} y \cdot p$ implies that $x \approx_{RK} y$.

Proof. (1) $\Rightarrow$ (2). This is an immediate consequence of Theorem 2.6.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Let $x, y \in \beta S$ and assume that $x \cdot p = y \cdot p$. Suppose that $x \neq y$ and pick disjoint subsets U, V of S with $U \in x$ and $V \in y$. Since $x \cdot p \in \overline{U \cdot p}$ and $y \cdot p \in \overline{V \cdot p}$, an application of Lemma 1.10 shows that we must have $s \cdot p = u \cdot p$ for some $s \in S$ and some $u \in \beta S$ with $u \neq s$. In particular $s \cdot p \approx_{RK} u \cdot p$ so that $s \approx_{RK} u$. But then $u \in S$, and hence by Lemma 1.7, $s = u$, a contradiction. □

Let S be a discrete semigroup. An idempotent $p \in S^*$ is said to be *regular* if the equation $x \cdot p = p$ has the unique solution $x = p$ in S^*. It was shown in [17] that Martin's Axiom implies that regular idempotents exist in $\mathbb{N}^*$, and I. Protasov has recently sent the authors a ZFC proof of this fact. The following theorem shows that it is possible for an ultrafilter $p \in S^*$ which is not right cancellable, to have the property that $q \leq_{RK} q \cdot p$ for every $q \in S^*$.

2.8 Theorem. *Let S be a discrete countable group and let $p \in S^*$ be a regular idempotent. Then $q \leq_{RK} q \cdot p$ for every $q \in S^*$. In fact, for every $q \in S^*$, either $q = q \cdot p$ or else $q <_{RK} q \cdot p$ and $p <_{RK} q \cdot p$.*

Proof. Let $q \in S^*$. We may suppose that $q \neq q \cdot p$. Then there exist disjoint subsets A and B of S such that $A \in q$ and $B \in q \cdot p$. We may suppose that $A \cdot p \subseteq \overline{B}$, because $\{a \in S : a \cdot p \in \overline{B}\} \in q$ and we may replace A by its intersection with this set. We claim that $A \cdot p$ is discrete and therefore strongly discrete. If $A \cdot p$ is not discrete, then $a \cdot p = x \cdot p$ for some $a \in A$ and some $x \neq a$ in $\overline{A}$. This implies that $a^{-1} \cdot x \cdot p = p$. Now

$a^{-1} \cdot x \notin S$ because $a^{-1} \cdot x$ is not isolated in βS, since x is not. Thus $a^{-1} \cdot x = p$ and so $x = a \cdot p$. This is a contradiction because $x \in \overline{A}$ and $a \cdot p \in \overline{B}$.

So $A \cdot p$ is discrete and Lemma 1.2 and Theorem 1.6 apply. □

We say that a semigroup S is *weakly right cancellative* if and only if whenever $x, y \in S$, $\{s \in S : sx = y\}$ is finite. Similarly a semigroup S is *weakly left cancellative* if and only if whenever $x, y \in S$, $\{s \in S : xs = y\}$ is finite.

2.9 Lemma. *Let S be an infinite right cancellative and weakly left cancellative semigroup. Let D be an infinite subset of S and let $\alpha = |D|$. Enumerate D as $\langle s_\sigma \rangle_{\sigma<\alpha}$. Then there is a sequence $\langle x_\tau \rangle_{\tau<\alpha}$ in D such that, whenever $\sigma < \tau < \alpha$ and $\delta < \gamma < \alpha$, if $(\sigma, \tau) \neq (\delta, \gamma)$, then $s_\sigma \cdot x_\tau \neq s_\delta \cdot x_\gamma$.*

Proof. Choose any $x_0, x_1 \in D$. Let $2 \leq \gamma < \alpha$ and assume that we have chosen $\langle x_\tau \rangle_{\tau<\gamma}$. Let $B_\gamma = \{s_\sigma \cdot x_\tau : \sigma < \tau < \gamma\}$ and note that $|B_\gamma| \leq |\gamma| \cdot |\gamma|$. For $\delta < \gamma$, let $C_{\delta,\gamma} = \{y \in S : s_\delta \cdot y \in B_\gamma\}$. Now, given $\delta < \gamma$ and $t \in B_\gamma$, $|\{y \in S : s_\delta \cdot y = t\}| < \omega$ by weak left cancellation so $|C_{\delta,\gamma}| \leq |\gamma| \cdot |\gamma| \cdot \omega$. Thus $|\bigcup_{\delta<\gamma} C_{\delta,\gamma}| \leq |\gamma| \cdot |\gamma| \cdot \omega \cdot |\gamma| < \alpha$ so pick $x_\gamma \in D \setminus \bigcup_{\delta<\gamma} C_{\delta,\gamma}$.

Suppose one has $\sigma < \tau < \alpha$ and $\delta < \gamma < \alpha$ such that $s_\sigma \cdot x_\tau = s_\delta \cdot x_\gamma$ and assume without loss of generality that $\tau \leq \gamma$. Suppose first that $\tau < \gamma$. Then $s_\sigma \cdot x_\tau \in B_\gamma$ and $x_\gamma \notin C_{\delta,\gamma}$ so $s_\sigma \cdot x_\tau \neq s_\delta \cdot x_\gamma$, a contradiction. Thus $\tau = \gamma$, so by right cancellation $s_\sigma = s_\delta$. □

The following result will be needed in the next section.

2.10 Theorem. *Let S be an infinite right cancellative and weakly left cancellative semigroup and let $\alpha = |S|$. Then there is a set P of uniform ultrafilters on S with the following properties:*

(1) $|P| = 2^{2^\alpha}$;
(2) For each pair of distinct elements $p, q \in P$, $\beta S \cdot p$ and $\beta S \cdot q$ are disjoint;
(3) For each $p \in P$, $S \cdot p$ is strongly discrete in βS;
(4) Each $p \in P$ is right cancellable in βS.

Proof. We apply Lemma 2.9 with $D = S$. Let $\langle x_\tau \rangle_{\tau<\alpha}$ be the sequence whose existence is guaranteed by this lemma. We take P to be the set of all uniform ultrafilters on $\{x_\tau : \tau < \alpha\}$.

Then (1) holds by [6, Corollary 7.8].

To prove (2), let $p, q \in P$ be distinct. We can choose disjoint $A, B \subseteq S$ with $A \in p$ and $B \in q$. For each $\sigma < \alpha$, let $A_\sigma = \{x_\tau \in A : \tau > \sigma\}$ and $B_\sigma = \{x_\tau \in B : \tau > \sigma\}$.

Then $A_\sigma \in p$ and $B_\sigma \in q$. So, for any $x, y \in \beta S$, $\bigcup_{\sigma<\alpha} s_\sigma \cdot A_\sigma \in x \cdot p$ and $\bigcup_{\sigma<\alpha} s_\sigma \cdot B_\sigma \in y \cdot q$. By Lemma 2.9 these sets are disjoint and so $x \cdot p \neq y \cdot q$.

To prove (3), let $p \in P$. For each $\sigma < \alpha$, let $X_\sigma = \{x_\tau : \tau > \sigma\}$. Then $s_\sigma \cdot X_\sigma \in s_\sigma \cdot p$ and the sets $s_\sigma \cdot X_\sigma$ are pairwise disjoint by Lemma 2.9.

Finally, to prove (4), let $p \in P$ and let x, y be distinct elements of βS. We can choose disjoint subsets U and V of S with $U \in x$ and $V \in y$. Then $\bigcup_{s_\sigma \in U} s_\sigma \cdot X_\sigma \in x \cdot p$ and $\bigcup_{s_\sigma \in V} s_\sigma \cdot X_\sigma \in y \cdot p$. Since these sets are disjoint, $x \cdot p \neq y \cdot p$. □

2.11 Corollary. *Let S be an infinite discrete right cancellative and weakly left cancellative semigroup with cardinality α. Then βS has 2^{2^α} minimal left ideals, and each minimal right ideal in βS contains 2^{2^α} idempotents.*

Proof. By Theorem 2.10, βS has 2^{2^α} disjoint left ideals and each of these contains a minimal left ideal (by [1, Proposition 2.4]). Furthermore, the intersection of every minimal right ideal and every minimal left ideal contains an idempotent (by [1, Theorem 2.11]). □

2.12 Lemma. *Let S be an infinite right cancellative and weakly left cancellative semigroup. Let D be an infinite subset of S and let $\alpha = |D|$. Enumerate D as $\langle s_\sigma \rangle_{\sigma<\alpha}$ and let $\langle x_\tau \rangle_{\tau<\alpha}$ be as guaranteed by Lemma 2.9. If p is any α-uniform ultrafilter with $\{x_\tau : \tau < \alpha\} \in p$, then $s \cdot p \neq t \cdot p$ whenever s and t are distinct members of D and $\{s \cdot p : s \in D\}$ is strongly discrete.*

Proof. For each $\sigma < \alpha$, let $B_{s_\sigma} = \{s_\sigma \cdot x_\tau : \sigma < \tau < \alpha\}$. Since p is α-uniform, $\{x_\tau : \sigma < \tau < \alpha\} \in p$ so $B_{s_\sigma} \in s_\sigma \cdot p$ for each $\sigma < \alpha$. By Lemma 2.9, if $\sigma \neq \delta$ then $B_{s_\sigma} \cap B_{s_\delta} = \emptyset$. □

2.13 Theorem. *Let S be an infinite discrete right cancellative and weakly left cancellative semigroup. There are no elements of βS which are maximal in the Comfort order.*

Proof. We apply Theorem 2.10. By this theorem, there is a subset P of βS with cardinality 2^{2^α}, such that $S \cdot p$ is strongly discrete for every $p \in P$. By Lemma 1.2 and Theorem 1.6, this implies that $q \leq_C q \cdot p$ for every $q \in \beta S$.

Let q be any member of βS. By Theorem 2.10, the left ideals $\beta S \cdot p$ are disjoint and hence the elements $q \cdot p$, with $p \in P$, are all distinct. Thus q has 2^{2^α} different $\leq_C$ successors in βS. By Theorem 1.19, q has at most 2^α $\leq_C$ predecessors in βS. Thus q must have $\leq_C$ successors which are not Comfort equivalent to q. □

2.14 Corollary. *Let S be any infinite set. Then there are no Comfort maximal ultrafilters on S.*

Proof. Given any infinite cardinal α, there is a group with cardinality α. For example, the direct sum of α copies of $\mathbb{Z}_2$ has cardinality α. $\square$

The following result contrasts with Theorem 2.1. Notice that some sort of cancellation assumptions are necessary in Theorem 2.15. For example, if S is a left zero semigroup (i.e. $x \cdot y = x$ for all $x, y \in S$) then so is βS and any nonempty subset of βS is a subsemigroup. The same remark applies to a right zero semigroup as well.

2.15 Theorem. *Let S be an infinite discrete cancellative semigroup. Then for each $p \in \beta S\backslash S$, $\{q \in \beta S : q \leq_{RK} p\}$ is not a subsemigroup of βS. In fact, for each $p \in \beta S\backslash S$ there exists $r \approx_{RK} p$ such that $r <_{RK} p \cdot r$. If $\min\{|D| : D \in p\} = |S|$, then r can be chosen so that r is right cancellable and for all $q \in \beta S\backslash S$, $r <_{RK} q \cdot r$.*

Proof. Let $p \in \beta S\backslash S$, let $\alpha = \min\{|D| : D \in p\}$, and pick $D \in p$ such that $|D| = \alpha$. If $\alpha = |S|$, require that $D = S$. Enumerate D as $\langle s_\sigma \rangle_{\sigma<\alpha}$ and let $\langle x_\tau \rangle_{\tau<\alpha}$ be as guaranteed by Lemma 2.9. Define $f : S \longrightarrow D$ by $f(s_\sigma) = x_\sigma$ and let $r = \overline{f}(p)$. Then r is an α-uniform ultrafilter and $p \approx_{RK} r$.

Then by Lemma 2.12, Lemma 1.2, and Theorem 1.6, for all $q \in \overline{D}\backslash S$, $r <_{RK} q \cdot r$. It follows that $p \cdot r \not\leq_{RK} p$ and hence that $\{x \in \beta S : x \leq_{RK}\}$ is not a subsemigroup of βS. If $D = S$, then by [16, Theorem 2.2], r is right cancellable. $\square$

3. Further Connections between Order Relations and Algebra in βS

We need the following well known result, whose proof we cannot find in the literature.

3.1 Lemma. *Let S be an infinite set and let $p \in \beta S$. Then there is a uniform ultrafilter q on S such that $p <_{RK} q$.*

Proof. If u is a uniform ultrafilter on S, it is clear that $p \otimes u$ is uniform. By Lemma 1.2, $p <_{RK} p \otimes u$. $\square$

3.2 Theorem. *Let S be an infinite, discrete and cancellative semigroup. For each $p \in \beta S$ there exists $q \in K(\beta S)$ such that $p <_C q$.*

Proof. Let $\alpha = |S|$ and let $p \in \beta S$. By Lemma 3.1 we may presume that p is a uniform ultrafilter. Pick by Theorem 2.15 some $r \approx_{RK} p$ such that r is right cancellable and for all $q \in \beta S \backslash S$, $r <_{RK} q \cdot r$ and consequently for all $q \in \beta S \backslash S$, $r \leq_C q \cdot r$

By Corollary 2.11 we have that $|K(\beta S)| = 2^{2^\alpha}$. Since r is right cancellable and $|K(\beta S)| = 2^{2^\alpha}$, we have $|\{q \cdot r : q \in K(\beta S)\}| = 2^{2^\alpha}$. By Theorem 1.19 $|\{q \in \beta S : q \leq_C r\}| \leq 2^\alpha$. Consequently, we may pick $q \in K(\beta S)$ such that $q \cdot r \not\leq_C r$. Then $p \approx_{RK} r <_C q \cdot r$. Since $q \in K(\beta S)$, $q \cdot r \in K(\beta S)$. □

We know from [5, Theorem 2.1] that if a point $p \in \beta\mathbb{N}$ is right cancellable in $(\beta\mathbb{N}, +)$, then for all $q \in \beta\mathbb{N}$, $q <_{RK} q + p$.

3.3 Theorem. *Let S be an infinite discrete semigroup, let $\alpha = |S|$ and assume that $2^\alpha < 2^\mathfrak{c}$. Let $p \in \beta S$ and assume that for all $q \in \beta S$, $q \leq_C q \cdot p$. Then p is weakly right cancellable in βS. That is, for each $r \in \beta S$, $\{q \in \beta S : q \cdot p = r\}$ is finite.*

Proof. Suppose we have some $r \in \beta S$ such that $\{q \in \beta S : q \cdot p = r\}$ is infinite. Now $\{q \in \beta S : q \cdot p = r\} = \rho_p^{-1}[\{r\}]$ and is therefore an infinite closed subset of βS. Thus by [13, 9H2], $|\{q \in \beta S : q \cdot p = r\}| \geq 2^c > 2^\alpha$. By Theorems 1.19 and 1.21, $\{q \in \beta S : q \leq_C r\} = \beta_r(S)$ and $|\beta_r(S)| \leq 2^\alpha$. This is a contradiction because $\{q \in \beta S : q \cdot p = r\} \subseteq \{q \in \beta S : q \leq_C r\}$. □

We now turn our attention to results about the semigroup $(\mathbb{N}, +)$. For each $k \in \mathbb{N}$, the natural homomorphism $q_k : \mathbb{Z} \to \mathbb{Z}_k$ has an extension $\bar{q}_k : \beta\mathbb{Z} \to \mathbb{Z}_k$ which is easily seen to be a homomorphism on $(\beta\mathbb{Z}, +)$. Given $x, y \in \beta\mathbb{N}$ and $k \in \mathbb{N}$, we say that $x \equiv y(\text{mod} k)$ if $\bar{q}_k(x) = \bar{q}_k(y)$. The proof of the following lemma is immediate.

3.4 Lemma. *Let $p, q, r \in \beta\mathbb{N}$ and let $k \in \mathbb{N}$.*

(a) If $p \equiv q(\text{mod} k)$, then $p + r \equiv q + r(\text{mod} k)$, $r + p \equiv r + q(\text{mod} k)$, and $r + p \equiv q + r(\text{mod} k)$.

(b) If $p + r \equiv q + r(\text{mod} k)$, $r + p \equiv r + q(\text{mod} k)$ or $r + p \equiv q + r(\text{mod} k)$, then $p \equiv q(\text{mod} k)$.

3.5 Lemma. *Let $A \subseteq \mathbb{N}$ and let $q \in \beta\mathbb{N}$. Then $A + q$ is discrete in $\beta\mathbb{N}$ if and only if $a + q \neq x + q$ whenever $a \in A$ and $x \in A^*$.*

Proof. $A + q$ fails to be discrete if and only if there exists $a \in A$ for which $a + q \in \overline{(A \setminus \{a\}) + q} = \overline{(A \setminus \{a\})} + q$. This is equivalent to asserting that $a + q = x + q$ for some $x \in \overline{A \setminus \{a\}}$. By Lemma 1.7, this implies that $x \in A^*$. □

3.6 Theorem. *Suppose that $p \in \beta\mathbb{N} \backslash \mathbb{N}$ has the property that, for some $A \in p$, whenever*

$x \in A^*$ and $k \in \mathbb{N}$, one has $x \equiv p(\text{mod} k)$. Then, for every $q \in \beta\mathbb{N}\setminus\mathbb{N}$, we have $p <_{RK} p + q$ and $q <_{RK} p + q$.

Proof. Let $q \in \mathbb{N}^*$. We show that we may suppose that $A + q$ is discrete. We observe that there is at most one $a \in A$ for which $a + q \in A^* + q$; for, if $a + q \in A^* + q$ and $b + q \in A^* + q$ where $a, b \in A$, we have $a \equiv b(\text{mod} k)$ for every $k \in \mathbb{N}$, and hence $a = b$. We delete this element (if it exists) from A, and then $A + q$ is discrete by Lemma 3.5.

We observe that any countable discrete set is strongly discrete, and so the result follows from Lemma 1.2 and Theorem 1.6. □

3.7 Corollary. *There is a dense open subset U of $\beta\mathbb{N}\setminus\mathbb{N}$ such that, for every $p \in U$ and every $q \in \beta\mathbb{N}\setminus\mathbb{N}$, we have $p <_{RK} p + q$ and $q <_{RK} p + q$.*

Proof. We define U by stating that an element $p \in \beta\mathbb{N}\setminus\mathbb{N}$ is in U if and only if there is a set $A \in p$ such that $x \in A^*$ implies that $x \equiv p(\text{mod} k)$ for every $k \in \mathbb{N}$. Given such A, $\overline{A}\setminus\mathbb{N} \subseteq U$ so U is open.

To see that U is dense, let $q \in \beta\mathbb{N}\setminus\mathbb{N}$ and let $Q \in q$. For each $k \in \mathbb{N}$, let

$$Q_k = \{b \in \mathbb{N} : b \equiv q(\text{mod} k)\}$$

and notice that $Q_k \in q$. For each $n \in \mathbb{N}$ choose $x_n \in Q \cap \bigcap_{k=1}^{n} Q_k$. Let $A = \{x_n : n \in \mathbb{N}\}$. Then $A^* \subseteq \overline{Q} \cap U$. □

3.8 Corollary. *If p is a P-point in $\beta\mathbb{N}\setminus\mathbb{N}$, then $p <_{RK} p+q$ and $q <_{RK} p+q$ for every $q \in \beta\mathbb{N}\setminus\mathbb{N}$.*

Proof. As in the proof of Corollary 3.7, for each $k \in \mathbb{N}$, let $Q_k = \{b \in \mathbb{N} : b \equiv p(\text{mod} k)\}$. Pick $A \in p$ such that $\overline{A}\setminus\mathbb{N} \subseteq \bigcap_{k=1}^{\infty} \overline{Q_k}$. □

We note that, by Ramsey's Theorem, every infinite sequence in $\mathbb{N}$ contains an infinite subsequence $\langle a_n \rangle_{n=1}^{\infty}$ satisfying the conditions of the following theorem.

3.9 Theorem. *Let $\langle a_n \rangle_{n=1}^{\infty}$ be an infinite increasing sequence in $\mathbb{N}$. Suppose that either of the two following conditions is satisfied:*

(i) For every $n \in \mathbb{N}$, a_{n+1} is a multiple of a_n.

(ii) For every $m, n \in \mathbb{N}$ with $m \neq n$, a_n is not a multiple of a_m.

Then, if $p \in \overline{\{a_n : n \in \mathbb{N}\}} \cap \mathbb{N}^*$, we have $p <_{RK} p+q$ and $q <_{RK} p+q$ for every $q \in \mathbb{N}^*$.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$.

We first consider the case in which $q \in \bigcap_{n\in\mathbb{N}} \overline{n\mathbb{N}}$.

For each $n \in \mathbb{N}$, let $B_n = \bigcap_{k\leq n+1}(a_k\mathbb{N})$. We observe that $B_n \in q$.

If condition (i) above is satisfied, we define $f : \mathbb{N} \to \mathbb{N}$ by stating that $f(n) = \max\{a_m : a_m|n\}$, defining f arbitrarily if no a_m divides n. Then, if $b \in B_n$, we have $f(a_n + b) = a_n$. It follows that $a_m + B_m$ and $a_n + B_n$ are disjoint and hence that $A + q$ is discrete, because $a_n + B_n \in a_n + q$ for every $n \in \mathbb{N}$. Thus Lemma 1.2 and Theorem 1.6 apply.

If condition (ii) is satisfied, we define $g : \mathbb{N} \to \mathbb{N}$ by stating that $g(n) = \min\{a_m : a_m|n\}$, defining g arbitrarily if no a_m divides n. Once again, if $b \in B_n$, we have $g(a_n + b) = a_n$ and can deduce that $A + q$ is discrete. Thus Lemma 1.2 and Theorem 1.6 apply.

Now let q be any element of $\mathbb{N}^*$. For each $n \in \mathbb{N}$, we can choose $b_n \in \mathbb{N}$ satisfying $b_n + q \equiv 0 \pmod{k}$ for every $k \in \{1, 2, \cdots, n\}$. To see this, let $-q \in \beta\mathbb{Z}$ be defined by $-q = \{-Q : Q \in q\}$. Then

$$\big\{b \in \mathbb{Z} : b + q \equiv 0(\mathrm{mod}k) \text{ for every } k \in \{1, 2, \cdots n\}\big\} \in -q$$

and is therefore non-empty. If b is in this set, so is $b + n!m$ for every $m \in \mathbb{Z}$, and thus this set contains positive integers.

Let $r \in \mathbb{N}^* \cap \overline{\{b_n : n \in \mathbb{N}\}}$. Then $r + q \in \bigcap_{n\in\mathbb{N}} \overline{n\mathbb{N}}$, because, for every $k \in \mathbb{N}$, we have $\bar{q}_k(b_n + q) = 0$ if $n > k$ and hence $\bar{q}_k(r + q) = 0$. This implies that $q + r \in \bigcap_{n\in\mathbb{N}} \overline{n\mathbb{N}}$.

By what we have already proved, with $q+r$ in place of q, we can assert that $A+q+r$ is discrete. By Lemma 3.5, this is equivalent to asserting that, for every $a \in A$ and every $x \in A^*$, $a + q + r \neq x + q + r$. This implies that $a + q \neq x + q$ and hence that $A + q$ is discrete. The required result again follows from Lemma 1.2 and Theorem 1.6. □

The set $\mathbb{H} \subseteq \beta\mathbb{N}$ is defined by $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$. Given $x \in \mathbb{N}$, we denote the binary support of x by $\mathrm{supp}(x)$. This is the subset of ω defined by the equation $x = \sum_{m\in\mathrm{supp}(x)} 2^m$.

The following theorem is not new – indeed it is a special case of [6, Theorem 10.9]. However, we give an algebraic proof which we believe to be new.

3.10 Theorem. *Let C be a subset of $\mathbb{N}^*$ with cardinality $\mathfrak{c}$. Then the elements of C have a common $\leq_{RK}$-successor in $\mathbb{H}$.*

Proof. We index C as $\langle p_x\rangle_{x\in\mathbb{R}}$. Let $\langle E_x\rangle_{x\in\mathbb{R}}$ be an almost disjoint family of subsets of $\{2^n : n \in \mathbb{N}\}$. For each $x \in \mathbb{R}$, choose $q_x \in \overline{E_x} \cap \mathbb{N}^*$ such that $q_x \approx_{RK} p_x$. For each finite non-empty subset F of $\mathbb{R}$, we put $s_F = \sum_{x\in F} q_x$ where the terms in the sum occur in the order of increasing indices. We order the set $\mathcal{P}_f(\mathbb{R})$ of finite nonempty subsets of $\mathbb{R}$ by set inclusion and choose q to be a limit point of the net $\langle s_F\rangle_{F\in\mathcal{P}_f(\mathbb{R})}$ in $\beta\mathbb{N}$.

For each $x \in \mathbb{R}$ define $f_x : \mathbb{N} \longrightarrow \mathbb{N}$ by $f_x(n) = \min\{2^m \in E_x : m \in \text{supp}(n)\}$ if $\{2^m \in E_x : m \in \text{supp}(n)\} \neq \emptyset$ and $f(n) = 1$ otherwise.

We shall show that $\overline{f}_x(q) = q_x$.

Suppose that this equation does not hold. Then we can choose $A \in q_x$ such that $f_x^{-1}[A] \notin q$, and so we can choose $R \in q$ such that $f_x^{-1}[A] \cap R = \emptyset$.

Let $F \in \mathcal{P}_f(\mathbb{R})$ satisfy $x \in F$ and $R \in s_F$. We can choose a disjoint family $\langle A_y \rangle_{y \in F}$ of subsets of $\mathbb{N}$ such that $A_x \subseteq E_x \cap A$ and, for every $y \in F$, $A_y \in q_y$ and $A_y \cap E_x = \emptyset$ if $y \neq x$. Let B be the set of all integers b of the form $b = \sum_{y \in F} n_y$ where $n_y \in A_y$ for all $y \in F$. We observe that this expression for b is unique and that $f_x(b) = n_x \in A$.

We claim that $B \in s_F$. To see this, we enumerate F in increasing order as $(y_1, y_2, \cdots, y_m)$. For each $i \in \{1, 2, \cdots, m\}$, we choose $n_i \in A_{y_i}$ and so $\sum_{i=1}^m n_i \in B$. Now we have

$$s_F = \lim_{n_1 \to q_{y_1}} \lim_{n_2 \to q_{y_2}} \cdots \lim_{n_m \to q_{y_m}} (\textstyle\sum_{i=1}^m n_i).$$

This shows that $s_F \in \overline{B}$ and hence that $B \in s_F$.

So we can then choose $b \in B \cap R$. Since $f_x(b) \in A$, it follows that $b \in f_x^{-1}[A] \cap R$, contradicting our assumption that this set is empty. □

3.11 Corollary. *The elements of any subset C of $\mathbb{N}^*$ with cardinality at most $\mathfrak{c}$ have a common $\leq_{RK}$ successor in any given minimal left ideal of $\beta\mathbb{N}$, and they also have a common $\leq_{RK}$ successor in any given minimal right ideal of $\beta\mathbb{N}$. Furthermore, there is a left ideal L of $\beta\mathbb{N}$ and a right ideal R of $\mathbb{N}$ such that $x <_{RK} y$ for every $x \in C$ and every $y \in L \cup R$.*

Proof. We know that the elements of C have a common $\leq_{RK}$ successor q in $\beta\mathbb{N}$. We can choose $p \in \overline{\{n! : n \in \mathbb{N}\}} \cap \mathbb{N}^*$ such that $q \approx_{RK} p$.

By the remark on p. 241 of [23], p is right cancellable in $\beta\mathbb{N}$. So, by [5, Theorem 2.1], $\mathbb{N} + p$ is strongly discrete and so by Lemma 1.2 and Theorem 1.6, for any $u \in \mathbb{N}^*$, $p <_{RK} u + p$. Also $p <_{RK} p + u$ (by Theorem 3.6). We can choose u to lie in any given minimal left ideal or in any given minimal right ideal.

Putting $L = \beta\mathbb{N} + p$ and $R = p + \beta\mathbb{N}$, we have $x <_{RK} y$ for every $x \in C$ and every $y \in L \cup R$. □

3.12 Corollary. *For each of the orders $\leq_{RK}$ and $\leq_C$, every minimal left ideal of $\beta\mathbb{N}$ contains an increasing $\mathfrak{c}^+$ chain and so does every minimal right ideal of $\beta\mathbb{N}$.*

Proof. This is proved by an obvious transfinite induction, using Corollary 3.11 and the fact that $\beta\mathbb{N}$ has no maximal $\leq_{RK}$ or maximal $\leq_C$ elements. □

3.13 Theorem. *There are at most $\mathfrak{c}$ elements of $\mathbb{N}^*$ whose $\leq_C$ successors form a subsemigroup of $\beta\mathbb{N}$.*

Proof. Suppose that $p \in \mathbb{N}^*$ has the property that its $\leq_C$ successors form a subsemigroup of $\beta\mathbb{N}$. We shall show that $p \leq_C q$ for every $q \in K(\beta\mathbb{N})$.

To see this, suppose that L is the minimal left ideal of $\beta\mathbb{N}$ for which $q \in L$. By Corollary 3.11, $p \leq_C r$ for some $r \in L$. Furthermore, there is a minimal left ideal M of $\beta\mathbb{N}$ such that $p \leq_C y$ for every $y \in M$. So $p \leq_C y + r$ for every $y \in M$. Now $M + r = L$ (by [1, Proposition 2.4]), and so $q \in M + r$ and hence $p \leq_C q$.

The result now follows from the fact that a given element q of $\beta\mathbb{N}$ can have at most $\mathfrak{c}$ $\leq_C$ predecessors (by Theorem 1.19). □

We can generalize part of Corollary 3.11 to semigroups of any cardinality.

3.14 Theorem. *Let $(S, \cdot)$ be an infinite discrete right cancellative and weakly left cancellative semigoup with cardinality α, and let C be a subset of S^* with cardinality at most 2^α. Then the elements of C have a common $\leq_{RK}$ successor in any given minimal right ideal of βS. Furthermore, there is a left ideal L of βS such that $x \leq_{RK} y$ for every $x \in C$ and every $y \in L$.*

Proof. By [6, Theorem 10.9], the elements of C have a common $\leq_{RK}$ successor q in βS. Let P be the set described in Theorem 2.9. By Lemma 3.1, we may suppose that q is uniform and hence that $q \approx_{RK} p$ for some $p \in P$. By Lemma 1.2 and Theorem 1.6, $p <_{RK} x \cdot p$ for every $x \in \beta S$ and we can choose x to lie in any given minimal right ideal.

If we put $L = \beta S \cdot p$, then $x <_{RK} y$ for every $x \in C$ and every $y \in L$. □

We conclude with some results about predecessors of P-points in $\mathbb{N}^*$. (The sets $A_\sigma(y, \mathbb{N})$ are defined in Lemma 1.18.)

We omit the proof of the following lemma, which can be proved by an obvious transfinite induction.

3.15 Lemma. *Let $a \in \mathbb{N}$ and let $x, y \in \mathbb{N}^*$. Then, for any ordinal σ satisfying $1 \leq \sigma < \omega_1$, $a + x \in A_\sigma(y, \mathbb{N})$ if and only if $x \in A_\sigma(y, \mathbb{N})$.*

3.16 Lemma. *Let p be a P-point in $\mathbb{N}^*$ and let $x \in \beta_p(\mathbb{N}) \cap \mathbb{N}^*$. Let σ be the first ordinal for which $x \in A_\sigma(p, \mathbb{N})$. Then x is a P-point in $\mathbb{N}^* \backslash A_{\sigma-1}(p, \mathbb{N})$. (We note that $\sigma - 1$ exists because σ is neither 0 nor a limit ordinal).*

Proof. We first deal with the case in which $\sigma = 1$. In this case, there is a function $f : \mathbb{N} \to \mathbb{N}$ for which $\overline{f}(p) = x$. Suppose that $\langle C_n \rangle_{n=1}^\infty$ is a sequence of compact subsets of $\mathbb{N}^*$ which do not contain x, for which $x \in \overline{\bigcup_{n=1}^{\infty} C_n}$. For each $n \in \mathbb{N}$, $p \notin \overline{f}^{-1}[C_n]$. Since p is a P-point, there is a set $P \in p$ for which $\overline{P} \cap \mathbb{N}^* \cap \overline{f}^{-1}[C_n] = \emptyset$ for every $n \in \mathbb{N}$.

We apply Lemma 1.10 with $A = f[P]$ and $B = \bigcup_{n=1}^{\infty} C_n$. We note that $A \cap \overline{B} = \emptyset$ and hence that there exists $y \in \overline{A} \cap B$. This implies that there is an element $q \in \overline{P}$ for which $\overline{f}(q) = y$ and that $y \in C_n$ for some $n \in \mathbb{N}$. So $q \in \overline{P} \cap \mathbb{N}^* \cap \overline{f}^{-1}[C_n]$, a contradiction.

We now suppose that $\sigma > 1$. We make the inductive assumption that, for every ordinal τ which is neither 0 nor a limit ordinal and satisfies $\tau < \sigma$, the points of $A_\tau(p,\mathbb{N})\backslash A_{\tau-1}(p,\mathbb{N})$ are P-points in $\mathbb{N}^*\backslash A_{\tau-1}(p,\mathbb{N})$.

Since $x \in A_\sigma(p,\mathbb{N})$, there is a function $g : \mathbb{N} \to A_{\sigma-1}(p,\mathbb{N})$ for which $\overline{g}(p) = x$. We may clearly suppose that $g[\mathbb{N}] \subseteq \mathbb{N}^*$. Suppose that $\langle D_n \rangle_{n=1}^{\infty}$ is a sequence of compact subsets of $\mathbb{N}^*\backslash A_{\sigma-1}(p,\mathbb{N})$ which do not contain x, such that $x \in \overline{\bigcup_{n=1}^{\infty} D_n}$. For each $n \in \mathbb{N}$, $p \notin \overline{g}^{-1}[D_n]$. So there is a set $Q \in p$ such that $\overline{Q} \cap \mathbb{N}^* \cap \overline{g}^{-1}[D_n] = \emptyset$ for every $n \in \mathbb{N}$. We apply Lemma 1.10 again, this time with $A = g[Q]$ and $B = \bigcup_{n=1}^{\infty} D_n$. We claim that $A \cap \overline{B} = \emptyset$. To see this, let $z \in A$ and let τ be the first ordinal for which $z \in A_\tau(p,\mathbb{N})$. Then τ is neither 0 nor a limit ordinal and satisfies $\tau < \sigma$. By our inductive assumption, z is a P-point in $\mathbb{N}^*\backslash A_{\tau-1}(p,\mathbb{N})$. Since $D_n \subseteq \mathbb{N}^*\backslash A_{\tau-1}(p,\mathbb{N})$ for every $n \in \mathbb{N}$, it follows that $z \notin \overline{B}$. So $\overline{A} \cap B \neq \emptyset$. However, just as before, this contradicts our assumption that $\overline{Q} \cap \mathbb{N}^* \cap \overline{g}^{-1}[D_n] = \emptyset$ for every $n \in \mathbb{N}$.

This establishes that x is a P-point in $\mathbb{N}^*\backslash A_{\sigma-1}(p,\mathbb{N})$ as claimed. □

Conclusion (i) of the following theorem is well known.

3.17 Theorem. *Let p be a P-point in $\mathbb{N}^*$ and let $x \in \mathbb{N}^*$. Then*
(i) If $x \leq_{RK} p$, x is a P-point in $\mathbb{N}^*$;
(ii) If $x \leq_C p$, x is right cancellable in $\beta\mathbb{N}$.

Proof. (i) follows from the case in which $\sigma = 1$ in Lemma 3.16.

To prove (ii), suppose that $x \leq_C p$ and that σ is the first ordinal for which $x \in A_\sigma(p,\mathbb{N})$. If x is not right cancellable, then $x = y + x$ for some $y \in \mathbb{N}^*$ (by [5, Theorem 2.1]). So $x \in \overline{\mathbb{N} + x}$. By Lemma 3.16, there must be an integer $a \in \mathbb{N}$ for which $a + x \in A_{\sigma-1}(p,\mathbb{N})$. By Lemma 3.15, this implies that $x \in A_{\sigma-1}(p,\mathbb{N})$, a contradiction. □

3.18 Corollary. *The Comfort type of any P-point in $\mathbb{N}^*$ is a subsemigroup of $\beta\mathbb{N}$.*

Proof. Let p be a P-point in $\mathbb{N}^*$ and suppose that $x, y \in \beta\mathbb{N}$ are Comfort equivalent to

p. Then $x + y \leq_C p$ by Theorem 2.1. By Theorem 3.17, y is right cancellable. So, by Corollary 1.9, we have $p \approx_C x \leq_C x + y$. □

Problems

We list some of the questions to which we do not know the answers.

(1) Can we characterize the ultrafilters p in $\beta\omega$ for which $\{q \in \beta\omega : q \approx_C p\}$ is a subsemigroup of $\beta\omega$?

(2) Are there any ultrafilters in $\mathbb{N}^*$ whose $\leq_C$ successors form a subsemigroup of $\mathbb{N}^*$?

(3) Is every $\leq_C$ minimal ultrafilter in $\mathbb{N}^*$ Comfort equivalent to a $\leq_{RK}$ minimal ultrafilter?

(4) Can the existence of $\leq_C$ minimal ultrafilters in $\mathbb{N}^*$ be demonstrated in ZFC?

(5) Let $p \in \mathbb{N}^*$. Are the two following statements equivalent?

For every $x, y \in \beta\mathbb{N}$, $p + x \leq p + y$ implies that $x \leq_{RK} y$.

p is left cancellable in $\beta\mathbb{N}$.

(6) Given $\{p_k : k \in \mathbb{N}\} \cup \{q\} \subseteq \mathbb{N}^*$, does there exist $r \in \mathbb{N}^*$ such that $r \leq_{RK} q \cdot p_k$ for all $k \in \mathbb{N}$ and $r \leq_{RK} q$?

(7) Given $\{p_k : k \in \mathbb{N}\} \cup \{q\} \subseteq \mathbb{N}^*$, does there exist $r \in \mathbb{N}^*$ such that $r \leq_{RK} p_k \cdot q$ for all $k \in \mathbb{N}$ and $r \leq_{RK} q$?

(8) Given a semigroup $(S, \cdot)$ and $u, v, p, q \in \beta S$ does $u \leq_C p$ and $v \leq_C q$ imply that $u \cdot v \leq p \cdot q$? Equivalently, is $\beta_p(S) \cdot \beta_q(S) \subseteq \beta_{p \cdot q}(S)$?

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