This paper appeared in Topology Proceedings **27** (2003), 479-496. To the best of my knowledge this is the final copy as it was submitted to the publisher. –NH

Topology Proceedings

ORDER COMPACTIFICATIONS OF DISCRETE SEMIGROUPS

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ABSTRACT. Given a partially ordered set $(X, \leq)$ one can construct the *order compactification* μX of X in the same fashion as Čech's construction of the Stone-Čech compactification, using the order preserving functions from X into the unit interval $[0,1]$. We consider a semigroup $(S, \cdot)$ which has an ordering which the semigroup respects in the sense that $x \leq y$ implies that $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $x, y, z \in S$. We show that the operation can be extended to μS making it into a right topological semigroup with S contained in the topological center such that both the left and right translations are order preserving. We then investigate the structure of μS for certain specific semigroups S.

1. Introduction

In his construction of the Stone-Čech compactification [2], E. Čech embedded a completely regular Hausdorff space X into the product ${}^{F}[0,1]$ where F is the set of all continuous functions from

2000 *Mathematics Subject Classification.* Primary 54D35; Secondary 22A15.

Key words and phrases. Order compactification, semigroup.

The first author acknowledges support received from the National Science Foundation via grant DMS 0243586.

The second author acknowledges support received from the United Kingdom Engineering and Physical Sciences Research Council, and the City University of New York.

X into $[0,1]$. It is well known that this approach can be generalized, and we summarize these well known facts now.

We shall be mainly interested in the case in which F is the set of order preserving functions from a partially ordered set to $[0,1]$. However some of the basic facts are valid in more generality. Below, we denote the set of functions from X to Y by ${}^{X}Y$.

Definition 1.1. Let X be a set. Let $F \subseteq {}^{X}[0,1]$ and let $Z = {}^{F}[0,1]$ with the product topology. Define the evaluation map $e : X \to Z$ by $e(x)(f) = f(x)$ and let $\mu X(F) = c\ell_Z e[X]$.

We shall usually write μX rather than $\mu X(F)$.

Theorem 1.2. *Let X be a set and let $F \subseteq {}^{X}[0,1]$. Then μX is a compact Hausdorff space. The evaluation e is injective if and only if F separates points of X. If $x \in X$ and there exist U open in $[0,1]$ and $f \in F$ such that $f^{-1}[U] = \{x\}$, then $e(x)$ is isolated in μX. Consequently, if for each $x \in X$ there exist U open in $[0,1]$ and $f \in F$ such that $f^{-1}[U] = \{x\}$, then $e[X]$ is open and discrete in μX, so that in particular $(e, \mu X)$ is a topological compactification of X with the discrete topology.*

Proof. The first and second conclusions are immediate. For the third, let $x \in X$ and pick U and f as guaranteed. Then ${\pi_f}^{-1}[U]$ is a neighborhood of $e(x)$. If $p \in {\pi_f}^{-1}[U]\backslash\{e(x)\}$ then ${\pi_f}^{-1}[U]\backslash\{e(x)\}$ is a neighborhood of p missing $e[X]$ so ${\pi_f}^{-1}[U] \cap \mu X = \{e(x)\}$. □

Regardless of the set F of functions there is a natural partial order induced on ${}^{F}[0,1]$.

Definition 1.3. Let X be a set and let $F \subseteq {}^{X}[0,1]$. For $p, q \in {}^{F}[0,1]$, we say that $p \leq q$ if and only if for each $f \in F$, $p(f) \leq q(f)$.

From this point on we shall be concerned with a specific class of functions from X to $[0,1]$.

Convention 1.4. Let $(X, \leq)$ be a partially ordered set. When F is mentioned without modification, it will be assumed that

$$F = \left\{ f \in {}^{X}[0,1] : \big(\forall x \in X\big)\big(\forall y \in X\big)\big(x \leq y \Rightarrow f(x) \leq f(y)\big)\right\}.$$

The compactification $\mu X(F)$ will be called the *order compactification* of X.

Theorem 1.5. *Let $(X, \leq)$ be a partially ordered set with the discrete topology. Then the order compactification $(e, \mu X)$ is a topological compactification in which $e[X]$ is open.*

Proof. By Theorem 1.2 it suffices to let $x \in X$ and produce U open in $[0,1]$ and $f \in F$ such that $f^{-1}[U] = \{x\}$. Let $U = (0,1)$ and define $f \in {}^X[0,1]$ by

$$f(y) = \begin{cases} 0 & \text{if } \neg(x \leq y) \\ 1/2 & \text{if } x = y \\ 1 & \text{if } x < y\,. \end{cases}$$

(When we write $x < y$ we of course mean that $x \leq y$ and $x \neq y$.) □

It is trivial that the evaluation map from a partially ordered set to its order compactification is order preserving. In fact a stronger conclusion holds.

Lemma 1.6. *Let $(X, \leq)$ be a partially ordered set and let $x, y \in X$. Then $x \leq y \Leftrightarrow e(x) \leq e(y)$.*

Proof. As we stated, the necessity is trivial. For the sufficiency assume that $\neg(x \leq y)$. Define $f \in {}^X[0,1]$ by

$$f(z) = \begin{cases} 0 & \text{if } \neg(x \leq z) \\ 1 & \text{if } x \leq z\,. \end{cases}$$

Then $f \in F$ and $e(y)(f) < e(x)(f)$ so $\neg\big(e(x) \leq e(y)\big)$. □

Notice that trivially $\leq$ is a partial order on ${}^F[0,1]$.

Lemma 1.7. *Assume that $\leq$ is a linear order on X and let $p, q \in \mu X$. Then*

$$p < q \Leftrightarrow \big(\exists f \in F\big)\big(p(f) < q(f)\big)\,.$$

In particular, μX is also linearly ordered by $\leq$.

Proof. The necessity is immediate.

For the sufficiency pick $f \in F$ such that $p(f) < q(f)$. Suppose that $\neg(p < q)$ and pick $g \in F$ such that $q(g) < p(g)$. Pick $a, b \in [0,1]$ such that $p(f) < a < q(f)$ and $q(g) < b < p(g)$. Then ${\pi_f}^{-1}\big[[0,a)\big] \cap {\pi_g}^{-1}\big[(b,1]\big]$ is a neighborhood of p and ${\pi_g}^{-1}\big[[0,b)\big] \cap {\pi_f}^{-1}\big[(a,1]\big]$ is a neighborhood of q so pick $x, y \in X$ such that

$$\begin{aligned} &e(x) \in {\pi_f}^{-1}\big[[0,a)\big] \cap {\pi_g}^{-1}\big[(b,1]\big] \text{ and} \\ &e(y) \in {\pi_g}^{-1}\big[[0,b)\big] \cap {\pi_f}^{-1}\big[(a,1]\big]\,. \end{aligned}$$

Then $f(x) < a < f(y)$ and $g(y) < b < g(x)$ so one has a contradiction regardless of whether $x \leq y$ or $y \leq x$. □

The following lemma provides a convenient description of the order on the order compactification.

Lemma 1.8. *Let $(X, \leq)$ be a partially ordered set and let $p, q \in \mu X$. Then*

$$p \leq q \Leftrightarrow (\forall U \in \mathcal{N}_p)(\forall V \in \mathcal{N}_q)(\exists x \in e^{-1}[U])(\exists y \in e^{-1}[V])(x \leq y)\,.$$

Proof. Necessity. Let $U \in \mathcal{N}_p$ and $V \in \mathcal{N}_q$ be given. Define $f \in {}^X[0,1]$ by

$$f(z) = \begin{cases} 1 & \text{if } (\exists x \in e^{-1}[U])(x \leq z) \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in F$ so $p(f) \leq q(f)$. Also $f\big[e^{-1}[U]\big] = \{1\}$ so $p(f) = 1$ and consequently $q(f) = 1$ so that ${\pi_f}^{-1}\big[(0,1]\big]$ is a neighborhood of q. So pick $y \in X$ such that $e(y) \in {\pi_f}^{-1}\big[(0,1]\big] \cap V$. Then $f(y) = 1$ so $(\exists x \in e^{-1}[U])(x \leq y)$.

Sufficiency. Assume that $\neg(p \leq q)$ and pick $f \in F$ such that $q(f) < p(f)$. Pick a such that $q(f) < a < p(f)$. Then ${\pi_f}^{-1}\big[(a,1]\big]$ is a neighborhood of p and ${\pi_f}^{-1}\big[[0,a)\big]$ is a neighborhood of q so pick $x \leq y$ such that $e(x) \in {\pi_f}^{-1}\big[(a,1]\big]$ and $e(y) \in {\pi_f}^{-1}\big[[0,a)\big]$. But then $f(y) < a < f(x)$ contradicting the fact that $f \in F$. □

As a topological compactification of the discrete space X, necessarily μX is a quotient of the Stone-Čech compactification βX of X. We see now that there is a significant amount of collapsing when this quotient is formed, unless there are no strictly monotonic sequences in X.

Theorem 1.9. *Let $(X, \leq)$ be a partially ordered set and let $\langle x_n \rangle_{n=1}^{\infty}$ be a monotonic sequence in X. Then $\langle e(x_n) \rangle_{n=1}^{\infty}$ converges in μX.*

More generally, let $D \subseteq X$ and assume that either $(D, \leq)$ or $(D, \geq)$ is a directed set. Then the net $\langle e(x) \rangle_{x \in D}$ converges in μX.

Proof. Assume without loss of generality that $(D, \leq)$ is a directed set. Define $r \in {}^F[0,1]$ by $r(f) = \text{lub}\{f(x) : x \in D\}$. We claim that $\langle e(x) \rangle_{x \in D}$ converges to r (and in particular $r \in \mu X$). So let a neighborhood U of r in ${}^F[0,1]$ be given. Pick a finite subset H of F and $\epsilon > 0$ such that

$$\bigcap_{h\in H} {\pi_h}^{-1}[(r(h)-\epsilon, r(h)+\epsilon)] \subseteq U\,.$$

For each $h \in H$ pick $x_h \in D$ such that $h(x_h) > r(h) - \epsilon$. Pick $x \in D$ such that $x_h \leq x$ for each $h \in H$. Then for each $h \in H$ and each $y \in D$ with $x \leq y$,

$$r(h) - \epsilon < h(x_h) \leq h(y) \leq r(h) < r(h) + \epsilon$$

and so $e(y) \in \bigcap_{h\in H} {\pi_h}^{-1}[(r(h)-\epsilon, r(h)+\epsilon)]$. □

We shall be concerned in the next section with the extension of the operation on an ordered semigroup S to μS and the following Theorem will be useful.

Theorem 1.10. *Let $(X, \leq)$ be a partially ordered set and let $f : X \to \mu X$ be an order preserving function. Then there exists a continuous $\widetilde{f} : \mu X \to \mu X$ such that $\widetilde{f} \circ e = f$.*

Proof. [6, Theorem 2(b)]. □

In Section 2 we shall show that for any ordered semigroup S for which the operation respects the order, the compactification μS is a compact right topological semigroup with respect to which both left and right translates preserve order. In Section 3 we shall investigate the structure of of μS for certain specific ordered semigroups.

2. The Order Compactification of a Discrete Semigroup

We shall see in this section if $(S, \cdot)$ is a semigroup which is also a partially ordered set for which the operation respects the order, then there is a natural extension of the operation to μS which makes $(\mu S, \cdot)$ into a right topological semigroup with $e[S]$ contained in its topological center. That is, for every $p \in \mu S$ the function $\rho_p : \mu S \to \mu S$ defined by $\rho_p(q) = q \cdot p$ is continuous. And, for every $x \in S$, the function $\lambda_{e(x)} : \mu S \to \mu S$ defined by $\lambda_{e(x)}(q) = e(x) \cdot q$ is continuous.

Given any semigroup S, equality is a partial order which the operation respects, and for this partial order $\mu S = \beta S$, the Stone-Čech compactification of S. Consequently one cannot hope in general to have μS as a semitopological semigroup, that is to have λ_p continuous for each $p \in \mu S$. (See [4, Section 4.2].) In fact, we shall see in the next section that for $(\mathbb{R},+)$ with its usual order, the topological center of $\mu\mathbb{R}$ is exactly $e[\mathbb{R}]$.

Theorem 2.1. *Let $(S,\cdot)$ be a semigroup and let $\leq$ be a partial order on S. Assume that for all $x,y,z\in S$, if $x\leq y$, then $x\cdot z\leq y\cdot z$ and $z\cdot x\leq z\cdot y$. Then there is a unique extension of $\cdot$ to μS such that $e:S\to\mu S$ is a homomorphism and $(\mu S,\cdot)$ is a compact right topological semigroup with $e[S]$ contained in its topological center.*

Proof. We establish existence and uniqueness at the same time, defining $\cdot$ on μS as we are forced to define it. First let $x\in S$ and define $l_x:S\to\mu S$ by $l_x(y)=e(x\cdot y)$. The requirements that $\lambda_{e(x)}$ be continuous and that e be a homomorphism say that $\lambda_{e(x)}$ must be a continuous function with the property that $\lambda_{e(x)}\circ e=l_x$. Since $e[S]$ is dense in μS, there can be at most one such continuous function. We claim that l_x is order preserving. So let $y,z\in S$ with $y\leq z$ and let $g\in F$ be given. Then $x\cdot y\leq x\cdot z$ so

$$l_x(y)(g)=e(x\cdot y)(g)=g(x\cdot y)\leq g(x\cdot z)=e(x\cdot z)(g)=l_x(z)(g)\,.$$

By Theorem 1.10, there is a continuous function $\widetilde{l_x}:\mu S\to\mu S$ such that $\widetilde{l_x}\circ e=l_x$. For each $p\in\mu S$, define $e(x)\cdot p=\widetilde{l_x}(p)$. Then we have that $\lambda_{e(x)}=\widetilde{l_x}$ and $e(x\cdot y)=e(x)\cdot e(y)$ for all $y\in S$.

At this stage $\cdot$ is defined (in the only way possible) on $e[S]\times\mu S$. Now let $p\in\mu S$ and define $r_p:S\to\mu S$ by $r_p(x)=\widetilde{l_x}(p)$. Then for $x\in S$, $e(x)\cdot p=\widetilde{l_x}(p)$ so ρ_p must be a continuous function such that $\rho_p\circ e=r_p$, and again we see that there can be at most one such function. We claim that r_p is order preserving. So let $x,y\in S$ with $x\leq y$ and let $g\in F$ be given. Suppose that $r_p(y)(g)<r_p(x)(g)$. Pick a such that $r_p(y)(g)<a<r_p(x)(g)$. That is, $\widetilde{l_y}(p)(g)<a<\widetilde{l_x}(p)(g)$. Then ${\pi_g}^{-1}\big[[0,a)\big]$ is a neighborhood of $\widetilde{l_y}(p)$ and ${\pi_g}^{-1}\big[(a,1]\big]$ is a neighborhood of $\widetilde{l_x}(p)$ so pick a neighborhood W of p such that $\widetilde{l_y}[W]\subseteq{\pi_g}^{-1}\big[[0,a)\big]$ and $\widetilde{l_x}[W]\subseteq{\pi_g}^{-1}\big[(a,1]\big]$. Pick $z\in S$ such that $e(z)\in W$. Then $x\cdot z\leq y\cdot z$ so $g(x\cdot z)\leq g(y\cdot z)$ and thus,

$$\begin{aligned} a\ &<\widetilde{l_x}\big(e(z)\big)(g)=l_x(z)(g)=e(x\cdot z)(g)=g(x\cdot z)\\ &\leq g(y\cdot z)=e(y\cdot z)(g)=l_y(z)(g)=\widetilde{l_y}\big(e(z)\big)(g)<a\,,\end{aligned}$$

a contradiction. Thus by Theorem 1.10, there is a continuous function $\widetilde{r_p}:\mu S\to\mu S$ such that $\widetilde{r_p}\circ e=r_p$. For $x\in S$ we have that

$\widetilde{r_p}\big(e(x)\big) = e(x)\cdot p$. For $q \in \mu S \setminus e[S]$, define $q \cdot p = \widetilde{r_p}(q)$. Now $\cdot$ has been defined on all of $\mu S \times \mu S$ and $\rho_p = \widetilde{r_p}$ for each $p \in \mu S$.

To complete the proof we show that $\cdot$ is associative on μS. To this end, let $p, q, r \in \mu S$. To see that $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ it suffices to show that $\rho_r \circ \rho_q$ and $\rho_{q\cdot r}$ agree on $e[S]$. So let $x \in S$. Then $(\rho_r \circ \rho_q)\big(e(x)\big) = (\rho_r \circ \lambda_{e(x)})(q)$ and $\rho_{q\cdot r}\big(e(x)\big) = (\lambda_{e(x)} \circ \rho_r)(q)$ so it suffices to show that $\rho_r \circ \lambda_{e(x)}$ and $\lambda_{e(x)} \circ \rho_r$ agree on $e[S]$. So let $y \in S$. Then $(\rho_r \circ \lambda_{e(x)})\big(e(y)\big) = (\lambda_{e(x)\cdot e(y)})(r) = (\lambda_{e(x\cdot y)})(r)$ and $(\lambda_{e(x)} \circ \rho_r)\big(e(y)\big) = (\lambda_{e(x)} \circ \lambda_{e(y)})(r)$ so it suffices to show that $\lambda_{e(x\cdot y)}$ and $\lambda_{e(x)} \circ \lambda_{e(y)}$ agree on $e[S]$. So let $z \in S$. Then

$$\begin{aligned}(\lambda_{e(x\cdot y)})\big(e(z)\big) &= e(x\cdot y)\cdot e(z) = e\big((x\cdot y)\cdot z\big) = e\big(x\cdot(y\cdot z)\big)\\ &= e(x)\cdot\big(e(y)\cdot e(z)\big) = (\lambda_{e(x)} \circ \lambda_{e(y)})\big(e(z)\big)\,.\end{aligned}$$

□

We establish now that for any $p \in \mu S$, both ρ_p and λ_p are order preserving (in spite of the fact that only ρ_p need be continuous).

Theorem 2.2. *Let $(S,\cdot)$ be a semigroup and let $\leq$ be a partial order on S. Assume that for all $x, y, z \in S$, if $x \leq y$, then $x\cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$. Let $\cdot$ be the operation on μS given in Theorem 2.1. Then for each $p \in \mu S$, both ρ_p and λ_p are order preserving.*

Proof. Let $p, q, s \in \mu S$ be given such that $q \leq s$.

Suppose first that $\neg\big(\rho_p(q) \leq \rho_p(s)\big)$. Pick $f \in F$ such that $\rho_p(q)(f) > \rho_p(s)(f)$. Pick a such that $\rho_p(s)(f) < a < \rho_p(q)(f)$. Then ${\pi_f}^{-1}\big[[0,a)\big]$ is a neighborhood of $\rho_p(s)$ and ${\pi_f}^{-1}\big[(a,1]\big]$ is a neighborhood of $\rho_p(q)$. Pick a neighborhood U of q and a neighborhood V of s such that

$$\rho_p[U] \subseteq {\pi_f}^{-1}\big[(a,1]\big] \text{ and } \rho_p[V] \subseteq {\pi_f}^{-1}\big[[0,a)\big]\,.$$

Pick by Lemma 1.8 $x \in e^{-1}[U]$ and $y \in e^{-1}[V]$ such that $x \leq y$. Then $e(x)\cdot p \in {\pi_f}^{-1}\big[(a,1]\big]$ and $e(y)\cdot p \in {\pi_f}^{-1}\big[[0,a)\big]$ so ${\pi_f}^{-1}\big[(a,1]\big]$ is a neighborhood of $\lambda_{e(x)}(p)$ and ${\pi_f}^{-1}\big[[0,a)\big]$ is a neighborhood of $\lambda_{e(y)}(p)$ so pick a neighborhood W of p such that $\lambda_{e(x)}[W] \subseteq {\pi_f}^{-1}\big[(a,1]\big]$ and $\lambda_{e(y)}[W] \subseteq {\pi_f}^{-1}\big[[0,a)\big]$. Pick $z \in e^{-1}[W]$. Then $f(x\cdot z) = e(x\cdot z)(f) = \big(e(x)\cdot e(z)\big)(f) > a$ and $f(y \cdot z) = e(y\cdot z)(f) = \big(e(y)\cdot e(z)\big)(f) < a$. But $x \leq y$ so $x \cdot z \leq y\cdot z$ and thus $f(x\cdot z) \leq f(y \cdot z)$, a contradiction.

Now suppose that $\neg\big(\lambda_p(q) \leq \lambda_p(s)\big)$. Pick $f \in F$ such that $\lambda_p(q)(f) > \lambda_p(s)(f)$. Pick a such that $\lambda_p(s)(f) < a < \lambda_p(q)(f)$. Then ${\pi_f}^{-1}\big[[0,a)\big]$ is a neighborhood of $\rho_s(p)$ and ${\pi_f}^{-1}\big[(a,1]\big]$ is a neighborhood of $\rho_q(p)$ so pick a neighborhood U of p such that $\rho_s[U] \subseteq {\pi_f}^{-1}\big[[0,a)\big]$ and $\rho_q[U] \subseteq {\pi_f}^{-1}\big[(a,1]\big]$. Pick $x \in e^{-1}[U]$. Then ${\pi_f}^{-1}\big[[0,a)\big]$ is a neighborhood of $\lambda_{e(x)}(s)$ and ${\pi_f}^{-1}\big[(a,1]\big]$ is a neighborhood of $\lambda_{e(x)}(q)$ so pick a neighborhood V of q and a neighborhood W of s such that $\lambda_{e(x)}[V] \subseteq {\pi_f}^{-1}\big[(a,1]\big]$ and $\lambda_{e(x)}[W] \subseteq {\pi_f}^{-1}\big[[0,a)\big]$. Pick by Lemma 1.8 $y \in e^{-1}[V]$ and $z \in e^{-1}[W]$ such that $y \leq z$. Then $x \cdot y \leq x \cdot z$ so $f(x \cdot y) \leq f(x \cdot z)$. On the other hand $f(x \cdot z) = e(x \cdot z)(f) < a < e(x \cdot y)(f) = f(x \cdot y)$, a contradiction. □

In fact, Theorem 2.2 is a corollary of the following more general result, whose statement and proof presume a knowledge of bitopological spaces.

Theorem 2.3. *Let (M, τ, τ^*) be a pH bitopological space, together with a semigroup operation, $\cdot$. Suppose that for each $p \in M$, $\rho_p : M \to M$ is pairwise continuous, and D is an ^{S}-dense subspace of M such that for each $x \in D$, λ_x is $\leq_\tau$-preserving. Then for each $p \in M$, ρ_p and λ_p are both $\leq_\tau$-preserving.*

Proof. Note first that each ρ_p preserves the specialization, $\leq_\tau$, as does any continuous function, by [6, Theorem 3(b)]. If λ_p were not $\leq_\tau$-preserving, then there would be $q, s \in M$ such that $q \leq_\tau s$ amd $\lambda_p(q) \not\leq_\tau \lambda_p(s)$. Since our bitopological space is pH, we can find disjoint $T \in \tau$, $U \in \tau^*$ such that $pq \in T$ and $ps \in U$. By the continuity of $\rho_q : (M, \tau) \to (M, \tau)$ and that of $\rho_s : (M, \tau^*) \to (M, \tau^*)$, we can then find $V \in \tau$ such that $p \in V$ and $Vq \subseteq T$ and $W \in \tau^*$ such that $p \in W$ and $Ws \subseteq U$. Since $p \in V \cap W$, D meets $V \cap W$; let $x \in D \cap V \cap W$. But then $xq \leq_\tau xs$ and $xs \in U$. Also $xs \in T$, and since $T \in \tau$, $xs \in T$ by [6, Theorem 3(a)], contradicting the disjointness of T and U. □

3. Order Compactifications of Certain Semigroups

Both $+$ and $\cdot$ respect the usual order on $\mathbb{N}$. Thus by Theorem 2.1 $(\mu\mathbb{N}, +)$ and $(\mu\mathbb{N}, \cdot)$ are compact right topological semigroups with $e[\mathbb{N}]$ contained in their topological centers. Unfortunately, this fact becomes significantly less interesting in view of the following theorem.

Theorem 3.1. *With the usual order on* $\mathbb{N}$, $\mu\mathbb{N}$ *is the one point compactification of* $\mathbb{N}$.

Proof. By Theorem 1.9, the sequence $\langle e(n)\rangle_{n=1}^{\infty}$ converges to a point p of $\mu\mathbb{N}$. By Theorem 1.2, $p \notin e[\mathbb{N}]$. Now let $q \in \mu\mathbb{N} \setminus \{p\}$ and pick disjoint neighborhoods U of p and V of q. Pick $m \in \mathbb{N}$ such that $\{e(n) : n \geq m\} \subseteq U$. Then $q \in c\ell\{e(n) : n < m\} = \{e(n) : n < m\}$. Therefore $\mu\mathbb{N} = e[\mathbb{N}] \cup \{p\}$. □

On the other hand, if $\mathbb{R}$ has its usual order, then $\mu\mathbb{R}$ has a more interesting structure. The referee has pointed out that the order compactifications underlying the examples of Theorems 3.2 and 3.4 have been considered, without the semigroup structure, before. See for example [1] and [5].

Notice the difference between the behavior of the points a^- and a^+ when compared to $-\infty$ and ∞; for example $a^- + b^+ = (a+b)^-$, while $\infty + -\infty = -\infty$.

Theorem 3.2. *Let* $\mathbb{R}$ *have its usual order. Define points* ∞, $-\infty$ *in* ${}^{F}[0,1]$, *and for each* $a \in \mathbb{R}$ *define points* a^+ *and* a^- *in* ${}^{F}[0,1]$ *by specifying that for* $f \in F$,

$$\begin{aligned} \infty(f) &= \text{lub}\{f(x) : x \in \mathbb{R}\} \\ (-\infty)(f) &= \text{glb}\{f(x) : x \in \mathbb{R}\} \\ a^+(f) &= \text{glb}\{f(x) : x \in \mathbb{R} \text{ and } x > a\} \\ a^-(f) &= \text{lub}\{f(x) : x \in \mathbb{R} \text{ and } x < a\}\,. \end{aligned}$$

Then $\mu\mathbb{R} = e[\mathbb{R}] \cup \{\infty, -\infty\} \cup \{a^+ : a \in \mathbb{R}\} \cup \{a^- : a \in \mathbb{R}\}$. *Also* $\mu\mathbb{R}$ *is linearly ordered and has the order topology. The point* ∞ *is the largest member of* $\mu\mathbb{R}$ *and the point* $-\infty$ *is the smallest member of* $\mu\mathbb{R}$. *Given* a, b *in* $\mathbb{R}$, $a^- < e(a) < a^+$, $a < b \Rightarrow a^+ < b^-$, *and*
(1) $a^- + b^- = a^- + e(b) = a^- + b^+ = e(a) + b^- = (a+b)^-$,
(2) $a^+ + b^- = a^+ + e(b) = a^+ + b^+ = e(a) + b^+ = (a+b)^+$,
(3) $\infty + \infty = \infty + a^- = \infty + e(a) = \infty + a^+ = a^- + \infty = e(a) + \infty = a^+ + \infty = -\infty + \infty = \infty$, *and*
(4) $-\infty + -\infty = -\infty + a^- = -\infty + e(a) = -\infty + a^+ = a^- + -\infty = e(a) + -\infty = a^+ + -\infty = \infty + -\infty = -\infty$.

Proof. We show first that $\{\infty, -\infty\} \cup \{a^+ : a \in \mathbb{R}\} \cup \{a^- : a \in \mathbb{R}\} \subseteq \mu\mathbb{R}$. To see that $\infty \in \mu\mathbb{R}$, let U be a neighborhood of ∞ and pick finite $H \subseteq F$ and $\epsilon > 0$ such that $\bigcap_{f\in H} {\pi_f}^{-1}[(\infty(f)-\epsilon, \infty(f)+\epsilon)] \subseteq U$. For each $f \in H$ pick $x_f \in \mathbb{R}$ such that $f(x_f) > \infty(f)-\epsilon$ and let $y = \max\{x_f : f \in H\}$. Then $e(y) \in \bigcap_{f\in H} {\pi_f}^{-1}[(\infty(f)-\epsilon, \infty(f)+\epsilon)]$. Similarly $-\infty \in \mu\mathbb{R}$.

Now let $a \in \mathbb{R}$. We show that $a^+ \in \mu\mathbb{R}$, the proof for a^- being similar. Let U be a neighborhood of a^+ and pick finite $H \subseteq F$ and $\epsilon > 0$ such that $\bigcap_{f\in H} {\pi_f}^{-1}[(a^+(f)-\epsilon, a^+(f)+\epsilon)] \subseteq U$. For each $f \in H$ pick $x_f \in \mathbb{R}$ such that $x_f > a$ and $f(x_f) < a^+(f)+\epsilon$. Let $y = \min\{x_f : f \in H\}$. Then $e(y) \in \bigcap_{f\in H} {\pi_f}^{-1}[(a^+(f)-\epsilon, a^+(f)+\epsilon)]$.

To see that $\mu\mathbb{R} \subseteq e[\mathbb{R}] \cup \{\infty, -\infty\} \cup \{a^+ : a \in \mathbb{R}\} \cup \{a^- : a \in \mathbb{R}\}$, let $p \in \mu\mathbb{R}$ and pick a net $\langle x_\alpha \rangle_{\alpha\in D}$ in $\mathbb{R}$ such that $\langle e(x_\alpha)\rangle_{\alpha\in D}$ converges to p in ${}^F[0,1]$. Let $\mathbb{R} \cup \{-\infty, \infty\}$ be the two point compactification of $\mathbb{R}$. By passing to a subnet, we may presume that the net $\langle x_\alpha \rangle_{\alpha\in D}$ converges in $\mathbb{R} \cup \{-\infty, \infty\}$.

Assume first that $\langle x_\alpha \rangle_{\alpha\in D}$ converges to ∞. We claim that $\langle e(x_\alpha)\rangle_{\alpha\in D}$ converges to ∞, so that $p = \infty$. So let U be a neighborhood of p and pick finite $H \subseteq F$ and $\epsilon > 0$ such that

$$\bigcap_{f\in H} {\pi_f}^{-1}[(\infty(f)-\epsilon, \infty(f)+\epsilon)] \subseteq U\,.$$

For each $f \in H$ pick $y_f \in \mathbb{R}$ such that $f(y_f) > \infty(f)-\epsilon$ and pick $\alpha_f \in D$ such that for all $\gamma \in D$, if $\gamma \geq \alpha_f$, then $x_\gamma \geq y_f$. Pick $\delta \in D$ such that for each $f \in H$, $\delta \geq \alpha_f$. If $\gamma \in D$ and $\gamma \geq \delta$, then for each $f \in H$, $x_\gamma \geq y_f$ and so $f(x_\gamma) \geq f(y_f) > \infty - \epsilon$ and thus $e(x_\gamma) \in \bigcap_{f\in H} {\pi_f}^{-1}[(\infty(f)-\epsilon, \infty(f)+\epsilon)]$.

Similarly, if $\langle x_\alpha \rangle_{\alpha\in D}$ converges to $-\infty$, then $p = -\infty$.

Now assume that $\langle x_\alpha \rangle_{\alpha\in D}$ converges to $a \in \mathbb{R}$. Let $E_1 = \{\alpha \in D : x_\alpha < a\}$, let $E_2 = \{\alpha \in D : x_\alpha = a\}$, and let $E_3 = \{\alpha \in D : x_\alpha > a\}$. If E_2 is cofinal in D, then a subnet of $\langle x_\alpha \rangle_{\alpha\in D}$ is constantly equal to a and therefore $p = e(a)$. We shall show that if E_1 is cofinal in D, then $\langle e(x_\alpha)\rangle_{\alpha\in D}$ clusters at a^- and thus $p = a^-$. So assume that E_1 is cofinal in D and let U be a neighborhood of a^- and let $\eta \in D$. Pick finite $H \subseteq F$ and $\epsilon > 0$ such that $\bigcap_{f\in H} {\pi_f}^{-1}[(a^-(f)-\epsilon, a^-(f)+\epsilon)] \subseteq U$. For each $f \in H$ pick $y_f \in \mathbb{R}$ such that $y_f < a$ and $f(y_f) > a^-(f)-\epsilon$ and pick $\alpha_f \in D$ such that whenever $\gamma \in D$ and $\gamma \geq \alpha_f$, one has $x_\gamma \in (y_f, a+1)$. Pick $\gamma \in E_1$ such that $\gamma \geq \eta$ and $\gamma \geq \alpha_f$ for each $f \in H$. Then for each $f \in H$ we have $y_f < x_\gamma < a$ so $a^-(f)-\epsilon < f(y_f) \leq f(x_\gamma) \leq a^-(f)$ so that $x_\gamma \in U$.

Similarly if E_3 is cofinal in D, then $p = a^+$.

By Lemma 1.7 and [6, Theorem 9], $\mu\mathbb{R}$ is linearly ordered and has the order topology.

Let $a, b \in \mathbb{R}$ with $a < b$ and pick $c \in \mathbb{R}$ with $a < c < b$. Define $f \in {}^{\mathbb{R}}[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x < a-1 \\ \frac{1}{7} & \text{if } a-1 \leq x < a \\ \frac{2}{7} & \text{if } x = a \\ \frac{3}{7} & \text{if } a < x \leq c \\ \frac{4}{7} & \text{if } c < x < b \\ \frac{5}{7} & \text{if } x = b \\ \frac{6}{7} & \text{if } b < x \leq b+1 \\ 1 & \text{if } x > b+1\,. \end{cases}$$

Then $f \in F$, $-\infty(f) = 0$, $a^-(f) = \frac{1}{7}$, $e(a)(f) = \frac{2}{7}$, $a^+(f) = \frac{3}{7}$, $b^-(f) = \frac{4}{7}$, $e(b)(f) = \frac{5}{7}$, $b^+(f) = \frac{6}{7}$, and $\infty(f) = 1$, so by Lemma 1.7 $-\infty < a^- < e(a) < a^+ < b^- < e(b) < b^+ < \infty$.

Notice that as a consequence of the order just displayed and the fact that $\mu\mathbb{R}$ has the order topology, one has for each $a \in \mathbb{R}$ that $\big\{\big(e(a-\epsilon), e(a)\big) : \epsilon > 0\big\}$ is a basic neighborhood system of a^-. Note also that $[a^-, a^+] = \{a^-, e(a), a^+\}$.

To establish (1), let $a, b \in \mathbb{R}$. Suppose first that $e(a) + b^- \neq (a+b)^-$ and pick disjoint neighborhoods U of $e(a) + b^-$ and V of $(a+b)^-$. Pick $\epsilon > 0$ such that $\big(e(a+b-\epsilon), e(a+b)\big) \subseteq V$ and pick a neighborhood W of b^- such that $e(a) + W \subseteq U$. Also $\big(e(b-\epsilon), e(b)\big)$ is a neighborhood of b^- so pick $x \in \mathbb{R}$ such that $e(x) \in \big(e(b-\epsilon), e(b)\big) \cap W$. Then $e(a+x) = e(a) + e(x) \in U$. But $b - \epsilon < x < b$ so $a + b - \epsilon < a + x < a + b$ and so $e(a+x) \in V$, a contradiction.

Now let $b^* \in \{b^-, e(b), b^+\}$ and suppose that $a^- + b^* \neq (a+b)^-$. Pick disjoint open neighborhoods U of $a^- + b^*$ and V of $(a+b)^-$. Pick $\epsilon > 0$ such that $\big(e(a+b-\epsilon), e(a+b)\big) \subseteq V$ and pick a neighborhood W of a^- such that $W + b^* \subseteq U$. Also $\big(e(a-\epsilon), e(a)\big)$ is a neighborhood of a^- so pick $x \in \mathbb{R}$ such that $e(x) \in \big(e(a-\epsilon), e(a)\big) \cap W$. Then $a - \epsilon < x < a$. Let $\delta = \min\{a - x, x - a + \epsilon\}$. Pick a neighborhood T of b^* such that $e(x) + T \subseteq U$. Also $\big(e(b-\delta), e(b+\delta)\big)$ is a neighborhood of b^*. Pick $y \in \mathbb{R}$ such that $e(y) \in \big(e(b-\delta), e(b+\delta)\big) \cap T$. Then $e(x+y) = e(x) + e(y) \in U$. But

$b-x+a-\epsilon \leq b-\delta < y < b+\delta \leq b+a-x$ so $b+a-\epsilon < x+y < b+a$ and so $e(x+y) \in V$, a contradiction.

The proof of conclusion (2) is essentially identical.

We conclude by verifying (3), the proof of (4) being similar. We first show that for any $p \in \mu\mathbb{R}$, $p + \infty = \infty$. Suppose instead we have $p \in \mu\mathbb{R}$ such that $p + \infty \neq \infty$ and pick disjoint open neighborhoods U of $p + \infty$ and V of ∞. Pick $x \in \mathbb{R}$ such that $(e(x), \infty] \subseteq V$. Pick a neighborhood W of p such that $W + \infty \subseteq U$ and pick $y \in \mathbb{R}$ such that $e(y) \in W$. Pick a neighborhood T of ∞ such that $e(y) + T \subseteq U$. Also $(e(x-y), \infty]$ is a neighborhood of ∞ so pick $z \in \mathbb{R}$ such that $e(z) \in (e(x-y), \infty] \cap T$. Then $e(y+z) = e(y) + e(z) \in U$. Also, $x < y + z$ so $e(y+z) \in V$, a contradiction.

Finally, let $a \in \mathbb{R}$ and let $a^* \in \{a^-, e(a), a^+\}$. Suppose that $\infty + a^* \neq \infty$ and pick disjoint open neighborhoods U of $\infty + a^*$ and V of ∞. Pick $x \in \mathbb{R}$ such that $(e(x), \infty] \subseteq V$ and pick a neighborhood W of ∞ such that $W + a^* \subseteq U$. Also $(e(x+1-a), \infty]$ is a neighborhood of ∞ so pick $y \in \mathbb{R}$ such that $e(y) \in (e(x+1-a), \infty] \cap W$. Pick a neighborhood T of a^* such that $e(y) + T \subseteq U$. Also $\big(e(a-1), e(a+1)\big)$ is a neighborhood of a^* so pick $z \in \mathbb{R}$ such that $e(z) \in \big(e(a-1), e(a+1)\big) \cap T$. Then $e(y+z) = e(y) + e(z) \in U$. But $a-1 < z$ and $x+1-a < y$ so $x < y+z$ and thus $e(y+z) \in V$, a contradiction. □

Corollary 3.3. *The topological center of* $(\mu\mathbb{R}, +)$ *is* $e[\mathbb{R}]$.

Proof. We have by Theorem 2.1 that $e[\mathbb{R}]$ is contained in the topological center of $\mu\mathbb{R}$. We have that $\lambda_\infty(-\infty) = -\infty$ while for any $p \in \mu\mathbb{R} \setminus \{-\infty\}$, $\lambda_\infty(p) = \infty$ and thus λ_∞ is not continuous at $-\infty$. Similarly, $\lambda_{-\infty}$ is not continuous at ∞.

Now let $a, b \in \mathbb{R}$. We claim that λ_{a^+} is not continuous at b^- and λ_{a^-} is not continuous at b^+. We write out the verification of the first of these assertions. We have that $\lambda_{a^+}(b^-) = (a+b)^+$ and $\big(e(a+b), e(a+b+1)\big)$ is a neighborhood of $(a+b)^+$. Suppose that we have a neighborhood U of b^- such that $\lambda_{a^+}[U] \subseteq \big(e(a+b), e(a+b+1)\big)$. Pick $c < b$ such that $\big(e(c), e(b)\big) \subseteq U$ and pick $x \in (c, b)$. Then $\lambda_{a^+}\big(e(x)\big) = (a+x)^+ \notin \big(e(a+b), e(a+b+1)\big)$. □

The proof as well as the statement of the following theorem are very similar to that of Theorem 3.2, so we omit the proof. Notice a contrast between the topology of $\mu\mathbb{R}$ and the topology of $\mu\mathbb{Q}$. The points of $\mu\mathbb{R} \setminus \{-\infty, \infty\}$ have a neighborhood base consisting of intervals of the form $\big(e(a), e(b)\big)$. But in $\mu\mathbb{Q}$, $\sqrt{2}^+$ is immediately preceeded by $\sqrt{2}^-$, and so $\sqrt{2}^+$ does not have a neighborhood base of the same form.

Theorem 3.4. *Let $\mathbb{Q}$ have its usual order. Define points ∞, $-\infty$ in ${}^F[0,1]$, and for each $a \in \mathbb{R}$ define points a^+ and a^- in ${}^F[0,1]$ by specifying that for $f \in F$,*

$$\begin{aligned} \infty(f) &= \text{lub}\{f(x) : x \in \mathbb{Q}\} \\ (-\infty)(f) &= \text{glb}\{f(x) : x \in \mathbb{Q}\} \\ a^+(f) &= \text{glb}\{f(x) : x \in \mathbb{Q} \text{ and } x > a\} \\ a^-(f) &= \text{lub}\{f(x) : x \in \mathbb{Q} \text{ and } x < a\}\,. \end{aligned}$$

Then $\mu\mathbb{Q} = e[\mathbb{Q}] \cup \{\infty, -\infty\} \cup \{a^+ : a \in \mathbb{R}\} \cup \{a^- : a \in \mathbb{R}\}$. Also $\mu\mathbb{Q}$ is linearly ordered and has the order topology. The point ∞ is the largest member of $\mu\mathbb{Q}$ and the point $-\infty$ is the smallest member of $\mu\mathbb{Q}$. Given a, b in $\mathbb{R}$, $a^- < e(a) < a^+$, $a < b \Rightarrow a^+ < b^-$, and
(1) $a^- + b^- = a^- + e(b) = a^- + b^+ = e(a) + b^- = (a+b)^-$,
(2) $a^+ + b^- = a^+ + e(b) = a^+ + b^+ = e(a) + b^+ = (a+b)^+$,
(3) $\infty + \infty = \infty + a^- = \infty + e(a) = \infty + a^+ = a^- + \infty = e(a) + \infty = a^+ + \infty = -\infty + \infty = \infty$, and
(4) $-\infty + -\infty = -\infty + a^- = -\infty + e(a) = -\infty + a^+ = a^- + -\infty = e(a) + -\infty = a^+ + -\infty = \infty + -\infty = -\infty$.
(In the above conclusions, it is intended that references to $e(x)$ be omitted in the event $x \notin \mathbb{Q}$.)

In $(\mathbb{R}, \cdot)$ with the usual order, the operation does not respect the order. However, letting $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, one has that both $+$ and $\cdot$ respect the order of $\mathbb{R}^+$. Further, since $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$ are isomorphic via an order preserving function, one has that $(\mu\mathbb{R}^+, \cdot)$ and $(\mu\mathbb{R}, +)$ are also isomorphic. In particular, given $a, b \in \mathbb{R}^+$ one has

(1) $a^- \cdot b^- = a^- \cdot e(b) = a^- \cdot b^+ = e(a) \cdot b^- = (a \cdot b)^-$,
(2) $a^+ \cdot b^- = a^+ \cdot e(b) = a^+ \cdot b^+ = e(a) \cdot b^+ = (a \cdot b)^+$,
(3) $\infty \cdot \infty = \infty \cdot a^- = \infty \cdot e(a) = \infty \cdot a^+ = a^- \cdot \infty = e(a) \cdot \infty = a^+ \cdot \infty = 0^+ \cdot \infty = \infty$, and
(4) $0^+ \cdot 0^+ = 0^+ \cdot a^- = 0^+ \cdot e(a) = 0^+ \cdot a^+ = a^- \cdot 0^+ = e(a) \cdot 0^+ = a^+ \cdot 0^+ = \infty \cdot 0^+ = 0^+$.

It is interesting to note that neither distributive law is satisfied in the system $(\mu\mathbb{R}^+,+,\cdot)$. Indeed, let $a,b\in\mathbb{R}^+$. Then $a^-\cdot(0^++b^+)=a^-\cdot b^+=(a\cdot b)^-$ while $a^-\cdot 0^++a^-\cdot b^+=0^++(a\cdot b)^-=(a\cdot b)^+$. Also if $a,b,c\in\mathbb{R}^+$, then $(e(a)+b^-)\cdot c^+=(a+b)^-\cdot c^+=\big((a+b)\cdot c\big)^-$ while $e(a)\cdot c^++b^-\cdot c^+=(a\cdot c)^+(b\cdot c)^-=(a\cdot c+b\cdot c)^+$. (On the other hand, for $p,q,r\in\mu\mathbb{R}^+\setminus\{0^+\}$ one does have that $p\cdot(q+r)=p\cdot q+p\cdot r$.)

With the usual ordering we have seen that $\mu\mathbb{R}$ and $\mu\mathbb{Q}$ are relatively small and that $\mu\mathbb{N}$ is positively tiny. Further, since we have explicitly described the operations we can tell exactly the smallest ideal, K, of each of $(\mu\mathbb{R},+)$, $(\mu\mathbb{Q},+)$, $(\mu\mathbb{N},+)$, $(\mu\mathbb{R}^+,\cdot)$, and $(\mu\mathbb{N},\cdot)$. For example, $K(\mu\mathbb{R},+)=\{-\infty,\infty\}$.

We now turn our attention to another ordering of $\mathbb{N}$ designed to reflect its multiplicative structure.

Definition 3.5. Let $x,y\in\mathbb{N}$. Then $x\preceq y$ if and only if x divides y.

One has that $\cdot$ respects the order $\preceq$. We shall see that $(\mu\mathbb{N},\preceq,\cdot)$ is as large as possible and has a significant amount of algebraic structure. In fact it contains a copy of the free semigroup on $2^{\mathfrak{c}}$ generators. On the other hand, the smallest ideal of $(\mu\mathbb{N},\preceq,\cdot)$ is trivial. (We remind the reader that members of the smallest ideals of $(\beta\mathbb{N},+)$ and $(\beta\mathbb{N},\cdot)$ are behind many of the most powerful combinatorial applications of those structures.)

Recall that if $p\in\beta\mathbb{N}$ and $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in a Hausdorff topological space X, then $p\text{-}\lim_{n\in\mathbb{N}} x_n=y$ if and only if for every neighborhood U of y, $\{n\in\mathbb{N}:x_n\in U\}\in p$, where we are taking p to be an ultrafilter on $\mathbb{N}$. (See [4, Section 3.5].)

We shall use our knowledge of the structure of $(\beta\mathbb{N},\cdot)$ to obtain information about $(\mu\mathbb{N},\preceq,\cdot)$.

Definition 3.6. Let $\mu\mathbb{N}$ be as determined by the order $\preceq$. Define $\varphi:\beta\mathbb{N}\to\mu\mathbb{N}$ by $\varphi(p)=p\text{-}\lim_{n\in\mathbb{N}} e(n)$.

Lemma 3.7. *The function φ is a continuous surjective homomorphism from $(\beta\mathbb{N},\cdot)$ to $(\mu\mathbb{N},\cdot)$.*

Proof. To see that φ is continuous, let $p\in\beta\mathbb{N}$ and let U be a neighborhood of $\varphi(p)$. Pick a neighborhood V of $\varphi(p)$ such that $c\ell V\subseteq U$. Then $e^{-1}[V]=\{n\in\mathbb{N}:e(n)\in V\}\in p$ so $c\ell(e^{-1}[V])$ is a neighborhood of p. And, if $q\in c\ell(e^{-1}[V])$, then $\varphi(q)\in U$.

Since $\varphi[\mathbb{N}] = e[\mathbb{N}]$ and $e[\mathbb{N}]$ is dense in $\mu\mathbb{N}$, we have $\varphi[\beta\mathbb{N}] = \mu\mathbb{N}$. (We are identifying the points of $\mathbb{N}$ with the principle ultrafilters, so if $n \in \mathbb{N}$, then $\varphi(n) = n\text{-}\lim\limits_{m\in\mathbb{N}} e(m) = e(n)$.)

Now φ is a continuous function extending e and $e[\mathbb{N}]$ is contained in the topological center of $\mu\mathbb{N}$ so by [4, Corollary 4.22], φ is a homomorphism. □

Theorem 3.8. *Let $\mathbb{P}$ be the set of primes. Then φ is injective on $c\ell(\mathbb{P})$. In particular, $|\mu\mathbb{N}| = 2^{\mathfrak{c}}$.*

Proof. It is well known that there are $2^{\mathfrak{c}}$ ultrafilters on a countably infinite set. (See [3] or [4].) And as a quotient of $\beta\mathbb{N}$, $|\mu\mathbb{N}| \leq 2^{\mathfrak{c}}$, so it suffices to establish the first assertion.

So let p and q be distinct members of $c\ell(\mathbb{P})$ and pick $A \in p \setminus q$. Define $f : \mathbb{N} \to [0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if for all } r \in A \cap \mathbb{P}\,,\ r \text{ does not divide } x \\ 1 & \text{if there exists } r \in A \cap \mathbb{P} \text{ such that } r \text{ divides } x\,. \end{cases}$$

Then $f \in F$. We claim first that $\varphi(p)(f) = 1$. So suppose instead that $\varphi(p)(f) < 1$. Then ${\pi_f}^{-1}\big[[0, 1)\big]$ is a neighborhood of $\varphi(p)$ so $\big\{n \in \mathbb{N} : e(n) \in {\pi_f}^{-1}\big[[0, 1)\big]\big\} \in p$. Pick $n \in A \cap \mathbb{P}$ such that $e(n) \in {\pi_f}^{-1}\big[[0, 1)\big]$. Then $f(n) < 1$, a contradiction. Similarly, using the fact that $\mathbb{P} \setminus A \in q$, one sees that $\varphi(q)(f) = 0$. □

We now set out to show in Theorem 3.11 that the elements of $\varphi[\,c\ell(\mathbb{P})\,] \setminus e[\mathbb{N}]$ generate a free semigroup.

Lemma 3.9. *Let $n \in \mathbb{N}$, let $p_1, p_2, \ldots, p_n \in \mu\mathbb{N} \setminus e[\mathbb{N}]$, and for each $i \in \{1, 2, \ldots, n\}$ let U_i be a neighborhood of p_i and let $C_i \subseteq \mathbb{N}$ such that $p_i \in c\ell e[C_i]$. Let*

$$\begin{aligned} B = \{\textstyle\prod_{i=1}^{n} x_i : \quad & x_1 < x_2 < \ldots < x_n \textit{ and} \\ & (\forall i \in \{1, 2, \ldots, n\})(e(x_i) \in U_i \textit{ and } x_i \in C_i)\}\,. \end{aligned}$$

Then $p_1 \cdot p_2 \cdots p_n \in c\ell e[B]$.

Proof. We proceed by induction on n. Assume first that $n = 1$. Let W be a neighborhood of p_1. Then $W \cap U_1$ is a neighborhood of p_1 so pick $y \in C_1$ such that $e(y) \in W \cap U_1$. Then $y \in B$ and $e(y) \in W$.

Now assume that $n > 1$ and the lemma is valid for $n - 1$. Let V be an open neighborhood of $p_1 \cdot p_2 \cdots p_n$ and pick a neighborhood W

of $p_1 \cdot p_2 \cdots p_{n-1}$ such that $W \cdot p_n \subseteq U$. By the induction hypothesis pick $x_1 < x_2 < \ldots < x_{n-1}$ such that for all $i \in \{1, 2, \ldots, n-1\}$, $e(x_i) \in U_i$ and $x_i \in C_i$, and $e(\prod_{i=1}^{n-1} x_i) \in W$. Pick a neighborhood R of p_n such that $e(\prod_{i=1}^{n-1} x_i) \cdot R \subseteq V$. Pick $x_n \in C_n$ such that

$$e(x_n) \in R \cap U_n \setminus \{e(y) : y \leq x_{n-1}\}\,.$$

Then $x_n > x_{n-1}$ and $e(\prod_{i=1}^{n} x_i) \in V$. □

Lemma 3.10. *For $x \in \mathbb{N}$, let $\ell(x)$ be the length of the prime factorization of x. Let $n \in \mathbb{N}$ and let $U_1, U_2, \ldots, U_n$ be open subsets of $\mu\mathbb{N}$ such that $U_i \cap e[\mathbb{P}] \neq \emptyset$ for each $i \in \{1, 2, \ldots, n\}$. Let*

$$B = \{\textstyle\prod_{i=1}^{n} x_i : \begin{array}{l} x_1 < x_2 < \ldots < x_n \text{ and} \\ (\forall i \in \{1, 2, \ldots, n\})(e(x_i) \in U_i \text{ and } x_i \in \mathbb{P})\}\,. \end{array}$$

Define $g_B : \mathbb{N} \to [0, 1]$ by

$$g_B(x) = \begin{cases} 1 & \text{if } \ell(x) > n \\ \frac{1}{2} & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then $g_B \in F$. Also

1. *if $p \in c\ell e[\{x \in \mathbb{N} : \ell(x) > n\}$, then $p(g_B) = 1$;*
2. *if $p \in c\ell e[B]$, then $p(g_B) = \frac{1}{2}$; and*
3. *if $p \in \mu\mathbb{N} \setminus (c\ell e[\{x \in \mathbb{N} : \ell(x) > n\}] \cup c\ell e[B])$, then $p(g_B) = 0$.*

Proof. It is routine to verify that $g_B \in F$, and conclusions (1) and (2) are immediate. To verify conclusion (3), let $p \in \mu\mathbb{N} \setminus (c\ell e[\{x \in \mathbb{N} : \ell(x) > n\}] \cup c\ell e[B])$ and suppose that $p(g_B) > 0$. Pick a neighborhood V of p such that

$$V \cap (e[\{x \in \mathbb{N} : \ell(x) > n\}] \cup e[B]) = \emptyset\,.$$

Also ${\pi_{g_B}}^{-1}\big[(0, 1]\big]$ is a neighborhood of p so pick $x \in \mathbb{N}$ such that

$$e(x) \in V \cap {\pi_{g_B}}^{-1}\big[(0, 1]\big]\,.$$

But then $g_B(x) \in \{\frac{1}{2}, 1\}$, a contradiction. □

Theorem 3.11. *Let $n, m \in \mathbb{N}$ and let $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m \in (c\ell e[\mathbb{P}]) \setminus e[\mathbb{N}]$ and assume that $p_1 \cdot p_2 \cdots p_n = q_1 \cdot q_2 \cdots q_m$. Then $m = n$ and for each $i \in \{1, 2, \ldots, n\}$, $p_i = q_i$.*

Proof. For each $r \in \{p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m\}$ pick an open neighborhood W_r of r such that $W_r \cap W_s = \emptyset$ whenever $r \neq s$. For $i \in \{1, 2, \ldots, n\}$, let $U_i = W_{p_i}$ and for $i \in \{1, 2, \ldots, m\}$, let $V_i = W_{q_i}$. Let

$$B = \{\textstyle\prod_{i=1}^n x_i : \; x_1 < x_2 < \ldots < x_n \text{ and } (\forall i \in \{1, 2, \ldots, n\})(e(x_i) \in U_i \text{ and } x_i \in \mathbb{P})\}$$

and let

$$C = \{\textstyle\prod_{i=1}^m x_i : \; x_1 < x_2 < \ldots < x_n \text{ and } (\forall i \in \{1, 2, \ldots, m\})(e(x_i) \in V_i \text{ and } x_i \in \mathbb{P})\}\,.$$

Then by Lemma 3.9 $p_1 \cdot p_2 \cdots p_n \in c\ell e[B]$ and $q_1 \cdot q_2 \cdots q_m \in c\ell e[C]$. Consequently by Lemma 3.10 $(p_1 \cdot p_2 \cdots p_n)(g_B) = \frac{1}{2}$ and $(q_1 \cdot q_2 \cdots q_m)(g_C) = \frac{1}{2}$. Thus ${\pi_{g_C}}^{-1}[(0,1)]$ is a neighborhood of $q_1 \cdot q_2 \cdots q_m = p_1 \cdot p_2 \cdots p_n \in c\ell e[B]$ so pick $x \in B$ such that $e(x) \in {\pi_{g_C}}^{-1}[(0,1)]$. Thus $g_C(x) = \frac{1}{2}$ so $x \in B \cap C$ so $x = \prod_{i=1}^n x_i = \prod_{i=1}^m y_i$ where $x_1 < x_2 < \ldots < x_n$, $y_1 < y_2 < \ldots < y_m$, for each $i \in \{1, 2, \ldots, n\}$, $x_i \in \mathbb{P}$ and $e(x_i) \in U_i$, and for each $i \in \{1, 2, \ldots, m\}$, $y_i \in \mathbb{P}$ and $e(y_i) \in V_i$. Then $\ell(x) = n = m$ and by the uniqueness of prime factorization $x_i = y_i$ for each $i \in \{1, 2, \ldots, n\}$. Thus for each $i \in \{1, 2, \ldots, n\}$, $U_i \cap V_i \neq \emptyset$; that is $W_{p_i} \cap W_{q_i} \neq \emptyset$, and consequently $p_i = q_i$. □

We have just established the existence of substantial algebraic structure in $(\mu\mathbb{N}, \preceq, \cdot)$. Unfortunately, we see that the structure of the smallest ideal is trivial.

Theorem 3.12. $\big|\varphi\big[\bigcap_{n=1}^{\infty} c\ell(n\mathbb{N})\big]\big| = 1$. *In particular,* $|K(\mu\mathbb{N})| = 1$.

Proof. For the "in particular" conclusion note that by Lemma 3.7 and [4, Exercise 1.7.3], $\varphi[K(\beta\mathbb{N}, \cdot)] = K(\mu\mathbb{N}, \cdot)$ and, since $c\ell(n\mathbb{N})$ is an ideal of $(\beta\mathbb{N}, \cdot)$, $K(\beta\mathbb{N}, \cdot) \subseteq \bigcap_{n=1}^{\infty} c\ell(n\mathbb{N})$.

Now let $p, q \in \bigcap_{n=1}^{\infty} c\ell(n\mathbb{N})$ and suppose that $\varphi(p) \neq \varphi(q)$. Pick $f \in F$ such that $\varphi(p)(f) \neq \varphi(q)(f)$ and assume without loss of generality that $b = \varphi(p)(f) < \varphi(q)(f) = a$ and let $\epsilon = a - b$. Now ${\pi_f}^{-1}[(a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2})]$ is a neighborhood of $\varphi(q)$ so

$$\{x \in \mathbb{N} : e(x) \in {\pi_f}^{-1}[(a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2})]\} \in q\,.$$

Pick $x \in \mathbb{N}$ such that $f(x) = e(x)(f) \in (a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2})$. Now ${\pi_f}^{-1}[(b - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})]$ is a neighborhood of $\varphi(p)$ so $A = \{y \in \mathbb{N} :$

$e(y) \in {\pi_f}^{-1}[(b - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})]\} \in p$. Also $x\mathbb{N} \in p$ so pick $y \in A \cap x\mathbb{N}$. Then $x \preceq y$ so $a - \frac{\epsilon}{2} < f(x) \leq f(y) < b + \frac{\epsilon}{2}$, a contradiction. □

References

1. J. Blatter, *Order compactifications of totally ordered topological spaces*, J. Approx. Theory **13** (1975), 56-65.
2. E. Čech, *On bicompact spaces*, Ann. of Math. **38** (1937), 823-244.
3. L. Gillman and M. Jerison, *Rings of continuous functions*, van Nostrand, Princeton, 1960.
4. N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.
5. D. Kent and T. Richmond, *Ordered compactification of totally ordered spaces*, Internat. J. Math. Math. Sci. **11** (1988), 683-694.
6. R. Kopperman, *The Stone-Čech compactification of a partially ordered set via bitopology,* manuscript.

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