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Prime Properties of the Smallest Ideal of $\beta\mathbb{N}$

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1. Introduction

It is well known that the semigroup operation defined on a discrete semigroup S can be extended in a natural way to the Stone-Čech compactification βS. This is done as follows: for each $s \in S$, the map $t \mapsto st$ from S to itself extends to a continuous map from βS to itself. The image of the element τ of βS under this extension is denoted by $s\tau$. Then, for each τ in βS, the map $s \mapsto s\tau$ again extends to a continuous map from βS to itself. The image of the element σ of βS under this second extension is denoted by $\sigma\tau$. Thus

$$\sigma\tau = Lim_\alpha Lim_\beta s_\alpha t_\beta$$

where $(s_\alpha), (t_\beta)$ denote nets in S converging to σ, τ respectively in βS.

The extended operation is associative, so that βS is again a semigroup. It is a compact right topological semigroup because, for each τ in βS, the map $\rho_\tau : \sigma \mapsto \sigma\tau$ from βS to itself is continuous. (At this point, the reader should be warned that the semigroup operation on S is frequently extended in the opposite order, making S a left topological semigroup).

Compact right topological semigroups have remarkable algebraic properties (Cf. [1] or [8]). The one with which we shall be concerned in this paper is that any such semigroup has a smallest ideal, which is the union of all the minimal left ideals and also of all the minimal right ideals. It is a union of groups; and so, for any element x in the smallest ideal, there will be an idempotent in the smallest ideal which is both a left identity and a right identity for x.

In this paper, we shall be concerned with $\beta\mathbb{N}$, the Stone-Čech compactification of the set $\mathbb{N}$ of positive integers. The operation which we extend is ordinary addition and we shall denote it additively even though it is wildly non-commutative.

The semigroup $(\beta\mathbb{N}, +)$ has interested several mathematicians. It is interesting in itself, as a natural extension of the most familiar of all semigroups. It is, in a sense, the largest possible extension of $(\mathbb{N}, +)$. It also has significant applications to combinatorial number theory [4] and to topological dynamics [3].

We remind the reader that the points of $\beta\mathbb{N}$ can be regarded as ultrafilters on $\mathbb{N}$, with the points of $\mathbb{N}$ itself corresponding to the principal ultrafilters. The topology of $\beta\mathbb{N}$ can be defined by taking the sets of the form

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$U_A = \{x \in \beta\mathbb{N} : A \in x\}$ as a base for the open sets, where A denotes a subset of $\mathbb{N}$. The sets U_A are then clopen, and, for every $A \subseteq \mathbb{N}$, $U_A = Cl_{\beta\mathbb{N}}A$.

In this paper, $\mathbb{N}^*$ will be used to denote $\beta\mathbb{N} \setminus \mathbb{N}$ and K will denote the smallest ideal of $\beta\mathbb{N}$.An idempotent in K will be called a minimal idempotent. For any $A \subseteq \mathbb{N}$, ClA or $\overline{A}$ will be used to denote $Cl_{\beta\mathbb{N}}A$. It is not hard to see that $\overline{K}$ is also an ideal of $\beta\mathbb{N}$.

It will be convenient to regard $\mathbb{N}$ as embedded in $\mathbb{Z}$ and $\beta\mathbb{N}$ as embedded in $\beta\mathbb{Z}$. We observe that $\mathbb{Z}$ lies in the centre of $\beta\mathbb{Z}$ and that $\mathbb{N}^*$ is a left ideal of $\beta\mathbb{Z}$.

We wish to explore the natural question of whether K or $\overline{K}$ is prime. Is it possible to have $q+p \in K$ $(\overline{K})$ if $q,p \in N^* \setminus K$ $(N^* \setminus \overline{K})$? We cannot answer this question, but can show that there are many elements p (q) of $\mathbb{N}^*$ with the property that $q+p \notin K$ whenever $q \in \mathbb{N}^* \setminus K$ $(p \in \mathbb{N}^* \setminus K)$, and that similar results are valid for $\overline{K}$.

Lemma 1. *For any countable subsets A and B of $\beta\mathbb{N}$, $\overline{A} \cap \overline{B} \neq \emptyset$ implies that $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.*

Proof. This theorem, which is valid in any F-space, is due to Frolík. A proof can be found in [9, Lemma 1]. ∎

2. Prime properties of K and $\overline{K}$

The following theorem is trivial and is obviously valid in any compact right topological semigroup.

Theorem 1. *Suppose that $q,p \in \mathbb{N}^* \setminus K$. If p is right cancellable in $\mathbb{N}^*$, $q+p \notin K$. If q is left cancellable in $\mathbb{N}^*$, we again have $q+p \notin K$.*

Proof. If $q+p \in K$, then $q+p = e+q+p$ for some idempotent $e \in K$. If p is right cancellable, this implies that $q = e+q \in K$.

Similarly, if q is left cancellable, it will follow that $p \in K$. ∎

Remark . Corresponding statements for $\overline{K}$ are less trivial. It is known that, if $p,q \in \mathbb{N}^* \setminus \overline{K}$ and if p is right cancellable, then $q+p \notin \overline{K}$ [2]. We shall give a short proof of this fact in Theorem 3, based on Theorem 2 below.

We do not know whether it is possible to have $p,q \in \mathbb{N}^* \setminus \overline{K}$ and $q+p \in \overline{K}$ if q is left cancellable. We shall show in Theorem 4 that it is not possible to have $p,q \in \mathbb{N}^* \setminus \overline{K}$ and $q+p \in \overline{K}$ if q has a neighbourhood in $\beta\mathbb{N}$ consisting of left cancellable elements.

We note that the set of elements of $\mathbb{N}^*$ which are both right and left cancellable contains a dense open subset of $\mathbb{N}^*$ [5, Corollary 4.5]. This gives us a rich set of elements p (q) with the property that $q \in \mathbb{N}^* \setminus K$ $(p \in \mathbb{N}^* \setminus K)$ implies that $q+p \notin K$. The corresponding remark for $\overline{K}$ is also valid.

The following theorem is our basic tool for much of the remainder of this paper.

Theorem 2. *Suppose that $p \in \mathbb{N}^* \setminus \overline{K}$ and that $B \subseteq \mathbb{N}$ satisfies $\overline{B} \cap K = \emptyset$ and $(\overline{B}+p) \cap \overline{K} \neq \emptyset$. Then there is a finite subset F of $\mathbb{N}$ for which $-F+\overline{B}+p$ contains a left ideal of $\beta\mathbb{N}$.*

Proof. Let V be a set containing $\overline{B}+p$ which is clopen in $\beta\mathbb{N}$. Then we will have $x \in V$ for some $x \in K$. We shall show that $\beta\mathbb{N}+x$ is covered by sets of the form $-n+V$, where $n \in \mathbb{N}$. To see this, let $y \in \beta\mathbb{N}+x$. Since $\beta\mathbb{N}+x$ is a minimal left ideal, $x \in \beta\mathbb{N} + y = Cl(\mathbb{N} + y)$. So $n + y \in V$ and $y \in -n + V$ for some $n \in \mathbb{N}$. This shows that $\beta\mathbb{N} + x$ is covered by the sets of the form $-n + V$ and is therefore covered by a finite number of these sets.

Thus for every clopen neighbourhood V of $\overline{B} + p$, there is a finite subset F of $\mathbb{N}$ for which $-F + V$ contains a left ideal of $\mathbb{N}^*$. Choose F_V such that $-F_V + V$ contains a left ideal of $\mathbb{N}^*$ and has $\max F_V$ as small as possible among all such sets.

We now claim that there is one finite subset F of $\mathbb{N}$ such that, for every clopen neighbourhood V of $\overline{B} + p$, $-F + V$ contains a left ideal of $\mathbb{N}^*$. Suppose instead that there is a sequence (V_n) for which $\max F_{V_n} \to \infty$. For each $n \in \mathbb{N}$, we define a function $\phi_n : B \to p$ by $\phi_n(b) = -b + (V_n \cap \mathbb{N})$. (If $b \in B$, $V_n \cap \mathbb{N} \in b + p$ and so $-b + (V_n \cap \mathbb{N}) \in p$.) Choose $A \in p$ for which $\overline{A} \cap K = \emptyset$ and define a function $\phi : B \to p$ by $\phi(b) = \left(\bigcap_{n \leq b} \phi_n(b)\right) \cap A$. Let $V = Cl\left(\bigcup_{b \in B}(b + \phi(b)\right)$. Then V is a clopen subset of $\beta\mathbb{N}$ containing $\overline{B} + p$. (To see this, let $q \in \overline{B}$. For each $b \in B$, $b + p \in Cl(b + \phi(b))$ because $p \in Cl\phi(b)$. Since $q + p \in Cl\{b + p : b \in B\}$, $q + p \in Cl\left(\bigcup_{b \in B}(b + \phi(b))\right)$.)

Observe that $K \cap Cl(r + \phi(b)) = \emptyset$ for every $r \in \mathbb{Z}$ and every $b \in B$. This follows from the fact that K is a left ideal in $\beta\mathbb{Z}$ and so $K = -r + K$, and we know that $K \cap Cl\phi(b) = \emptyset$.

Then $-F_V + V$ contains a minimal left ideal L of $\mathbb{N}^*$. Since $L \subseteq K$,

$$L \cap Cl\Big(\bigcup_{b \in B, b \leq n} (-F_V + b + \phi(b))\Big) = \emptyset$$

for every $n \in \mathbb{N}$. However, if $b > n$, $\phi(b) \subseteq \phi_n(b)$ and so

$$L \subseteq Cl\Big(\bigcup_{b \in B, b > n} (-F_V + b + \phi_n(b))\Big) \subseteq -F_V + V_n.$$

Since $\max F_{V_n}$ was chosen to be as small as possible, this implies that $\max F_{V_n} \leq \max F_V$, contradicting the assumption that $\max F_{V_n} \to \infty$.

We therefore have a finite subset F of $\mathbb{N}$ such that, for every clopen neighbourhood V of $\overline{B} + p$, $-F + V$ contains a left ideal of $\mathbb{N}^*$, as claimed.

For each such V, there will be a maximum left ideal of $\beta\mathbb{N}$, $L_V \subseteq -F + V$, because a union of left ideals is again a left ideal. L_V will be closed, because the closure of a left ideal of $\beta\mathbb{N}$ is also a left ideal. (This can be seen as follows: Let M be a left ideal of $\beta\mathbb{N}$. For each $n \in \mathbb{N}$, $n + \overline{M} = Cl(n + M) \subseteq \overline{M}$, and so $\beta\mathbb{N} + \overline{M} = Cl(\mathbb{N} + \overline{M}) \subseteq \overline{M}$.) Clearly, if V and V' are clopen neighbourhoods of $\overline{B} + p$, we will have $L_V \cap L_{V'} \subseteq L_{V \cap V'}$. Thus the sets L_V have the finite intersection property and hence have a non-empty intersection. This will be a left ideal contained in $\bigcap_V (-F + V) = -F + \bigcap_V V = -F + \overline{B} + p$. ∎

We now observe that we can replace $-F$ in Theorem 2 by F.

Corollary 1. *Under the conditions of Theorem 2, there is a finite subset F of $\mathbb{N}$ for which $F + \overline{B} + p$ contains a left ideal.*

Proof. There will be a finite subset G of $\mathbb{N}$ for which $-G + \overline{B} + p$ contains a left ideal, and which therefore contains a minimal left ideal L (by Theorem 2). Choose $r \in \mathbb{N}$ satisfying $r > \max G$. Since r is in the centre of $\beta\mathbb{N}$, $r + L$ is a left ideal contained in L and so $L = r + L \subseteq r - G + \overline{B} + p$. Hence we can choose $F = r - G \subseteq \mathbb{N}$. ∎

Corollary 2. *Let $p \in \mathbb{N}^* \setminus \overline{K}$. If there exists $q \in \mathbb{N}^* \setminus \overline{K}$ such that $q + p \in \overline{K}$, then there exists $q' \in \mathbb{N}^* \setminus \overline{K}$ such that $q' + p$ is a minimal idempotent.*

Proof. Pick $B \in q$ such that $\overline{B} \cap K = \emptyset$. Pick finite $F \subseteq \mathbb{N}$ and a minimal left ideal L of $\mathbb{N}^*$ with $L \subseteq -F + \overline{B} + p$, by Theorem 2. Pick $n \in F$ and $x \in \overline{B}$ such that $-n + x + p$ is an idempotent in L. Let $q' = -n + x$. Then $q' \notin \overline{K}$ since $n + q' = x \notin \overline{K}$. ■

Theorem 3. *Suppose that $p \in \mathbb{N}^* \setminus \overline{K}$ is right cancellable in $\mathbb{N}^*$. Then, if $q \in \mathbb{N}^* \setminus \overline{K}$, it follows that $q + p \notin \overline{K}$.*

Proof. Let $B \in q$ satisfy $\overline{B} \cap K = \emptyset$. If $q + p \in \overline{K}$, Theorem 2 implies that there will be a finite subset F of $\mathbb{N}$ for which $-F + \overline{B} + p$ intersects K. Thus $x + p \in K$ for some $x \in -F + \overline{B}$. Since $x + p = e + x + p$ for some minimal idempotent e and since p is right cancellable, this implies that $x = e + x \in K$. However, this is a contradiction, because the fact that $\overline{B} \cap K = \emptyset$ implies that $(-F + \overline{B}) \cap K = \emptyset$. ■

Theorem 4. *Suppose that there is a set $B \in q$ with the property that every element of $\overline{B}$ is left cancellable in $\mathbb{N}^*$. Then, if $p \in \mathbb{N}^* \setminus \overline{K}$, it follows that $q + p \notin \overline{K}$.*

Proof. If $q + p \in \overline{K}$, we must have $-n + x + p \in K$ for some $n \in \mathbb{N}$ and some $x \in \overline{B}$ (by Theorem 2). Thus $-n + x + p = -n + x + p + e$ for some minimal idempotent e. Since $-n$ and x are left cancellable, it follows that $p = p + e \in K$ – a contradiction. ■

Recall that a p-point of $\mathbb{N}^*$ is a point q with the property that the intersection of countably many neighbourhoods of q is again a neighbourhood of q (in $\mathbb{N}^*$).

Corollary 3. *Let p be a p-point of $\mathbb{N}^*$.*
(a) If $q \notin K$, then $p + q \notin K$ and $q + p \notin K$.
(b) If $q \notin \overline{K}$, then $p + q \notin \overline{K}$ and $q + p \notin \overline{K}$.

Proof. By [9, Theorem 1] p is right cancellable, since $p \notin p + \beta\mathbb{N}$ (if it were, it would be an accumulation point of the countable subset $\mathbb{N} + p$ of $\mathbb{N}^*$). By the proof of [5, Theorem 4.7] there is a member B of p such that every member of $\overline{B}$ is left cancellable. (Be cautioned that in [5] $\beta\mathbb{N}$ is taken to be left rather than right topological.) Since no point of K is left cancellable, one has $\overline{B} \cap K = \emptyset$ and hence $p \notin \overline{K}$. Thus Theorems 1, 3 and 4 apply. ■

Theorem 5. *If $\overline{K}$ is not prime, then K is not prime.*

Proof. Suppose that $p, q \in \mathbb{N}^* \setminus \overline{K}$ and $q + p \in \overline{K}$. Pick $B \in q$ such that $\overline{B} \cap K = \emptyset$. By Theorem 2 pick $x \in \overline{B}$ and $n \in \mathbb{N}$ with $-n + x + p \in K$. Since $x \in \overline{B}$, $x \notin K$ and so $-n + x \notin K$. ■

Theorem 6. *K and $\overline{K}$ both have the property that they are prime if and only if they are semi-prime.*

Proof. Suppose that $p, q \in \mathbb{N}^* \setminus K$ and that $q + p \in K$. It was shown in [10, Theorem 2] that there would be an element x of $\mathbb{N}^*$ for which $x + q$ would be right cancellable. So we may replace q by $x + q$ and suppose that q is right cancellable. Hence $p + q \notin K$ (by Theorem 1). However, $(p + q) + (p + q) \in K$ and so K is not semi-prime.

Now suppose that $p, q \in \mathbb{N}^* \setminus \overline{K}$ and that $q + p \in \overline{K}$. Since $q \notin K$, it follows from [10, Theorem 2] that there is an infinite increasing sequence $\langle m_n \rangle_{n=1}^{\infty}$ of positive integers such that $m_n | m_{n+1}$ for each n and $x + q$ is right cancellable for every $x \in Cl\{m_n : n \in \mathbb{N}\}$. By [5, Corollary 4.4] every such x is left cancellable. Thus $x + q \notin \overline{K}$ by Theorem 4. Thus we may presume that q is right cancellable and hence that $p + q \notin \overline{K}$ (by Theorem 3). Then the fact that $(p + q) + (p + q) \in \overline{K}$ shows that $\overline{K}$ is not semi-prime. ■

Let T denote $\bigcap_{n=1}^{\infty} Cl(\mathbb{N}n)$.

Lemma 2. *For every $x \in \mathbb{N}^*$ there exists $y \in \mathbb{N}^*$ satisfying:*
i) $y + x \in T$;
ii) $x + y \in T$;
iii) y is right cancellable;
iv) y has a neighbourhood consisting of left cancellable elements.

Proof. For each $n \in \mathbb{N}$, q_n will denote the canonical homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}_n$. It is not hard to see that $q_n^{\beta} : \beta\mathbb{Z} \to \mathbb{Z}_n$ is also a homomorphism.

We can inductively choose a sequence $\langle m_n \rangle_{n=1}^{\infty}$ of positive integers satisfying $m_{n+1} - m_n \to \infty$ and $q_k(m_n) = -q_k^{\beta}(x)$ for every $k = 1, 2, .., n$. That this is possible can be seen as follows: given $n \in \mathbb{N}$, $\{r \in \mathbb{N} : q_k(r) = q_k^{\beta}(x)$ for every $k = 1, 2, .., n\}$ is a member of x. Pick r in this set. Then, for any $m \in \mathbb{N}$, $q_k(m \cdot n! - r) = -q_k^{\beta}(x)$.

Now let $y \in \mathbb{N}^* \cap Cl\{m_n : n \in \mathbb{N}\}$. Since $q_k^{\beta}(y) = -q_k^{\beta}(x)$ for every $k \in \mathbb{N}$, $x + y \in T$ and $y + x \in T$.

By the remark on p.241 of [9], $y \notin \mathbb{N}^* + \mathbb{N}^*$ and y is therefore right cancellable.

It remains to show that any member of $Cl\{m_n : n \in \mathbb{N}\}$ is left cancellable. Since every member of $\mathbb{N}$ is left cancellable and we have assumed nothing about y except that it is in $Cl\{m_n : n \in \mathbb{N}\}$, it is sufficient to show that y is left cancellable.

We first note that that there is at most one $a \in \mathbb{N}$ satisfying $q_k(a) = -q_k^{\beta}(x)$ for every $k \in \mathbb{N}$. For, if $a, b \in \mathbb{N}$ and $a < b$, then $q_k(a) \neq q_k(b)$ if $k \geq b$. Thus we suppose that the sequence $\langle m_n \rangle_{n=1}^{\infty}$ does not contain an integer a with this property, because we could delete such an integer if necessary.

Suppose that $y + u = y + v$ for some $u, v \in \beta\mathbb{N}$. This implies that $q_k(u) = q_k(v)$ for every $k \in \mathbb{N}$. Since $y + u \in Cl\{m_n + u : n \in \mathbb{N}\}$ and $y + v \in Cl\{m_n + v : n \in \mathbb{N}\}$, an application of Lemma 1 allows us to suppose that $m_n + u = y' + v$ for some $n \in \mathbb{N}$ and some $y' \in Cl\{m_n : n \in \mathbb{N}\}$. We then have $q_k(m_n) = q_k(y')$ for every $k \in \mathbb{N}$. We have, however, ruled out this possibility, and so $y' \in \mathbb{N}$. But then $y' = m_n$ and therefore $u = v$ because integers are cancellable in $\beta\mathbb{Z}$. ■

Theorem 7. *If $q, p \in \mathbb{N}^* \setminus K$ and $q + p \in K$, then $q + p' \in K$ for some $p' \in T \cap (\mathbb{N}^* \setminus K)$ and $q' + p \in K$ for some $q' \in T \cap (\mathbb{N}^* \setminus K)$.*

The same statement holds if K is replaced by $\overline{K}$.

Proof. The proof is an immediate consequence of Lemma 2 and Theorems 1, 3 and 4. ■

Corollary 4. *K $(\overline{K})$ is prime if and only if $K \cap T$ $(\overline{K} \cap T)$ is prime.* ■

In a similar vein, we have the following result, which we can only establish for K and for the left hand argument.

Theorem 8. *If $q,p\in\mathbb{N}^*\setminus K$ and $q+p\in K$, then there is an idempotent $q'\in\mathbb{N}^*\setminus K$ such that $q'+p\in K$.*

Proof. Pick a minimal left ideal L with $q+p\in L$. Then $\{x\in\mathbb{N}^*: x+p\in L\}$ is a compact semigroup which contains a right cancellable element x. (By [10, Theorem 2] there will be some y such that $y+q$ is right cancellable. Let $x=y+q$.) The smallest compact semigroup containing x misses K [6, Theorem 2.3]. This semigroup necessarily contains idempotents [3, Corollary 2.10]. ∎

Theorem 9. *There are elements q of $\mathbb{N}^*$ which are not left cancellative and have the property that $q+p\notin\overline{K}$ whenever $p\in\mathbb{N}^*\setminus\overline{K}$. In fact, there are idempotents with this property.*

Proof. We choose an infinite increasing sequence $\langle x_n\rangle_{n=1}^\infty$ of positive integers for which $x_1=1$ and x_n is a factor of x_{n+1} for every $n=1,2,3,\ldots$. Every positive integer m has a unique expression as $m=\sum_1^\infty a_nx_n$, where each a_n is an integer satisfying $0\le a_n<x_{n+1}/x_n$. (This can be seen as follows: a_1 is the remainder obtained when m is divided by x_2 ; then a_2x_2 is the remainder obtained when $m-a_1x_1$ is divided by x_3, etc.) If m is expressed in this way, $\text{supp}m$ will denote $\{n\in\mathbb{N}: a_n\neq 0\}$.

Now let M be any infinite subset of $\mathbb{N}$ for which $\mathbb{N}\setminus M$ is infinite, and let $B=\{m\in\mathbb{N}:\text{supp }m\cap M=\emptyset\}$.

We observe that $\overline{B}\cap T$is a semigroup. To see this, suppose that $x,y\in\overline{B}\cap T$. For any $m,n\in B$ satisfying $\min(\text{supp }n)>\max(\text{supp }m)$, $m+n\in B$. Allowing n to converge to y shows that $m+y\in\overline{B}$. Then allowing m to converge to x shows that $x+y\in\overline{B}$. Thus $\overline{B}$ will contain idempotents.

We claim that for any element $q\in\overline{B}$ and any $p\in\mathbb{N}^*\setminus\overline{K}$, $q+p\notin\overline{K}$.

In the light of Lemma 2 and Theorem 3, it suffices to prove our claim under the additional assumption that $p\in T$. (For pick right cancellable y with $p+y\in T$. By theorem 3, $p+y\notin\overline{K}$. If $q+p+y\notin\overline{K}$, then of course $q+p\notin\overline{K}$.)

If $q+p\in\overline{K}$, there will be a finite subset F of $\mathbb{N}$ for which $F+\overline{B}+p$ contains a left ideal L of $\beta\mathbb{N}$ (by Corollary 1). There will clearly be a positive integer k with the property that $\text{supp }n\cap M\cap(k,\infty)=\emptyset$ for every $n\in F+B$. Let $m\in M$ satisfy $m>k$ and let $A=\{a\in\mathbb{N}:\text{supp }a\cap M\cap(k,m]=\emptyset\}$. We claim that $F+\overline{B}+p\subseteq\overline{A}$. To see this, let $x\in F+\overline{B}$. If $n,s\in\mathbb{N}$ satisfy $n\in F+B$ and $\min(\text{supp }s)>m$ then $m+s\in A$. We can allow s to converge to p (since $p\in T$) and then allow n to converge to x, and hence deduce that $x+p\in\overline{A}$. However, $(x_m+A)\cap A=\emptyset$ and so $(x_m+\overline{A})\cap\overline{A}=\emptyset$. This contradicts the inclusions $x_m+L\subseteq x_m+\overline{A}$ and $x_m+L\subseteq L\subseteq\overline{A}$. ∎

In the following theorem we show that there are elements p of $\mathbb{N}^*$ which are not right cancellable and have the property that $q+p\notin K$ if $q\in\mathbb{N}^*\setminus K$, as well as the property that $q+p\notin\overline{K}$ if $q\in\mathbb{N}^*\setminus\overline{K}$. Indeed, there idempotents p for which these statements hold.

Following [8], we define a quasi-order $\le_R$ on the elements of $\mathbb{N}^*$ by stating that $x\le_R y$ if $y+x=x$. It was shown in [8] that every idempotent in $\mathbb{N}^*$ is dominated in this quasi-order by an idempotent which is $\le_R$-maximal. If p is a $\le_R$-maximal idempotent, then, by [7, Lemma 3.1 and Theorem 3.5] there is an idempotent q with $p\le_R q$ such that the set $C=\{x\in\mathbb{N}^*:x+q=q\}$ is a finite right zero semigroup. But since p is $\le_R$-maximal, one has that $C=\{x\in\mathbb{N}^*:x+p=p\}$. (We note that this implies that $p\notin K$, because any idempotent in K has $2^{\mathfrak{c}}$ left identities.) Furthermore, the map ρ_p is one-one on $\beta\mathbb{N}\setminus(\beta\mathbb{N}+C)$.

Theorem 10. *Let p be a $\leq_R$-maximal idempotent of $\mathbb{N}^*$. If $q \in \beta\mathbb{N} \setminus K$, $q + p \notin K$.*

Proof. Suppose, on the contrary, that $q + p \in K$. Then $q + p = e + q + p$ for some minimal idempotent e.

Let $C = \{x \in \mathbb{N}^* : x + p = p\}$. As observed above, C is finite and, for all $x, y \in C$, $x + y = y$. Note first that $q \notin \beta\mathbb{N} + C$. For, if we had $q = u + c$ for some $u \in \beta\mathbb{N}$ and some $c \in C$, we would have $q = q + c = q + (p + c) \in K$. Thus we may pick $X \in q$ such that $\overline{X} \cap (\beta\mathbb{N} + C) = \emptyset$. (Since C is finite, $\beta\mathbb{N} + C$ is compact.)

Next we claim that $e + q \notin \beta\mathbb{N} + C$. Indeed, if $e + q \in \beta\mathbb{N} + c$ for some $c \in C$, then by [9, Theorem 2], $q \in \beta\mathbb{N} + c$ or $c \in \beta\mathbb{N} + q$. We have already seen that the first of these alternatives is impossible. But if $c \in \beta\mathbb{N} + q$, then $p = c + p \in \beta\mathbb{N} + q + p \subseteq K$. Thus the second alternative is ruled out too.

Since $e+q \notin \beta\mathbb{N}+C$, we can choose $Y \in e+q$ such that $\overline{Y} \cap (\beta\mathbb{N}+C) = \emptyset$. Since $e + q \in K$ and $q \notin K$, we may assume that $X \cap Y = \emptyset$.

Since $q+p \in Cl(X+p)$ and $e+q+p \in Cl(Y+p)$ it follows from Lemma 1 that $n + p = y + p$ for some $n \in X$ and some $y \in \overline{Y}$, or else $x + p = n + p$ for some $x \in X$ and some $y \in \overline{Y}$. Since $m + p \neq n + p$ if $m, n \in \mathbb{N}$ and $m \neq n$ (as can be seen by considering congruences), we can take y or x to be in $\mathbb{N}^*$.

But the first possibility implies that $p = -n + y + p$ and hence that $-n + y \in C$, contradicting our assumption that $Y \cap (\beta\mathbb{N} + C) = \emptyset$. Similarly, the second possibility contradicts the assumption that $\overline{X} \cap (\beta\mathbb{N} + C) = \emptyset$. ∎

We have already noted that, if e is any idempotent in $\mathbb{N}^*$, there is a $\leq_R$-maximal idempotent p satisfying $e \leq_R p$. If $e \notin \overline{K}$ then $p \notin \overline{K}$. So there are $\leq_R$-maximal idempotents in $\mathbb{N}^* \setminus \overline{K}$.

Theorem 11. *Suppose that p is a $\leq_R$-maximal idempotent in $\mathbb{N}^* \setminus \overline{K}$. Then, if $q \in \mathbb{N}^* \setminus \overline{K}$, it follows that $q + p \notin \overline{K}$.*

Proof. Exactly as in the proof of Theorem 10, we can show that $q \notin \beta\mathbb{Z} + C$. Hence there is a set $X \in q$ for which $\overline{X} \cap (\beta\mathbb{Z} + C) = \emptyset$ and $\overline{X} \cap \overline{K} = \emptyset$. We note that $(n + \overline{X}) \cap (\beta\mathbb{Z} + C) = \emptyset$ for every $n \in \mathbb{Z}$.

Suppose that $q + p \in \overline{K}$. There will be a finite subset F of $\mathbb{N}$ for which $-F + \overline{X} + p$ contains a minimal left ideal L (by Theorem 2). Observe that $(-F + \overline{X}) \cap (\beta\mathbb{Z} + C) = \emptyset$.

Let $x \in L$. Then $x = y + p$ for some $y \in -F + \overline{X}$. For each $n \in \mathbb{N}$, $n + x \in L$ and so $n + x = y_n + p$ for some $y_n \in -F + \overline{X}$. Now $n + y \notin \beta\mathbb{N} + C$ and $y_n \notin \beta\mathbb{N} + C$. Since ρ_p is one-one on $\beta\mathbb{N} \setminus (\beta\mathbb{N} + C)$, it follows that $n + y = y_n$. This implies that $n + y \in -F + \overline{X}$ and hence that $\beta\mathbb{N} + y \subseteq -F + \overline{X}$. But this contradicts our assumption that $(-F + \overline{X}) \cap K = \emptyset$. ∎

The following theorem is trivial, but seems worth noting.

Theorem 12. *If $p \in \mathbb{N}^* \setminus K$ $(\mathbb{N}^* \setminus \overline{K})$, then $\mathbb{N}^* + p$ is not contained in K $(\overline{K})$.*

Proof. Suppose that $p \in \mathbb{N}^* \setminus K$. If p is not right cancellable, $p \in \mathbb{N}^* + p$ [9, Theorem 1] and so $\mathbb{N}^* + p$ is not contained in K.

Otherwise, if p is right cancellable, $q + p \notin K$ (by Theorem 1) if $q \in \mathbb{N}^* \setminus K$. So we can again assert that $\mathbb{N}^* + p$ is not contained in K.

The corresponding theorem for $\overline{K}$ follows in the same way from Theorem 3. ∎

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