This paper was published in *Discrete Math.* **240** (2001), 45-70. To the best of my knowledge, this is the final version as it was submitted to the publisher.

# VIP Systems in Partial Semigroups

**Neil Hindman**[1]

and

**Randall McCutcheon**[2]

**Abstract**. A *VIP system* is a polynomial type generalization of the notion of an IP system, i.e., a set of finite sums. We extend the notion of VIP system to commutative *partial semigroups* and obtain an analogue of Furstenberg's central sets theorem for these systems which extends the polynomial Hales-Jewett Theorem of Bergelson and Leibman. Several Ramsey Theoretic consequences, including the central sets theorem itself, are then derived from these results.

## 1. Introduction

Given a set $A$, we write $\mathcal{P}_f(A)$ for the set of finite nonempty subsets of $A$. We use the special notation $\mathcal{F} = \mathcal{P}_f(\mathbb{N})$. Also, if $\langle H_n \rangle_{n=1}^\infty$ is a sequence of sets, then $FU(\langle H_n \rangle_{n=1}^\infty) = \{\bigcup_{n\in\alpha} H_n : \alpha \in \mathcal{F}\}$. An *IP system* in a commutative semigroup $(S, +)$ is an indexed family $\langle v_\alpha \rangle_{\alpha\in\mathcal{F}}$ where there exists a sequence $\langle y_n \rangle_{n=1}^\infty$ such that for each $\alpha \in \mathcal{F}$, $v_\alpha = \sum_{n\in\alpha} y_n$. (Equivalently, $v_{\alpha\cup\beta} = v_\alpha + v_\beta$ for all $\alpha, \beta \in \mathcal{F}$ with $\alpha \cap \beta = \emptyset$.) An *IP ring* $\mathcal{F}^{(1)}$ is a set of the form $\mathcal{F}^{(1)} = FU(\langle \alpha_n \rangle_{n=1}^\infty)$ where $\langle \alpha_n \rangle_{n=1}^\infty$ is a sequence of members of $\mathcal{F}$ such that $\max \alpha_n < \min \alpha_{n+1}$ for each $n$.

In [3] V. Bergelson and A. Leibman established strong generalizations of van der Waerden's Theorem and Szemerédi's Theorem. Our starting point is their generalization of van der Waerden's Theorem.

**1.1 Theorem**. *Let $k \in \mathbb{N}$ and suppose $\{p_1(x), \cdots, p_k(x)\} \subseteq \mathbb{Z}[x]$ are polynomials having zero constant term. Let $\langle n_\alpha \rangle_{\alpha\in\mathcal{F}}$ be an IP system. For any finite coloring of $\mathbb{N}$ there exists $a \in \mathbb{N}$ and $\alpha \in \mathcal{F}$ such that*

$$\{a, a + p_1(n_\alpha), \cdots, a + p_k(n_\alpha)\}$$

*is monochromatic.*

---

[1] This author acknowledges support received from the National Science Foundation via grant DMS-0070593.

[2] This author acknowledges support received from the National Science Foundation via a post doctoral fellowship administered by the University of Maryland.

**Proof**. This is an unstated combinatorial corollary to [3, Corollary 1.9]. (For a simple algebraic proof of this result see [8].) $\square$

To see that Theorem 1.1 implies van der Waerden's theorem, let $k \in \mathbb{N}$ and for each $t \in \{1, 2, \ldots, k\}$, let $p_t(x) = t \cdot x$. Then Theorem 1.1 tells one that, not only can one always find a monochrome arithmetic progression of length $k+1$, but in fact the increment can be chosen in any prespecified IP system.

The following infinitary generalization of Theorem 1.1 was obtained in [10]. (The linear case of Theorem 1.2 follows from Furstenberg's central sets theorem [6, Proposition 8.21].)

**1.2 Theorem**. *Let $k \in \mathbb{N}$ and suppose $\{p_1(x), \ldots, p_k(x)\} \subseteq \mathbb{Z}[x]$ are polynomials having zero constant term. Let $\langle n_\alpha \rangle_{\alpha \in \mathcal{F}}$ be an IP set. If $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r C_i$ then there exists an IP ring $\mathcal{F}^{(1)}$, an IP system $\langle a_\alpha \rangle_{\alpha \in \mathcal{F}^{(1)}}$, and some $j$ with $1 \leq j \leq r$ such that for all $\alpha \in \mathcal{F}^{(1)}$,*

$$\big\{a_\alpha, a_\alpha + p_1(n_\alpha), \ldots, a_\alpha + p_k(n_\alpha)\big\} \subseteq C_j.$$

**Proof**. [10, Theorem 1.6]. $\square$

Our goal in this paper is to obtain a similar infinitary generalization of a "set polynomial" extension of the Hales-Jewett Theorem, also due to Bergelson and Leibman. To discuss this we need to introduce some terminology. (For a statement of the Hales-Jewett Theorem itself, see Section 4.) Let $l \in \mathbb{N}$. A *set-monomial* (over $\mathbb{N}^l$) in the variable $X$ is an expression $m(X) = S_1 \times S_2 \times \ldots \times S_l$, where for each $i \in \{1, 2, \ldots, l\}$, $S_i$ is either the symbol $X$ or a nonempty singleton subset of $\mathbb{N}$ (these are called *coordinate coefficients*). The *degree* of the monomial is the number of times the symbol $X$ appears in the list $S_1, \ldots, S_l$. For example, taking $l = 3$, $m(X) = \{5\} \times X \times X$ is a set-monomial of degree 2, while $m(X) = X \times \{17\} \times \{2\}$ is a set-monomial of degree 1. A *set-polynomial* is an expression of the form $p(X) = m_1(X) \cup m_2(X) \cup \ldots \cup m_k(X)$, where $k \in \mathbb{N}$ and $m_1(X), \ldots, m_k(X)$ are set-monomials. The degree of a set-polynomial is the largest degree of its set-monomial "summands", and its *constant term* consists of the "sum" of those $m_i$ that are constant, i.e. of degree zero.

A polynomial $p(A)$ determines a function from $\mathcal{P}_f(\mathbb{N})$ to $\mathcal{P}_f(\mathbb{N}^l)$ in the obvious way (interpreting the symbol $\times$ as Cartesian product and the symbol $\cup$ as union). Here now is the polynomial Hales-Jewett theorem of Bergelson and Leibman.

**1.3 Theorem**. *Let $l \in \mathbb{N}$ and let $\mathcal{P}$ be a finite family of set-polynomials over $\mathbb{N}^l$ whose constant terms are empty. Let $I \subseteq \mathbb{N}$ be any finite set and let $r \in \mathbb{N}$. There exists a finite set $N \subseteq \mathbb{N}$, with $N \cap I = \emptyset$, having the property that if $\mathcal{P}_f\big(\bigcup_{P(X) \in \mathcal{P}} P(N)\big) = \bigcup_{i=1}^r C_i$*

*then there exist $i \in \{1,2,\ldots,r\}$, some nonempty $B \subseteq N$, and some $A \subseteq \bigcup_{P(X)\in\mathcal{P}} P(N)$ such that $A \cap P(B) = \emptyset$ for all $P \in \mathcal{P}$ and*

$$\big\{A \cup P(B) : P(X) \in \mathcal{P}\big\} \subseteq C_i.$$

**Proof**. [4, Theorem 3.5]. $\square$

We shall utilize the following special case of Theorem 1.3.

**1.4 Corollary**. *Let $k, r, d \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that if*

$$\mathcal{P}_f\big(\{1,2,\ldots,k\} \times \{1,2,\ldots,N\}^d\big) = \textstyle\bigcup_{i=1}^r C_i$$

*then there exists $A \in \mathcal{P}_f\big(\{1,2,\ldots,k\} \times \{1,2,\ldots,N\}^d\big)$ and $B \in \mathcal{P}_f(\{1,2,\ldots,N\})$, with $A \cap (\{1,2,\ldots,k\} \times B^d) = \emptyset$, and $j \in \{1,2,\ldots,r\}$, such that*

$$\big\{A \cup (E \times B^d) : E \subseteq \{1,2,\ldots,k\}\big\} \subseteq C_j.$$

**Proof**. This is the special case of Theorem 1.3 corresponding to the set-polynomials $P(X) = E \times X^d$, $E \subseteq \{1,2,\ldots,k\}$. $\square$

We shall obtain in Theorem 4.4 a result which generalizes Theorem 1.3 in much the same way that Theorem 1.2 generalizes Theorem 1.1. On the way to this result, we shall need the notions of *VIP system* and *partial semigroup*. The notion of VIP system was introduced in [2]. Recall that given a set $A$ and a cardinal number $\kappa$, $[A]^\kappa = \{B \subseteq A : |B| = \kappa\}$.

**1.5 Definition**. Let $(G, +)$ be an abelian group. A sequence $\langle v_\alpha\rangle_{\alpha\in\mathcal{F}}$ in $G$ is called a *VIP system* if there exists some non-negative integer $d$ (the least such $d$ is called the *degree* of the system) such that for every pairwise disjoint $\alpha_0, \alpha_1, \ldots, \alpha_d \in \mathcal{F}$ we have

$$\sum_{t=1}^{d+1} (-1)^t \sum_{\mathcal{B}\in[\{\alpha_0,\ldots,\alpha_d\}]^t} v_{\cup\mathcal{B}} = 0\,.$$

"Degree" suggests that VIP systems have a "polynomial" nature; and indeed they do. Notice that the VIP systems of degree 1 (i.e. "linear" VIP systems) are precisely the IP systems (the above equation in this case takes the form $v_{\alpha_0\cup\alpha_1} - v_{\alpha_0} - v_{\alpha_1} = 0$). More generally, it is easy to verify that, given any sequence $\langle x_n\rangle_{n=1}^\infty$ in $G$, and any $n \in \mathbb{N}$, if $v_\alpha = (\sum_{i\in\alpha} x_i)^n$, then $\langle v_\alpha\rangle_{\alpha\in\mathcal{F}}$ is a VIP system of degree n.

There is an alternate characterization of VIP systems that is often simpler to work with. For $d \in \mathbb{N}$, let $\mathcal{F}_d$ denote the family of nonempty subsets of $\mathbb{N}$ having cardinality at most $d$.

**1.6 Theorem**. *Let $G$ be an additive abelian group and let $d \in \mathbb{N}$. A sequence indexed by $\mathcal{F}$, $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$, in $G$ is a VIP system of degree at most $d$ if and only if there exists a function from $\mathcal{F}_d$ to $G$, written $\gamma \to n_\gamma$, $\gamma \in \mathcal{F}_d$, such that*

$$v_\alpha = \sum_{\gamma \subseteq \alpha, \gamma \in \mathcal{F}_d} n_\gamma$$

*for all $\alpha \in \mathcal{F}$.*

**Proof**. [10, Proposition 2.5]. $\square$

This characterization may be used to show, for example, that if $R$ is a commutative ring, $k, d \in \mathbb{N}$, and $p \in R[x_1, x_2, \ldots, x_k]$ is a polynomial of degree $d$ with coefficients in $R$ and with $p(0, \ldots, 0) = 0$, and if $\langle n_\alpha^{(i)} \rangle_{\alpha \in \mathcal{F}}$ are IP systems in $R$, $1 \leq i \leq k$, then letting $v_\alpha = p(n_\alpha^{(1)}, n_\alpha^{(2)}, \ldots, n_\alpha^{(k)})$, $\alpha \in \mathcal{F}$, the resulting sequence $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ is a VIP system of degree at most $d$. (For a proof, see [10, Proposition 2.6].)

In this paper we shall extend (in Section 3) the definition of VIP system to partial semigroups. In so doing, we must make a choice between trying to mimic either Definition 1.5 or the characterization provided by Theorem 1.6 (these two approaches, though equivalent for groups, yield different notions when naturally applied to semigroups). We choose to follow the characterization of Theorem 1.6, as this is easier and encompasses all of the interesting examples of which we are aware.

We shall be using the Stone-Čech compactification $\beta S$ of a discrete space $S$. We take the points of $\beta S$ to be the ultrafilters on $S$, the principal ultrafilters being identified with the points of $S$. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of $\beta S$.

If $S$ is a semigroup, then there is a natural extension of the operation of $S$ to $\beta S$, customarily denoted by the same symbol, making $\beta S$ a compact right topological semigroup with $S$ contained in its topological center. (If the operation is "$\cdot$", this says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$.) See [9] for an elementary introduction to the semigroup $\beta S$ as well as for any unfamiliar algebraic assertions encountered here.

## 2. Partial Semigroups

There are many cases in which one is interested in a set with a natural operation in which it is convenient to not have the operation defined for all pairs of members of the set. This can arise in basically two ways. The simpler of the two ways is the situation in which the natural operation does not satisfy the closure property. For example, given a sequence $\langle x_n\rangle_{n=1}^\infty$ in a semigroup $(S,\cdot)$, let $FP(\langle x_n\rangle_{n=1}^\infty)=\{\prod_{n\in\alpha}\ x_n:\alpha\in\mathcal{F}\}$, where the products are taken in increasing order of indices. Then, for example $x_1\cdot x_3\in FP(\langle x_n\rangle_{n=1}^\infty)$ and $x_3\cdot x_5\in FP(\langle x_n\rangle_{n=1}^\infty)$ but it is not likely that $x_1\cdot x_3\cdot x_3\cdot x_5\in FP(\langle x_n\rangle_{n=1}^\infty)$. (Of course, it is possible that $x_1\cdot x_3\cdot x_3\cdot x_5=x_7\in FP(\langle x_n\rangle_{n=1}^\infty)$.) On the other hand, if one requires that $\alpha<\beta$ (meaning $\max\alpha<\min\beta$), or in the event that $S$ is commutative simply that $\alpha\cap\beta=\emptyset$, then $\prod_{n\in\alpha}\ x_n\cdot\prod_{n\in\beta}\ x_n=\prod_{n\in\alpha\cup\beta}\ x_n\in FP(\langle x_n\rangle_{n=1}^\infty)$.

The other way in which the problem can arise is the situation in which the natural operation does yield another member of the set, but does not behave as one wants it to. For example, consider the semigroups $(\mathcal{F},\cup)$ and $(\mathbb{N},+)$ and the function $c:\mathcal{F}\to\mathbb{N}$ defined by $c(\alpha)=|\alpha|$. Then $c$ is not a homomorphism, but it behaves like one on disjoint sets.

We address these problems by using the notion of partial semigroup introduced in [1] to obtain some Ramsey Theoretic results about variable words and sequences of variable words.

**2.1 Definition**. A *partial semigroup* is a pair $(S,*)$ where $*$ maps a subset of $S\times S$ to $S$ and for all $a,b,c\in S$, $(a*b)*c=a*(b*c)$ in the sense that if either side is defined, then so is the other and they are equal.

**2.2 Definition**. Let $(S,*)$ be a partial semigroup.
(a) For $s\in S$, $\varphi(s)=\{t\in S:s*t$ is defined$\}$.
(b) For $H\in\mathcal{P}_f(S)$, $\sigma(H)=\bigcap_{s\in H}\ \varphi(s)$.
(c) $\sigma(\emptyset)=S$.
(d) For $s\in S$ and $A\subseteq S$, $s^{-1}A=\{t\in\varphi(s):s*t\in A\}$.
(e) $(S,*)$ is *adequate* if and only if $\sigma(H)\neq\emptyset$ for all $H\in\mathcal{P}_f(S)$.

Given a partial semigroup $S$ and $a,b,c\in S$, one thus has that the statements
(i) $b\in\varphi(a)$ and $c\in\varphi(a*b)$
and
(ii) $c\in\varphi(b)$ and $b*c\in\varphi(a)$
are equivalent and imply that $(a*b)*c=a*(b*c)$.

Notice that, just as in a semigroup, only even more strongly, the notation $s^{-1}A$ should not be read as suggesting that there is some object $s^{-1}\in S$. We do see however, that in some sense the behavior resembles the case in which such objects exist.

**2.3 Lemma**. *Let $(S,*)$ be a partial semigroup, let $A\subseteq S$ and let $a,b,c\in S$. Then*

$$c\in b^{-1}(a^{-1}A)\Leftrightarrow b\in\varphi(a)\ \textit{and}\ c\in(a*b)^{-1}A\,.$$

*In particular, if $b\in\varphi(a)$, then $b^{-1}(a^{-1}A)=(a*b)^{-1}A$.*

**Proof**.

$$\begin{array}{rcl} c \in b^{-1}(a^{-1}A) & \Leftrightarrow & c \in \varphi(b) \text{ and } b * c \in a^{-1}A \\ & \Leftrightarrow & c \in \varphi(b) \text{ and } b * c \in \varphi(a) \text{ and } a * (b * c) \in A \\ & \Leftrightarrow & b \in \varphi(a) \text{ and } c \in \varphi(a * b) \text{ and } (a * b) * c \in A \\ & \Leftrightarrow & b \in \varphi(a) \text{ and } c \in (a * b)^{-1}A\,. \end{array}$$ □

We now introduce formally the product of more than one elements of $S$. (If the operation of $S$ is denoted "+", then $\prod$ will be replaced by $\sum$.)

**2.4 Definition**. Let $(S, *)$ be a partial semigroup. Given $k \in \mathbb{N}$ and $x_1, x_2, \ldots, x_k \in S$, we define $\prod_{i=1}^{k} x_i$ inductively by:

$$\begin{array}{rcl} \prod_{i=1}^{1} x_i & = & x_1 \\ \prod_{i=1}^{k+1} x_i & = & \begin{cases} (\prod_{i=1}^{k} x_i) * x_{k+1} & \text{if } \prod_{i=1}^{k} x_i \text{ is defined and } x_{k+1} \in \varphi(\prod_{i=1}^{k} x_i) \\ \text{undefined} & \text{otherwise.} \end{cases} \end{array}$$

Notice that if $\prod_{i=1}^{k} x_i$ is defined and $t < k$, then necessarily $\prod_{i=1}^{t} x_i$ is defined.

We are interested in adequate partial semigroups because they give rise to a natural *subsemigroup* of $\beta S$. (That is, a subset of $\beta S$ which is, in a natural way, a semigroup.)

**2.5 Definition**. Let $(S, *)$ be an adequate partial semigroup. Then

$$\delta S = \bigcap_{a \in S} \overline{\varphi(a)}\,.$$

Notice that the fact that $S$ is adequate is exactly what is needed to guarantee that $\delta S \neq \emptyset$. Notice also that if $S$ is in fact a semigroup, then $\delta S = \beta S$.

Recall from [9, Theorem 4.12] that if $(S, \cdot)$ is a semigroup, $A \subseteq S$, $a \in S$, and $p, q \in \beta S$, then

$$A \in a \cdot q \Leftrightarrow a^{-1}A \in q$$

and

$$A \in p \cdot q \Leftrightarrow \{a \in S : a^{-1}A \in q\} \in p\,.$$

Motivated by these characterizations, we extend the partial operation $*$ to as much of $\beta S$ as we can reasonably hope to be sensible.

**2.6 Definition**. Let $(S, *)$ be an adequate partial semigroup.

(i) For $a \in S$ and $q \in \overline{\varphi(a)}$, $a * q = \{A \subseteq S : a^{-1}A \in q\}$.

(ii) For $p \in \beta S$ and $q \in \delta S$, $p * q = \{A \subseteq S : \{a \in S : a^{-1}A \in q\} \in p\}$.

**2.7 Lemma**. *Let $(S, *)$ be an adequate partial semigroup.*

*(i) If $a \in S$ and $q \in \overline{\varphi(a)}$, then $a * q \in \beta S$.*

*(ii) If $p \in \beta S$ and $q \in \delta S$, then $p * q \in \beta S$.*

*(iii) Let $p \in \beta S$, $q \in \delta S$, and $a \in S$. Then $\varphi(a) \in p * q$ if and only if $\varphi(a) \in p$.*
*(iv) If $p, q \in \delta S$, then $p * q \in \delta S$.*

**Proof**. (i). We need to show that $a * q$ is an ultrafilter on $S$, that is, $a * q$ has the finite intersection property and given any set $A \subseteq S$, either $A \in a * q$ or $S \backslash A \in a * q$. For the first assertion, let $\mathcal{H} \in \mathcal{P}_f(a * q)$. Then $\bigcap_{A \in \mathcal{H}} a^{-1}A \in q$ so pick $b \in \bigcap_{A \in \mathcal{H}} a^{-1}A$. Then $b \in \varphi(a)$ and $a * b \in \bigcap \mathcal{H}$.

For the second assertion, let $A \subseteq S$ and assume that $A \notin a * q$. Then $S \backslash a^{-1}A \in q$, $\varphi(a) \in q$, and $(S \backslash a^{-1}A) \cap \varphi(a) = a^{-1}(S \backslash A)$.

(ii). Let $\mathcal{H} \in \mathcal{P}_f(p * q)$, pick $a \in \bigcap_{A \in \mathcal{H}} \{a \in S : a^{-1}A \in q\}$, and pick $b \in \bigcap_{A \in \mathcal{H}} a^{-1}A$. Then $b \in \varphi(a)$ and $a * b \in \bigcap \mathcal{H}$.

Now let $A \subseteq S$ and assume that $A \notin p * q$. Let $B = S \backslash \{a \in S : a^{-1}A \in q\}$. Then $B \in p$. We claim that $B \subseteq \{a \in S : a^{-1}(S \backslash A) \in q\}$ (so that $S \backslash A \in p * q$). Let $a \in B$. Then $a^{-1}A \notin q$ and so $A \notin a * q$. Since $\varphi(a) \in q$ we have by part (i) that $S \backslash A \in a * q$ and thus $a^{-1}(S \backslash A) \in q$ as required.

(iii). Necessity. Assume that $\varphi(a) \in p * q$ so that $\{b \in S : b^{-1}\varphi(a) \in q\} \in p$. We show that $\{b \in S : b^{-1}\varphi(a) \in q\} \subseteq \varphi(a)$. So let $b^{-1}\varphi(a) \in q$. Pick $c \in b^{-1}\varphi(a)$. Then $c \in \varphi(b)$ and $b * c \in \varphi(a)$ so $a * (b * c)$ is defined and thus $a * (b * c) = (a * b) * c$ and in particular $b \in \varphi(a)$.

Sufficiency. Assume that $\varphi(a) \in p$. We claim that $\varphi(a) \subseteq \{b \in S : b^{-1}\varphi(a) \in q\}$ so that $\varphi(a) \in p * q$. Let $b \in \varphi(a)$. Since $q \in \delta S$, $\varphi(a * b) \in q$. Therefore it suffices to show that $\varphi(a * b) \subseteq b^{-1}\varphi(a)$. Let $c \in \varphi(a * b)$. Then $(a * b) * c = a * (b * c)$ so $c \in \varphi(b)$ and $b * c \in \varphi(a)$. That is, $c \in b^{-1}\varphi(a)$ as required.

(iv). This is an immediate consequence of part (iii). □

**2.8 Lemma**. *Let $(S, *)$ be an adequate partial semigroup and let $q \in \delta S$. Then the function $\rho_q : \beta S \to \beta S$ defined by $\rho_q(p) = p * q$ is continuous.*

**Proof**. Note that by Lemma 2.7, the function $\rho_q$ does take $\beta S$ to $\beta S$. Let $p \in \beta S$ and let $A \in p * q$ (so that $\overline{A}$ is a basic neighborhood of $\rho_q(p)$ ). Let $B = \{a \in S : a^{-1}A \in q\}$. Then $B \in p$ and $\rho_q[\overline{B}] \subseteq \overline{A}$. □

**2.9 Lemma**. *Let $p \in \beta S$ and let $q, r \in \delta S$. Then $p * (q * r) = (p * q) * r$.*

**Proof**. Notice that by Lemma 2.7, both $p * (q * r)$ and $(p * q) * r$ are in $\beta S$. Suppose that $p*(q*r) \neq (p*q)*r$ and pick $A \in p*(q*r) \backslash (p*q)*r$. Let $B = \{a \in S : a^{-1}(S \backslash A) \in r\}$. Then $B \in p * q$ so $\{b \in S : b^{-1}B \in q\} \in p$. Also, $\{b \in S : b^{-1}A \in q * r\} \in p$ so pick $b \in S$ such that $b^{-1}B \in q$ and $b^{-1}A \in q * r$. Then $\{c \in S : c^{-1}(b^{-1}A) \in r\} \in q$ so pick $c \in b^{-1}B$ such that $c^{-1}(b^{-1}A) \in r$. Then $c \in \varphi(b)$ and $b * c \in B$ so $(b * c)^{-1}(S \backslash A) \in r$. Since $c \in \varphi(b)$ we have by Lemma 2.3 that $c^{-1}(b^{-1}A) = (b*c)^{-1}A$. Since $\big((b*c)^{-1}A\big) \cap \big((b*c)^{-1}(S\backslash A)\big) = \emptyset$, we have a contradiction. $\square$

**2.10 Theorem**. *Let $(S,*)$ be an adequate partial semigroup. Then $(\delta S,*)$ is a compact Hausdorff right topological semigroup.*

**Proof**. Lemmas 2.7(iv), 2.8, and 2.9 and the fact that $\delta S$ is a closed subset of $\beta S$. $\square$

As a compact Hausdorff right topological semigroup, $\delta S$ is guaranteed the structure common to all such objects. In particular, it has a smallest two sided ideal $K(\delta S)$ which is the union of all of the minimal left ideals of $\delta S$ as well as the union of all of the minimal right ideals of $\delta S$. (A subset $L$ of a semigroup $(S,\cdot)$ is a left ideal of $S$ if and only if $L \neq \emptyset$ and $S \cdot L \subseteq L$. Similarly, a right ideal $R$ satisfies $R \cdot S \subseteq R$. A two sided ideal is both a right and a left ideal.) Further, given a minimal left ideal $L$ of $\delta S$ and a minimal right ideal $R$ of $\delta S$, $L \cap R$ is a group (so in particular $L$ and $R$ each have idempotents). (See [9, Theorems 2.7 and 2.8].)

**2.11 Definition**. Let $p = p * p \in \delta S$ and let $A \in p$. Then $A^\star = \{x \in A : x^{-1}A \in p\}$.

Given an idempotent $p \in \delta S$ and $A \in p$, it is immediate that $A^\star \in p$.

**2.12 Lemma**. *Let $p = p * p \in \delta S$, let $A \in p$, and let $x \in A^\star$. Then $x^{-1}(A^\star) \in p$.*

**Proof**. Since $x \in A^\star$, $x^{-1}A \in p$ and so $(x^{-1}A)^\star \in p$. Thus it suffices to show that $(x^{-1}A)^\star \subseteq x^{-1}(A^\star)$. (In fact equality holds.) Let $y \in (x^{-1}A)^\star$. Then $y \in x^{-1}A$ and $y^{-1}(x^{-1}A) \in p$. Then $y \in \varphi(x)$ and $x * y \in A$ and, by Lemma 2.3, $(x * y)^{-1}A = y^{-1}(x^{-1}A) \in p$. That is $y \in \varphi(x)$ and $x * y \in A^\star$ as required. $\square$

Some important notions of largeness in a semigroup $S$ can be characterized in terms of $K(\beta S)$. It is our intention to utilize $K(\delta S)$ to obtain appropriate analogues of these notions for partial semigroups.

Recall that a subset $A$ of a semigroup $S$ is *syndetic* if and only if there exists some $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} t^{-1}A$. This has an equivalent formulation in terms of $\beta S$, namely $\beta S = \bigcup_{t \in H} \overline{t^{-1}A}$. Given a point $p \in \beta S$, one has that $p \in K(\beta S)$ if and only if for every $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is syndetic [9, Theorem 4.39]. We introduce now a notion of "syndetic" in a partial semigroup $S$, and provide evidence that it is an appropriate notion by verifying the corresponding results for $\delta S$.

Notice that one certainly wants an adequate partial semigroup $S$ to be syndetic in itself (since, after all, we are concerned with notions of largeness). In the simple examples of partial semigroups already presented (all of which are adequate) one does not have a finite subset $H$ of $S$ with $S \subseteq \bigcup_{t \in H} \varphi(t)$. Consequently one cannot hope to have the verbatim definition of "syndetic" apply to partial semigroups. The modification needed turns out to be quite minor.

**2.13 Definition**. Let $(S,*)$ be a partial semigroup and let $A\subseteq S$. Then $A$ is *syndetic* if and only if there is some $H\in\mathcal{P}_f(S)$ such that $\sigma(H)\subseteq\bigcup_{t\in H}\ t^{-1}A$.

Notice that if $S$ is a semigroup, Definition 2.10 agrees with the standard definition of "syndetic".

**2.14 Lemma**. *Let $(S,*)$ be an adequate partial semigroup and let $A\subseteq S$. Then $A$ is syndetic if and only if there exists $H\in\mathcal{P}_f(S)$ such that $\delta S\subseteq\bigcup_{t\in H}\ \overline{t^{-1}A}$.*

**Proof**. Necessity. Pick $H\in\mathcal{P}_f(S)$ such that $\sigma(H)\subseteq\bigcup_{t\in H}\ t^{-1}A$. Then $\delta S\subseteq\overline{\sigma(H)}\subseteq\overline{\bigcup_{t\in H}\ t^{-1}A}=\bigcup_{t\in H}\ \overline{t^{-1}A}$.

Sufficiency. Suppose that for each $H\in\mathcal{P}_f(S)$, $\sigma(H)\backslash\bigcup_{t\in H}\ t^{-1}A\neq\emptyset$. Let $\mathcal{A}=\{\varphi(t)\backslash t^{-1}A:t\in S\}$. Then, given $H\in\mathcal{P}_f(S)$, $\sigma(H)\backslash\bigcup_{t\in H}\ t^{-1}A\subseteq\bigcap_{t\in H}\ (\varphi(t)\backslash t^{-1}A)$ so $\mathcal{A}$ has the finite intersection property. Pick $q\in\beta S$ such that $\mathcal{A}\subseteq q$ and note that $\{\varphi(t):t\in S\}\subseteq q$ so that $q\in\delta S$. Pick $H\in\mathcal{P}_f(S)$ such that $\delta S\subseteq\bigcup_{t\in H}\ \overline{t^{-1}A}$ and pick $t\in H$ such that $q\in\overline{t^{-1}A}$. This is a contradiction. □

**2.15 Theorem**. *Let $(S,*)$ be an adequate partial semigroup and let $p\in\delta S$. The following statements are equivalent.*

*(a) $p\in K(\delta S)$.*

*(b) For all $A\in p$, $\{x\in S:x^{-1}A\in p\}$ is syndetic.*

*(c) For all $q\in\delta S$, $p\in\delta S*q*p$.*

**Proof**. (a) implies (b). Let $A\in p$ and let $B=\{x\in S:x^{-1}A\in p\}$. Let $L$ be the minimal left ideal of $\delta S$ such that $p\in L$. We claim that $L\subseteq\bigcup_{t\in S}\ \overline{t^{-1}A}$. To see this, let $q\in L$. Then $\delta S*q$ is a left ideal contained in $L$ so $L=\delta S*q$. Consequently $p\in L=\delta S*q$ so pick $r\in\delta S$ such that $p=r*q$. Then $\{t\in S:t^{-1}A\in q\}\in r$ so pick $t\in S$ such that $t^{-1}A\in q$. Now $L=\rho_p[\delta S]$ so $L$ is compact. Pick $H\in\mathcal{P}_f(S)$ such that $L\subseteq\bigcup_{t\in H}\ \overline{t^{-1}A}$. We claim that $\delta S\subseteq\bigcup_{t\in H}\ \overline{t^{-1}B}$ so that, by Lemma 2.14, $B$ is syndetic.

Let $r\in\delta S$. Then $r*p\in L\subseteq\bigcup_{t\in H}\ \overline{t^{-1}A}$ so pick $t\in H$ such that $t^{-1}A\in r*p$. Then $\{x\in S:x^{-1}(t^{-1}A)\in p\}\in r$ and $\varphi(t)\in r$. We claim that

$$\varphi(t)\cap\{x\in S:x^{-1}(t^{-1}A)\in p\}\subseteq t^{-1}B$$

so that $t^{-1}B\in r$ as required. So let $x\in\varphi(t)$ such that $x^{-1}(t^{-1}A)\in p$. Then by Lemma 2.3, $x^{-1}(t^{-1}A)=(t*x)^{-1}A$ so that $t*x\in B$.

(b) implies (c). Let $q\in\delta S$. For $A\in p$, let $B(A)=\{x\in S:x^{-1}A\in q*p\}$. We claim that $\{B(A):A\in p\}$ has the finite intersection property. Since, given $A_1$ and $A_2$,

$B(A_1 \cap A_2) = B(A_1) \cap B(A_2)$, it suffices to show that each $B(A) \neq \emptyset$. To this end, let $A \in p$, let $C = \{x \in S : x^{-1}A \in p\}$, and pick $H \in \mathcal{P}_f(S)$ such that $\sigma(H) \subseteq \bigcup_{t \in H} t^{-1}C$. For each $y \in \sigma(H)$, pick $t_y \in H$ such that $t_y * y \in C$. Now $\sigma(H) \in q$ and $\sigma(H) = \bigcup_{t \in H} \{y \in \sigma(H) : t_y = t\}$ so pick $t \in H$ such that $\{y \in \sigma(H) : t_y = t\} \in q$. We show that $t \in B(A)$. For this it suffices to show that $\{y \in \sigma(H) : t_y = t\} \subseteq \{y \in S : y^{-1}(t^{-1}A) \in p\}$. So let $y \in \sigma(H)$ such that $t_y = t$. Then $t * y \in C$ so $(t * y)^{-1}A \in p$. Since $y \in \varphi(t)$, $(t * y)^{-1}A = y^{-1}(t^{-1}A)$ by Lemma 2.3.

Since $\{B(A) : A \in p\}$ has the finite intersection property, pick $r \in \beta S$ such that $\{B(A) : A \in p\} \subseteq r$. Then for all $A \in p$, $\{x \in S : x^{-1}A \in q * p\} \in r$ so $p = r * (q * p)$. Since $p \in \delta S$, for each $a \in S$, $\varphi(a) \in p$ and consequently, by Lemma 2.7(iii), for each $a \in S$, $\varphi(a) \in r$. That is $r \in \delta S$.

(c) implies (a). Pick $q \in K(\delta S)$. Then $\delta S * q * p \subseteq K(\delta S)$. □

Two other important notions of largeness in a semigroup $S$ are the notions of *piecewise syndetic* sets and *central* sets. Both of these notions have simple characterizations in terms of $\beta S$. A subset $A$ of $S$ is piecewise syndetic if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$ [9, Theorem 4.40] and $A$ is central if and only if there is an idempotent $p \in K(\beta S)$ such that $A \in p$ [9, Definition 4.42].

**2.16 Definition**. Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$.

(a) The set $A$ is *piecewise syndetic* in $S$ if and only if $\overline{A} \cap K(\delta S) \neq \emptyset$.

(b) The set $A$ is *central* in $S$ if and only if there is some idempotent $p$ in $K(\delta S)$ such that $A \in p$.

Notice that (unlike the notion of syndetic), both "piecewise syndetic" and "central" are partition regular notions. That is, if a finite union of sets has one of these properties, then some one of them does. (This fact is immediate from the definitions.)

The following result will be needed in the next section.

**2.17 Lemma**. *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$ be piecewise syndetic. There exists $H \in \mathcal{P}_f(S)$ such that for every finite nonempty set $T \subseteq \sigma(H)$, there exists $x \in \sigma(T)$ such that $T * x \subseteq \bigcup_{t \in H} t^{-1}A$.*

**Proof**. Pick $q \in \overline{A} \cap K(\delta S)$ and let $B = \{x \in S : x^{-1}A \in q\}$. By Theorem 2.15, $B$ is syndetic, so pick $H \in \mathcal{P}_f(S)$ such that $\sigma(H) \subseteq \bigcup_{t \in H} t^{-1}B$. Let $T \in \mathcal{P}_f\big(\sigma(H)\big)$. For each $y \in T$ pick $t_y \in H$ such that $t_y * y \in B$ and thus $(t_y * y)^{-1}A \in q$. Pick $x \in \bigcap_{y \in T} (t_y * y)^{-1}A$. Given any $y \in T$, $y \in \varphi(t_y)$ and $x \in \varphi(t_y * y)$ so that $x \in \varphi(y)$. That is $x \in \sigma(T)$. Further, given $y \in T$, $t_y * (y * x) = (t_y * y) * x \in A$ and thus $y * x \in {t_y}^{-1}A \subseteq \bigcup_{t \in H} t^{-1}A$. □

## 3. VIP systems in Partial Semigroups

We shall be concerned with extending the notion of *VIP system* to an arbitrary (partial) semigroup. This notion involves equations involving a large number of sums, and we do not wish to be concerned about the order in which these sums are taken. (An already complicated situation would be made more complicated by such considerations.) Consequently, we shall restrict our attention from now on to commutative semigroups and commutative partial semigroups. (When we say that a partial semigroup $(S,+)$ is commutative, we mean that for all $a,b \in S$, $a+b$ is defined if and only if $b+a$ is defined, and of course, if defined $a+b=b+a$.)

In particular, we shall use additive notation. If we are speaking of an additive semigroup or partial semigroup $S$ with an identity, we shall denote that identity by 0, in which case $S \cup \{0\} = S$. If $S$ does not have an identity, then $S \cup \{0\}$ denotes $S$ with an identity adjoined.

We begin by recording an observation, whose simple proof we omit.

**3.1 Lemma**. *Let $(S,+)$ be a commutative partial semigroup, let $k \in \mathbb{N}$, and let $x_1, x_2, \ldots, x_k \in S$. If $\sum_{i=1}^{k} x_i$ is defined, $t \in \{1,2,\ldots,k\}$, and $s_1, s_2, \ldots, s_t$ are distinct members of $\{1,2,\ldots,k\}$, then $\sum_{i=1}^{t} x_{s_i}$ is defined. If $t=k$, then $\sum_{i=1}^{k} x_{s_i} = \sum_{i=1}^{k} x_i$.*

Here now is the definition of VIP system for partial semigroups.

**3.2 Definition**. Let $(S,+)$ be a commutative partial semigroup. Let $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ be an $\mathcal{F}$-sequence in $S$. $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ is called a *VIP system* if there exist some $d \in \mathbb{N}$ and a function from $\mathcal{F}_d$ to $S \cup \{0\}$, written $\gamma \to m_\gamma$, $\gamma \in \mathcal{F}_d$, such that

$$v_\alpha = \sum_{\gamma \subseteq \alpha, \gamma \in \mathcal{F}_d} m_\gamma \tag{3.1}$$

for all $\alpha \in \mathcal{F}$. (In particular, the sum is always defined.) The sequence $\langle m_\gamma \rangle_{\gamma \in \mathcal{F}_d}$ is said to *generate* the VIP system $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$

We shall also refer to a family $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}^{(1)}}$ where $\mathcal{F}^{(1)}$ is an IP ring and the obvious analogue of (3.1) is satisfied, as a VIP system.

Notice that, given any finite $\mathcal{G} \subseteq \mathcal{F}_d$, $\sum_{\gamma \in \mathcal{G}} m_\gamma$ is defined. To see this, let $\alpha = \bigcup \mathcal{G}$. Then $\sum_{\gamma \in \mathcal{G}} m_\gamma$ is a sum of terms included in $\sum_{\gamma \subseteq \alpha, \gamma \in \mathcal{F}_d} m_\gamma$.

It is easily shown that for cancellative semigroups, the sequence of generators $\langle m_\gamma \rangle_{\gamma \in \mathcal{F}_d}$ of a VIP system $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ is unique. (Assume $\langle n_\gamma \rangle_{\gamma \in \mathcal{F}_d}$ also generates $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$. One obtains $m_\alpha = n_\alpha$ for $\alpha \in \mathcal{F}_d$ by induction on $|\alpha|$, using (3.1).) In the non-cancellative case, however, generators need not be unique. For example, let $S = \{1,2,3\}$ and define

$x * y = \min\{x+y, 3\}$ for $x, y \in S$. For $\alpha \in \mathcal{F}$, put $v_\alpha = 1$ if $|\alpha| = 1$ and $v_\alpha = 3$ if $|\alpha| > 1$. Then one easily shows that $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ is a VIP system of degree 2 in $S$. If $\langle m_\gamma \rangle_{\gamma \in \mathcal{F}_2}$ generates $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ then $m_\gamma = 1$ for all singletons $\gamma$, but for $|\gamma| = 2$, $m_\gamma$ can be anything.

If a semigroup $S$ is commutative and cancellative, then $S$ can be embedded in a group $G$, its so called "group of quotients". In this case it may happen that a VIP system in $G$ that is entirely contained in $S$ fails to be a VIP system in $S$. Consider for example the semigroup $(\mathbb{N}, +)$ and define, for $\alpha \in \mathcal{F}$, $v_\alpha = \left(\sum_{n \in \alpha} (-1)^n\right)^2$. Then $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ is a VIP system of degree 2 in $(\mathbb{Z}, +)$, but is not a VIP system of any degree in $(\mathbb{N}, +)$. The reason for this, of course, is that the function $\gamma : \mathcal{F}_2 \to \mathbb{Z}$ for which (3.1) holds is given by $m_{\{i\}} = 1$ and $m_{\{i,j\}} = 2(-1)^{i+j}$ for $i, j \in \mathbb{N}$, $i \neq j$. In particular, not all the generators are contained in $\mathbb{N}$ even though the system itself is.

This motivates the following definition.

**3.3 Definition**. Let $S$ be a commutative, cancellative semigroup and let $G$ be the group of quotients of $S$. An $\mathcal{F}$-sequence $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ in $S$ is called a *weak VIP system* if it is a VIP system in $G$.

In a semigroup $S$, if $\langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{F}}$ are VIP systems, $1 \leq i \leq k$, then for every finite coloring of $S$ there exists a monochromatic configuration of the form $\{a + v_\alpha^{(i)} : 1 \leq i \leq k\}$, where $a \in S$ and $\alpha \in \mathcal{F}$. (This is a consequence of Corollary 1.4; see the proof of Theorem 3.7 below.) In adequate partial semigroups, the situation is somewhat more complicated.

**3.4 Theorem**. *There exists a commutative adequate partial semigroup $(S, +)$, a VIP system $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}} \subseteq S$ of degree 2, a VIP system $\langle u_\alpha \rangle_{\alpha \in \mathcal{F}} \subseteq S$ of degree 1, and a 2-cell partition of $S$ such that there exists no monochromatic configuration of the form $\{a, a+v_\alpha, a+u_\alpha\}$.*

**Proof**. Let $S$ consist of all ordered pairs $(A, \beta)$, where $A$ is a finite (possibly empty) subset of $\mathbb{N} \times \mathbb{N}$, $\beta$ is a finite (possibly empty) subset of $\mathbb{N}$, and $A \subseteq (\mathbb{N} \setminus \beta)^2$.

For $(A, \alpha)$ and $(B, \beta)$ in $X$, define $(A, \alpha) + (B, \beta) = (A \cup B, \alpha \cup \beta)$ if and only if $(A \cup B, \alpha \cup \beta) \in S$ (otherwise the sum is undefined). Then $S$ is a commutative adequate partial semigroup. For $\alpha \in \mathcal{F}$ let $v_\alpha = (\alpha \times \alpha, \emptyset)$ and let $u_\alpha = (\emptyset, \alpha)$. Then $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ is a VIP system of degree 2 (with $m_{\{i\}} = (\{(i,i)\}, \emptyset)$ and, for $i \neq j$, $m_{\{i,j\}} = (\{(i,j), (j,i)\}, \emptyset)$ ) and $\langle u_\alpha \rangle_{\alpha \in \mathcal{F}}$ is a VIP system of degree 1.

Suppose $A \in \mathcal{P}_f(\mathbb{N}^2)$. Let a subset $E$ of $A$ be called a maximal subsquare of $A$ if $E$ is a nonempty perfect square, $E \subseteq A$, and if for every perfect square $F$ with $E \subseteq F \subseteq A$ it is the case that $E = F$. Let $m_A$ be the number of maximal subsquares of $A$. One may check that if $A \subseteq (\mathbb{N} \setminus \alpha)^2$ then $m_{A \cup \alpha^2} = m_A + 1$ if $\alpha \neq \emptyset$. Let $C_1$ (respectively $C_2$) consist of all $(A, \beta) \in S$ with $m_A$ odd (respectively even).

Any configuration $\{a, a+v_\alpha, a+u_\alpha\} \subseteq S$. has the form

$$\{(A,\beta),(A\cup(\alpha\times\alpha),\beta),(A,\beta\cup\alpha)\}.$$

Moreover $A \subseteq (\mathbb{N}\setminus(\beta\cup\alpha))^2$ (this comes from the fact that the third element in the configuration is in $S$). But this means that $m_{A\cup(\alpha\times\alpha)} = m_A + 1$, so that the first two elements of the configuration cannot possibly be contained in the same cell. $\square$

In order to obtain the desired monochromatic configurations in partial semigroups, we are forced to restrict attention to a special class of finite families of VIP systems.

**3.5 Definition**. Let $(S,+)$ be a commutative adequate partial semigroup. A finite set $\{\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}} : 1 \le i \le k\}$ of VIP systems is said to be *adequate* if there exist $d, t \in \mathbb{N}$, a set $\{\langle m_\gamma^{(i)}\rangle_{\gamma\in\mathcal{F}_d} : 1 \le i \le k\}$, a set of VIP systems $\{\langle u_\alpha^{(i)} = \sum_{\gamma\subseteq\alpha,\gamma\in\mathcal{F}_d} n_\gamma^{(i)}\rangle_{\alpha\in\mathcal{F}} : 1 \le i \le t\}$, and sets $E_1, E_2, \ldots, E_k \subseteq \{1,2,\ldots,t\}$ such that:

(1) For each $i \in \{1,2,\ldots,k\}$, $\langle m_\gamma^{(i)}\rangle_{\gamma\in\mathcal{F}_d}$ generates $\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}}$.

(2) For every $H \in \mathcal{P}_f(S)$, there exists $m \in \mathbb{N}$ such that for every $l \in \mathbb{N}$ and pairwise distinct $\gamma_1, \ldots, \gamma_l \in \mathcal{F}_d$ with each $\gamma_i \not\subseteq \{1,2,\ldots,m\}$, $\sum_{i=1}^t \sum_{j=1}^l n_{\gamma_j}^{(i)} \in \sigma(H)\cup\{0\}$. (In particular, the sum is defined.)

(3) $m_\gamma^{(i)} = \sum_{t\in E_i} n_\gamma^{(t)}$ for all $i \in \{1,2,\ldots,k\}$ and all $\gamma \in \mathcal{F}_d$.

Notice that if $S$ is a semigroup then any finite set $\{\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}} : 1 \le i \le k\}$ of VIP systems is adequate (taking $t = k$ and $u_\alpha^{(i)} = v_\alpha^{(i)}$). Notice also that all subsums of $\sum_{i=1}^t \sum_{j=1}^l n_{\gamma_j}^{(i)}$ are in $\sigma(H)\cup\{0\}$.

As a consequence of Definition 3.5(3), notice that for all $i \in \{1,2,\ldots,k\}$ and all $\alpha \in \mathcal{F}$, $v_\alpha^{(i)} = \sum_{t\in E_i} u_\alpha^{(t)}$.

We shall take certain liberties with the application of Definition 3.5. For example, in the proof of Theorem 3.10 we shall replace $\{1,2,\ldots,t\}$ by another finite set and replace $\mathcal{F}$ by $\{\beta\in\mathcal{F} : \beta > \alpha\}$ for some $\alpha \in \mathcal{F}$.

**3.6 Definition**. Let $S$ be a commutative adequate partial semigroup and let $\mathcal{A} \subseteq \mathcal{P}_f(S)$. $\mathcal{A}$ is said to be *adequately partition regular* if for every finite subset $H$ of $S$ and every $r \in \mathbb{N}$, there exists a finite set $F \subseteq \sigma(H)$ having the property that if $F = \bigcup_{i=1}^r C_i$ then for some $j \in \{1,2,\ldots,r\}$, $C_j$ contains a member of $\mathcal{A}$. $\mathcal{A}$ is said to be *shift invariant* if for all $A \in \mathcal{A}$ and all $x \in \sigma(A)$, $A + x = \{a + x : a \in A\} \in \mathcal{A}$.

**3.7 Theorem**. *Let $(S,+)$ be a commutative adequate partial semigroup and let $k \in \mathbb{N}$. If $\{\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}} : 1 \le i \le k\}$ is an adequate set of VIP systems in $S$, and $\beta \in \mathcal{F}$, then the family*

$$\mathcal{A} = \big\{\{a, a+v_\alpha^{(1)}, a+v_\alpha^{(2)}, \ldots, a+v_\alpha^{(k)}\} : a \in \sigma(\{v_\alpha^{(1)}, v_\alpha^{(2)}, \ldots, v_\alpha^{(k)}\}),\ \alpha\in\mathcal{F},\ \textit{and } \alpha > \beta\big\}$$

*is adequately partition regular.*

**Proof**. Since $\{\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}} : 1\leq i\leq k\}$ is an adequate set of VIP systems, choose $d,t\in\mathbb{N}$, a set $\{\langle m_\gamma^{(i)}\rangle_{\gamma\in\mathcal{F}_d} : 1\leq i\leq k\}$, a set of VIP systems $\{\langle u_\alpha^{(i)} = \sum_{\gamma\subseteq\alpha,\gamma\in\mathcal{F}_d} n_\gamma^{(i)}\rangle_{\alpha\in\mathcal{F}} : 1\leq i\leq t\}$, and sets $E_1,E_2,\ldots,E_k\subseteq\{1,2,\ldots,t\}$ such that conditions (1), (2), and (3) of Definition 3.5 are satisfied.

Let $H\in\mathcal{P}_f(S)$ and let $r\in\mathbb{N}$ be given. Let $m\in\mathbb{N}$ be as guaranteed by condition (2) of Definition 3.5. We may assume that $m>\max\beta$. By Corollary 1.4, choose $N\in\mathbb{N}$ such that for any $(r+1)$-coloring of $\mathcal{P}_f\big(\{1,2,\ldots,t\}\times\{1,2,\ldots,N\}^d\big)$, there exists a monchromatic configuration of the form $\big\{A\cup(E\times B^d) : E\subseteq\{1,2,\ldots,t\}\big\}$, with $A\in\mathcal{P}_f\big(\{1,2,\ldots,t\}\times\{1,2,\ldots,N\}^d\big)$, $B\in\mathcal{P}_f(\{1,2,\ldots,N\})$, and $A\cap(\{1,2,\ldots,k\}\times B^d)=\emptyset$.

We now seek to define a function $\mu:\mathcal{P}_f\big(\{1,2,\ldots,t\}\times\{1,2,\ldots,N\}^d\big)\to S\cup\{0\}$ with the property that $\mu(A\cup B)=\mu(A)+\mu(B)$ whenever $A\cap B=\emptyset$. Clearly it suffices to define $\mu$ on singletons, provided any finite sum of the images of these singletons exists. Assume then that $x=(i,a_1,a_2,\ldots,a_d)\in\{1,2,\ldots,t\}\times\{1,2,\ldots,N\}^d$. We let $\mu(x)=0$ if it is not that case that there exists $l$ such that $a_1<a_2<\ldots<a_l=a_{l+1}=\ldots=a_d$, otherwise we let $\mu(x)=n^{(i)}_{\{m+a_1,m+a_2,\ldots,m+a_l\}}$. By condition (2) of Definition 3.5, $\mu$ may be extended additively to all of $\mathcal{P}_f\big(\{1,2,\ldots,t\}\times\{1,2,\ldots,N\}^d\big)$. If $0\in S$, let $F=\mu[\mathcal{P}_f\big(\{1,2,\ldots,t\}\times\{1,2,\ldots,N\}^d\big)]$. If $0\notin S$, let $F=\mu[\mathcal{P}_f\big(\{1,2,\ldots,t\}\times\{1,2,\ldots,N\}^d\big)]\backslash\{0\}$. Then $F\subseteq\sigma(H)$. One easily checks that for all $B\in\mathcal{P}_f(\{1,2,\ldots,N\})$, $\mu(\{i\}\times B^d)=u_\alpha^{(i)}$, where $\alpha=\{b+m : b\in B\}$.

Assume that $F=\bigcup_{i=1}^r C_i$. Construct a partition $\mathcal{P}_f\big(\{1,2,\ldots,t\}\times\{1,2,\ldots,N\}^d\big)=\bigcup_{i=1}^{r+1} D_i$ by the rule $E\in D_i$ if $\mu(E)\in C_i$. If $\mu(E)=0$ and $0\notin S$, let $E\in D_{r+1}$. By the choice of $N$, pick $A\in\mathcal{P}_f\big(\{1,2,\ldots,t\}\times\{1,2,\ldots,N\}^d\big)$, $B\in\mathcal{P}_f(\{1,2,\ldots,N\})$, and $j\in\{1,2,\ldots,r+1\}$ such that $A\cap(\{1,2,\ldots,k\}\times B^d)=\emptyset$ and $\big\{A\cup(E\times B^d) : E\subseteq\{1,2,\ldots,t\}\big\}\subseteq D_j$.

Let $a=\mu(A)$ and let $\alpha=\{m+b : b\in B\}$. Given $i\in\{1,2,\ldots,k\}$,

$$\begin{aligned}\mu\big(A\cup(E_i\times B^d)\big) &= \mu(A)+\textstyle\sum_{n\in E_i}\mu(\{n\}\times B^d)\\ &= a+\textstyle\sum_{n\in E_i} u_\alpha^{(n)}\\ &= a+v_\alpha^{(i)}\,.\end{aligned}$$

Since $v_\alpha^{(1)}\in S$, $\mu\big(A\cup(E_1\times B^d)\big)\in S$ and consequently $j\neq r+1$.

Thus $\{a+v_\alpha^{(1)},a+v_\alpha^{(2)},\ldots,a+v_\alpha^{(k)}\}=\mu[\{A\cup(E_i\times B^d) : 1\leq i\leq k\}]\subseteq C_j$. Also $a=\mu\big(A\cup(\emptyset\times B^d)\big)\in C_j$. Finally $a\in\sigma(\{v_\alpha^{(1)},v_\alpha^{(2)},\ldots,v_\alpha^{(k)}\})$ and $\min\alpha>m>\max\beta$. $\square$

**3.8 Theorem**. *Let $(S,+)$ be an adequate commutative partial semigroup and let $\mathcal{A}$ be a shift invariant, adequately partition regular family of finite subsets of $S$. Let $E\subseteq S$ be piecewise syndetic. Then $E$ contains a member of $\mathcal{A}$.*

**Proof**. Pick, by Lemma 2.17, $H \in \mathcal{P}_f(S)$ such that for every finite $T \subseteq \sigma(H)$, there exists $x \in \sigma(T)$ such that

$$T + x \subseteq \bigcup_{f \in H} -f + E\,.$$

Let $r = |H|$ and let $T \subseteq \sigma(H)$ be chosen so that if $T = \bigcup_{i=1}^{r} C_i$, then some $C_j$ contains a member of $\mathcal{A}$. Choose $x \in \sigma(T)$ such that $T + x \subseteq \bigcup_{f \in H} -f + E$. For $f \in H$ let $C_f = \{t \in T : t + x \in -f + E\}$. Then $T = \bigcup_{f \in H} C_f$ so pick $A \in \mathcal{A}$ and $f \in H$ such that $A \subseteq C_f$. We have that $A + x \subseteq -f + E$ so $(A + x) + f \subseteq E$, while $A + x + f \in \mathcal{A}$. $\square$

**3.9 Lemma**. *Let $(S, +)$ be a commutative partial semigroup, let $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$ be a VIP system in $S$ where, for each $\alpha \in \mathcal{F}$, $v_\alpha = \sum_{\gamma \subseteq \alpha, \gamma \in \mathcal{F}_d} m_\gamma$. Fix $\alpha \in \mathcal{F}$ and for $\beta \in \mathcal{F}$, $\beta > \alpha$, let $q_\beta = \sum_{\varphi \subseteq \beta,\, \varphi \in \mathcal{F}_d} b_\varphi$, where for $\varphi > \alpha$, $b_\varphi = \sum_{\psi \subseteq \alpha,\, |\psi| \leq d - |\varphi|} m_{\varphi \cup \psi}$. Then $\langle q_\beta \rangle_{\beta \in \mathcal{F}, \beta > \alpha}$ is a VIP system and $q_\beta + v_\alpha = v_{\alpha \cup \beta}$ for all $\beta \in \mathcal{F}$ with $\beta > \alpha$.*

**Proof**. The first assertion is obvious. (Notice that (1) each $n_\varphi$ is a sum of $m_\gamma$'s and (2) if $\varphi \neq \varphi'$, $\psi \subseteq \alpha$, $|\psi| \leq d - |\varphi|$, and $\psi' \subseteq \alpha$, $|\psi'| \leq d - |\varphi'|$, then $\varphi \cup \psi \neq \varphi' \cup \psi'$. Consequently, we have that all sums of $n_\varphi$'s are defined.) For $\beta > \alpha$, one has

$$\begin{aligned} v_{\alpha \cup \beta} &= \textstyle\sum_{\gamma \subseteq \alpha \cup \beta,\, \gamma \in \mathcal{F}_d} m_\gamma \\ &= \textstyle\sum_{\gamma \subseteq \alpha,\, \gamma \in \mathcal{F}_d} m_\gamma + \sum_{\varphi \subseteq \beta,\, \varphi \in \mathcal{F}_d} \sum_{\psi \subseteq \alpha,\, |\psi| \leq d - |\varphi|} m_{\varphi \cup \psi} \\ &= \textstyle\sum_{\gamma \subseteq \alpha,\, \gamma \in \mathcal{F}_d} m_\gamma + \sum_{\varphi \subseteq \beta,\, \varphi \in \mathcal{F}_d} b_\varphi = v_\alpha + q_\beta\,. \end{aligned}$$

$\square$

We will agree to denote the VIP system $\langle q_\beta \rangle_{\beta \in \mathcal{F}, \beta > \alpha}$ constructed in Lemma 3.9 by $\langle v_{\alpha \cup \beta} - v_\alpha \rangle_{\beta \in \mathcal{F}, \beta > \alpha}$. (This does not imply that subtraction makes sense in $S$.) Note that $\langle v_{\alpha \cup \beta} - v_\alpha \rangle_{\beta \in \mathcal{F}, \beta > \alpha}$ depends crucially on a chosen set of generators for $\langle v_\alpha \rangle_{\alpha \in \mathcal{F}}$. Different generators, if they exist, may not give the same thing.

In the following theorem, notice that if $F \neq \emptyset$, then $\langle q_\beta^{(i,F)} \rangle_{\beta \in \mathcal{F}, \beta > \alpha_s} = \big\langle v^{(i)}_{\beta \cup \bigcup_{j \in F} \alpha_j} - v^{(i)}_{\bigcup_{j \in F} \alpha_j} \big\rangle_{\beta \in \mathcal{F}, \beta > \alpha_s}$.

**3.10 Theorem**. *Let $\big\{ \langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{F}} : 1 \leq i \leq k \big\}$ be an adequate set of VIP systems and pick $d, t \in \mathbb{N}$, a set $\{ \langle m_\gamma^{(i)} \rangle_{\gamma \in \mathcal{F}_d} : 1 \leq i \leq k \}$, a set of VIP systems*

$$\{ \langle u_\alpha^{(i)} = \textstyle\sum_{\gamma \subseteq \alpha, \gamma \in \mathcal{F}_d} n_\gamma^{(i)} \rangle_{\alpha \in \mathcal{F}} : 1 \leq i \leq t \}\,,$$

*and sets $E_1, E_2, \ldots, E_k \subseteq \{1, 2, \ldots, t\}$ satisfying conditions (1), (2), and (3) of Definition 3.5. Let $\alpha_1, \ldots, \alpha_s \in \mathcal{F}$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_s$. For $F \subseteq \{1, 2, \ldots, s\}$, $i \in \{1, 2, \ldots, k\}$ and $\varphi \in \mathcal{F}_d$ with $\varphi > \alpha_s$, and $1 \leq i \leq k$, let*

$$b_\varphi^{(i,F)} = \textstyle\sum_{\psi \subseteq \bigcup_{j \in F} \alpha_j,\, |\psi| \leq d - |\varphi|} \; m_{\varphi \cup \psi}^{(i)}\,.$$

*For $F \subseteq \{1, 2, \ldots, s\}$, $i \in \{1, 2, \ldots, k\}$, and $\beta \in \mathcal{F}_d$ with $\beta > \alpha_s$, let*

$$q_\beta^{(i,F)} = \textstyle\sum_{\varphi \subseteq \beta,\, \varphi \in \mathcal{F}_d} \; b_\varphi^{(i,F)}\,.$$

*Then* $\{\langle q_\beta^{(i,F)}\rangle_{\beta\in\mathcal{F},\beta>\alpha_s} : 1\le i\le k,\ F\subseteq\{1,2,\ldots,s\}\}$ *is an adequate set of VIP systems.*

**Proof**. Let $K=\bigcup_{i=1}^s \alpha_i$. For $\psi\subseteq K$, write $\mathrm{supp}(\psi)=\{i:\psi\cap\alpha_i\neq\emptyset\}$. For $T\subseteq\{1,2,\ldots,s\}$, $l\in\{1,2,\ldots,t\}$, and $\varphi\in\mathcal{F}_d$ with $\varphi>\alpha_s$, put

$$w_\varphi^{(l,T)}=\sum_{\psi\subseteq K,\,\mathrm{supp}(\psi)=T,\,|\psi|\le d-|\varphi|}\ n_{\varphi\cup\psi}^{(l)}\,.$$

For $\beta\in\mathcal{F}$ with $\beta>\alpha_s$, let $r_\beta^{(l,T)}=\sum_{\varphi\subseteq\beta,\,\varphi\in\mathcal{F}_d}\ w_\varphi^{(l,T)}$. For $i\in\{1,2,\ldots,k\}$ and $F\subseteq\{1,2,\ldots,s\}$, let $D_{i,F}=\{(l,T):l\in E_i \text{ and } F\subseteq\{1,2,\ldots,s\}\}$.

We claim that (with the finite set $\{1,2,\ldots,k\}\times\mathcal{P}(\{1,2,\ldots,s\})$ replacing $\{1,2,\ldots,k\}$ and the finite set $\{1,2,\ldots,t\}\times\mathcal{P}(\{1,2,\ldots,s\})$ replacing $\{1,2,\ldots,t\}$) the sets

$$\begin{array}{l}\{\langle b_\varphi^{(i,F)}\rangle_{\varphi\in\mathcal{F}_d,\,\varphi>\alpha_s} : i\in\{1,2,\ldots,k\} \text{ and } F\subseteq\{1,2,\ldots,s\}\}\,,\\ \{\langle r_\beta^{(l,T)}=\sum_{\varphi\subseteq\beta,\,\varphi\in\mathcal{F}_d}\ w_\varphi^{(l,T)}\rangle_{\beta\in\mathcal{F},\,\beta>\alpha_s} : l\in\{1,2,\ldots,t\} \text{ and } T\subseteq\{1,2,\ldots,s\}\}\,, \text{ and}\\ \{D_{i,F} : i\in\{1,2,\ldots,k\} \text{ and } F\subseteq\{1,2,\ldots,s\}\}\end{array}$$

satisfy conditions (1), (2), and (3) of Definition 3.5.

Condition (1) holds by the definition of $q_\beta^{(i,F)}$.

To verify condition (2), let $H\in\mathcal{P}_f(S)$ and pick $m\in\mathbb{N}$ such that for every $g\in\mathbb{N}$ and pairwise distinct $\gamma_1,\gamma_2,\ldots,\gamma_g\in\mathcal{F}_d$ with each $\gamma_j\not\subseteq\{1,2,\ldots,m\}$, $\sum_{l=1}^t\sum_{j=1}^g n_{\gamma_j}^{(l)}\in\sigma(H)\cup\{0\}$. Let $\varphi_1,\varphi_2,\ldots,\varphi_g$ be pairwise distinct members of $\mathcal{F}$ with each $\varphi_j\not\subseteq\{1,2,\ldots,m\}$ and each $\varphi_j>\alpha_s$. Then

$$\begin{array}{l}\sum_{l=1}^t\sum_{T\subseteq\{1,2,\ldots,s\}}\sum_{j=1}^g w_\varphi^{(l,T)} =\\ \sum_{l=1}^t\sum_{T\subseteq\{1,2,\ldots,s\}}\sum_{j=1}^g\sum_{\psi\subseteq K,\,\mathrm{supp}(\psi)=T,\,|\psi|\le d-|\varphi_j|} n_{\varphi_j\cup\psi}^{(l)}\,.\end{array}$$

It thus suffices to observe that if $T,T'\subseteq\{1,2,\ldots,s\}$, $j,j'\in\{1,2,\ldots,g\}$, $\psi,\psi'\subseteq K$, $\mathrm{supp}(\psi)=T$, $\mathrm{supp}(\psi')=T'$, $|\psi|\le d-|\varphi_j|$, $|\psi'|\le d-|\varphi_{j'}|$, and $(T,j,\psi)\neq(T',j',\psi')$, then $\varphi_j\cup\psi\neq\varphi_{j'}\cup\psi'$.

To verify condition (3), let $i\in\{1,2,\ldots,k\}$, let $F\subseteq\{1,2,\ldots,s\}$, and let $\varphi\in\mathcal{F}_d$ with $\varphi>\alpha_s$. Then

$$\begin{array}{rcl} b_\varphi^{(i,F)} &=& \sum_{\psi\subseteq\bigcup_{j\in F}\alpha_j,\,|\psi|\le d-|\varphi|}\ m_{\varphi\cup\psi}^{(i)}\\ &=& \sum_{\psi\subseteq\bigcup_{j\in F}\alpha_j,\,|\psi|\le d-|\varphi|}\sum_{l\in E_i} n_{\varphi\cup\psi}^{(l)}\\ &=& \sum_{T\subseteq F}\sum_{\psi\subseteq K,\,\mathrm{supp}(\psi)=T,\,|\psi|\le d-|\varphi|}\sum_{l\in E_i} n_{\varphi\cup\psi}^{(l)}\\ &=& \sum_{(l,T)\in D_{i,F}}\ w_\varphi^{(l,T)}\,.\end{array}$$

□

**3.11 Theorem**. *Let $(S,+)$ be a commutative adequate partial semigroup and let $C\subseteq S$ be a central set. Suppose that $\{\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}} : 1\le i\le k\}$ is an adequate set of VIP systems. Then there exist sequences $\langle a_n\rangle_{n=1}^\infty$ in $S$ and $\langle\alpha_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that $\alpha_n<\alpha_{n+1}$ for each*

*$n$ and for every $F \in \mathcal{F}$, if $\gamma = \bigcup_{t \in F} \alpha_t$, then*

$$\big\{ \textstyle\sum_{t \in F} a_t \big\} \cup \big\{ \textstyle\sum_{t \in F} a_t + v_\gamma^{(i)} : 1 \leq i \leq k \big\} \subseteq C \,.$$

**Proof**. Pick an idempotent $p \in K(\delta S)$ such that $C \in p$. Let $C^\star = \{x \in C : -x + C \in p\}$. Then for each $x \in C^\star$, $-x + C^\star \in p$ by Lemma 2.12.

Let

$$\mathcal{A} = \big\{ \{a, a + v_\alpha^{(1)}, a + v_\alpha^{(2)}, \ldots, a + v_\alpha^{(k)}\} : \alpha \in \mathcal{F} \text{ and } a \in \sigma(\{v_\alpha^{(1)}, v_\alpha^{(2)}, \ldots, v_\alpha^{(k)}\}) \big\} \,.$$

By Theorem 3.7, $\mathcal{A}$ is adequately partition regular and $\mathcal{A}$ is trivially shift invariant. Since $C^\star \in p$ and $p \in K(\delta S)$, $C^\star$ is piecewise syndetic. So pick by Theorem 3.8 some $a_1 \in S$ and $\alpha_1 \in \mathcal{F}$ such that $\{a_1, a_1 + v_{\alpha_1}^{(1)}, a_1 + v_{\alpha_1}^{(2)}, \ldots, a_1 + v_{\alpha_1}^{(k)}\} \subseteq C^\star$.

Inductively, let $n \in \mathbb{N}$ and assume that we have chosen $\langle a_t \rangle_{t=1}^n$ in $S$ and $\langle \alpha_t \rangle_{t=1}^n$ in $\mathcal{F}$ such that

(1) for $t \in \{1, 2, \ldots, n-1\}$, if any, $\alpha_t < \alpha_{t+1}$, and
(2) for $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$, if $\gamma = \bigcup_{t \in F} \alpha_t$, then $\sum_{t \in F} a_t \in C^\star$ and for each $i \in \{1, 2, \ldots, k\}$, $\sum_{t \in F} a_t + v_\gamma^{(i)} \in C^\star$.

For each $\gamma \in FU(\langle \alpha_t \rangle_{t=1}^n)$ and each $i \in \{1, 2, \ldots, k\}$, let

$$\langle q_\beta^{(i,\gamma)} \rangle_{\beta \in \mathcal{F}} = \langle v_{\gamma \cup \beta}^{(i)} - v_\gamma^{(i)} \rangle_{\beta \in \mathcal{F}, \beta > \alpha_n} \,.$$

By Theorem 3.10 the family

$$\big\{ \langle q_\beta^{(i,\gamma)} \rangle_{\beta \in \mathcal{F}, \beta > \alpha_t} : 1 \leq i \leq k,\ \gamma \in FU(\langle \alpha_t \rangle_{t=1}^n) \big\} \cup \big\{ \langle v_\beta^{(i)} \rangle_{\beta \in \mathcal{F}} : 1 \leq i \leq k \big\}$$

is an adequate set of VIP systems. Let

$$\begin{aligned} \mathcal{B} = \big\{ &\{a\} \cup \big\{ a + v_\alpha^{(i)} : i \in \{1, 2, \ldots, k\} \big\} \cup \textstyle\bigcup_{\gamma \in FU(\langle \alpha_t \rangle_{t=1}^n)} \big\{ a + q_\alpha^{(i,\gamma)} : i \in \{1, 2, \ldots, k\} \big\} : \\ &\alpha \in \mathcal{F},\ \alpha > \alpha_n, \text{ and } a \in \sigma(\{v_\alpha^{(i)} : 1 \leq i \leq k\} \cup \{q_\alpha^{(i,\gamma)} : 1 \leq i \leq k,\ \gamma \in FU(\langle \alpha_t \rangle_{t=1}^n)\}) \big\} \,. \end{aligned}$$

By Theorem 3.7, $\mathcal{B}$ is adequately partition regular. Let

$$\begin{aligned} D \;=\; & C^\star \cap \textstyle\bigcap \big\{ -\sum_{t \in H} a_t + C^\star : \emptyset \neq H \subseteq \{1, 2, \ldots, n\} \big\} \cap \\ & \textstyle\bigcap \{ -(\sum_{t \in H} a_t + v_\gamma^{(i)}) + C^\star : \emptyset \neq H \subseteq \{1, 2, \ldots, n\} \text{ and } \gamma = \bigcup_{t \in H} \alpha_t \} \,. \end{aligned}$$

Then $D \in p$ and so $D$ is piecewise syndetic.

Pick by Theorem 3.8 some $\alpha_{n+1} \in \mathcal{F}$ such that $\alpha_{n+1} > \alpha_n$ and some

$$a_{n+1} \in \sigma(\{v_{\alpha_{n+1}}^{(i)} : 1 \leq i \leq k\} \cup \{q_{\alpha_{n+1}}^{(i,\gamma)} : 1 \leq i \leq k \text{ and } \gamma \in FU(\langle al_t \rangle_{t=1}^n)\})$$

such that

$$\begin{aligned} &\{a_{n+1}\} \cup \big\{ a_{n+1} + v_{\alpha_{n+1}}^{(i)} : i \in \{1, 2, \ldots, k\} \big\} \cup \\ &\textstyle\bigcup_{\gamma \in FU(\langle \alpha_t \rangle_{t=1}^n)} \big\{ a_{n+1} + q_{\alpha_{n+1}}^{(i,\gamma)} : i \in \{1, 2, \ldots, k\} \big\} \\ &\subseteq D \,. \end{aligned}$$

Induction hypothesis (1) trivially holds. To verify (2), let $\emptyset \neq F \subseteq \{1,2,\ldots,n+1\}$ and let $\gamma = \bigcup_{t\in F} \alpha_t$. If $n+1 \notin F$, the conclusion holds by assumption. If $F = \{n+1\}$, then we have $\{a_{n+1}\} \cup \big\{a_{n+1} + v^{(i)}_{\alpha_{n+1}} : i \in \{1,2,\ldots,k\}\big\} \subseteq D \subseteq C^\star$.

So assume that $\{n+1\} \subsetneq F$, let $H = F\backslash\{n+1\}$, and let $\mu = \bigcup_{t\in F} \alpha_t$. Then $a_{n+1} \in D \subseteq -\sum_{t\in H} a_t + C^\star$ so $\sum_{t\in F} a_t \in C^\star$.

Let $\gamma = \bigcup_{t\in H} \alpha_t$ and let $i \in \{1,2,\ldots,k\}$. Then

$$a_{n+1} + q^{(i,\gamma)}_{\alpha_{n+1}} \in D \subseteq -(\sum_{t\in H} a_t + v^{(i)}_\gamma) + C^\star$$

and so $(\sum_{t\in H} a_t + v^{(i)}_\gamma) + (a_{n+1} + q^{(i,\gamma)}_{\alpha_{n+1}}) \in C^\star$. That is,

$$\textstyle\sum_{t\in F} a_t + v^{(i)}_\mu = \big(\sum_{t\in H} a_t + a_{n+1}\big) + \big(v^{(i)}_\gamma + q^{(i,\gamma)}_{\alpha_{n+1}}\big) \in C^\star\,. \qquad \square$$

Here is a shorter formulation of Theorem 3.11 that makes use of the IP ring terminology. A special case of the result (for groups, coloristically formulated) appears as [10, Theorem 2.8].

**3.12 Theorem**. *Let $(S,+)$ be a commutative adequate partial semigroup and let $C \subseteq S$ be a central set. Suppose that $\big\{\langle v^{(i)}_\alpha\rangle_{\alpha\in\mathcal{F}} : 1 \leq i \leq k\big\}$ is an adequate set of VIP systems. Then there exists an IP ring $\mathcal{F}^{(1)}$ and an IP system $\langle b_\alpha\rangle_{\alpha\in\mathcal{F}^{(1)}}$ in $S$ such that for all $\alpha \in \mathcal{F}^{(1)}$, $\{b_\alpha, b_\alpha + v^{(i)}_\alpha, \ldots, b_\alpha + v^{(k)}_\alpha\} \subseteq C$.*

**Proof**. Choose $\langle a_n\rangle_{n=1}^\infty$ and $\langle \alpha_n\rangle_{n=1}^\infty$ as in Theorem 3.11. Let $\mathcal{F}^{(1)} = FU\big(\langle\alpha_n\rangle_{n=1}^\infty\big)$. For $F \in \mathcal{F}$, let $\alpha = \bigcup_{t\in F} \alpha_t$ and put $b_\alpha = \sum_{t\in F} a_t$. $\square$

The proof of Theorem 3.11 actually gives a stronger conclusion, wherein at each stage of forming a sum one is allowed to choose a different value of $i$. Since it is not as clean to state, we formulate it separately.

**3.13 Theorem**. *Let $(S,+)$ be a commutative adequate partial semigroup and let $C \subseteq S$ be a central set. Suppose that $\big\{\langle v^{(i)}_\alpha\rangle_{\alpha\in\mathcal{F}} : 1 \leq i \leq k\big\}$ is an adequate set of VIP systems. Then there exist sequences $\langle a_n\rangle_{n=1}^\infty$ in $S$ and $\langle\alpha_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that $\alpha_n < \alpha_{n+1}$ for each $n$ and such that for every $F \in \mathcal{F}$, $\sum_{t\in F} a_t \in C$ and if $\beta_1 < \beta_2 < \ldots < \beta_s$, where each $\beta_j \subseteq F$, and $i_1, \ldots, i_s \in \{1,2,\ldots,k\}$, then writing $\gamma_j = \bigcup_{t\in\beta_j} \alpha_t$, for $j \in \{1,2,\ldots,s\}$, we have $\sum_{t\in F} a_t + \sum_{j=1}^s v^{(i_j)}_{\gamma_j} \in C$.*

**Proof**. Modify the proof of Theorem 3.11 as follows. First, replace induction hypothesis (2) by:

(2) for $\emptyset \neq F \subseteq \{1,2,\ldots,n\}$, $\sum_{t\in F}\ a_t \in C^\star$ and, if $\beta_1 < \beta_2 < \ldots < \beta_s$, where each $\beta_j \subseteq F$, and $i_1,\ldots,i_s \in \{1,2,\ldots,k\}$, and for $j \in \{1,2,\ldots,s\}$, $\gamma_j = \bigcup_{t\in\beta_j}\alpha_t$, then $\sum_{t\in F} a_t + \sum_{j=1}^s v_{\gamma_j}^{(i_j)} \in C^\star$.

Second, replace the set $D$ by

$$\begin{aligned} D \;=\; & C^\star \cap \bigcap\{-\textstyle\sum_{t\in H} a_t + C^\star : \emptyset \neq H \subseteq \{1,2,\ldots,n\}\}\cap \\ & \bigcap\{-(\textstyle\sum_{t\in H} a_t + \sum_{j=1}^s v_{\gamma_j}^{(i_j)}) + C^\star : \emptyset \neq H \subseteq \{1,2,\ldots,n\},\ s\in\mathbb{N}, \\ & \beta_1 < \beta_2 < \ldots < \beta_s,\ \textstyle\bigcup_{j=1}^s \beta_j \subseteq H, \text{ and for } j\in\{1,2,\ldots,s\},\ \gamma_j = \bigcup_{t\in\beta_j}\alpha_t\}\,. \end{aligned}$$

We leave the verification of the details of the proof to the reader. □

**3.14 Corollary**. *Let $(S,+)$ be a commutative cancellative semigroup, let $C$ be central in $S$, and let $\{\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}} : 1 \leq i \leq k\}$ be a set of weak VIP systems in $S$. Then there exist sequences $\langle a_n\rangle_{n=1}^\infty$ in $S$ and $\langle \alpha_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that $\alpha_n < \alpha_{n+1}$ for each $n$ and for every $F \in \mathcal{F}$ and every $i \in \{1,2,\ldots,k\}$, if $\gamma = \bigcup_{t\in F}\alpha_t$, then $\sum_{t\in F} a_t + v_\gamma^{(i)} \in C$.*

**Proof**. Let $G$ be the group of quotients of $S$. Then, with subtraction in $G$, we have $G = \{a - b : a, b \in S\}$.

We claim that $S$ is piecewise syndetic in $G$. That is, there exists $H \in \mathcal{P}_f(G)$ such that for each $F \in \mathcal{P}_f(G)$, there exists $x \in G$ such that $F + x \subseteq \bigcup_{t\in H}(-t + S)$. Indeed, let $H = \{0\}$ and let $F \in \mathcal{P}_f(G)$ be given. Pick $l \in \mathbb{N}$ and $a_1, a_2, \ldots, a_l, b_1, b_2, \ldots, b_l$ in $S$ such that $F = \{a_i - b_i : 1 \leq i \leq l\}$. Let $x = \sum_{i=1}^l b_i$. Then $F + x \subseteq S = -0 + S$. (We have in fact shown that $S$ is "thick" in $G$.)

Since $S$ is piecewise syndetic, $\overline{S} \cap K(\beta G) \neq \emptyset$ by [9, Theorem 4.40] and consequently $K(\beta S) = \overline{S} \cap K(\beta G)$ by [9, Theorem 1.65]. Since $C$ is central in $S$, by definition there is some idempotent $p \in K(\beta S)$ such that $C \in p$. But then $p \in K(\beta G)$ and thus $C$ is central in $G$.

Also, for each $i \in \{1,2,\ldots,k\}$, $\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}}$ is a weak VIP system in $S$ and is therefore a VIP system in $G$. Thus, $\{\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}} : 1 \leq i \leq k\}$ is an adequate set of VIP systems in $G$ so by Theorem 3.11, there exist sequences $\langle a_n\rangle_{n=1}^\infty$ in $G$ and $\langle \alpha_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that $\alpha_n < \alpha_{n+1}$ for each $n$ and for every $F \in \mathcal{F}$, if $\gamma = \bigcup_{t\in F}\alpha_t$, then

$$\{\textstyle\sum_{t\in F} a_t\} \cup \{\textstyle\sum_{t\in F} a_t + v_\gamma^{(i)} : 1 \leq i \leq k\} \subseteq C\,.$$

In particular, each $a_t$ is in $C \subseteq S$ so $\langle a_n\rangle_{n=1}^\infty$ is a sequence in $S$ as required. □

A stronger version of Corollary 3.14, based on Theorem 3.13, can also be proved in the same way.

Many variations on the theme of Theorem 3.11 are possible. As a matter of fact, one can formulate the following "VIP-free" version and prove it in nearly the same way.

**3.15 Theorem**. *Let $(S,+)$ be a commutative adequate partial semigroup, let $U$ be a set, and for each $s \in U$, let $T_s$ be a set. For each $s \in U$ and each $t \in T_s$, let $A_{s,t} \in \mathcal{P}_f(S)$, such that the family $\mathcal{A}_s = \{A_{s,t} : t \in T_s\}$ is shift invariant and adequately partition regular. Let $s_1 \in U$ and suppose $\phi : \bigcup_{s \in U}(\{s\} \times T_s) \to U$ is a function. If $C \subseteq S$ is a central set then there exist sequences $\langle s_n \rangle_{n=2}^{\infty}$ in $U$ and $\langle t_n \rangle_{n=1}^{\infty}$ with each $t_n \in T_{s_n}$ such that $\phi(s_{n-1}, t_{n-1}) = s_n$ for $n \geq 2$ and such that if $n_1 < \ldots < n_m$ and for each $i \in \{1, 2, \ldots, m\}$, $x_{n_i} \in A_{s_{n_i}, t_{n_i}}$, then $(x_{n_1} + x_{n_2} + \ldots + x_{n_m}) \in C$. (In particular, the sum is defined.)*

**Proof**. The proof of Theorem 3.11 needs to be modified as follows: Having chosen $\langle s_i \rangle_{i=1}^{n}$ and $\langle t_i \rangle_{i=1}^{n-1}$, replace the adequately partition regular family $\mathcal{B}$ constructed in the proof of Theorem 3.11 by $\mathcal{A}_{s_n}$ and replace the piecewise syndetic set $D$ by

$$\begin{aligned} D' \quad = \quad & C^\star \cap \bigcap \{-(x_{n_1} + x_{n_2} + \ldots + x_{n_m}) + C^\star : \\ & \qquad n_1 < n_2 < \ldots < n_m < n \text{ and each } x_{n_i} \in A_{s_{n_i}, t_{n_i}}\}. \end{aligned}$$

Then one chooses $t_n$ so that $A_{s_n, t_n} \subseteq D'$ and lets $s_{n+1} = \phi(s_n, t_n)$. □

## 4. Applications

In this section we shall give a few applications of Theorem 3.11. The first two are quite simple and will give some indication of where the result stands in relation to prior results. First, we show that Theorem 1.2 from the introduction can be obtained as a corollary of Theorem 3.11.

**Proof of Theorem 1.2**. We work in the semigroup $(\mathbb{N}, +)$. Without loss of generality, we will assume that one of the polynomials $p_i(x)$ is the zero polynomial. For $1 \leq i \leq k$, let $v_\alpha^{(i)} = p_i(n_\alpha)$ for $\alpha \in \mathcal{F}$. Then by Theorem 1.6, each $\langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{F}}$ is a VIP system. Therefore (since we are working in a semigroup, as opposed to a partial semigroup), the family $\{\langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{F}} : 1 \leq i \leq k\}$ is an adequate set of VIP systems. The result now follows immediately from the conclusion of Theorem 3.11. □

Next we observe that the central sets theorem for commutative semigroups [9, Theorem 14.11] is a consequence of Theorem 3.13. (The version stated in [9] is slightly more general than that given here, because it deals with infinitely many sequences. The corresponding version of Theorem 3.13 is routine to establish.)

**4.1 Theorem**. *Let $(S,+)$ be a commutative semigroup and let $C \subseteq S$ be a central set. Suppose that $\{\langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{F}} : 1 \leq i \leq k\}$ is a set of IP systems. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $S$ and $\langle \alpha_n \rangle_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\alpha_n < \alpha_{n+1}$ for each $n$ and such that for every $F \in \mathcal{F}$,*

*$\sum_{t\in F} a_t \in C$ and if $\beta_1 < \beta_2 < \ldots < \beta_s$, where each $\beta_j \subseteq F$, and $i_1, \ldots, i_s \in \{1, 2, \ldots, k\}$, then writing $\gamma_j = \bigcup_{t\in\beta_j} \alpha_t$, for $j \in \{1, 2, \ldots, s\}$, we have $\sum_{t\in F} a_t + \sum_{j=1}^{s} v_{\gamma_j}^{(i_j)} \in C$.*

**Proof**. In a commutative semigroup, any set of IP systems is an adequate set of VIP systems, so Theorem 3.13 applies directly. $\square$

Our next application is a proof of a version of a theorem of Carlson and Simpson [5, Theorem 6.3] which provides an infinitary extension of the Hales-Jewett Theorem [7]. Let $\Gamma$ be the free semigroup with identity on the alphabet $\{1, 2, \ldots, k\}$. That is, $\Gamma$ is the set of finite words $w = w_1 w_2 \cdots w_n$, with each $w_j \in \{1, 2, \ldots, k\}$, together with the empty word. A *variable word* is a word over the alphabet $\{1, 2, \ldots, k\} \cup \{v\}$ in which $v$ actually occurs, where $v$ is a "variable" not in $\{1, 2, \ldots, k\}$. Given a variable word $w(v)$ and some $i \in \{1, 2, \ldots, k\}$, of course $w(i)$ is the member of $\Gamma$ resulting from replacing each occurrence of $v$ with $i$. The Hales-Jewett Theorem is then the assertion that whenever $\Gamma$ is finitely colored, there must exist some variable word $w(v)$ such that $\{w(1), w(2), \ldots, w(k)\}$ is monochrome. (The Hales-Jewett Theorem is a generalization of van der Waerden's Theorem. This can be seen by mapping a word $w = w_1 w_2 \cdots w_n$ to $\sum_{i=1}^{n} w_i$.)

**4.2 Theorem**. *For any finite coloring of $\Gamma$, there exists a sequence $\langle w_n(v)\rangle_{n=1}^{\infty}$ of variable words such that the set of all finite products $w_{n_1}(i_1) w_{n_2}(i_2) \cdots w_{n_m}(i_m)$, where $n_1 < n_2 < \ldots < n_m$ and $\{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, k\}$, is monochromatic.*

**Proof**. Let $W = \mathcal{P}_f(\{0, 1, 2, \ldots, k-1\} \times \mathbb{N})$, and for $A, B \in W$, define $A + B = A \cup B$ if $A \cap B = \emptyset$ (otherwise $A + B$ is undefined). Then $(W, +)$ is a commutative adequate partial semigroup. Define a map $c : W \to \Gamma$ as follows. Given $A \in W$, pick $l \in \mathbb{N}$ such that $A \subseteq (\{1, 2, \ldots, k\} \times \{1, 2, \ldots, l\})$. For every $n \in \{1, 2, \ldots, l\}$ put $A_n = \big\{i \in \{0, 1, 2, \ldots, k-1\} : (i, n) \in A\big\}$. Next let $d_n = i$ if $A_n = \{i\}$, $i \in \{0, 1, 2, \ldots, k-1\}$. Otherwise, let $d_n = \emptyset$. Finally let $c(A)$ be the word $d_1 d_2 d_3 \cdots d_l$. Letting $\Gamma = \bigcup_{i=1}^{r} C_i$ be any finite partition, we have $W = \bigcup_{i=1}^{r} D_i$, where $D_i = c^{-1}[C_i]$. For some $j$, $D_j$ is central.

For $i \in \{0, 1, 2, \ldots, k-1\}$ and $\alpha \in \mathcal{F}$ put $v_\alpha^{(i)} = \{i\} \times \alpha$. Then each $\langle v_\alpha^{(i)}\rangle_{\alpha\in\mathcal{F}}$ is a VIP system (of degree 1) in $W$. Moreover one easily sees that these VIP systems form an adequate set (taking $u_\alpha^{(i)} = v_\alpha^{(i)}$ and $m_{\{j\}}^{(i)} = n_{\{j\}}^{(i)} = \{i, j\}$). Therefore, Theorem 3.13 provides sequences $\langle a_n\rangle_{n=1}^{\infty}$ in $W$ and $\langle \alpha_n\rangle_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\alpha_n < \alpha_{n+1}$ for each $n$ and such that for every $F \in \mathcal{F}$, $\sum_{t\in F} a_t \in D_j$ and, if $i_t \in \{1, 2, \ldots, k\}$ for each $t \in F$, then $\sum_{t\in F}(a_t + v_{\alpha_t}^{(i_t)}) \in D_j$.

For $A, B \in W$ write $A < B$ if there exists $N \in \mathbb{N}$ such that $A \subseteq \{0, 1, 2, \ldots, k-1\} \times \{1, 2, \ldots, N\}$ and $B \subseteq \{0, 1, 2, \ldots, k-1\} \times \{N+1, N+2, \ldots\}$. Since $\sum_{t\in F} a_t$ is always defined, we have that the $a_n$'s are pairwise disjoint, and consequently we may pass to a subsequence of indices $n$ and assume that $a_n < a_{n+1}$ and $a_n < v^{(i)}_{\alpha_{n+1}}$ for all $n \in \mathbb{N}$. This condition implies that for every $F \in \mathcal{F}$ and elements $i_t \in \{0, 1, 2, \dots, k-1\}$, $t \in F$,

$$\prod_{t\in F} c\big(a_t + v^{(i_t)}_{\alpha_t}\big) = c\Big(\sum_{t\in F}\big(a_t + v^{(i_t)}_{\alpha_t}\big)\Big) \in C_j.$$

For each $t \in \mathbb{N}$ and each $i \in \{1, 2, \dots, k\}$ we have that $a_t + v^{(i)}_{\alpha_t}$ is defined. Consequently, each $a_t \cap (\{1, 2, \dots, k\} \times \alpha) = \emptyset$.

Given $t \in \mathbb{N}$, define $w_t(v)$ as follows. Pick $l \in \mathbb{N}$ such that $\alpha_t \subseteq \{1, 2, \dots, l\}$ and $a_t \subseteq (\{1, 2, \dots, k\} \times \{1, 2, \dots, l\})$. For every $n \in \{1, 2, \dots, l\}$ put $A_{t,n} = \big\{i \in \{0, 1, 2, \dots, k-1\} : (i, n) \in a_t\big\}$. Next let $d_{t,n} = i$ if $A_{t,n} = \{i\}$, and let $d_{t,n} = v$ if $n \in \alpha_t$. Otherwise, let $d_n = \emptyset$. Then let $w_t(v) = d_{t,1} d_{t,2} \cdots d_{t,l}$. Then, for each $i \in \{1, 2, \dots, k\}$, $w_t(i) = c(a_t + v^{(i)}_{\alpha_t})$, so we are done. □

We shall prove in Theorem 4.4 an infinitary version of Theorem 1.3. For $A, B \in \mathcal{P}_f(\mathbb{N}^l)$, write $A + B = A \cup B$ if $A \cap B = \emptyset$ (otherwise $A + B$ is undefined). Then $(\mathcal{P}_f(\mathbb{N}^l), +)$ is a commutative adequate partial semigroup.

**4.3 Lemma**. *Let $l \in \mathbb{N}$ and let $\mathcal{P}$ be a finite family of set-polynomials over $(\mathcal{P}_f(\mathbb{N}^l), +)$ whose constant terms are empty. Then there exist $q \in \mathbb{N}$ and an IP ring $\mathcal{F}^{(1)} = \{\alpha \in \mathcal{F} : \min \alpha > q\}$ such that $\{\langle P(\alpha)\rangle_{\alpha\in\mathcal{F}^{(1)}} : P(X) \in \mathcal{P}\}$ is an adequate set of VIP systems.*

**Proof**. We first establish some notation. For $P(X) \in \mathcal{P}$, let $E_{P(X)} = \{Q(X) : Q(X)$ is a monomial summand of $P(X)\}$ and let $\mathcal{R} = \bigcup_{P(X)\in\mathcal{P}} E_{P(X)}$. Given $Q(X) \in \mathcal{R}$, write

$$Q(X) = S_1^{Q(X)} \times S_2^{Q(X)} \times \ldots \times S_l^{Q(X)}\,.$$

Let $D_{Q(X)} = \{j \in \{1, 2, \dots, l\} : S_j^{Q(X)} = X\}$. (Then $|D_{Q(X)}|$ is the degree of $Q(X)$.)

Let $q = \max\{i :$ there exists $Q(X) \in \mathcal{R}$ such that $\{i\}$ is a coordinate coefficient of $Q(X)\}$. (If the specified set is empty, that is if $\mathcal{P} = \{X \times X \times \ldots \times X\}$, let $q = 1$.) Let $\mathcal{F}^{(1)} = \{\alpha \in \mathcal{F} : \min \alpha > q\}$. Let $d$ be the largest degree of members of $\mathcal{P}$ (which is the same as the largest degree of members of $\mathcal{R}$). For $\gamma \in \mathcal{F}^{(1)}_d = \{\alpha \in \mathcal{F}^{(1)} : |\alpha| \le d\}$ and $Q(X) \in \mathcal{R}$, let

$$n_\gamma^{Q(X)} = \{(x_1, x_2, \dots, x_l) \in Q(\gamma) : \{x_j : j \in D_{Q(X)}\} = \gamma\}\,.$$

(Notice that if $|\gamma|$ is greater than the degree of $Q(X)$, then $n_\gamma^{Q(X)} = \emptyset$.)

Next observe that if $n_\gamma^{Q(X)} \ne \emptyset$, then both $\gamma$ and $Q(X)$ are uniquely determined by any member of $n_\gamma^{Q(X)}$. Indeed, if $(x_1, x_2, \dots, x_l) \in n_\gamma^{Q(X)}$, then

$$D_{Q(X)} = \{j \in \{1, 2, \dots, l\} : x_j > q\}\,, \gamma = \{x_j : j \in D_{Q(X)}\}\,,$$

and for $j \in \{1,2,\ldots,l\}\backslash D_{Q(X)}$, if any, $S_j^{Q(X)} = \{x_j\}$. Consequently, if $Q_1(X), Q_2(X) \in \mathcal{R}$, $\gamma_1, \gamma_2 \in \mathcal{F}_d^{(1)}$, and $n_{\gamma_1}^{Q_1(X)} \cap n_{\gamma_2}^{Q_2(X)} \neq \emptyset$, then $Q_1(X) = Q_2(X)$ and $\gamma_1 = \gamma_2$. This fact tells us that all sums of distinct terms of the form $n_\gamma^{Q(X)}$ are defined.

Now for $\gamma \in \mathcal{F}_d^{(1)}$ and $P(X) \in \mathcal{P}$, let $m_\gamma^{P(X)} = \sum_{Q(X)\in E_{P(X)}} n_\gamma^{Q(X)}$. Then condition (3) of Definition 3.5 is satisfied directly.

We show next that for $\alpha \in \mathcal{F}_d^{(1)}$ and $Q(X) \in \mathcal{R}$, $Q(\alpha) = \sum_{\gamma\in\mathcal{F}_d^{(1)},\gamma\subseteq\alpha} n_\gamma^{Q(X)}$. Given $\gamma \in \mathcal{F}_d^{(1)}$ with $\gamma \subseteq \alpha$, $n_\gamma^{Q(X)} \subseteq Q(\gamma) \subseteq Q(\alpha)$. For the reverse inclusion, let $(x_1, x_2, \ldots, x_l) \in Q(\alpha)$ and let $\gamma = \{x_i : i \in \{1,2,\ldots,l\}$ and $x_i > q\}$. Then $(x_1, x_2, \ldots, x_l) \in n_\gamma^{Q(X)}$.

It thus remains to establish conditions (1) and (2) of Definition 3.5. For condition (1), let $P(X) \in \mathcal{P}$ and let $\alpha \in \mathcal{F}^{(1)}$. Then

$$\begin{aligned} P(\alpha) &= \textstyle\sum_{Q(X)\in E_{P(X)}} Q(\alpha) \\ &= \textstyle\sum_{Q(X)\in E_{P(X)}} \sum_{\gamma\in\mathcal{F}_d^{(1)},\gamma\subseteq\alpha} n_\gamma^{Q(X)} \\ &= \textstyle\sum_{\gamma\in\mathcal{F}_d^{(1)},\gamma\subseteq\alpha} \sum_{Q(X)\in E_{P(X)}} n_\gamma^{Q(X)} \\ &= \textstyle\sum_{\gamma\in\mathcal{F}_d^{(1)},\gamma\subseteq\alpha} m_\gamma^{P(X)} . \end{aligned}$$

Finally, to establish condition (2), let $H \in \mathcal{P}_f\big(\mathcal{P}_f(\mathbb{N}^l)\big)$. Pick $m \geq q$ such that for all $A \in H$, $A \subseteq \{1,2,\ldots,m\}^l$. Let $p \in \mathbb{N}$ and let $\gamma_1, \gamma_2, \ldots, \gamma_p$ be pairwise distinct elements of $\mathcal{F}^{(1)}$ with each $\gamma_i \not\subseteq \{1,2,\ldots,m\}$. We need to show that $\sum_{Q(X)\in\mathcal{R}} \sum_{j=1}^p n_{\gamma_j}^{Q(X)} \in \sigma(H) \cup \{\emptyset\}$. (We have already observed that the sum is defined.) That is, we need to see that for each $A \in H$, $A \cap \sum_{Q(X)\in\mathcal{R}} \sum_{j=1}^p n_{\gamma_j}^{Q(X)} = \emptyset$. Suppose instead that we have $Q(X) \in \mathcal{R}$, $j \in \{1,2,\ldots,p\}$, and $(x_1, x_2, \ldots, x_l) \in A \cap n_{\gamma_j}^{Q(X)}$. Since $A \subseteq \{1,2,\ldots,m\}^l$ and $\gamma_j \subseteq \{x_1, x_2, \ldots, x_l\}$ one has $\gamma_j \subseteq \{1,2,\ldots,m\}$, a contradiction. $\square$

**4.4 Theorem**. *Let $l \in \mathbb{N}$ and let $\mathcal{P}$ be a finite family of set-polynomials over $(\mathcal{P}_f(\mathbb{N}^l), +)$ whose constant terms are empty. If $D \subseteq \mathcal{P}_f(\mathbb{N}^l)$ is a central set then there exist sequences $\langle A_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N}^l)$ and $\langle \alpha_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that $\alpha_n < \alpha_{n+1}$ for each $n$ and for every $F \in \mathcal{F}$ we have $\{A_\gamma\} \cup \{A_\gamma + P(\gamma) : P \in \mathcal{P}\} \subseteq C$, where $\gamma = \bigcup_{t\in F} \alpha_t$ and $A_\gamma = \sum_{t\in F} A_t$.*

**Proof**. By Lemma 4.3 there is an IP ring $\mathcal{F}^{(1)}$ such that $\{\langle P(\alpha)\rangle_{\alpha\in\mathcal{F}^{(1)}} : P(X) \in \mathcal{P}\}$ is an adequate set of VIP systems. Thus Theorem 3.11 applies. $\square$

Let us consider now the "finite unions" formulation of the finite sums theorem: for any finite partition of an IP ring $\mathcal{F}^{(1)}$, there exists a monochromatic IP ring $\mathcal{F}^{(2)}$. (See [9, Corollary 5.17].) Thus IP rings have a sort of "chromatic indestructability" property, and indeed this property embodies completely the finite sums (or finite unions) theorem.

Several natural multidimensional analogs of IP rings fail to have the chromatic indestructability property. For example, $(\mathcal{F}^{(1)})^2$, where $\mathcal{F}^{(1)}$ is an IP ring, and the finite unions of a 2-dimensional lattice of sets $FU(\langle A_i \times A_j \rangle_{i,j\in\mathbb{N}})$, where the $A_i$ are pairwise disjoint subsets of $\mathbb{N}$, are objects which can be finitely partitioned in such a way that no cell of the partition contains an object of the same kind. The Milliken-Taylor theorem states that for any IP ring $\mathcal{F}^{(1)}$ and any finite partition of the object $\{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i \in \mathcal{F}^{(1)}, \alpha_1 < \alpha_2 < \ldots < \alpha_n\}$, there exists an IP ring $\mathcal{F}^{(2)}$ such that $\{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i \in \mathcal{F}^{(2)}, \alpha_1 < \alpha_2 < \ldots < \alpha_n\}$ is monochromatic. ([12, Theorem 2.2] and [13, Lemma 2.2], or see [9, Corollary 18.9].) In restricting to such a special class of $n$-tuples, it can be argued that this theorem is merely "almost multidimensional".

Part of the difficulty seems to be that the finite sums theorem is *projective* rather than *affine*; in particular the configurations it guarantees are not shift invariant. Results such as the central sets theorem have both a projective and affine component. It seems unlikely that the projective portion of the central sets theorem can be meaningfully "polynomialized". The affine portion however can be, as Theorem 4.4 attests. In our final application, we offer a different "almost multidimensional" version of the finite sums theorem by defining a partition regular class of structures vaguely similar to 2-dimensional lattices of sets.

Suppose we are given pairwise disjoint sequences $\langle A_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N}^2)$ and $\langle B_n \rangle_{n=1}^{\infty}$ in $\mathcal{F}$ with the additional property that $A_n \cap (B_i \times B_j) = \emptyset$ for all $i, j, n \in \mathbb{N}$. Next, for $E \in \mathcal{P}_f(\mathbb{N}^2)$, let $\mathrm{supp}(E) = \{x \in \mathbb{N} : \text{there exists } y \in \mathbb{N} \text{ such that } (x, y) \in E \text{ or } (y, x) \in E\}$. Then $\mathrm{supp}(E)$ is the smallest set $B$ such that $E \subseteq B^2$. The family of sets

$$\{\bigcup_{n\in\mathrm{supp}(E)} A_n \cup \bigcup_{(i,j)\in E} B_i \times B_j : E \in \mathcal{P}_f(\mathbb{N}^2)\}$$

will be called an *affine* 2*-ring*. Notice the similarity between affine 2-rings and the finite unions of a lattice of sets mentioned earlier. However, affine 2-rings have the advantage of partition regularity.

**4.5 Theorem**. *For any finite partition of an affine* 2*-ring* $\mathcal{A}_1$*, there exists a monochromatic affine* 2*-ring* $\mathcal{A}_2$*.*

**Proof**. Let $S = \mathcal{P}_f(\mathbb{N}^2)$ and denote disjoint union on $S$ by $+$. Then $(S, +)$ is an adequate partial semigroup. Let

$$\mathcal{A}_1 = \{\bigcup_{n\in\mathrm{supp}(E)} A_n \cup \bigcup_{(i,j)\in E} B_i \times B_j : E \in S\} = \bigcup_{i=1}^{r} C_i \,.$$

For $E \in S$ let

$$\Gamma(E) = \bigcup_{n\in\mathrm{supp}(E)} A_n \cup \bigcup_{(i,j)\in E} (B_i \times B_j)\,,$$

and put $E \in D_i$ if and only if $\Gamma(E) \in C_i$. Then $S = \bigcup_{i=1}^{r} D_i$, so one of the cells, say $D_l$, is central.

We claim there exist sequences $\langle a_n\rangle_{n=1}^\infty$ in $S$ and $\langle \alpha_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that

$$\mathcal{A} = \{\textstyle\sum_{n\in\mathrm{supp}(E)} a_n + \sum_{(i,j)\in E}(\alpha_i \times \alpha_j) : E \in S\} \subseteq D_l.$$

(In particular, so that all such sums exist. That is, the sequences $\langle a_n\rangle_{n=1}^\infty$ and $\langle \alpha_n\rangle_{n=1}^\infty$ are pairwise disjoint and $a_n \cap (\alpha_i \times \alpha_j) = \emptyset$ for all $i, j, n \in \mathbb{N}$.)

Having established the claim, $\Gamma(\mathcal{A}) = \mathcal{A}_2$ will be contained in $C_l$. It is routine, though admittedly somewhat tedious, to verify that $\mathcal{A}_2$ is the affine 2-ring generated by the sequences $\langle A'_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N}^2)$ and $\langle B'_n\rangle_{n=1}^\infty$ in $\mathcal{F}$, where

$$A'_n = \textstyle\bigcup_{k\in\mathrm{supp}(a_n)\cup\alpha_n} A_k \cup \bigcup_{(i,j)\in a_n}(B_i \times B_j) \text{ and } B'_n = \bigcup_{k\in\alpha_n} B_k\,.$$

The claim may be obtained from Theorem 3.15 as follows. Let

$$U = \{\emptyset\}\cup\{(\alpha_1, \alpha_2, \ldots, \alpha_s) : k \in \mathbb{N},\ \alpha_i \in \mathcal{F} \text{ for } i \in \{1, 2, \ldots, k\}, \text{ and } \alpha_i\cap\alpha_j = \emptyset \text{ if } i \neq j\}\,.$$

Let $\mathcal{R}_\emptyset = \{\emptyset, X \times X\}$ and for $k \in \mathbb{N}$ and $s = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in U$, let

$$\mathcal{R}_s = FU(\langle X \times X, \alpha_1 \times X, \ldots, \alpha_k \times X, X \times \alpha_1, \ldots, X \times \alpha_k\rangle) \cup \{\emptyset\}$$

so that $\mathcal{R}_s$ is a family of $2^{2k+1}$ set polynomials.

For each $s \in U$, pick by Lemma 4.3 some $q_s \in \mathbb{N}$ and an IP ring $\mathcal{F}^{(s)} = \{\alpha \in \mathcal{F} : \min\alpha > q_s\}$ such that $\{\langle P(\alpha)\rangle_{\alpha\in\mathcal{F}^{(s)}} : P \in \mathcal{R}_s\}$ is an adequate set of VIP systems. If $k \in \mathbb{N}$ and $s = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we may presume that $q_s > \max\alpha_i$ for each $i \in \{1, 2, \ldots, k\}$. For each $s \in U$, let $T_s = \{(A, \alpha) : \alpha \in \mathcal{F}^{(s)}, A \in S$, and for each $P \in \mathcal{R}_s$, $A \cap P(\alpha) = \emptyset\}$.

For $s \in U$ and $t = (A, \alpha) \in T_s$, let $A_{s,t} = \{A + P(\alpha) : P \in \mathcal{R}_s\}$ and let $\mathcal{A}_s = \{A_{s,t} : t \in T_s\}$. By Theorem 3.7, we have that

$$\mathcal{A}_s = \big\{\{A + P(\alpha) : P \in \mathcal{R}_s\,, \alpha \in \mathcal{F}\,, \min\alpha > q_s\,, \text{ and } A \cap P(\alpha) = \emptyset \text{ for each } P \in \mathcal{R}_s\}$$

is adequately partition regular.

Define $\phi : \bigcup_{s\in U}(\{s\} \times T_s) \to U$ by $\phi(\emptyset, t) = \phi\big(\emptyset, (A, \alpha)\big) = (\alpha)$ and $\phi(s, t) = \phi\big((\alpha_1, \alpha_2, \ldots, \alpha_k), (A, \alpha)\big) = (\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha)$.

Let $s_1 = \emptyset$. By Theorem 3.15, pick sequences $\langle s_n\rangle_{n=2}^\infty$ in $U$ and $\langle t_n\rangle_{n=1}^\infty$, with each $t_n = (a_n, \alpha_n) \in T_{s_n}$, such that $\phi(s_{n-1}, t_{n-1}) = s_n$ for $n \geq 2$ and such that if $n_1 < \ldots < n_m$ and for each $i \in \{1, 2, \ldots, m\}$, $x_{n_i} \in A_{s_{n_i}, t_{n_i}}$, then $x_{n_1} + x_{n_2} + \ldots + x_{n_m} \in D_l$.

One may establish inductively that $s_{n+1} = (\alpha_1, \ldots, \alpha_n)$ for $n \in \mathbb{N}$, so that

$$\begin{aligned} A_{s_n,t_n} =& \{a_n + P(\alpha_n) : P \in \mathcal{R}_{s_n}\} \\ =& \{a_n\} \cup a_n + FS(\langle \alpha_n \times \alpha_n, \alpha_n \times \alpha_1, \ldots, \alpha_n \times \alpha_{n-1}, \alpha_1 \times \alpha_n, \ldots, \alpha_{n-1} \times \alpha_n\rangle)\,. \end{aligned}$$

Now to establish the claim, let $E \in S$ be given and let $\mathrm{supp}(E) = \{n_1, n_2, \ldots, n_m\}$ where $n_1 < n_2 < \ldots < n_m$. For each $i \in \{1, 2, \ldots, m\}$, let

$$x_{n_i} = a_{n_i} + \sum_{k \leq n_i, (k, n_i) \in E} (\alpha_k \times \alpha_{n_i}) + \sum_{k < n_i, (n_i, k) \in E} (\alpha_{n_i} \times \alpha_k)$$

and note that $x_{n_i} \in A_{s_{n_i}, t_{n_i}}$.

Then

$$\sum_{n \in \mathrm{supp}(E)} a_n + \sum_{(i,j) \in E} (\alpha_i \times \alpha_j) = x_{n_1} + x_{n_2} + \ldots + x_{n_m} \in D_l\,. \qquad \square$$

Theorem 4.5 is related to some results in spaces of matrices obtained in [11].

## References


[1] V. Bergelson, A. Blass, and N. Hindman, Partition theorems for spaces of variable words, *Proc. London Math. Soc.* **68** (1994), 449-476.

[2] V. Bergelson, H. Furstenberg and R. McCutcheon, IP sets and polynomial recurrence, *Ergodic Theory and Dynamical Systems* **16** (1996) 963-974.

[3] V. Bergelson and A. Leibman, Polynomial extensions of van der Waerden's and Szemerédi's theorems, *Journal Amer. Math. Soc.* **9** (1996) 725-753.

[4] V. Bergelson and A. Leibman, Set polynomials and a polynomial extension of Hales-Jewett theorem, *Annals of Math.* **150** (1999), 33-75.

[5] T. Carlson and S. Simpson, A dual form of Ramsey's theorem, *Adv. in Math.* **53** (1984) 265-290.

[6] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

[7] A. Hales and R. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222-229.

[8] N. Hindman, Problems and new results on the algebra of $\beta\mathbb{N}$ and its application to Ramsey Theory, in *Unsolved problems in mathematics for the* $21^{\mathrm{st}}$ *century – a tribute to Kiyoshi Iséki's* $80^{\mathrm{th}}$ *birthday*, J. Abe and S. Tanaka, eds., IOS Press, Amsterdam, to appear.

[9] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification–theory and applications*, Walter de Gruyter, Berlin, 1998.

[10] R. McCutcheon, An infinitary polynomial van der Waerden theorem, *J. Combinatorial Theory* (Series A) **86** (1999), 214-231.

[11] R. McCutcheon, An infinitary polynomial Hales-Jewett theorem, *Israel J. Math.*, to appear.

[12] K. Milliken, Ramsey's Theorem with sums or unions, *J. Comb. Theory* (Series A) **18** (1975), 276-290.

[13] A. Taylor, A canonical partition relation for finite subsets of $\omega$, *J. Comb. Theory* (Series A) **21** (1976), 137-146.

Neil Hindman
Department of Mathematics
Howard University
Washington, DC 20059
`nhindman@fac.howard.edu`
`nhindman@aol.com`
`http://members.aol.com/nhindman/`

Randall McCutcheon
Department of Mathematics
University of Maryland
College Park, MD 20742
`randall@math.umd.edu`