This paper was published in *J. Combinatorics* **4** (2013), 81-93. To the best of my knowledge, this is the final version as it was submitted to the publisher. –NH

Some new additive and multiplicative Ramsey numbers

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For $a, r \in \mathbb{N}$, the set of positive integers, define $FSP_2(a, r)$ (respectively $SP_2(a, r)$) to be the first $n \in \mathbb{N}$, if such exists, such that whenever $\{1, 2, \ldots, n\}$ is r-colored, there exist x and y with $a \leq x < y$ such that $\{x, y, x + y, xy\}$ is monochromatic (respectively $\{x + y, xy\}$ is monochromatic). If no such n exists, the number is defined to be infinite. It is an old result of R. Graham that $SP_2(a, 2)$ is finite for all a. With that exception, the only cases (with $r > 1$) for which $FSP_2(a, r)$ or $SP_2(a, r)$ are known to be finite are those for which explicit values have been computed. In this paper we provide exact values of $FSP_2(a, 2)$ for $a \leq 5$ (of which $FSP_2(1, 2)$ and $FSP_2(2, 2)$ were previously known). We provide exact values of $SP_2(a, 3)$ for $a \leq 9$ and exact values of $SP_2(a, 2)$ for $a \leq 105$. We also compute upper and lower bounds for $SP_2(a, 2)$.

1. Introduction

In [1, Theorem 2.6] it was shown that if $r \in \mathbb{N}$ and $\mathbb{N}$ is r-colored, there exist sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty})$ is monochromatic. Here $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$ and $FP(\langle y_n \rangle_{n=1}^{\infty}) = \{\prod_{t \in F} y_t : F \in \mathcal{P}_f(\mathbb{N})\}$, where $\mathcal{P}_f(\mathbb{N})$ is the set of finite nonempty subsets of $\mathbb{N}$. At that time, it was not known whether one could always choose one sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle x_n \rangle_{n=1}^{\infty})$ monochromatic. However, it was shown in [3] that there is a 7-coloring of $\mathbb{N}$ such that there is no sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $PS(\langle x_n \rangle_{n=1}^{\infty}) \cup PP(\langle x_n \rangle_{n=1}^{\infty})$ monochromatic, where $PS(\langle x_n \rangle_{n=1}^{\infty}) = \{x_n + x_m : n \neq m\}$ and $PP(\langle x_n \rangle_{n=1}^{\infty}) = \{x_n x_m : n \neq m\}$.

*This author acknowledges support received from the National Science Foundation via Grant DMS-1160566.

2010 *Mathematics Subject Classification*. Primary 05D10

Given a finite sequence $\langle x_t\rangle_{t=1}^k$, the notations $FS(\langle x_t\rangle_{t=1}^k)$, $FP(\langle x_t\rangle_{t=1}^k)$, $PS(\langle x_t\rangle_{t=1}^k)$, and $PP(\langle x_t\rangle_{t=1}^k)$ have their obvious meanings. The first author of this paper has maintained for a few decades that the following is a fact.

Let $r, k \in \mathbb{N}$ and let $\mathbb{N}$ be r-colored. There exists a sequence $\langle x_t\rangle_{t=1}^k$ such that $FS(\langle x_t\rangle_{t=1}^k) \cup FP(\langle x_t\rangle_{t=1}^k)$ is monochromatic.

It should be understood that he has not claimed that he can prove this fact. The only case (with $r > 1$) with a proof is $r = k = 2$. And this is a computer generated result. R. Graham sometime in the mid 1970's used a computer program to verify that if $\{1, 2, \ldots, 252\}$ is 2-colored, then there exist $x \neq y$ such that $\{x, y, x + y, xy\}$ is monochromatic, and that the corresponding statement fails for $\{1, 2, \ldots, 251\}$. He noted that the fact that $x = 1$ is allowed is crucial because, if y and 1 are the same color, then $y + 1$ must be the opposite color. Accordingly, one is led to define $FSP_2(a, r)$ as in the abstract, and ask whether one can establish that $FSP_2(a, r)$ is finite for all a and r.

In [1] the fact that $FSP_2(1, 2) = 252$ was verified and it was established that $FSP_2(2, 2) = 990$. These results were obtained using a Fortran program on the IBM mainframe computer in use at SUNY Binghamton in 1976. The result for $FSP_2(1, 2)$ took approximately two seconds of computer time and the result for $FSP_2(2, 2)$ took approximately four minutes of computer time. The author of [1] did not try to find $FSP_2(3, 2)$, feeling that the amount of computer time required would not be feasible. In 2012, running a Pascal program on a desktop computer with an Intel Cor2Duo CPU E8400 processor operating at 3.00 GHz, the result for $FSP_2(2, 2)$ took less than 2 seconds of computer time. And the fact that $FSP_2(3, 2) = 3150$ was established using less than 52 seconds of computer time. We no longer have the program used in Binghamton, and we were able to significantly improve the efficiency of the current program by some modifications, but we suspect that the main difference in time consumption comes from improvement in processor speeds.

In Section 2 of this paper we discuss the basic coloring algorithm used to compute $FSP_2(a, 2)$, $SP_2(a, 2)$, and $SP_2(a, 3)$ as well as the limits on extending the results to higher values of a.

In the case of $SP_2(a, 2)$ we have a good deal of additional information. As we mentioned in the abstract, R. Graham proved that $SP_2(a, 2)$ is finite for all a. (This proof was presented with his permission as [2, Theorem 3.3].) We have computed the upper bounds that are provided by this original proof, and also have computed a very slight improvement. Both of these bounds are on the order of $\frac{8}{9}a^4$. We also prove a lower bound on the order of a^3. These results are presented in Section 3.

We conclude this introduction by presenting the exact numbers that we have obtained. As we have remarked, the fact that $FSP_2(1,2) = 252$ is due to R. Graham. The fact that $SP_2(1,3) = 100$ was established by D. Tang in [4], where a detailed proof that any 3-coloring of $\{1, 2, \ldots, 100\}$ has a monochromatic $\{x + y, xy\}$ with $x < y$ was presented.

a	$FSP_2(a, 2)$
1	252
2	990
3	3150
4	5600
5	14364

Table 1: Values of $FSP_2(a, 2)$

a	$SP_2(a, 3)$
1	100
2	216
3	774
4	3504
5	11100
6	28260
7	62034
8	122304
9	222264

Table 2: Values of $SP_2(a, 3)$

2. The coloring algorithm

Flow charts for the basic coloring algorithm used to compute values for $FSP_2(a, 2)$, $SP_2(a, 2)$, and $SP_2(a, 3)$ are in Figures 1, 2, 3, and either 4 (for $FSP_2(a, 2)$ and $SP_2(a, 2)$) or 5 (for $SP_2(a, 3)$). For $SP_2(a, r)$ one is dealing with a graph with vertex set $V = \{a, a + 1, \ldots, \max\}$ and edges of the form $\{x + y, xy\}$ where $a \leq x < y$. If one can r-color V with no

a	$SP_2(a,2)$	a	$SP_2(a,2)$	a	$SP_2(a,2)$
1	8	36	68688	71	514182
2	24	37	80771	72	523584
3	54	38	83752	73	554216
4	128	39	86697	74	564028
5	250	40	96000	75	573750
6	432	41	109265	76	583376
7	686	42	112896	77	592900
8	1024	43	116487	78	638820
9	1377	44	120032	79	661546
10	1800	45	125550	80	672000
11	2662	46	152352	81	721710
12	3168	47	156839	82	766536
13	3887	48	161280	83	778457
14	4312	49	180075	84	790272
15	5625	50	185000	85	867000
16	6912	51	189873	86	880124
17	8959	52	205504	87	893142
18	9720	53	210675	88	906048
19	11552	54	215784	89	926757
20	12400	55	235950	90	939600
21	14553	56	241472	91	1010282
22	16456	57	272916	92	1100320
23	20102	58	292668	93	1115721
24	21312	59	299366	94	1139844
25	26875	60	306000	95	1155200
26	28392	61	320006	96	1170432
27	29889	62	326740	97	1213761
28	32928	63	333396	98	1229312
29	38686	64	360448	99	1244727
30	40500	65	367575	100	1280000
31	48050	66	400752	101	1366934
32	50176	67	430944	102	1383732
33	52272	68	439280	103	1495869
34	60112	69	447534	104	1514240
35	66150	70	455700	105	1543500

Table 3: Values of $SP_2(a,2)$

monochromatic edges, then $SP_2(a, r) > \max$. In discussing $FSP_2(a, 2)$ we will use hypergraph terminology, so that an "edge" is a set of the form $\{x, y, x + y, xy\}$ where $a \le x < y$.

For each of the calculations one inputs the numbers a and m. If there is an edge-free coloring of V the algorithm will find one. If there is none, the algorithm amounts to a proof by cases that no such coloring exists.

The "preliminary calculations" referred to in Figure 1 involve such things as computing the degrees of vertices and sorting so that the higher degree vertices come first. Here also some tables may be computed that allow one to quickly find which vertices a given vertex is connected to. In the case of the computation of $SP_2(a, 3)$, one enters by hand the vertices of a triangle, assigning each such vertex to one color. For example, when $a = 3$, the values 32, 87, and 252 are vertices of a triangle. ($32 = 3 + 29$ and $87 = 3 \cdot 29$ so $\{32, 87\}$ is an edge. $32 = 14 + 18$ and $252 = 14 \cdot 18$ so $\{32, 252\}$ is an edge. $87 = 3+84$ and $252 = 3\cdot 84$ so $\{87, 252\}$ is an edge.) Of these, only $\{32, 252\}$ remains an edge when $a = 4$. The smallest triangles for $a = 3$ through 9 are given in Table 4.

a	Triangle
4	{59, 220, 864}
5	{94,445,2200}
6	{137,786,4680}
7	{188,1267,8820}
8	{247,1912,15232}
9	{314,2745,24624}

Table 4: Smallest Triangles

There are three variables and three arrays that are updated throughout the program. The array entry `assgclr[i]` is 0 if `i` has not been assigned to a color, and otherwise is the number of the color to which `i` has been assigned; `assglist[i]` is the `i`$^{\text{th}}$ vertex assigned; and `listlevel[i]` is the address in `assglist` of the first item assigned when `level=i`. The variable `level` records the number of free assignments made; `listtop` is the address in `assglist` of the last assignment made; and `checktop` is the address in `assglist` of the last entry which has been checked for forcing or contradictions resulting from its assignment.

In Figure 2 the "first" unassigned is one of the highest degree vertices which have not been assigned. One such is assigned to color 1. Then, as de-

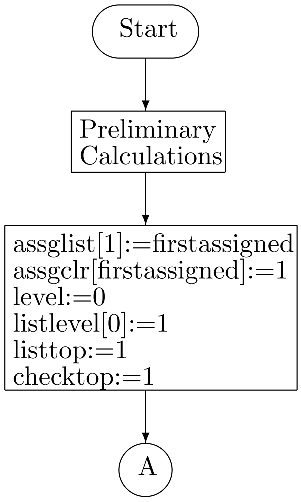

Figure 1: Start of Coloring Algorithm

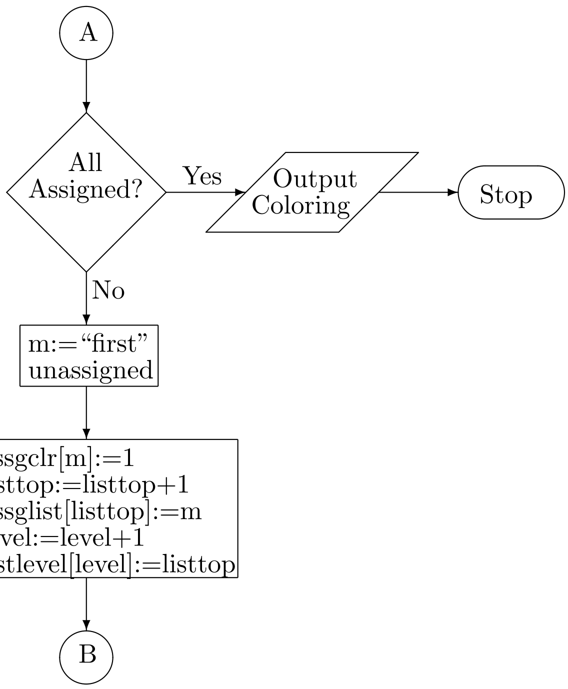

Figure 2: Free Assignments

scribed in Figure 3 one checks whether this assignment forces a monochromatic edge. If not, one finds all vertices which are forced to some other color (called "kolor" in Figure 3). The process is iterated to see what is forced by each of the new assignments.

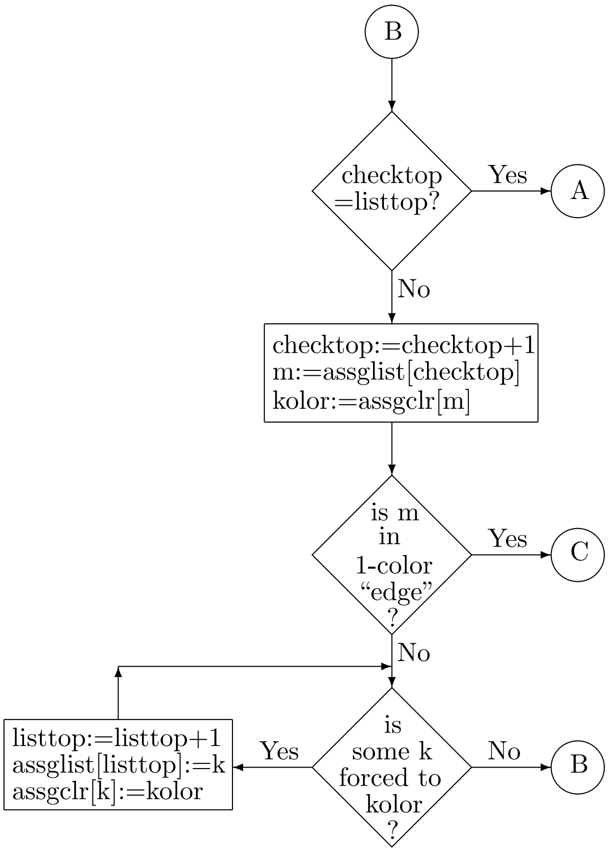

Figure 3: Forcing

If there is a monochromatic edge, then one proceeds to the "contradiction" segment. What is done here depends on whether one is considering 2-colorings (Figure 4) or 3-colorings (Figure 5). In either case, let `k` be the vertex which was freely assigned at the current value of `level` and let `color` be the color to which it was assigned. In the case of 2-colorings, necessarily `color` $= 1$ and, since this assignment led to a contradiction, `k` is forced to color 2 at the previous value of `level` (unless `level` $= 0$, in which case no good coloring is possible).

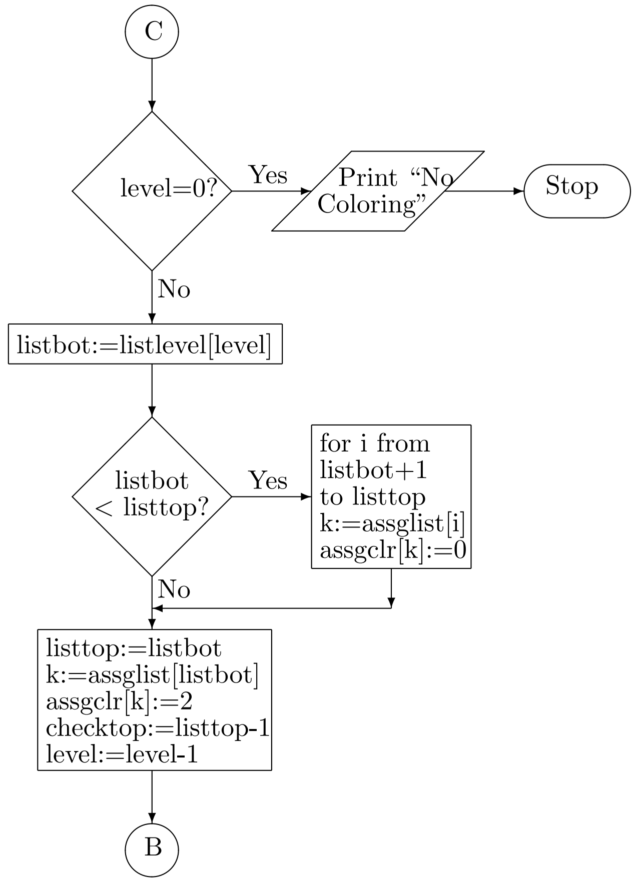

Figure 4: Contradiction for 2-coloring

In the case of 3-colorings, if `color` $= 1$, then one needs to consider the assigment of `k` to color 2. If `color` $= 2$, then since both the assignment of `k` to color 1 and to color 2 has led to a contradiction, `k` is forced to color 3 at the previous value of `level` (again unless `level` $= 0$, in which case no good coloring is possible).

In terms of time consumption, the most critical part of the algorithm is in Figure 3 where one determines whether `m` is in a monochromatic edge and whether the assignment of `m` to `kolor` forces some other vertex to a different color. Consider for example the program for computing $SP_2(a, 3)$. Computing from scratch each time whether there is an edge from `m` to `k` would, of course, be very time consuming. Initially, among the preliminary calculations we computed a $\{a, a+1, a+2, \dots, \max\} \times \{a, a+1, a+2, \dots, \max\}$ array `edges` with `edges[i,j]` $= 1$ if there is an edge between `i` and `j` and `edges[i,j]` $= 0$ otherwise. One then checked individually for each value of `k` whether there is an edge from `m` to `k`. Running this version of the algorithm on a laptop computer with an Intel Cor2Duo CPU P8400 processor operating at 2.26 GHz with $a = 6$ and $\max = 28259$ it took 246 seconds to find a coloring with no edges and with $\max = 28260$ it took 350 seconds to determine that all colorings have a monochromatic edge.

Motivated partly by the fact that the array `edges` was taking up a huge amount of RAM as a increased, we changed `edges` to a one dimensional array (with size twice the total number of edges) and introduced a new $\{a, a+1, a+2, \dots, \max\} \times \{1, 2\}$ array `inedges`. We set `inedges[i,1]` $= 0$ if the degree of `i` is 0. Otherwise we set `inedges[i,1]` $= k$ and `inedges[i,2]` $= m$ where `edges[k]`, `edges[k+1]`, ... ,`edges[m]` are the other ends of edges with `i`. This significantly reduced the amount of RAM being used, and allowed to only check values of `k` which do form an edge with `m`. Running the revised program on the same laptop with $a = 6$ and $\max = 28259$ it took 7 seconds to find a coloring with no edges and with $\max = 28260$ it took 5 seconds to determine that all colorings have a monochromatic edge.

A word about our confidence in our results is in order. In principle, when the program declares that there are no good colorings, one could have the computer print the cases considered and use those to write out a proof. This was in fact done in [1, Theorem 4.3]. But for larger numbers, this is not feasible. What we are very confident of is the fact that each of the listed numbers is at least as large as stated. This is because we had the program in each instance produce a file which had the colors to which each number was assigned. This file was then used as input for a separate program which, with a very simple and transparent algorithm, verified that the coloring had no monochromatic edges.

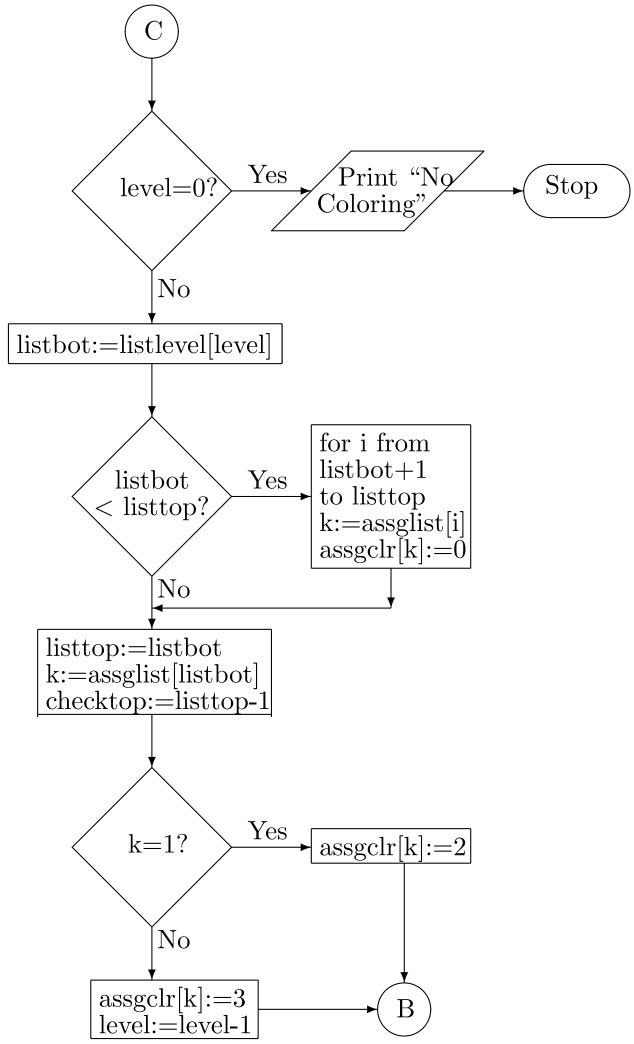

Figure 5: Contradiction for 3-coloring

We conclude this section with a brief discussion of what made us stop when we did. For the computation of $FSP_2(a, 2)$ the reason was simply time consumption. Running on the same laptop mentioned above, it took 3 hours, 47 minutes, and 19 seconds to find a 2-coloring of $\{5, 6, \ldots, 14363\}$ with no $x < y$ such that $\{x, y, x + y, xy\}$ is monochromatic and it took 3 hours, 54 minutes, and 7 seconds to show that any 2-coloring of $\{5, 6, \ldots, 14364\}$ does have a monochromatic edge. This contrasts with times of 40 seconds for $\{4, 5, \ldots, 5599\}$ and 13 minutes and 44 seconds for $\{4, 5, \ldots, 5600\}$. It seems clear that finding $FSP_2(6, 2)$ using this program is not feasible. (Don't forget that the program needs to be run once for each value of max that is tested.)

For the computation of $SP_2(a, 3)$, the constraint was available RAM. The revised program mentioned three paragraphs above required twice the storage (as 4 byte numbers) as the number of edges. The number of edges in $\{7, 8, \ldots, 62033\}$ is 194985. The number of edges in $\{8, 9, ..., 122303\}$ is 408521. The number of edges in $\{9, 10, ..., 222263\}$ is 781075.

For the computation of $SP_2(a, 2)$, the constraint was largely available RAM, mixed in with a feeling of diminishing returns. The main reason we continued all the way to 105 was the fact, which we will discuss further in the next section, that in each case we kept finding that $SP_2(a, 2)$ is divisible by a^2.

3. Bounds for $SP_2(a, 2)$

In this section, we compute the upper bound on $SP_2(a, 2)$ which results from Graham's original proof that the number is finite, and establish a slight improvement on that bound. Both of these upper bounds are on the order of $\frac{8}{9}a^4$. And we establish a lower bound of $a^2(a + \lfloor 2\sqrt{a} \rfloor)$.

For $a \in \{3, 4, 5, 6, 7, 8, 11\}$, $SP_2(a, 2) = 2a^3$ and for all computed cases with $a > 2$, $SP_2(a, 2) \leq 2a^3$. We have not been able to prove that $2a^3$ is an upper bound.

We begin with the very simple proof of our lower bound.

Theorem 3.1. *Let $a \in \mathbb{N}$. Then $SP_2(a, 2) \geq a^2(a + \lfloor 2\sqrt{a} \rfloor)$.*

Proof. The conclusion holds for $a = 1$ and $a = 2$, so assume that $a \geq 3$. Let $k = a + \lfloor 2\sqrt{a} \rfloor$ and note that $\left(\frac{a+k}{2}\right)^2 \leq a(k + 1)$. Let

$$A_1 = \{a, a + 1, \ldots, a + k\} \cup \{a(k + 1), a(k + 1) + 1, \ldots, a^2k - 1\}$$

and let $A_2 = \{a + k + 1, a + k + 2, \ldots, a(k + 1) - 1\}$. We need to show that there do not exist x and y with $a \leq x < y$ and $i \in \{1, 2\}$ such that $\{x + y, xy\} \subseteq A_i$. So suppose instead we have such x, y, and i.

If $x+y \geq a(k+1)$, then, since $x \geq a$, $xy \geq a(x+y-a) \geq a^2k$ so $xy \notin (A_1 \cup A_2)$. Suppose $x+y \in \{a, a+1, \ldots, a+k\}$. Then $xy \geq a(a+1)$ and $a(a+1) > a+k$ since $a > 2$. Also $xy < \left(\frac{a+k}{2}\right)^2 \leq a(k+1)$ so $xy \notin A_1$.

Finally suppose that $x+y \in A_2$. Then $xy \geq a(x+y-a) \geq a(k+1)$ so $xy \notin A_2$. □

Now we present the upper bound which is given by Graham's original argument.

Theorem 3.2. *Let $a \in \mathbb{N}$ and let $t = \lfloor \frac{a}{3} \rfloor$.*

(1) If $a = 3t$, then $SP_2(a,2) \leq 72t^4 + 72t^3 + 6t^2 - 6t - 4$.
(2) If $a = 3t+1$, then $SP_2(a,2) \leq 72t^4 + 216t^3 + 222t^2 + 90t + 8$.
(3) If $a = 3t+2$, then $SP_2(a,2) \leq 72t^4 + 288t^3 + 420t^2 + 264t + 56$.

Proof. We shall establish (1), the other proofs being very similar. The case $t = 1$ holds by hand computation, so assume $t > 1$. Let $k = t^2 + 3t$. We are claiming that $SP_2(a,2) \leq 2(k-2)(k+1)$. Let $\varphi : \{1, 2, \ldots, 2(k-2)(k+1)\} \to \{1, 2\}$.

Case 1. $\varphi\big(3(3t+1)\big) = \big(3(3t+2)\big) = \ldots = \varphi(3k)$. Let $x = 3t$ and let $y = 6t+3$. Then $a = x < y$, $x+y = 3(3t+1)$, and $xy = 3k < 2(k-2)(k+1)$.

Case 2. Not case 1. Pick $d \in \{3t, 3t+1, \ldots, k-2\}$ such that $\varphi(3d+3) \neq \varphi(3d+6)$. Then $d(2d+6) = 2d(d+3)$. If $\varphi\big(d(2d+6)\big) = \varphi(3d+6)$, let $x = d$ and let $y = 2d+6$. If $\varphi\big(2d(d+3)\big) = \varphi(3d+3)$, let $x = d+3$ and $y = 2d$. Since $t > 1$, $x < y$. □

Graham's original argument was based on the fact that if all of the multiples of 3 are the same color, then one trivially gets arbitrarily large $x < y$ such that $x+y$ and xy are the same color. Our slight improvement is based on the fact that if all of the elements which are not multiples of 3 are the same color, the same conclusion holds.

Theorem 3.3. *Let $a \in \mathbb{N}$ and let $t = \lfloor \frac{a}{3} \rfloor$.*

(1) If $a = 3t$, then $SP_2(a,2) \leq 72t^4 + 72t^3 - 6t^2 - 12t$.
(2) If $a = 3t+1$, then $SP_2(a,2) \leq 72t^4 + 144t^3 + 96t^2 + 24t$.
(3) If $a = 3t+2$, then $SP_2(a,2) \leq 72t^4 + 216t^3 + 234t^2 + 108t + 8$.

Proof. We shall establish (2), the other proofs being very similar. (Again, for (1), the case $t = 1$ needs separate verification.) Let $k = 6t^2 + 6t$. We are claiming that $SP_2(a,2) \leq 2k(k+2)$. Let $\varphi : \{1, 2, \ldots, 2k(k+2)\} \to \{1, 2\}$.

Case 1. There is some $m \in \{a, a+1, \ldots, k\}$ such that $\varphi(3m+1) \neq \varphi(3m+2)$. Then $2m(m+1) < 2k(k+2)$. If $\varphi\big(2m(m+1)\big) = \varphi(3m+1)$, let $x = m+1$ and $y = 2m$. If $\varphi(m(2m+2)) = \varphi(3m+2)$, let $x = m$ and $y = 2m+2$.

Case 2. There is some $m \in \{a, a+1, \ldots, k\}$ such that $\varphi(3m+2) \neq \varphi(3m+4)$. Then $2m(m+2) \leq 2k(k+2)$. If $\varphi(2m(m+2)) = \varphi(3m+2)$, let $x = m+2$ and $y = 2m$. If $\varphi(m(2m+4)) = \varphi(3m+4)$, let $x = m$ and $y = 2m+2$.

Case 3. For all $m \in \{a, a+1, \ldots, k\}$, $\varphi(3m+1) = \varphi(3m+2)$ and for all $m \in \{a, a+1, \ldots, k\}$, $\varphi(3m+2) = \varphi(3m+4)$. Then $\varphi(3a+1) = \varphi(3k+4)$. Let $x = a+1$ and $y = 2a$. □

As we noted earlier, in all computed cases, $SP_2(a, 2)$ is divisible by a^2. As a increases, this seems less likely to be a random occurrence. But we have been unable even to prove that $SP_2(a, 2)$ is divisible by a.

Conjecture 3.4. For all $a \in \mathbb{N}$, $SP_2(a, 2)$ is divisible by a.

We close with two experimental observations. The first, which allowed us to compute $SP_2(a, 2)$ with very small time consumption, is the fact that in each case the graph with edges $\{x+y, xy\}$ for $a \leq x < y$ consisted of the component of $2a+1$ together with a graph which is trivially bipartite. (As a consequence, to verify on the laptop described previously that $\{105, 106, \ldots, 1543500\}$ could not be two colored without monochromatic edges took less than 13 seconds of processor time while to show that the graph on $\{105, 106, \ldots, 1543499\}$ is bipartite took 6 seconds of processor time.) In most cases the remainder either consisted of a large number of degree 0 vertices and a few (at most 15) isolated edges, or it consisted of the degree 0 vertices, some isolated edges, and one or two disjoint stars. The most complicated configuration of the remainder after deleting the component of $2a+1$ occurred when $a = 53$. Then this remainder had 6 isolated edges and stars centered at 129, 130, 131, and 133 whose centers had degrees of 12, 12, 13, and 14 respectively. The stars centered at 129 and 130 had the vertex 4104 in common; the stars centered at 130 and 131 had the vertex 4200 in common; the stars centered at 131 and 133 had the vertex 4104 in common; and the stars centered at 129 and 133 had the vertex 4158 in common.

Our second experimental observation is that in most, but not all, cases the 2-coloring A_1, A_2 of $\{a, a+1, \ldots, SP_2(a, 2) - 1\}$ which our program found was very similar to the coloring in Theorem 3.1. That is, there was some number m such that, letting $k = SP_2(a, 2)$, letting $b = a + \frac{k}{a}$, and letting $B = \{xy : a \leq x < y\,,\, x+y \leq m, \text{ and } xy \geq b\}$, one had

$$A_1 = (\{a, a+1, \ldots, m\} \cup \{b, b+1, \ldots, k-1\}) \setminus B$$

and $A_2 = \{m+1, m+2, \ldots, b-1\} \cup B$.

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