This paper was published in *Semigroup Forum* **51** (1995), 299-318. To the best of my knowledge, this is the final version as it was submitted to the publisher.–NH

Nearly Prime Subsemigroups of $\beta\mathbb{N}$

Neil Hindman and Dona Strauss

Communicated by Karl H. Hofmann

Abstract: We show that there exist relatively small subsemigroups M of $\beta\mathbb{N}$ with the property that if $p+q$ and $q+p$ are in M then both p and q are in $M+\mathbb{Z}$. We also show that it is consistent with the usual axioms of set theory that there is some idempotent e in $\beta\mathbb{N}$ such that if $p+q=e$, then both p and q are in $e+\mathbb{Z}$.

1. Introduction

In [11] it was shown that it is consistent with the usual axioms of set theory that $\beta\mathbb{N}$, the Stone-Čech compactification of the positive integers, has maximal groups that are as small as possible. That is they are just copies of the group $\mathbb{Z}$ of integers. It was shown in fact that if e is the identity of such a group and $p+q=q+p=e$, then p and q are in $e+\mathbb{Z}$.

Two natural questions are thus raised. The first is whether one can prove the existence of such small groups without making any special set theoretic assumptions. The second is whether these identities can be written in any nontrivial way as a sum.

In Section 2 we address the first of these questions, producing without special assumptions semigroups M that are in some senses "small" (though not in cardinality) with the property that if $p+q\in M$ and $q+p\in M$, then p and q are in $M+\mathbb{Z}$.

In Sections 4 and 5 we answer the second of these questions. That is, we show that the continuum hypothesis (or even only Martin's Axiom) implies the existence of an idempotent e such that if $q+p=e$ one has both q and p are in $e+\mathbb{Z}$. Additional consistency results are derived in these sections.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, the principal ultrafilters being identified with the points of $\mathbb{N}$. As is well known (see for example [14]), there is an operation $+$ on $\beta\mathbb{N}$ making $(\beta\mathbb{N},+)$ a right topological semigroup with $\mathbb{N}$ contained in, in fact equal to, its topological center. (By "right topological" we mean ρ_p is continuous for each $p\in\mathbb{N}$, where $\rho_p(q)=q+p$. By "topological center" we mean the set of points x for which λ_x is continuous, where $\lambda_x(p)=x+p$.) Given p and q in $\beta\mathbb{N}$, the sum $p+q$ is characterized as follows. For $A\subseteq\mathbb{N}$, $A\in p+q$ if and only if $\{x\in\mathbb{N}:A-x\in q\}\in p$ where $A-x=\{y\in\mathbb{N}:y+x\in A\}$. Alternatively if $\langle x_i\rangle_{i\in I}$ and $\langle y_j\rangle_{j\in J}$ are nets in $\mathbb{N}$ converging to p and q respectively, then $p+q=\lim_{i\in I}\lim_{j\in J}(x_i+y_j)$.

The first author acknowledges support received from the National Science Foundation (USA) via grant DMS90-25025.

The special idempotents that we utilize are called "strongly summable ultrafilters" and "divisibly strongly summable ultrafilters". (The names come from the fact that they are generated by special sets of sums, obviously not from the fact that they are hard to write as sums.) The existence of these ultrafilters follows from the continuum hypothesis, or even Martin's Axiom. (See [12] and Section 5 of this paper.) On the other hand their existence cannot be proved in ZFC. (See [19] or [5].)

Following [16] we define pre-orders (i.e. reflexive and transitive relations) on the elements of $\beta\mathbb{N}$ by $p \leq_R q$ if and only if either $p = q$ or $p = q + p$ and $p \leq_L q$ if and only if either $p = q$ or $p = p + q$.

It was shown by Ruppert [20, Theorem I.2.7] that any compact right topological semigroup has $\leq_R$-maximal elements. As we shall see in Section 4, all strongly summable ultrafilters are $\leq_R$-maximal. In fact if p is a strongly summable ultrafilter, then $\{q \in \beta\mathbb{N} : p \leq_R q\}$ is finite.

The reader should be cautioned when looking at references [11], [12], [14], and [15] that there $(\beta\mathbb{N}, +)$ is taken to be left topological rather than right topological. To make matters worse, in referring to [20], the semigroup $(\beta\mathbb{N}, +)$ is *called* right topological but has the continuity making it what *we* call left topological.

We write $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$. Recall that the ordinal $\omega = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$. We shall occasionally refer to $\beta\mathbb{Z}$. We will brush over the distinction between ultrafilters on $\mathbb{N}$ and ultrafilters on $\mathbb{Z}$ with $\mathbb{N}$ as a member and thereby pretend that $\beta\mathbb{N} \subseteq \beta\mathbb{Z}$ (just as we are pretending that $\mathbb{N} \subseteq \beta\mathbb{N}$).

We shall have need of the following lemma, apparently due originally to Frolík.

Lemma 1.1. *Let A and B be countable subsets of $\beta\mathbb{N}$. If $c\ell(A) \cap c\ell(B) \neq \emptyset$, then either $A \cap c\ell(B) \neq \emptyset$ or $B \cap c\ell(A) \neq \emptyset$.*

Proof. See [22, Lemma 1]. ∎

2. Divisible Sequences and Nearly Prime Subsemigroups

We will be concerned here with ultrafilters living on the tails of finite sums of sequences, especially of divisible sequences.

Definition 2.1. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$.

(a) For each $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty}) = \{\Sigma_{n\in F}\, x_n : F$ *is a finite nonempty subset of* $\mathbb{N}$ *and* $\min F \geq m\}$.

(b) The sequence $\langle x_n \rangle_{n=1}^{\infty}$ is *divisible* if and only if for each $n \in \mathbb{N}$, $x_{n+1} > x_n > 1$ and x_n divides x_{n+1}.

(c) $M(\langle x_n \rangle_{n=1}^{\infty}) = \bigcap_{m=1}^{\infty} c\ell(FS(\langle x_n \rangle_{n=m}^{\infty}))$.

It is especially convenient to work with divisible sequences for the following well known reason. If $\langle x_n \rangle_{n=1}^{\infty}$ is a divisible sequence and $x_0 = 1$, then each $y \in \mathbb{N}$ has a unique representation of the form $\Sigma_{n\in F}\, b_n x_n$ where each $b_n \in \{1, 2, \ldots, \frac{x_{n+1}}{x_n} - 1\}$ and F is a finite nonempty subset of $\omega = \mathbb{N} \cup \{0\}$. Further, given such an expression, x_t divides y if and only if $\min F \geq t$. (Thus most of one's intuition gained from years of dealing with the special sequence $x_n = 10^n$ remains valid.)

It is well known and easy to see that if for each n, $x_{n+1} > \Sigma_{t=1}^{n}\, x_t$, then expressions in $FS(\langle x_n \rangle_{n=1}^{\infty})$ are unique. That is, if $\Sigma_{n\in F}\, x_n = \Sigma_{n\in G}\, x_n$, then $F = G$. We shall have need of the following stronger result. (We take $\Sigma_{t\in\emptyset}\, x_t = 0$.)

Lemma 2.2. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that for each n, $x_{n+1} > \Sigma_{t=1}^n x_t$. Let $y \in \mathbb{N}$ and let $r \in \mathbb{N}$ with $x_r \geq y$. If F and G are finite subsets of $\mathbb{N}$ and $y + \Sigma_{t\in G}\, x_t = \Sigma_{t\in F}\, x_t$ and either $G = \emptyset$ or $\min G > r$, then $G \subseteq F$ and $y = \Sigma_{t\in F\setminus G}\, x_t$.*

Proof. We proceed by induction on $|G|$, the case $|G| = 0$ being immediate. So assume $|G| > 0$ and the result holds for smaller sets. Trivially $F \neq \emptyset$. Let $m = \max G$ and $n = \max F$. If we had $m < n$, then we would have

$$\Sigma_{t\in F}\, x_t \geq x_n > \Sigma_{t=1}^{n-1}\, x_t \geq x_r + \Sigma_{t\in G}\, x_t \geq y + \Sigma_{t\in G}\, x_t = \Sigma_{t\in F}\, x_t,$$

a contradiction. If we had $m > n$ then we would have

$$y + \Sigma_{t\in G}\, x_t > x_m > \Sigma_{t=1}^{m-1}\, x_t \geq \Sigma_{t\in F}\, x_t = y + \Sigma_{t\in G},$$

a contradiction. Thus $m = n$. Let $F^* = F\setminus\{m\}$ and $G^* = G\setminus\{m\}$. Then $y + \Sigma_{t\in G^*}\, x_t = \Sigma_{t\in F^*}\, x_t$ so by our induction hypothesis $G^* \subseteq F^*$ and $y = \Sigma_{t\in F^*\setminus G^*}\, x_t = \Sigma_{t\in F\setminus G}\, x_t$. ∎

We need no divisibility assumptions for our first theorem.

Theorem 2.3. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ and let $M = M(\langle x_n\rangle_{n=1}^\infty)$. Then M is a compact subsemigroup of $\beta\mathbb{N}$. Assume $x_{n+1} > \Sigma_{t=1}^n\, x_t$ for each n and let $p, q \in \beta\mathbb{N}$. If $q \in M$ and $p + q \in M$, then $p \in M$.*

Proof. For each $m \in \mathbb{N}$, let $B_m = FS(\langle x_n\rangle_{n=m}^\infty)$. To see that M is a subsemigroup, let $r, s \in M$ and let $m \in \mathbb{N}$ be given. We show that $B_m \subseteq \{y \in \mathbb{N} : B_m - y \in s\}$ so that $B_m \in r + s$. Let $y \in B_m$ and pick F with $\min F \geq m$ such that $y = \Sigma_{t\in F}\, x_t$. Let $k = \max F + 1$. Then $B_k \subseteq B_m - y$ so $B_m - y \in s$.

Now assume $q \in M$ and $p + q \in M$. Let $m \in M$ be given. We show $B_m \in p$. We know $B_m \in p + q$ so if $C = \{y \in \mathbb{N} : B_m - y \in q\}$, then $C \in p$. We show $C \subseteq B_m$ so let $y \in C$. Pick $k \in \mathbb{N}$ such that $x_k \geq y$. Now $B_{k+1} \in q$ and $B_m - y \in q$ so pick $z \in B_{k+1} \cap (B_m - y)$. Since $z \in B_{k+1}$, pick G with $\min G \geq k+1$ such that $z = \Sigma_{t\in G}\, x_t$. Since $z + y \in B_m$, pick F with $\min F \geq m$ such that $z + y = \Sigma_{t\in F}\, x_t$. Then by Lemma 2.2 we have $y = \Sigma_{t\in F\setminus G}\, x_t$ so that $y \in B_m$ as required. ∎

Observe that any divisible sequence satisfies the hypothesis of Theorem 2.3. Observe also that if $\langle x_n\rangle_{n=1}^\infty$ is any divisible sequence then $M(\langle x_n\rangle_{n=1}^\infty) \subseteq \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$. Equality is possible, for example if $x_n = 2^n$ for each n or more generally if $\frac{x_{n+1}}{x_n}$ is eventually equal to 2. If on the other hand infinitely often $\frac{x_{n+1}}{x_n} \geq 3$ the inclusion will be proper.

We are grateful to the referee for pointing out the following algebraic characterizations of $M(\langle x_n\rangle_{n=1}^\infty)$ and $\bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$. The algebraic proofs of several of the remaining results of this section are also due to the referee.

Given a divisible sequence $\langle x_n\rangle_{n=1}^\infty$, let $a_0 = x_1$ and for each $n \in \mathbb{N}$, let $a_n = \frac{x_{n+1}}{x_n}$, and let $\vec{a} = \langle a_n\rangle_{n=0}^\infty$. Then the product space $\Delta_{\vec{a}} = \prod_{n=0}^\infty \{0, 1, \ldots, a_n - 1\}$ is the familiar ring of $\vec{a}$-adic integers. (See [9, §10, pp. 107-114].) We will only be concerned with the additive group thereof, where addition is defined as in ordinary arithmetic. That is, given $\vec{x}$ and $\vec{y}$ in $\Delta_{\vec{a}}$, one adds coordinate by coordinate, starting at the lower order coordinates, and reduces the sum by a_n and carries 1 to the next coordinate, whenever the n^{th} coordinate sum is at least a_n. Then the non-negative integers are naturally identified with the members of $\Delta_{\vec{a}}$ with finitely many non-zero coordinates. Thus, given $n \in \mathbb{N}$ if n is identified with $\alpha(n)$ one has $n = \Sigma_{t=0}^\infty\, \alpha_t(n) \cdot x_t$ where $x_0 = 1$.

In the following definition we suppress the dependence of α, μ, $\vec{a}$, and *supp* on the particular divisible sequence because we never work with more than one at a time.

Definition 2.4. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a divisible sequence and let $a_0 = x_1$ and $a_n = \frac{x_{n+1}}{x_n}$ for each $n \in \mathbb{N}$. Then $\alpha : \beta\mathbb{N} \longrightarrow \Delta_{\vec{a}}$ is the continuous extension of the natural inclusion of $\mathbb{N}$ in $\Delta_{\vec{a}}$. For $p \in \beta\mathbb{N}$, $\mathrm{supp}(p) = \{t \in \omega : \alpha_t(p) \neq 0\}$. Let $\mathbb{N}^{\infty}$ denote the one point compactification $\mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$ and define $\mu : \mathbb{N} \longrightarrow \mathbb{N}^{\infty}$ by $\mu(n) = \max\{\alpha_t(n) : t \in \omega\}$. Denote also by μ its continuous extension from $\beta\mathbb{N}$ to $\mathbb{N}^{\infty}$.

It should be observed that for $p \in \mathbb{N}^*$, $\mu(p)$ need not equal $\sup\{\alpha_t(p) : t \in \omega\}$. To see this, let p be a cluster point of the sequence $\langle x_n\rangle_{n=1}^{\infty}$. Given $t, k \in \mathbb{N}$ one has $\alpha_k(x_t)$ is 1 if $k = t$ and 0 otherwise. So $\mu(p) = 1$. But $\alpha_t(p) = 0$ for all t.

Lemma 2.5. *The function μ is a homomorphism from $\alpha^{-1}[\{0\}]$ to the semigroup $(\mathbb{N}^{\infty}, \max)$.*

Proof. If $m, n \in \mathbb{N}$ and $\mathrm{supp}(m) \cap \mathrm{supp}(n) = \emptyset$, then for each t, $\alpha_t(m+n) = \max\{\alpha_t(m), \alpha_t(n)\}$. ∎

Lemma 2.6. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a divisible sequence.*

(a) $\bigcap_{n=1}^{\infty} c\ell(\mathbb{N}x_n) = \alpha^{-1}[\{0\}]$.

(b) $M(\langle x_n\rangle_{n=1}^{\infty}) = \alpha^{-1}[\{0\}] \cap \mu^{-1}[\{1\}]$.

(c) If $p, q \in \beta\mathbb{N}$ and any two of p, q, and $p+q$ are in $\alpha^{-1}[\{0\}]$ so is the third.

(d) If $p, q \in \beta\mathbb{N}$ and any two of p, q, and $p+q$ are in $M(\langle x_n\rangle_{n=1}^{\infty})$ so is the third.

(e) If $p+q \in M(\langle x_n\rangle_{n=1}^{\infty})$ and either $p \in \alpha^{-1}[\{0\}]$ or $q \in \alpha^{-1}[\{0\}]$ then both p and q are in $M(\langle x_n\rangle_{n=1}^{\infty})$.

Proof. Both (a) and (b) are routine computations and (c) is easy since α is a homomorphism to a group. To prove (d) and (e), use (c) and the fact that μ is a homomorphism from $\alpha^{-1}[\{0\}]$ to $(\mathbb{N}^{\infty}, \max)$. ∎

Definition 2.7. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a divisible sequence.

(a) For $A \subseteq \mathbb{N}$, $S_A = \{n \in \mathbb{N} : \mathrm{supp}(n) \subseteq A\}$.

(b) For $s \in \mathbb{N}^*$, $M_s = \bigcap_{A \in s}(M(\langle x_n\rangle_{n=1}^{\infty}) \cap c\ell S_A)$.

Lemma 2.8. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a divisible sequence and let $s \in \mathbb{N}^*$.*

(a) M_s is a compact subsemigroup of $\beta\mathbb{N}$.

(b) If $p \in \beta\mathbb{N}, q \in \alpha^{-1}[\{0\}]$, and $p+q \in M_s$, then p and q are in M_s.

Proof. (a). Given $m, n \in S_A$ with $\mathrm{supp}(m) \cap \mathrm{supp}(n) = \emptyset$, one has that $m+n \in S_A$. Thus $\alpha^{-1}[\{0\}] \cap S_A$ is a subsemigroup of $\beta\mathbb{N}$. Further if $B \subseteq A$, then $S_B \subseteq S_A$ so $M_s \neq \emptyset$, and is thus a compact semigroup.

(b) By Lemma 2.6(e) we have that $p, q \in M(\langle x_n\rangle_{n=1}^{\infty})$. Take nets $\langle u_i\rangle_{i\in I}$ and $\langle v_j\rangle_{j\in J}$ in $\mathbb{N}$ such that $\lim_{i\in I} u_i = p$ and $\lim_{j\in J} v_j = q$. Take $A \in s$. We can find $i_0 \in I$ such that for $i \geq i_0, u_i + q \in c\ell S_A$ (for $c\ell S_A$ is a clopen set containing $p+q$). For each i, we can then find $j(i) \in J$ such that for $j \geq j(i)$ both $u_i + v_j \in c\ell S_A$ and $\max(\mathrm{supp}(v_j)) > \max(\mathrm{supp}(u_i))$ (the latter since $\alpha_t(q) = 0$ for all t and $\lim_{j\in J} \alpha_t(v_j) = \alpha_t(q)$ for each t). Fixing i for the moment, we see that for $j \geq j(i)$, since $\mathrm{supp}(u_i) \cap \mathrm{supp}(v_j) = \emptyset$ and

$u_i + v_j \in \mathbb{N} \cap c\ell S_A = S_A$ we have $\mathrm{supp}(u_i) \subseteq A$ and $\mathrm{supp}(v_j) \subseteq A$. As q is the limit of the subnet $\langle v_j \rangle_{j \geq j(i)}$, we have $q \in c\ell S_A$. In a similar way, for $i \geq i_0$, using $\mathrm{supp}(u_i) \subseteq A$, we get $p \in c\ell S_A$. So for each $a \in S$, we have $p, q \in M(\langle x_n \rangle_{n=1}^{\infty}) \cap c\ell S_A$, and our conclusion follows. ∎

Lemma 2.9. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a divisible sequence and assume $a_n \geq 3$ for all n. Assume $p, q \in \beta\mathbb{N}$ and $p + q \in \mu^{-1}[\{1\}]$. Then there is some $r \in \mathbb{N}$ such that either $\alpha_n(q) = a_n - 1$ for all $n \geq r$ or $\alpha_n(q) \leq 1$ for all $n \geq r$.*

Proof. For $q \in \beta\mathbb{N}$, we write $\ell(q) = \sup\{t \in \omega : \alpha_t(q) \neq 0\}$ (where we allow $\ell(q) = \infty$ and put $\ell(q) = -1$ if $\alpha_t(q) = 0$ for every t).

Now $\mu^{-1}[\{1\}]$ is a neighborhood of $p + q = \rho_q(p)$ so pick $u \in \mathbb{N}$ such that $\mu(u+q) = 1$. Since $\alpha_r(u) = 0$ when $r > \ell(u)$, for such values of r we have:

(a) If there is no carrying into the r^{th} place, $\alpha_r(u + q) = \alpha_r(q)$

(b) If there is carrying into the r^{th} place, $\alpha_r(u + q) \equiv 1 + \alpha_r(q) \pmod{a_r}$.

Consequently, if there is any $r > \ell(u)$ for which there is no carrying into the r^{th} place we have from (a) and the fact that $\mu(u + q) = 1$, that eventually $\alpha_t(q) \leq 1$.

Alternately, for all $r \geq \ell(u)$ one has carrying into and out of the r^{th} place which means $\alpha_r(q) = a_r - 1$. ∎

The assumption that $a_n \geq 4$ is not necessary for the following result. In fact necessary and sufficient is the assertion that a_n is either eventually greater than 2 or frequently greater than 3. The proof under these assumptions is more complicated however.

Theorem 2.10. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a divisible sequence and assume that $a_n \geq 4$ for all $n \in \mathbb{N}$. Let $p, q \in \beta\mathbb{N}$ and let $M = M(\langle x_n \rangle_{n=1}^{\infty})$. If $p + q \in M$ and $q + p \in M$, then $p, q \in M + \mathbb{Z}$*

Proof. The hypotheses say that $\alpha(q + p) = \alpha(p + q) = 0$ and $\mu(q + p) = \mu(p + q) = 1$. Pick by Lemma 2.9 some $r \in \mathbb{N}$ such that either

$$\text{(i) } \alpha_n(q) = a_n - 1 \text{ for all } n \geq r \text{ or}$$

$$\text{(ii) } \alpha_n(q) \leq 1 \text{ for all } n \geq r.$$

We may assume also that p satisfies either statement (i) or (ii) for the same value of r.

We show first that we can't have both p and q satisfying (ii). Suppose they do. Then given $n \geq r$ there is no carrying out of position n since $1+\alpha_n(p)+\alpha_n(q) \leq 3 < a_n$. Thus for $n > r, \alpha_n(p + q) = \alpha_n(p) + \alpha_n(q)$ so $\alpha_n(p) = \alpha_n(q)$. But then we pick $k, m \in \mathbb{N}$ with $\alpha(k) = \alpha(p)$ and $\alpha(m) = \alpha(q)$ so

$$\alpha(k + m) = \alpha(k) + \alpha(m) = \alpha(p) + \alpha(q) = \alpha(p + q) = 0$$

so $k + m = 0$, a contradiction.

Thus we can assume without loss of generality that q satisfies (i). We claim that $\alpha_n(p) = 0$ for all but finitely many values of n. This will suffice since that implies $\alpha(p) = \alpha(m)$ for some $m \in \mathbb{N}$ and hence $\alpha(-m + p) = 0$. Since $(-m + p) + (m + q) = p + q \in M$ we then have that $-m + p$ and $m + q$ are in M by Lemma 2.6(e). To establish the claim, assume that we have some $n > r$ with $\alpha_n(p) > 0$. Then there is carrying out of position n. Further

$$0 = \alpha_{n+1}(p + q) \equiv 1 + \alpha_{n+1}(p) + a_{n+1} - 1 \pmod{a_{n+1}}$$

so $\alpha_{n+1}(p) = 0$ and there is carrying out of position $n + 1$. By induction $\alpha_k(p) = 0$ and there is carrying out of position k for all $k > n$. The claim is established. ∎

Corollary 2.11. *Let $\langle x_n\rangle_{n=1}^\infty$ be a divisible sequence such that $a_n \geq 4$ for all n, let $M = M(\langle x_n\rangle_{n=1}^\infty)$, let e be an idempotent in M, and let H be the largest group with e as identity. Then $H = \mathbb{Z} + (H \cap M)$.*

Proof. That $\mathbb{Z} + (H \cap M) \subseteq H$ is immediate. For the reverse inclusion let $p \in H$ and pick q such that $p + q = q + p = e$. Then by 2.10, pick $n \in \mathbb{Z}$ such that $p - n \in M$. Then also $e - n \in H$ so $p - n = p + (e - n) \in H$ so $p - n \in H \cap M$. ∎

The conclusion of Corollary 2.11 need not hold if $\frac{x_{n+1}}{x_n}$ is eventually equal to 2. For then $M(\langle x_n\rangle_{n=1}^\infty) = \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$ so intersects the smallest ideal and hence contains a minimal idempotent e. Then $H = e + \beta\mathbb{N} + e$ (from the standard structure theorem–see [2, Theorem I.2.12]) and $e + \beta\mathbb{N} + e \neq \mathbb{Z} + (H \cap M(\langle x_n\rangle_{n=1}^\infty))$, by the following result.

Theorem 2.12. *Let $\langle x_n\rangle_{n=1}^\infty$ be a divisible sequence and let e be any idempotent. Then $e + \beta\mathbb{N} + e \nsubseteq \mathbb{Z} + \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$.*

Proof. Let $\mathcal{A} = \{\mathbb{N}x_{2n+1} + \Sigma_{k=1}^n\, x_{2k} : n \in \mathbb{N}\}$. Then $\mathcal{A}$ has the finite intersection property since it is nested. Pick $p \in \beta\mathbb{N}$ with $\mathcal{A} \subseteq p$. Now given any $n \in \mathbb{N}$,

$$x_{2n+1} - \Sigma_{k=1}^n\, x_{2k} > x_{2n+1} - x_{2n} - x_{2n-1}.$$

Consequently $p \notin \mathbb{Z} + \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$. By considering the natural homomorphisms from $\beta\mathbb{N}$ to the integers mod x_n, one sees that $e + p + e \notin \mathbb{Z} + \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$. ∎

The following lemma is stated in greater generality than needed here because we will use it again in Section 5.

Lemma 2.13. *Let $\langle x_n\rangle_{n=1}^\infty$ be a divisible sequence in $\mathbb{N}$ such that a_n is frequently greater than 2 and let S be a compact subsemigroup of $\beta\mathbb{N}$ satisfying the following three statements:*

(1) $S \subseteq M(\langle x_n\rangle_{n=1}^\infty)$.

(2) If $p, q \in \beta\mathbb{N}$, $q + p \in S$, and either $p \in \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$ or $q \in \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$, then $p \in S$ and $q \in S$.

(3) Given any $p \in S$ and any infinite $L \subseteq \mathbb{N}$ there is a sequence $\langle z_n\rangle_{n=1}^\infty$ such that $FS(\langle z_n\rangle_{n=1}^\infty) \subseteq FS(\langle x_n\rangle_{n=1}^\infty)$, $FS(\langle z_n\rangle_{n=1}^\infty) \in p$, and $L \setminus \bigcup\{\mathrm{supp}(y) : y \in FS(\langle z_n\rangle_{n=1}^\infty)\}$ is infinite.

Then given any $p, q \in \mathbb{N}^*$ if $q + p \in S$, then $p \in S + \mathbb{Z}$ and $q \in S + \mathbb{Z}$.

Proof. Let $p, q \in \mathbb{N}^*$ be given with $q + p \in S$. It suffices to show that $p \in \mathbb{Z} + \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$. (Indeed assume we have $n \in \mathbb{Z}$ with $p - n \in \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$. Then $(q + n) + (p - n) = q + p \in S$ so by assumption (2), $p - n \in S$ and $q + n \in S$.) So we suppose instead that $p \notin \mathbb{Z} + \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$.

By Lemma 2.9 and assumption (1) $\alpha_n(p)$ is eventually $a_n - 1$ or is eventually in $\{0,1\}$. Since $p \notin \mathbb{Z} + \bigcap_{n=1}^\infty c\ell(\mathbb{N}x_n)$, we must have $\alpha_n(p)$ is eventually in $\{0,1\}$ and is not eventually equal to 0. Since a_n is frequently greater than 2 and $\alpha(q + p) = 0$ a simple consideration of carrying possibilities shows that $\alpha_n(p)$ is not eventually equal to 1. Thus we have that $L = \{t \in \mathbb{N} : \alpha_t(p) = 1 \textit{ and } \alpha_{t-1}(p) = 0\}$ is infinite. Pick a sequence $\langle z_n\rangle_{n=1}^\infty$ as guaranteed for L and $q + p$ by assumption (3). Now $FS(\langle z_n\rangle_{n=1}^\infty) \in q + p$ so pick $k \in \mathbb{N}$ such that $FS(\langle z_n\rangle_{n=1}^\infty) - k \in p$. Pick m and t in L such that $x_m > k$ and $t > m$ and $t \notin \bigcup\{\mathrm{supp}(y) : y \in FS(\langle z_n\rangle_{n=1}^\infty)\}$. Let $V = \{y \in \mathbb{N} : \textit{for all } n \in \{1, 2, \ldots, t\}\ \alpha_n(y) = \alpha_n(p)\}$. Then $V \in p$ so pick $y \in V \cap (FS(\langle z_n\rangle_{n=1}^\infty) - k)$. Since $y \in V$ and $t \in L$, $\alpha_t(y) = 1$ and $\alpha_{t-1}(y) = 0$. Since $k < x_m$ and $m < t$ there will be no carrying out of position $t - 1$ when k and y are added, so $\alpha_t(y + k) = 1$. But $y + k \in FS(\langle z_n\rangle_{n=1}^\infty)$ so $t \notin \mathrm{supp}(y + k)$, a contradiction. ∎

Theorem 2.14. *Let $\langle x_n\rangle_{n=1}^\infty$ be a divisible sequence such that a_n is frequently greater than 2 and let $s \in \mathbb{N}^*$. Given any $p, q \in \mathbb{N}^*$, if $q+p \in M_s$, then $p \in M_s + \mathbb{Z}$ and $q \in M_s + \mathbb{Z}$.*

Proof. Assumption (1) of Lemma 2.13 holds trivially and assumption (2) holds by Lemma 2.8(b). (If $q \in \alpha^{-1}[\{0\}]$, then since $M_s \subseteq \alpha^{-1}[\{0\}]$, one has also that $p \in \alpha^{-1}[\{0\}]$.) It thus suffices to show that assumption (3) also holds.

To this end let $p \in M$ and let an infinite $L \subseteq \mathbb{N}$ be given. Pick an infinite subset B of L such that $B \notin s$ and let $A = \mathbb{N}\backslash B$. Let $\langle z_n\rangle_{n=1}^\infty$ enumerate $\{x_n : n \in A\}$ in increasing order. Then $B \subseteq L\backslash \bigcup\{\text{supp}(y) : y \in FS(\langle z_n\rangle_{n=1}^\infty)\}$ so $\langle z_n\rangle_{n=1}^\infty$ is as required. ■

In view of our results about elements of $M(\langle x_n\rangle_{n=1}^\infty)$, it is useful to know when we can guarantee $p \in M(\langle x_n\rangle_{n=1}^\infty)$ from some weaker assumptions. Recall that if $p = p+p$, then any $A \in p$ contains $FS(\langle y_n\rangle_{n=1}^\infty)$ for some sequence $\langle y_n\rangle_{n=1}^\infty$. In particular, p will satisfy the final hypothesis of the following lemma.

Lemma 2.15. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that for each n, $x_{n+1} > 2\cdot\Sigma_{t=1}^n x_t$ and let $p \in \beta\mathbb{N}$. If $FS(\langle x_n\rangle_{n=1}^\infty) \in p$ and for each $A \in p$ there exist $y, z \in A$ with $y+z \in A$, then $p \in M(\langle x_n\rangle_{n=1}^\infty)$.*

Proof. Let $m \in \mathbb{N}$ be given and suppose $FS(\langle x_n\rangle_{n=m}^\infty) \notin p$. Now

$$FS(\langle x_n\rangle_{n=1}^\infty) = FS(\langle x_t\rangle_{t=1}^m) \cup \bigcup_{F\subseteq\{1,2,\ldots,m-1\}} (\Sigma_{t\in F}\, x_t + FS(\langle x_n\rangle_{n=m}^\infty))$$

(where $\Sigma_{t\in\emptyset}\, x_t = 0$) and $FS(\langle x_t\rangle_{t=1}^m)$ is finite. So pick a nonempty $F \subseteq \{1,2,\ldots,m-1\}$ such that $A = FS(\langle x_n\rangle_{n=m}^\infty) + \Sigma_{t\in F}\, x_t \in p$. Pick $y, z \in A$ such that $y+z \in A$. Pick G, H, and K contained in $\{m, m+1, m+2, \ldots\}$ such that $y = \Sigma_{t\in G}\, x_t + \Sigma_{t\in F}\, x_t$, $z = \Sigma_{t\in H}\, x_t + \Sigma_{t\in F}\, x_t$, and $y+z = \Sigma_{t\in K}\, x_t + \Sigma_{t\in F}\, x_t$. By [5, Lemma 1C], linear combinations of x_t's with coefficients 0, 1, or 2 are unique. In particular $F \subseteq K$ while $F \cap K = \emptyset$, a contradiction. ■

We close this section with an additional contrast between minimal idempotents and those living on $FS(\langle x_n\rangle_{n=1}^\infty)$ for a thin sequence $\langle x_n\rangle_{n=1}^\infty$. From [1, Theorem 5.15] we have that if p is any minimal idempotent in $\beta\mathbb{N}$, there is another minimal idempotent q of $\beta\mathbb{N}$ such that $p = -q+p$. (From [11, Theorem 4.2] we know $p \neq -p+p$.) By $-q$ we mean of course $\{-A : A \in q\}$. Be cautioned that $-q+q \neq 0$ unless $q \in \mathbb{Z}$.

Theorem 2.16. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that $\lim_{n\to\infty}(x_{n+1} - \Sigma_{t=1}^n x_t) = \infty$. If $p \in M(\langle x_n\rangle_{n=1}^\infty)$ and $q \in \mathbb{N}^*$, then $p \neq -q+p$.*

Proof. Suppose we have such q. Then $\{y \in \mathbb{N} : FS(\langle x_n\rangle_{n=1}^\infty) + y \in p\} \in q$ so pick $y \in \mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^\infty) + y \in p$. Pick m such that for all $\ell \geq m$, $x_{\ell+1} - \Sigma_{t=1}^\ell x_t > y$. Then $FS(\langle x_n\rangle_{n=1}^\infty) \cap (FS(\langle x_n\rangle_{n=1}^\infty) + y) \in p$. So we may pick $z = \Sigma_{n\in F}\, x_n$ in this intersection such that $\min F > m$ and $|F|$ is a minimum among all members of this intersection. Pick $G \subseteq \mathbb{N}$ such that $z = \Sigma_{n\in G}\, x_n + y$. Let $k = \max F$. If $k > \max G$, then

$$\Sigma_{n\in F}\, x_n \geq x_k > \Sigma_{t=1}^{k-1} x_t \geq \Sigma_{n\in G}\, x_n + y = \Sigma_{n\in F}\, x_n,$$

a contradiction. If $k < \max G = \ell$, then

$$\Sigma_{n\in G}\, x_n + y > x_\ell > \Sigma_{t=1}^{\ell-1} x_t \geq \Sigma_{n\in F}\, x_n = \Sigma_{n\in G}\, x_n + y,$$

again a contradiction. Thus $k = \max G$. Let $F^* = F\backslash\{k\}$ and let $G^* = G\backslash\{k\}$. Then $\Sigma_{n\in F^*}\, x_n = \Sigma_{n\in G^*}\, x_n + y$. By the minimality of $|F|$ we must have $F^* = \emptyset$ or $G^* = \emptyset$. Since $y > 0, F^* \neq \emptyset$ so $G^* = \emptyset$ so $y = \Sigma_{n\in F^*}\, x_n$. But $\Sigma_{n\in F^*}\, x_n \geq x_{m+1} > y$, a contradiction. ■

Corollary 2.17. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that $x_{n+1} > 2\cdot\Sigma_{t=1}^{n} x_t$ for all n. If $p\in\beta\mathbb{N}$, $FS(\langle x_n\rangle_{n=1}^{\infty})\in p$ and for all $A\in p$ there exist $y,z\in A$ with $y+z\in A$ (in particular if $p+p=p$), then there does not exist $q\in\mathbb{N}^*$ with $p=-q+p$.*

Proof. By Lemma 2.15, $p\in M(\langle x_n\rangle_{n=1}^{\infty})$ so Theorem 2.16 applies. ■

3. Points with Few $\leq_R$ Successors

Recall that for elements p and q of $\beta\mathbb{N}$ we write $p\leq_R q$ when $p=q+p$ or $p=q$. We establish in this section additional contrasts between idempotents in the smallest ideal and some far away from it.

If p is an element in the smallest ideal of $\beta\mathbb{N}$, then for each idempotent q of $p+\beta\mathbb{N}$, one has $p\leq_R q$ so $\{q\in\beta\mathbb{N}: p\leq_R q\}$ has 2^c elements. As we shall see in Section 4, if p is a strongly summable ultrafilter, then $\{q\in\beta\mathbb{N}: p\leq_R q\}=\{p\}$. We show in Theorem 3.5 below that we can get (in ZFC) idempotents p with $\{q\in\beta\mathbb{N}: p\leq_R q\}$ finite.

As we remarked earlier, Ruppert [20] established the existence of $\leq_R$-maximal elements of $\beta\mathbb{N}$. We shall see in Theorem 3.2 that they are plentiful.

Note that if $p,q\in\beta\mathbb{N}$, $p\neq q$, and $p\leq_R q$, then $p\in\beta\mathbb{N}+p$ so that p is not right cancellable [4, Theorem 2.1]. We see now that any element of $\beta\mathbb{N}$ which is not right cancellable lies below an idempotent which is $\leq_R$-maximal (among idempotents) and more.

Lemma 3.1. *Let $p\in\beta\mathbb{N}$ such that $p\in\beta\mathbb{N}+p$. Then there is an idempotent e of $\beta\mathbb{N}$ such that $p\leq_R e$ and whenever $f\in\beta\mathbb{N}$ with $f\in\beta\mathbb{N}+f$ and $e\leq_R f$ one has $f\leq_R e$.*

Proof. Let $\mathcal{N}=\{f\in\beta\mathbb{N}: f\in\beta\mathbb{N}+f\}$. For each $f\in\mathcal{N}$, let $B_f=\{q\in\beta\mathbb{N}: f\leq_R q\}$. Then $B_f\neq\emptyset$ and $B_f=\rho_f^{-1}[\{f\}]\cup\{f\}$ so B_f is a compact subsemigroup of $\beta\mathbb{N}$. Let $\mathcal{G}=\{\Gamma:\Gamma$ *is a* $\leq_R$*-chain in* $\mathcal{N}$ *and* $p\in\Gamma\}$. Then $\mathcal{G}\neq\emptyset$ since $\{p\}\in\mathcal{G}$ so pick a maximal member Γ of $\mathcal{G}$. Given $f\leq_R g$ in Γ one has $B_g\subseteq B_f$ so $L=\bigcap_{f\in\Gamma} B_f\neq\emptyset$.

Pick by [7, Corollary 2.10] an idempotent e in L. Then $e\in B_p$ so $p\leq_R e$. Given $f\in\mathcal{N}$ with $e\leq_R f$ one has for all $q\in\Gamma$, $q\leq_R f$ so $f\in\Gamma$ so $f\leq_R e$. ■

Theorem 3.2. *There are 2^c $\leq_R$-maximal idempotents in $\beta\mathbb{N}$.*

Proof. Define $\phi:\mathbb{N}\longrightarrow\mathbb{N}$ by $\phi(\Sigma_{n\in F}2^n)=2^{\min F}$. Denote also by ϕ its continuous extension from $\beta\mathbb{N}$ to $\beta\mathbb{N}$. We claim that given any $p\in\beta\mathbb{N}$ and any $q\in\bigcap_{n=1}^{\infty} c\ell\mathbb{N}2^n$, one has $\phi(p+q)=\phi(p)$. For this it suffices to show that for any $x\in\mathbb{N}$ and any $q\in\bigcap_{n=1}^{\infty} c\ell\mathbb{N}2^n$, $\phi(x+q)=\phi(x)$, for then $\phi\circ\rho_q$ and ϕ agree on $\mathbb{N}$. To see this, given $x=\Sigma_{n\in F}2^n$ let $\ell=\min F$. Then for all $y\in\mathbb{N}2^{\ell+1}$, $\phi(x+y)=\phi(x)$. We also observe that for $p\in c\ell\{2^n:n\in\mathbb{N}\}$, $\phi(p)=p$.

Now we show that for each $p\in\mathbb{N}^*\cap c\ell\{2^n:n\in\mathbb{N}\}$, there is a $\leq_R$-maximal idempotent f with $\phi(f)=p$. Since $|\mathbb{N}^*\cap c\ell\{2^n:n\in\mathbb{N}\}|=2^c$ [8, 9.12] this will complete the proof. Given $p\in\mathbb{N}^*\cap c\ell\{2^n:n\in\mathbb{N}\}$, $p+\beta\mathbb{N}$ is a right ideal of $\beta\mathbb{N}$ which therefore contains a minimal right ideal and hence contains an idempotent e [2, Corollary 1.3.12 and Theorem 1.3.11]. Pick $r\in\beta\mathbb{N}$ with $e=p+r$ and note that both e and p are in $\bigcap_{n=1}^{\infty} c\ell\mathbb{N}2^n$ and consequently $r\in\bigcap_{n=1}^{\infty} c\ell\mathbb{N}2^n$. Pick by Lemma 3.1 an idempotent $f\in\beta\mathbb{N}$ such that $e\leq_R f$ and f is $\leq_R$-maximal. Then

$$\phi(f)=\phi(f+e)=\phi(e)=\phi(p+r)=\phi(p)=p.$$ ■

Let $H = \bigcap_{n=1}^{\infty} c\ell\mathbb{N}2^n$. Then H contains all of the idempotents of $\beta\mathbb{N}$ (since any idempotent is in the kernel of the natural homomorphism from $\beta\mathbb{N}$ to $\mathbb{Z}_{2^n}$). We use this fact to show that $\leq_R$-maximal idempotents are plentiful very close to any idempotent in $\beta\mathbb{N}$.

Corollary 3.3. *Let p be any idempotent in $\beta\mathbb{N}$ and let $\langle A_n\rangle_{n=1}^{\infty}$ be a sequence of members of p. There exist c $\leq_R$-maximal idempotents in $\bigcap_{n=1}^{\infty} c\ell A_n$.*

Proof. Let $B_1 = A_1$ and let $C_1 = \{x \in \mathbb{N} : B_1 - x \in p\}$. Then $C_1 \in p$ so pick $x_1 \in B_1 \cap C_1$. Let $B_2 = A_2 \cap (B_1 - x_1) \cap \mathbb{N}4x_1$. Inductively, given $B_n = A_n \cap (B_{n-1} - x_{n-1}) \cap \mathbb{N}4x_{n-1}$, let $C_n = \{x \in \mathbb{N} : B_n - x \in p\}$ and pick $x_n \in B_n \cap C_n$.

Then an easy induction (see for example [14, Theorem 8.6]) establishes that whenever F is a finite nonempty subset of $\mathbb{N}$ and $m \leq \min F$ one has $\Sigma_{n\in F}\, x_n \in B_m$. Consequently $M = M(\langle x_n\rangle_{n=1}^{\infty}) \subseteq \bigcap_{n=1}^{\infty} c\ell A_n$. Trivially $\langle x_n\rangle_{n=1}^{\infty}$is a divisible sequence. The map $\tau : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $\tau(\Sigma_{n\in F}\, 2^n) = \Sigma_{n\in F}\, x_n$ extends to a map from $\beta\mathbb{N}$ to $\beta\mathbb{N}$whose restriction to $H = \bigcap_{n=1}^{\infty} c\ell\mathbb{N}2^n$ is an isomorphism onto M. Thus by Theorem 3.2 M contains 2^c idempotents which are $\leq_R$-maximal in M. Given any one such, say q and given any idempotent r with $q\leq_R r$ one has by Lemma 2.6(d) that $r \in M$ and hence $r\leq_R q$. ∎

Even though the operation in $\beta\mathbb{N}$ that we are using is "$+$" we use the multiplicative terminology "right zero" semigroup to describe a semigroup in which each element is a left identity.

Lemma 3.4. *Let C be a compact right zero subsemigroup of $\beta\mathbb{N}$. Then C is finite.*

Proof. This follows from [6, Theorem 8.4] but it has a simple self contained proof, so we present it. Suppose C is infinite and pick an infinite discrete sequence $\langle p_n\rangle_{n=1}^{\infty}$ in C. Pick an accumulation point q of the sequence $\langle p_n\rangle_{n=1}^{\infty}$. Then $q = q + q \in \beta\mathbb{N} + q = c\ell(\mathbb{N} + q)$ so $c\ell(\mathbb{N} + q) \cap c\ell\{p_n : n \in \mathbb{N}\} \neq \emptyset$ so by Lemma 1.1, either $(\mathbb{N}+q)\cap c\ell\{p_n : n \in \mathbb{N}\} \neq \emptyset$ or $\{p_n : n \in \mathbb{N}\}\cap c\ell(\mathbb{N}+q) = \emptyset$.

Assume first we have some $m \in \mathbb{N}$ with $m + q \in c\ell\{p_n : n \in \mathbb{N}\}$. Now each $p_n = p_n + p_n$ and $q = q + q$ so $\{q\} \cup \{p_n : n \in \mathbb{N}\} \subseteq c\ell\mathbb{N}(m+1)$. (Consider the natural homomorphism from $\beta\mathbb{N}$ to $\mathbb{Z}_{m+1}$.) But then $m \equiv 0 \bmod (m+1)$, a contradiction.

Thus we have some $n \in \mathbb{N}$ and some $r \in \beta\mathbb{N}$ such that $p_n = r + q$. But then $q = p_n + q = r + q + q = r + q = p_n$, a contradiction. ∎

Lemma 3.1 guarantees a plentiful supply of elements satisfying the hypothesis of the following theorem.

Theorem 3.5. *Let $e = e + e$ in $\beta\mathbb{N}$ and assume that whenever $f \in \beta\mathbb{N}$ with $f \in \beta\mathbb{N} + f$ and $e\leq_R f$ one has $f\leq_R e$. Let $C = \{f \in \beta\mathbb{N} : e\leq_R f\}$. Then C is a finite right zero semigroup.*

Proof. We have that C is a compact subsemigroup of $\beta\mathbb{N}$. As a compact right topological semigroup, C has a smallest ideal K. (See [2, Theorem 1.3.11].) We first observe that $e \in K$. Indeed, pick any $g \in K$. Then $e = g+e \in K+C \subseteq K$. Also $\{e\} = C + e$ so one, and hence all, of the minimal ideals are singletons.

We now claim that $C = K$. Suppose instead that we have some $q \in C\backslash K$. Then $q \notin \beta\mathbb{N} + q$. (For if we had $q \in \beta\mathbb{N} + q$, by hypothesis we would have $q\leq_R e$ so $q = e + q \in K + C \subseteq K$.) But then by [18, Theorem 3.3] there exist idempotents f and g in the smallest compact semigroup containing q, and hence in C, with $f = g + f = f + g \neq g$. Since $g = g + g$ we have (as in the parenthetical line above) that $g \in K$. But then $C + g$ is a minimal left ideal so $C + g = \{g\}$ while $f \in C + g$ and $f \neq g$, a contradiction.

Since $K = C$ we have for each $g \in C$ that $C + g = \{g\}$ and hence C is a right zero semigroup. By Lemma 3.4, C is finite. ∎

If e and C are as in Theorem 3.5 we have for each $g \in C$ that $\{f \in C : f + e = g\}$ is empty if $g \neq e$ and equals C if $g = e$. We shall see in Theorem 3.7 that a similar behavior applies throughout $\beta\mathbb{N}$.

Lemma 3.6. *Let $e = e + e$ in $\beta\mathbb{N}$, let $C = \{f \in \beta\mathbb{N} : e = f + e\}$ and assume C is finite. Let $p, q \in \beta\mathbb{N}$. If $p + e = q + e$ and $p \neq q$, then $p \in q + C$ or $q \in p + C$.*

Proof. Let $C = \{c_1, c_2, \ldots, c_n\}$. Suppose $p \notin q + C$ and $q \notin p + C$. Pick $X_0 \in p$ and $Y_0 \in q$ such that $X_0 \cap Y_0 = \emptyset$ and for each $i \in \{1, 2, \ldots, n\}$, $X_0 \notin q + c_i$ and $Y_0 \notin p + c_i$. (So given $i \in \{y \in \mathbb{N} : (\mathbb{N} \backslash Y_0) - y \in c_i\} \in q$ and $\{y \in \mathbb{N} : (\mathbb{N} \backslash Y_0) - y \in c_i\} \in p$.) Let $X = X_0 \cap \bigcap_{i=1}^n \{y \in \mathbb{N} : (\mathbb{N} \backslash Y_0) - y \in c_i\}$ and let $Y = Y_0 \cap \bigcap_{i=1}^n \{y \in \mathbb{N} : (\mathbb{N} \backslash X_0) - y \in c_i\}$.

Now $p + e \in (c\ell X) + e = c\ell(X + e)$ and $q + e \in c\ell(Y + e)$ so by Lemma 1.1 we have either $(X + e) \cap c\ell(Y + e) \neq \emptyset$ or $(Y + e) \cap c\ell(X + e) \neq \emptyset$. We assume without loss of generality that we have some $r \in c\ell X$ and some $m \in Y$ such that $m + e = r + e$.

One cannot have $r \in \mathbb{N}$ since then one would have $r = m$. (Consider congruence classes mod $\max\{r, m\}$.). But $r \in c\ell X$ and $m \in Y$. Thus $r \in (c\ell X) \backslash \mathbb{N}$. But now $e = (r - m) + e$, so $r - m \in C$ by definition. Pick $i \in \{1, 2, \ldots, n\}$ such that $r = m + c_i$. Since $m \in Y$, we have $(\mathbb{N} \backslash X_0) - m \in c_i$. Since $r \in c\ell X$ we have $X - m \in c_i$. But $(X - m) \cap ((\mathbb{N} \backslash X_0) - m) = \emptyset$, a contradiction. ∎

Again we remark that Lemma 3.1 and Theorem 3.5 provide a plentiful source of idempotents e satisfying the hypothesis of the following theorem.

Theorem 3.7. *Let $e = e + e$ in $\beta\mathbb{N}$, let $C = \{f \in \beta\mathbb{N} : e = f + e\}$ and assume C is a finite right zero semigroup. Let $n = |C|$. Then for each $p \in \beta\mathbb{N}$, $|\{f \in \beta\mathbb{N} : f + e = p\}|$ is $0, n$, or $n + 1$.*

Proof. Let $p \in \beta\mathbb{N}$ and assume $\{f \in \beta\mathbb{N} : f + e = p\} \neq \emptyset$. By Lemma 3.6 there is at most one $f \in \beta\mathbb{N} \backslash (\beta\mathbb{N} + C)$ such that $f + e = p$. Also if $c_1 \neq c_2$ in C we claim $(\beta\mathbb{N} + c_1) \cap (\beta\mathbb{N} + c_2) = \emptyset$. Indeed if $(\beta\mathbb{N} + c_1) \cap (\beta\mathbb{N} + c_2) \neq \emptyset$, a routine application of Lemma 1.1 yields that $c_1 \in \beta\mathbb{N} + c_2$ or $c_2 \in \beta\mathbb{N} + c_1$. But if, say, $c_1 \in \beta\mathbb{N} + c_2$ then $c_1 = c_1 + c_2$ while, since C is a right zero semigroup, $c_1 + c_2 = c_2$. Thus it suffices to show that for each $c \in C$ there is a unique $q \in \beta\mathbb{N} + c$ with $p = q + e$, so let $c \in C$ be given. Now $p + c \in \beta\mathbb{N} + c$ and $p + c + e = p + e = p$ (since $p \in \beta\mathbb{N} + e$). Now assume $q \in \beta\mathbb{N} + c$ and $p = q + e$. Then $q = q + c$ since c is an idempotent. So $q = q + c = q + e + c = p + c$. ∎

4. Properties of Strongly Summable Ultrafilters

We begin by formally defining the special kinds of ultrafilters with which we are concerned.

Definition 4.1. Let $p \in \beta\mathbb{N}$.

(a) p is a *strongly summable* ultrafilter if and only if for every $A \in p$ there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$ and $FS(\langle x_n \rangle_{n=1}^{\infty}) \in p$.

(b) p is a *divisibly strongly summable* ultrafilter if and only if for every $A \in p$ there is a divisible sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$ and $FS(\langle x_n \rangle_{n=1}^{\infty}) \in p$.

(c) p is a *special divisibly stronly summable* ultrafilter if and only if, letting $x_n = (n+1)!$, $p \in M(\langle x_n \rangle_{n=1}^{\infty})$ and p is divisibly strongly summable and for each infinite $L \subseteq \mathbb{N}$ there is a sequence $\langle z_n \rangle_{n=1}^{\infty}$ such that $FS(\langle z_n \rangle_{n=1}^{\infty}) \subseteq FS(\langle x_n \rangle_{n=1}^{\infty})$ and $FS(\langle z_n \rangle_{n=1}^{\infty}) \in p$ and $L \backslash \bigcup \{\mathrm{supp}(y) : y \in FS(\langle z_n \rangle_{n=1}^{\infty})\}$ is infinite (where *supp* is defined with respect to $\langle x_n \rangle_{n=1}^{\infty}$).

We discuss in this section properties of these ultrafilters in the order of their definitions, postponing to Section 5 discussions of their existence.

Recall from [12, Theorem 2.3] that strongly summable ultrafilters are idempotents. We show now that they are $\leq_R$-maximal in a strong sense.

Theorem 4.2. *Let p be a strongly summable ultrafilter. Then $\{q \in \beta\mathbb{N} : q + p = p\} = \{p\}$. Given any $r \in \beta\mathbb{N}$, $\{q \in \beta\mathbb{N} : q + p = r\}$ has $0, 1$, or 2 members.*

Proof. The first assertion is [4, Theorem 3.3]. The second assertion then follows from Theorem 3.7. ■

When we say $\langle x_n \rangle_{n=1}^{\infty}$ has "distinct finite sums" we mean that $\Sigma_{n \in F}\, x_n = \Sigma_{n \in G}\, x_n$ implies $F = G$. This holds in particular if $x_{n+1} > \Sigma_{t=1}^{n}\, x_t$ for each $n \in \mathbb{N}$.

Lemma 4.3. *Let $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ be sequences in $\mathbb{N}$ with distinct finite sums. Define $\tau : \mathbb{N} \longrightarrow \mathbb{N}$ as follows. If F is a finite nonempty subset of $\mathbb{N}$, then $\tau(\Sigma_{n \in F}\, y_n) = \Sigma_{n \in F}\, x_n$. If $z \in \mathbb{N} \backslash FS(\langle y_n \rangle_{n=1}^{\infty})$, then $\tau(z) = 1$. Denote also by τ its continuous extension from $\beta\mathbb{N}$ to $\beta\mathbb{N}$. The restriction of τ to $M(\langle y_n \rangle_{n=1}^{\infty})$ is an isomorphism onto $M(\langle x_n \rangle_{n=1}^{\infty})$.*

Proof. Define analogously $\delta(\Sigma_{n \in F}\, x_n) = \Sigma_{n \in F}\, y_n$. Then $\delta \circ \tau$ is the identity on $FS(\langle y_n \rangle_{n=1}^{\infty})$ so τ is one-to-one on $c\ell\ FS(\langle y_n \rangle_{n=1}^{\infty})$. That τ is a homomorphism on $M(\langle y_n \rangle_{n=1}^{\infty})$ follows from [15, Lemma 2.2]. Given $p \in M(\langle y_n \rangle_{n=1}^{\infty})$ one immediately concludes that $\tau(p) \in M(\langle x_n \rangle_{n=1}^{\infty})$. Given $q \in M(\langle x_n \rangle_{n=1}^{\infty})$, one has that $\delta(q) \in M(\langle y_n \rangle_{n=1}^{\infty})$ and $\tau(\delta(q)) = q$. ■

Our next result is a consequence of the existence of strongly summable ultrafilters.

Theorem 4.4. *Let $E = \{p \in \beta\mathbb{N} : p$ is strongly summable and for all $q \neq p,\ q + p \neq p$ and $p + q \neq p\}$ and assume there exists some strongly summable ultrafilter. Then E is dense in the set of idempotents. In particular, the set of idempotents which are both $\leq_L$-maximal and $\leq_R$-maximal is dense in the set of all idempotents.*

Proof. Using [11, Lemma 2.5] one sees that
(*) if for each $n \in \mathbb{N}$, $x_{n+1} > 4 \cdot \Sigma_{t=1}^{n}\, x_t$ and $\langle z_n \rangle_{n=1}^{\infty}$ is a sequence with $FS(\langle z_n \rangle_{n=1}^{\infty}) \subseteq FS(\langle x_n \rangle_{n=1}^{\infty})$, there exists a sequence $\langle F_m \rangle_{m=1}^{\infty}$ of pairwise disjoint finite sets such that $z_m = \Sigma_{t \in F_m}\, x_t$ for each $m \in \mathbb{N}$.

Let r be an idempotent of $\beta\mathbb{N}$ and let $V \in r$. Pick (by [13, Theorem 3.3]) a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq V$. By thinning we may presume $x_{n+1} > 4 \cdot \Sigma_{t=1}^{n}\, x_t$ for all $n \in \mathbb{N}$. Pick a strongly summable ultrafilter s and pick by [11,Lemma 2.4] a sequence $\langle y_n \rangle_{n=1}^{\infty}$ with $y_{n+1} > 4 \cdot \Sigma_{t=1}^{n}\, y_t$ for each n and with $FS(\langle y_n \rangle_{n=1}^{\infty}) \in s$. Let τ be as in Lemma 4.3 and let $p = \tau(s)$. Using (*) one easily sees that p is strongly summable.

Since p is strongly summable we have by Theorem 4.2 that for all $q \neq p$, $q+p \neq p$.

Now let $q \neq p$ be given and suppose that $p+q=p$. Since p is idempotent we have that for all n, $\mathbb{N}n \in p$ and consequently for all n, $\mathbb{N}n \in q$. Pick $A \in p \backslash q$. Since p is strongly summable, pick by [12, Lemma 2.2] a sequence $\langle z_n \rangle_{n=1}^{\infty}$ with $p \in M(\langle z_n \rangle_{n=1}^{\infty})$ and $FS(\langle z_n \rangle_{n=1}^{\infty}) \subseteq A \cap FS(\langle x_n \rangle_{n=1}^{\infty})$. Let $B = FS(\langle z_n \rangle_{n=1}^{\infty})$. Then $B \in p = p+q$ so $\{y \in \mathbb{N} : B - y \in q\} \in p$ so pick $y \in B$ such that $b - y \in q$. Now $y = \Sigma_{n \in F}\, z_n$ for some F. Note that if $n \neq m$ then $\mathrm{supp}(z_n) \cap \mathrm{supp}(z_m) = \emptyset$. (For if we had $t \in \mathrm{supp}(z_n) \cap \mathrm{supp}(z_m)$ we would have $\alpha_t(z_n + z_m) = 2$.) Let $H = \bigcup_{n \in F} \mathrm{supp}(z_n)$. Then $H = \mathrm{supp}(y)$ and in fact $y = \Sigma_{t \in H}\, x_t$. Let $k = \max H + 1$. Then $\mathbb{N}x_k \in q$ so pick $w \in \mathbb{N}x_k \cap (B - y) \cap (\mathbb{N} \backslash A)$. Since x_k divides w we have $\min \mathrm{supp}(w) > k$ so $\mathrm{supp}(w + y) = \mathrm{supp}(w) \cup \mathrm{supp}(y)$. Since $w + y \in B$ we have for each $t \in \mathrm{supp}(w+y)$ that $\alpha_t(w+y) = 1$. We also have for some $L \subseteq \mathbb{N}$, $w + y = \Sigma_{n \in L}\, z_n$. Then $\mathrm{supp}(w+y) = \bigcup_{n \in L} \mathrm{supp}(z_n)$. Consequently we have $F \subseteq L$ and $w = \Sigma_{n \in L \backslash F}\, z_n \in A$, a contradiction. ■

We now turn our attention to divisibly strongly summable ultrafilters.

Lemma 4.5. *Let p be a divisibly strongly summable ultrafilter and let $A \in p$. There is a divisible sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$, $p \in M(\langle x_n \rangle_{n=1}^{\infty})$, and for each n, $x_{n+1} > 4 \cdot \Sigma_{t=1}^{n}\, x_t$.*

Proof. This may be taken nearly verbatim from the proof of [11, Lemma 2.4]. ■

Theorem 4.6. *Let p be a divisibly strongly summable ultrafilter and let $q, r \in \beta\mathbb{N}$. If $r \in \bigcap_{n=1}^{\infty} c\ell\mathbb{N}n$ and $p = q + r$, then q=r=p. In particular, if $q \neq p$, then $p + q \neq p$.*

Proof. It suffices by Theorem 4.2 to show $r = p$. Suppose instead that $r \neq p$ and pick $A \in p \backslash r$. Pick a divisible sequence $\langle x_n \rangle_{n=1}^{\infty}$ as guaranteed by Lemma 4.5. By Lemma 2.6(e), $r \in M(\langle x_n \rangle_{n=1}^{\infty})$ so $A \in r$, a contradiction. ■

Finally we see that special divisibly strongly summable ultrafilters can only be written as sums in a trivial fashion.

Theorem 4.7. *Let p be a special divisibly strongly summable ultrafilter and let q and r be in $\beta\mathbb{N}$. If $p = q + r$, then for some $n \in \mathbb{Z}$, $q = p + n$ and $r = p - n$.*

Proof. For each n let $x_n = (n+1)!$. Let $S = \{p\}$. Then hypotheses (1) and (3) of Lemma 2.13 hold immediately. Now $\bigcap_{n=1}^{\infty} c\ell\mathbb{N}n = \bigcap_{n=1}^{\infty} c\ell\mathbb{N}x_n$ so hypothesis (2) of Lemma 2.13 holds by Theorem 4.6 (because if $q \in \bigcap_{n=1}^{\infty} c\ell\mathbb{N}n$ and $p = q + r$, then $r \in \bigcap_{n=1}^{\infty} c\ell\mathbb{N}n$). The hypotheses being satisfied, Lemma 2.13 yields the desired result. ■

Corollary 4.8. *Let p be a special divisibly strongly summable ultrafilter. Then $p + \mathbb{N}^*$ is maximal among right ideals of the form $q + \mathbb{N}^*$ and $\mathbb{N}^* + p$ is maximal among left ideals of the form $\mathbb{N}^* + q$.*

Proof. We prove only the assertion about right ideals since the other proof is nearly identical. Assume $p + \mathbb{N}^* \subseteq q + \mathbb{N}^*$. Then $p \in q + \mathbb{N}^*$ so $p = q + r$ for some $r \in \mathbb{N}^*$. By Theorem 4.7, $q = p + n$ for some $n \in \mathbb{Z}$. Thus $q + \mathbb{N}^* \subseteq p + \mathbb{N}^*$. ■

5. Existence of Divisibly Strongly Summable Ultrafilters

We show in this section that Martin's Axiom implies the existence of divisibly strongly summable ultrafilters very close (i.e. sharing any fewer than c prespecified members) to any idempotent. Similarly Martin's Axiom also implies the existence of special divisibly strongly summable ultrafilters very close to any idempotent in $M(\langle n!\rangle_{n=1}^\infty)$. (As was shown in [19] and [5] the existence of strongly summable ultrafilters cannot be proved in ZFC.)

Since the continuum hypothesis (CH) implies Martin's Axiom (MA) these results also follow from CH. However, because CH constructions are more familiar to many people, we present in Theorem 5.2 a CH construction of a special divisibly strongly summable ultrafilter.

The following lemma is well known among aficianados.

Lemma 5.1. *Let $\langle x_n\rangle_{n=1}^\infty$ be an increasing sequence in $\mathbb{N}$, let $r\in\mathbb{N}$, and assume $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq\bigcup_{i=1}^r C_i$. There exist $i\in\{1,2,\ldots,r\}$ and a sequence $\langle y_n\rangle_{n=1}^\infty$ such that*

(1) $FS(\langle y_n\rangle_{n=1}^\infty)\subseteq C_i\cap FS(\langle x_n\rangle_{n=1}^\infty)$,

(2) for each n, $y_n>2\cdot\Sigma_{t=1}^n y_t$,

(3) for each n, $FS(\langle y_t\rangle_{t=n}^\infty)\subseteq FS(\langle x_t\rangle_{t=n}^\infty)$,

(4) for each n, $y_n|y_{n+1}$, and

(5) if for some m, $FS(\langle x_n\rangle_{n=m}^\infty)\subseteq FS(\langle (n+1)!\rangle_{n=1}^\infty)$, then $FS(\langle y_n\rangle_{n=1}^\infty)\subseteq FS(\langle (n+1)!\rangle_{n=1}^\infty)$.

Proof. If there is some integer m such that $FS(\langle x_n\rangle_{n=m}^\infty)\subseteq FS(\langle (n+1)!\rangle_{n=1}^\infty)$, pick the first such and replace each x_n by x_{n+m-1}. For each $i\in\{1,2,\ldots,r\}$, let $E_i=\{F: F$ is a finite nonempty subset of $\mathbb{N}$ and $\Sigma_{n\in F}\,x_n\in C_i\}$. By [10, Corollary 3.2], pick an infinite sequence $\langle H_n\rangle_{n=1}^\infty$ of pairwise disjoint finite nonempty subsets of $\mathbb{N}$ and $i\in\{1,2,\ldots,r\}$ such that $\bigcup_{n\in F} H_n\in E_i$ whenever F is a finite nonempty subset of $\mathbb{N}$. By discarding some of the H_n's we may presume that for each n, $\max H_n<\min H_{n+1}$. Now let $y_1=\Sigma_{t\in H_1}\,x_t$. Given $y_n=\Sigma_{k\in G_n}\Sigma_{t\in H_k}\,x_t$, pick ℓ such that $x_\ell>2\cdot\Sigma_{k=1}^n y_k$. Choose a subset G_{n+1} of $\{\ell+1,\ell+2,\ldots\}$ such that $|G_{n+1}|=y+n$ and so that for $k,s\in G_{n+1}$, $\Sigma_{t\in H_k}\,x_t\equiv\Sigma_{t\in H_s}\,x_t \pmod{y_n}$. Then letting $y_{n+1}=\Sigma_{k\in G_{n+1}}\Sigma_{t\in H_k}\,x_t$, we have $y_n|y_{n+1}$ and $y_{n+1}>2\cdot\Sigma_{t=1}^n y_t$. ■

Theorem 5.2. *Assume the continuum hypothesis. There exists a special divisibly strongly summable ultrafilter.*

Proof. Let $x_n=(n+1)!$ for each n and let α and *supp* be as given in Definition 2.4 for $\langle x_n\rangle_{n=1}^\infty$. Well order $\mathcal{P}(\mathbb{N})$ as $\langle A_\sigma\rangle_{\sigma<\omega_1}$ with $A_0=\mathbb{N}$. Let $Z_0=A_0$ and let $\Pi_0=\{FS(\langle (2t)!\rangle_{t=m}^\infty): m\in\mathbb{N}\}\cup\{\mathbb{N}\}$. Let $\delta<\omega_1$ be given and assume we have chosen Z_σ and Π_σ for $\sigma<\delta$ satisfying the following induction hypotheses.

(1) $Z_\sigma=A_\sigma$ or $Z_\sigma=\mathbb{N}\backslash A_\sigma$;

(2) $Z_\sigma\in\Pi_\sigma$;

(3) $|\Pi_\sigma|=\omega$;

(4) Π_σ is closed under finite intersections;

(5) if $\sigma<\tau$, then $\Pi_\sigma\subseteq\Pi_\tau$;

(6) for each $B\in\Pi_\sigma$ there is a divisible sequence $\langle y_n\rangle_{n=1}^\infty$ such that $FS(\langle y_n\rangle_{n=1}^\infty)\subseteq B$ and for each $m\in\mathbb{N}$, $FS(\langle y_n\rangle_{n=m}^\infty)\in\Pi_\sigma$;

(7) if A_σ is infinite, then there is a sequence $\langle z_n\rangle_{n=1}^\infty$ such that $FS(\langle z_n\rangle_{n=1}^\infty)\in\Pi_\sigma$ and $FS(\langle z_n\rangle_{n=1}^\infty)\subseteq FS(\langle x_n\rangle_{n=1}^\infty)$ and $A_\sigma\backslash\bigcup\{\text{supp}(y): y\in FS(\langle z_n\rangle_{n=1}^\infty)\}$ is infinite.

We observe that all hypotheses are satisfied at $\sigma = 0$.

Now let $\Pi_\delta' = \bigcup_{\sigma<\delta} \Pi_\sigma$ and note that Π_δ' satisfies hypotheses (3), (4), and (6). Let $\langle V_n\rangle_{n=1}^\infty$ enumerate Π_δ' with $V_1 = FS(\langle(2t)!\rangle_{t=1}^\infty)$. Let $U_1 = V_1$. Pick $\langle y_{1,n}\rangle_{n=1}^\infty$ as guaranteed by (6) for U_1, let $w_1 = y_{1,1}$, and let $U_2 = U_1 \cap V_2 \cap FS(\langle y_{1,k}\rangle_{k=2}^\infty)$. Note that $U_2 \in \Pi_\delta'$.

Inductively, given $U_n \in \Pi_\delta'$, pick $\langle y_{n,k}\rangle_{k=1}^\infty$ as guaranteed by (6) for U_n. Let $w_n = y_{n,1}$, and let $U_{n+1} = U_n \cap V_{n+1} \cap FS(\langle y_{n,k}\rangle_{k=2}^\infty)$.

Now we observe that for each n, $U_{n+1} \subseteq U_n - w_n$. Indeed, given $z \in U_{n+1}$ we have $z \in FS(\langle y_{n,k}\rangle_{k=2}^\infty)$ so $z + w_n \in FS(\langle y_{n,k}\rangle_{k=2}^\infty) \subseteq U_n$. Thus one routinely proves by induction on $|F|$ that if F is a finite subset of $\mathbb{N}$ and $\ell = \min F$, then $\Sigma_{t\in F}\, w_t \in U_\ell$ (and hence $FS(\langle w_n\rangle_{n=\ell}^\infty) \subseteq U_\ell$).

Let $C_0 = A_\delta$ and $C_1 = \mathbb{N}\backslash A_\delta$. By Lemma 5.1, pick an increasing sequence $\langle u_n\rangle_{n=1}^\infty$ and $i \in \{0,1\}$ such that $FS(\langle u_n\rangle_{n=1}^\infty) \subseteq FS(\langle w_n\rangle_{n=1}^\infty) \cap C_i$ and for each n, $u_n | u_{n+1}$ and for each n, $FS(\langle u_t\rangle_{t=n}^\infty) \subseteq FS(\langle w_t\rangle_{t=n}^\infty)$.

If $i = 0$, let $Z_\delta = A_\delta$. If $i = 1$, let $Z_\delta = \mathbb{N}\backslash A_\delta$. If A_δ is finite, let $z_n = u_n$ for each n.

If A_δ is infinite, proceed as follows. Observe that $FS(\langle u_n\rangle_{n=1}^\infty) \subseteq FS(\langle w_n\rangle_{n=1}^\infty) \subseteq U_1 \subseteq FS(\langle x_n\rangle_{n=1}^\infty)$. Let $z_1 = u_1$. Inductively given $z_n = u_{\ell(n)}$, let $t = \max\ \mathrm{supp}(z_n)$, pick $j \in A_\delta$ with $j > t$ and pick $\ell(n+1)$ such that $\min\ \mathrm{supp}(u_{\ell(n+1)}) > j$. Let $z_{n+1} = u_{\ell(n+1)}$.

Let $\Pi_\delta^* = \Pi_\delta' \cup \{Z_\delta\} \cup \{FS(\langle z_m\rangle_{n=m}^\infty) : m \in \mathbb{N}\}$ and let $\Pi_\delta = \{\bigcap \mathcal{F} : \mathcal{F}$ *is a finite nonempty subset of* $\Pi_\delta^*\}$. Then hypotheses (1), (2), (3), (4), (5), and (7) follow immediately.

To verify hypothesis (6), let $B \in \Pi_\delta$ be given. Then pick $W \in \Pi_\delta'$ and $m \in \mathbb{N}$ such that $W \cap FS(\langle z_k\rangle_{k=m}^\infty) \subseteq B$. Further $W = V_n$ for some n so let $\ell = \max\{n,m\}$ and let $y_k = z_{\ell+k}$ for all $k \in \mathbb{N}$. Then given $r \in \mathbb{N}$, $FS(\langle y_k\rangle_{k=r}^\infty) = FS(\langle z_k\rangle_{k=\ell+r}^\infty) \in \Pi_\alpha$. Also $FS(\langle y_k\rangle_{k=1}^\infty) \subseteq FS(\langle z_k\rangle_{k=m}^\infty)$ and

$$FS(\langle y_k\rangle_{k=1}^\infty) \subseteq FS(\langle z_k\rangle_{k=n}^\infty) \subseteq FS(\langle u_k\rangle_{k=n}^\infty) \subseteq FS(\langle w_k\rangle_{k=n}^\infty) \subseteq U_n \subseteq V_n.$$

Thus $FS(\langle y_k\rangle_{k=1}^\infty) \subseteq B$ so (6) holds.

Let $p = \bigcup_{\delta<\omega_1} \Pi_\delta$. Then p is a special divisibly strongly summable ultrafilter. ■

We now turn our attention to an MA construction of divisibly strongly summable ultrafilters and special divisibly strongly summable ultrafilters. The reader unfamiliar with MA is referred to [17, pp. 53-61] for an elementary introduction (with the caution that an inequality is reversed in the definition of "filter").

Definition 5.3. (a) $\mathcal{P}_f(A)$ is the set of finite nonempty subsets of A and $[A]^\omega$ is the set of countably infinite subsets of A.

(b) $(\mathcal{A}, f)$ is a *divisibly strongly summable pair* if and only if

(1) $\emptyset \neq \mathcal{A} \subseteq [\mathbb{N}]^\omega$,

(2) $f : \mathcal{P}_f(\mathcal{A}) \longrightarrow [\mathbb{N}]^\omega$,

(3) if $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$ and $\langle y_n\rangle_{n=1}^\infty$ enumerates $f(\mathcal{F})$ in increasing order then

(i) $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq \bigcap \mathcal{F}$

(ii) $\{FS(\langle y_n\rangle_{n=m}^\infty) : m \in \mathbb{N}\} \subseteq \mathcal{A}$

(iii) for all n, $y_n | y_{n+1}$.

(c) $(\mathcal{A},f)$ is a *weakly summable pair* if and only if

(1) $\emptyset \neq \mathcal{A} \subseteq [\mathbb{N}]^\omega$,

(2) $f : \mathcal{P}_f(\mathcal{A}) \longrightarrow [\mathbb{N}]^\omega$,

(3) for all $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$,

(i) $f(\mathcal{F}) \subseteq \bigcap \mathcal{F}$, and

(ii) for all $x \in f(\mathcal{F})$ there exists $B \in \mathcal{A}$ such that $B \subseteq (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{F} - x)$.

(d) Given $\mathcal{A} \subseteq [\mathbb{N}]^\omega$, $Q(\mathcal{A}) = \{\langle s, \mathcal{F}\rangle : s \in \mathcal{P}_f(\mathbb{N})$ and $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})\}$.

(e) Given $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ and $\langle s, \mathcal{F}\rangle$ and $\langle s', \mathcal{F}'\rangle$ in $Q(\mathcal{A})$, $\langle s', \mathcal{F}'\rangle \leq \langle s, \mathcal{F}\rangle$ if and only if

(1) $s \subseteq s'$,

(2) $\mathcal{F} \subseteq \mathcal{F}'$,

(3) for all $y \in s'\backslash s$ and all $x \in s$, $x < y$,

(4) there exists $g : s'\backslash s \longrightarrow \mathcal{F}'$ such that for x and y in $s'\backslash s$

(i) $x \in \bigcap \mathcal{F}$ and $g(x) \subseteq (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{F} - x)$, and

(ii) if $x < y$, then $y \in g(x)$ and $g(y) \subseteq g(x) \cap (g(x) - y)$.

(f) Given $\mathcal{A} \subseteq [\mathbb{N}]^\omega$, $V \in \mathcal{A}$, and $n \in \mathbb{N}$,

(1) $D(V) = \{\langle s, \mathcal{F}\rangle \in Q(\mathcal{A}) : V \in \mathcal{F}\}$,

(2) $E(n) = \{\langle s, \mathcal{F}\rangle \in Q(\mathcal{A}) : s\backslash\{1, 2, \ldots, n\} \neq \emptyset\}$.

Observe that a divisibly strongly summable pair is also a weakly summable pair. (To verify (c) (3) (ii), given $f(\mathcal{F}) = \{y_n : n \in \mathbb{N}\}$ one has for each m that $FS(\langle y_n\rangle_{n=m+1}^\infty) \subseteq (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{F} - y_m)$.)

Lemma 5.4. *Let $\omega \leq \kappa < c$ and assume MA(κ). Let $(\mathcal{A}, f)$ be a weakly summable pair with $|\mathcal{A}| = \kappa$ and let $C \subseteq \mathbb{N}$. There exists a divisibly strongly summable pair ($\mathcal{B}$,g) such that*

(1) $\mathcal{A} \subseteq \mathcal{B}$,

(2) $|\mathcal{B}| = \kappa$,

(3) $C \in \mathcal{B}$ or $\mathbb{N}\backslash C \in \mathcal{B}$,

(4) there is a sequence $\langle y_n\rangle_{n=1}^\infty$ such that

(i) $FS(\langle y_n\rangle_{n=1}^\infty) \in \mathcal{B}$

(ii) for each n, $y_{n+1} > 2 \cdot \Sigma_{t=1}^n y_t$,

(iii) for each $\mathcal{H} \in \mathcal{P}_f(\mathcal{B})$ there exists $m \in \mathbb{N}$ such that $FS(\langle y_n\rangle_{n=m}^\infty) \subseteq \bigcap \mathcal{H}$, and

(iv) if $FS(\langle (n+1)!\rangle_{n=1}^\infty) \in \mathcal{A}$ and $|C| = \omega$, then $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq FS(\langle (n+1)!\rangle_{n=1}^\infty)$ and $|C\backslash \bigcup_{n=1}^\infty \mathrm{supp}(y_n)| = \omega$, where supp is as given in Definition 2.4 for $x_n = (n+1)!$.

(5) if $(\mathcal{A}, f)$ is a divisibly strongly summable pair, then $f \subseteq g$.

Proof. By [12, Lemma 3.7], $Q(\mathcal{A})$ is a c.c.c partial order. By [12, Lemmas 3.5 and 3.6] $\{D(V) : V \in \mathcal{A}\} \cup \{E(n) : n \in \mathbb{N}\}$ is a set of κ dense subsets of $Q(\mathcal{A})$. Pick by MA(κ), a filter $\mathcal{G}$ in $Q(\mathcal{A})$ such that $\mathcal{G} \cap D(V) \neq \emptyset$ for each $V \in \mathcal{A}$ and $\mathcal{G} \cap E(n) \neq \emptyset$ for each $n \in \mathbb{N}$. Let $A = \bigcup\{s :$ *for some* $\mathcal{F}$, $\langle s, \mathcal{F}\rangle \in \mathcal{G}\}$. Since $\mathcal{G} \cap E(n) \neq \emptyset$ for each n we have that A is infinite. Let $\langle z_n\rangle_{n=1}^\infty$ enumerate A in increasing order

We now show that if $FS(\langle (n+1)!\rangle_{n=1}^\infty) \in \mathcal{A}$, then there is some ℓ such that $FS(\langle z_n\rangle_{n=\ell}^\infty) \subseteq FS(\langle (n+1)!\rangle_{n=1}^\infty)$. To this end, let $W = FS(\langle (n+1)!\rangle_{n=1}^\infty)$ and assume $W \in \mathcal{A}$. Pick $\langle s_W, \mathcal{F}_W\rangle \in \mathcal{G} \cap D(W)$. Let $\ell = \max(s_W)+1$. To see that $FS(\langle z_n\rangle_{n=\ell}^\infty) \subseteq W$, let $L \in \mathcal{P}_f(\mathbb{N})$ with $\min L \geq \ell$. For each $n \in L$, pick $\langle s_n, \mathcal{F}_n\rangle \in \mathcal{G}$ with $z_n \in s_n$. Pick $\langle s, \mathcal{F}\rangle \in \mathcal{G}$ such that $\langle s, \mathcal{F}\rangle \leq \langle s_W, \mathcal{F}_W\rangle$

and $\langle s,\mathcal{F}\rangle \leq \langle s_n,\mathcal{F}_n\rangle$ for each $n\in L$. Then $\{z_n : n\in L\}\subseteq s\backslash s_W$. Now by [12, Lemma 3.7(b)], $\Sigma_{n\in L}\, z_n \in \bigcap \mathcal{F}_W \subseteq W$.

Let $C_0 = C$ and $C_1 = \mathbb{N}\backslash C$. Pick by Lemma 5.1 some $i\in\{0,1\}$ and a sequence $\langle w_n\rangle_{n=1}^\infty$ such that $FS(\langle w_n\rangle_{n=1}^\infty)\subseteq C_i\cap FS(\langle z_n\rangle_{n=1}^\infty)$, for each n $w_{n+1} > 2\cdot\Sigma_{t=1}^n\, w_t$, $FS(\langle w_t\rangle_{t=n}^\infty)\subseteq FS(\langle z_t\rangle_{t=n}^\infty)$, and $w_n|w_{n+1}$ and so that if for some ℓ, $FS(\langle z_t\rangle_{t=\ell}^\infty)\subseteq FS(\langle (n+1)!\rangle_{n=1}^\infty)$, then $FS(\langle w_n\rangle_{n=1}^\infty)\subseteq FS(\langle (n+1)!\rangle_{n=1}^\infty)$.

If $FS(\langle (n+1)!\rangle_{n=1}^\infty)\notin\mathcal{A}$ or C is finite let $y_n = w_n$ for each n. Otherwise proceed as follows. We have shown that for some ℓ, $FS(\langle z_n\rangle_{n=\ell}^\infty)\subseteq FS(\langle (n+1)!\rangle_{n=1}^\infty)$ so we have that $FS(\langle w_t\rangle_{t=1}^\infty)\subseteq FS(\langle (t+1)!\rangle_{t=1}^\infty)$. Let $y_1 = w_1$ and inductively assume we have $y_n = w_{\ell(n)}$. Let $t = \max\ \mathrm{supp}(y_n)$, pick $j\in C$ with $j>t$, pick $\ell(n+1)$ with $\min\ \mathrm{supp}(w_{\ell(n+1)}) > j$. (Since $FS(\langle w_t\rangle_{t=1}^\infty)\subseteq FS(\langle (t+1)!\rangle_{t=1}^\infty)$, we know $\mathrm{supp}(w_r)\cap\mathrm{supp}(w_s)=\emptyset$ for $r\neq s$.) Let $y_{n+1} = w_{\ell(n+1)}$. Observe that conclusion (4) (iv) holds for $\langle y_n\rangle_{n=1}^\infty$.

Let $\mathcal{B} = \mathcal{A}\cup\{C_i\}\cup\{FS(\langle y_n\rangle_{n=m}^\infty) : m\in\mathbb{N}\}$. Then conclusions (1), (2), (3), (4) (i), (4) (ii), and (4) (iv) hold immediately. We claim it suffices to establish (4) (iii). Indeed, assume we have done this. Given $\mathcal{H}\in\mathcal{P}_f(\mathcal{B})$, pick $m\in\mathbb{N}$ such that $FS(\langle y_n\rangle_{n=m}^\infty)\subseteq\bigcap\mathcal{H}$. If $\mathcal{H}\subseteq\mathcal{A}$ and $(\mathcal{A},f)$ is a divisibly strongly summable pair, let $g(\mathcal{H}) = f(\mathcal{H})$ (so that (5) will hold). Otherwise $g(\mathcal{H}) = \{y_n : n\geq m\}$. We claim that $(\mathcal{B},g)$ is a divisibly strongly summable pair. Certainly requirements (1) and (2) of the definition are immediate. Let $\mathcal{H}\in\mathcal{P}_f(\mathcal{B})$. If $\mathcal{H}\subseteq\mathcal{A}$ and $(\mathcal{A},f)$ is a divisibly strongly summable pair, then (3) holds for $(\mathcal{B},g)$ because it holds for $(\mathcal{A},f)$. Otherwise, $g(\mathcal{H}) = \{y_n : n\geq m\}$ and conclusions (3) (i), (ii), and (iii) hold for $y'_k = y_{m+k-1}$.

It thus remains to establish statement (4) (iii) of this lemma. If $\mathcal{H}\cap\mathcal{A} = \emptyset$, then $\mathcal{H}$ is nested so we easily pick m with $FS(\langle y_n\rangle_{n=m}^\infty)\subseteq\bigcap\mathcal{H}$. So we assume $\mathcal{H}\cap\mathcal{A}\neq\emptyset$. If $\mathcal{H}\backslash\mathcal{A}\neq\emptyset$, pick m such that $FS(\langle y_n\rangle_{n=m}^\infty)\subseteq\bigcap(\mathcal{H}\backslash\mathcal{A})$. If $\mathcal{H}\subseteq\mathcal{A}$, let $m=1$.

For each $V\in\mathcal{H}\cap\mathcal{A}$, pick $\langle s_V,\mathcal{F}_V\rangle\in\mathcal{G}\cap D(V)$. Pick $\langle s,\mathcal{F}\rangle\in\mathcal{G}$ such that $\langle s,\mathcal{F}\rangle\leq\langle s_V,\mathcal{F}_V\rangle$ for each $V\in\mathcal{H}\cap\mathcal{A}$. Let $\ell = \max(s\cup\{m\})+1$. We claim that $FS(\langle y_n\rangle_{n=\ell}^\infty)\subseteq\bigcap\mathcal{H}$. Since $FS(\langle y_n\rangle_{n=\ell}^\infty)\subseteq FS(\langle y_n\rangle_{n=m}^\infty)$ and since $\mathcal{H}\cap\mathcal{A}\subseteq\mathcal{F}$, it suffices to show $FS(\langle y_n\rangle_{n=\ell}^\infty)\subseteq\bigcap\mathcal{F}$. Since $FS(\langle y_n\rangle_{n=\ell}^\infty)\subseteq FS(\langle w_n\rangle_{n=\ell}^\infty)\subseteq FS(\langle z_n\rangle_{n=\ell}^\infty)$, it suffices to show $FS(\langle z_n\rangle_{n=\ell}^\infty)\subseteq\bigcap\mathcal{F}$. So let $T\in\mathcal{P}_f(\mathbb{N})$ with $\min T\geq\ell$. For each $n\in T$ pick $\langle s_n,\mathcal{F}_n\rangle\in\mathcal{G}$ with $z_n\in s_n$. Pick $\langle s',\mathcal{F}'\rangle\in\mathcal{G}$ with $\langle s',\mathcal{F}'\rangle\leq\langle s,\mathcal{F}\rangle$ and for each $n\in T$, $\langle s',\mathcal{F}'\rangle\leq\langle s_n,\mathcal{F}_n\rangle$. Now given $n\in T$, $z_n\geq n\geq\ell>\max s$ so $\{z_n : n\in T\}\subseteq s'\backslash s$. Thus by [12, Lemma 3.7(b)], $\Sigma_{n\in T}\, z_n\in\bigcap\mathcal{F}$ as required. ∎

Recall that MA is the assertion that for all $\kappa < c$, MA(κ) holds.

Theorem 5.5. *Assume MA. Let $p\in\beta\mathbb{N}$ with $p+p=p$ and let $\mathcal{A}\subseteq p$ with $|\mathcal{A}|<c$. There exists a divisibly strongly summable ultrafilter q such that $\mathcal{A}\subseteq q$. If $FS(\langle (n+1)!\rangle_{n=1}^\infty)\in p$, then q can be chosen to be a divisibly strongly summable ultrafilter.*

Proof. If $FS(\langle (n+1)!\rangle_{n=1}^\infty)\in p$, then we may presume that $FS(\langle (n+1)!\rangle_{n=1}^\infty)\in\mathcal{A}$. We may also presume $|\mathcal{A}|\geq\omega$. By [12, Lemma 3.10], pick a weakly summable pair $(\mathcal{C},f)$ such that $\mathcal{A}\subseteq\mathcal{C}$ and $|\mathcal{C}| = |\mathcal{A}|$. Well order $\mathcal{P}(\mathbb{N})$ as $\langle C_\sigma\rangle_{\sigma<c}$. Pick a divisibly strongly summable pair $(\mathcal{B}_0,g_0)$ as guaranteed by Lemma 5.4 for $(\mathcal{A},f)$ and C_0.

Inductively, let $\sigma<c$ be given and assume for $\delta<\sigma$ we have chosen $(\mathcal{B}_\delta,g_\delta)$ such that:

(1) $(\mathcal{B}_\delta,g_\delta)$ is a divisibly strongly summable pair,

(2) if $\tau<\delta$, then $\mathcal{B}_\tau\subseteq\mathcal{B}_\delta$ and $g_\tau\subseteq g_\delta$,

(3) $C_\delta \in \mathcal{B}_\delta$ or $\mathbb{N}\backslash C_\delta \in \mathcal{B}_\delta$,

(4) $|\mathcal{B}_\delta| \leq \max\{|\mathcal{C}|, |\delta|\}$, and

(5) if $FS(\langle (n+1)! \rangle_{n=1}^\infty) \in \mathcal{A}$ and $|C_\delta| = \omega$, then there is a sequence $\langle y_n \rangle_{n=1}^\infty$ such that $FS(\langle y_n \rangle_{n=1}^\infty) \in \mathcal{B}_\delta$ and $FS(\langle y_n \rangle_{n=1}^\infty) \subseteq FS(\langle (n+1)! \rangle_{n=1}^\infty)$ and $|C_\delta \backslash \bigcup_{n=1}^\infty \mathrm{supp}(y_n)| = \omega$.

Let $\mathcal{B}'_\sigma = \bigcup_{\delta<\sigma} \mathcal{B}_\delta$ and let $g'_\sigma = \bigcup_{\delta<\sigma} g_\delta$ and note that $(\mathcal{B}'_\sigma, g'_\sigma)$ is a divisbly strongly summable pair. Choose a divisibly strongly summable pair $(\mathcal{B}_\sigma, g_\sigma)$ as guaranteed by Lemma 5.4 for $(\mathcal{B}'_\sigma, g'_\sigma)$ and C_σ. All induction hypotheses are satisfied.

Let $q = \bigcup_{\sigma<c} \mathcal{B}_\sigma$. By hypothesis (3), q is an ultrafilter. To see that q is divisibly strongly summable, let $A \in q$ and pick $\sigma < c$ such that $A \in \mathcal{B}_\sigma$. Let $\langle y_n \rangle_{n=1}^\infty$ enumerate $g_\sigma(\{A\})$ in increasing order. Then $FS(\langle y_n \rangle_{n=2}^\infty) \subseteq A$, $FS(\langle y_n \rangle_{n=2}^\infty) \in \mathcal{B}_\sigma \subseteq q$, and for each n, $y_n | y_{n+1}$. (The reason for starting the sequence at y_2 rather than y_1 is to handle the minor technical requirement that the first term of a divisible sequence is bigger than 1.)

Finally, assume that $FS(\langle (n+1)! \rangle_{n=1}^\infty) \in \mathcal{A}$. We need to show that $q \in M(\langle (n+1)! \rangle_{n=1}^\infty)$ and that given any $L \in [\mathbb{N}]^\omega$, there is a divisible sequence $\langle y_n \rangle_{n=1}^\infty$ with $FS(\langle y_n \rangle_{n=1}^\infty) \in q$, $FS(\langle y_n \rangle_{n=1}^\infty) \subseteq FS(\langle (n+1)! \rangle_{n=1}^\infty)$, and $L \backslash \bigcup\{\mathrm{supp}(z) : z \in FS(\langle y_n \rangle_{n=1}^\infty)\}$ is infinite. The second assertion follows from hypothesis (5) at stage δ where $L = C_\delta$. (Since $FS(\langle y_n \rangle_{n=1}^\infty) \subseteq FS(\langle (n+1)! \rangle_{n=1}^\infty)$ we have for $n \neq m$ that $\mathrm{supp}(y_n) \cap \mathrm{supp}(y_m) = \emptyset$ so $\bigcup\{\mathrm{supp}(z) : z \in FS(\langle y_n \rangle_{n=1}^\infty)\} = \bigcup_{n=1}^\infty \mathrm{supp}(y_n)$.) To see that $q \in M(\langle (n+1)! \rangle_{n=1}^\infty)$, let $m \in \mathbb{N}$ be given and suppose that $FS(\langle (n+1)! \rangle_{n=m}^\infty) \notin q$. Now q is divisibly strongly summable so $q = q+q$. Since $FS(\langle (n+1)! \rangle_{n=1}^\infty) \in q$ and

$$FS(\langle (n+1)! \rangle_{n=1}^\infty) = FS(\langle (n+1)! \rangle_{n=1}^{m-1}) \cup FS(\langle (n+1)! \rangle_{n=m}^\infty) \cup$$

$$\bigcup\{FS(\langle (n+1)! \rangle_{n=m}^\infty) + \Sigma_{n\in F}\ (n+1)! : \emptyset \neq F \subseteq \{1, 2, \ldots, m-1\}\}.$$

The first of these sets is finite and the second is not in q by assumption so pick a nonempty $F \subseteq \{1, 2, \ldots, m-1\}$ such that $FS(\langle (n+1)! \rangle_{n=m}^\infty) + \Sigma_{n\in F}\ (n+1)! \in q$. Since $q+q = q$, pick y and z such that $\{y, z, y+z\} \subseteq FS(\langle (n+1)! \rangle_{n=m}^\infty) + \Sigma_{n\in F}\ (n+1)!$. Then pick $t \in F$. One has $\alpha_t(y) = \alpha_t(z) = 1$ so $\alpha_t(y+z) = 2$, a contradiction. (Here α is as defined for the sequence $x_n = (n+1)!$.) ∎

It is easy (assuming MA) to produce divisibly strongly summable ultrafilters which are not special divisibly strongly summable; just pick any idempotent p with $FS(\langle (n+1)! \rangle_{n=1}^\infty) \notin p$, let $\mathcal{A} = \{\mathbb{N}\backslash FS(\langle (n+1)! \rangle_{n=1}^\infty)\}$, and apply Theorem 5.5. However it is conceivable that all strongly summable ultrafilters are in fact divisibly strongly summable. We conclude by showing that this is not the case (again assuming MA). We first introduce some notions from [3].

Definition 5.6. (a) Given a sequence $\langle F_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$, $FU(\langle F_n \rangle_{n=1}^\infty) = \{\bigcup_{n\in G} F_n : G \in \mathcal{P}_f(\mathbb{N})\}$.

(b) $\mathcal{U}$ is a union ultrafilter if an only if $\mathcal{U}$ is an ultrafilter on $\mathcal{P}_f(\mathbb{N})$ and for all $\mathcal{A} \in \mathcal{U}$, there exists a sequence $\langle F_n \rangle_{n=1}^\infty$ of pairwise disjoint members of $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle F_n \rangle_{n=1}^\infty) \in \mathcal{U}$ and $FU(\langle F_n \rangle_{n=1}^\infty) \subseteq \mathcal{A}$.

(c) $\mathcal{U}$ is an ordered union ultrafilter if and only if $\mathcal{U}$ is an ultrafilter on $\mathcal{P}_f(\mathbb{N})$ and for all $\mathcal{A} \in \mathcal{U}$, there exists a sequence $\langle F_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max F_n < \min F_{n+1}$ and $FU(\langle F_n \rangle_{n=1}^\infty) \in \mathcal{U}$ and $FU(\langle F_n \rangle_{n=1}^\infty) \subseteq \mathcal{A}$.

Lemma 5.7. *Define a sequence inductively by* $x_1 = 2$, $x_2 = 8$, *and for* $n > 2$, $x_n = \Pi\{\Sigma_{t\in F}\ x_t : \emptyset \neq F \subseteq \{1, 2, \ldots, n-1\}\}$. *If* $F, G \in \mathcal{P}_f(\mathbb{N})$, $F \cap G = \emptyset$, *and* $\Sigma_{t\in G}\ x_t | \Sigma_{t\in F}\ x_t$, *then* $\max G < \min F$.

Proof. Let $r = \max G$ and suppose that $\min F < r$. Let $F_1 = F \cap \{1, 2, \ldots, r\}$ and $F_2 = F \backslash F_1$. Let $b = \Sigma_{t\in F_1}\ x_t$ and let $a = \Sigma_{t\in G}\ x_t$. If $F_2 = \emptyset$ we have directly that $a|b$. If $F_2 \neq \emptyset$, let $c = \Sigma_{t\in F_2}\ x_t$ and note that by the construction of the sequence $a|c$ so again we conlude that $a|b$. But $0 < b < x_r \leq a$, a contradiction. ∎

Theorem 5.8. *Assume MA. There is a strongly summable ultrafilter which is not divisibly strongly summable.*

Proof. Let $\langle x_n\rangle_{n=1}^\infty$ be the sequence which is defined in Lemma 5.7. Define $\gamma : \mathcal{P}_f(\mathbb{N}) \longrightarrow \mathbb{N}$ by $\gamma(F) = \Sigma_{n\in F}\ x_n$. Pick by [5, Theorem 4] a union ultrafilter $\mathcal{U}$ which is not an ordered union ultrafilter. Let $p = \{A \subseteq \mathbb{N} : \gamma^{-1}[A] \in \mathcal{U}\}$. Then p is an ultrafilter on $\mathbb{N}$. To see thatt p is strongly summable, let $A \in p$ be given. Pick a sequence $\langle F_n\rangle_{n=1}^\infty$ of pairwise disjoint members of $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle F_n\rangle_{n=1}^\infty) \in \mathcal{U}$ and $FU(\langle F_n\rangle_{n=1}^\infty) \subseteq \gamma^{-1}[A]$. For each n, let $y_n = \Sigma_{t\in F_n}\ x_t$. Then $FS(\langle y_n\rangle_{n=1}^\infty) = \gamma[FU(\langle F_n\rangle_{n=1}^\infty)]$ so $FS(\langle y_n\rangle_{n=1}^\infty) \in p$ and $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq A$.

To see that p is not divisibly strongly summable, pick $\mathcal{A} \in \mathcal{U}$ such that there is no sequence $\langle F_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ with $\max F_n < \min F_{n+1}$ for each n and $FU(\langle F_n\rangle_{n=1}^\infty) \subseteq \mathcal{A}$ and $FU(\langle F_n\rangle_{n=1}^\infty) \in \mathcal{U}$. Let $A = \{\Sigma_{t\in F}\ x_t : F \in \mathcal{A}\}$. Then $A \in p$. Suppose we have a divisible sequence $\langle y_n\rangle_{n=1}^\infty$ with $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq A$ and $FS(\langle y_n\rangle_{n=1}^\infty) \in p$. For each n, let $F_n = \mathrm{supp}(y_n)$, i.e. $y_n = \Sigma_{t\in F_n}\ x_t$. Observe that $F_n \cap F_m = \emptyset$ for $n \neq m$. (If we had $t \in F_n \cap F_m$ we would have $\alpha_t(y_n + y_m) = 2$, while $y_n + y_m \in A$) Then $FU(\langle F_n\rangle_{n=1}^\infty) = \gamma^{-1}[FS(\langle y_n\rangle_{n=1}^\infty)]$ so $FU(\langle F_n\rangle_{n=1}^\infty) \in \mathcal{U}$ and $FU(\langle F_n\rangle_{n=1}^\infty) \subseteq \mathcal{A}$. Also given $n \in \mathbb{N}$ we have by Lemma 5.7 that $\max F_n < \min F_{n+1}$. This contradiction completes the proof. ∎

References

[1] Bergelson, V., N. Hindman, and B. Kra, *Iterated spectra of numbers – elementary, dynamical, and algebraic approaches,* manuscript.

[2] Berglund, J., H. Junghenn, and P. Milnes,"Analysis on Semigroups", Wiley, New York, 1989.

[3] Blass, A., *Ultrafilters related to Hindman's finite-unions theorem and its extensions,* in "Logic and Combinatorics" (S. Simpson, ed.), Contemporary Math. **65**(1987), 89-124.

[4] Blass, A., and N. Hindman, *Sums of ultrafilters and the Rudin-Keisler and Rudin-Frolík orders,* in "General Topology and Applications" (R. Shortt, ed.), Lecture Notes in Pure and Applied Math. **123**(1990), 59-70.

[5] Blass, A., and N. Hindman, *On strongly summable ultrafilters and union ultrafilters,* Trans. Amer. Math. Soc. **304**(1987), 83-99.

[6] Budak, T., N. Isik, and J. Pym, *Subsemigroups of Stone-Čech compactifications,* Proc. Cambr. Phil. Soc. **116**(1994), 99-118.

[7] Ellis, R., "Lectures on Topological Dynamics", Benjamin, New York, 1969.

[8] Gillman, L., and M. Jerison, "Rings of Continuous Functions", D. van Nostrand, Princeton, 1960.

[9] Hewitt, E., and K. Ross, "Abstract Harmonic Analysis, I", Springer-Verlag, New York, 1963.

[10] Hindman, N., *Finite sums from sequences within cells of a partition of* $\mathbb{N}$, J. Comb. Theory (Series A) **17**(1974), 1-11.

[11] Hindman, N., *Strongly summable ultrafilters on* $\mathbb{N}$ *and small maximal subgroups of* $\beta\mathbb{N}$, Semigroup Forum **42**(1991), 63-75.

[12] Hindman, N., *Summable ultrafilters and finite sums,* in "Logic and Combinatorics" (S. Simpson, ed.) Contemporary Math. **65**(1987), 263-274.

[13] Hindman, N., *The existence of certain ultrafilters on* $\mathbb{N}$ *and a conjecture of Graham and Rothschild,* Proc. Amer. Math. Soc. **36**(1972), 341-346.

[14] Hindman, N., *Ultrafilters and combinatorial number theory,* in "Number Theory, Carbondale 1974" (M. Nathanson, ed.), Lecture Notes in Math. **751**(1979), 119-184.

[15] Hindman, N., and J. Pym, *Free groups and semigroups in* $\beta\mathbb{N}$*,* Semigroup Forum **30**(1984), 177-193.

[16] Hofmann, K., and P. Mostert, "Elements of Compact Semigroups", C. E. Merrill, Columbus, 1967.

[17] Kunen, K., "Set Theory: An Introduction to Independence Proofs", North Holland, Amsterdam, 1980.

[18] el Mabhouh, A., J. Pym, and D. Strauss, *Subsemigroups of* $\beta\mathbb{N}$*,* Topology and its Applications, to appear.

[19] Matet, P., *Some filters of partitions*, J. Symbolic Logic **53**(1988), 540-553.

[20] Ruppert, W., "Compact Semitopological Semigroups: An Intrinsic Theory", Lecture Notes in Math. **1079**, Springer-Verlag, Berlin, 1984.

[21] Strauss, D., *Ideals and commutativity in* $\beta\mathbb{N}$*,* Topology and its Applications, to appear.

[22] Strauss, D. *Semigroup structures on* $\beta\mathbb{N}$*,* Semigroup Forum **44**(1992), 238-244.

Department of Mathematics
Howard University
Washington, DC 20059
USA

Department of Pure Mathematics
University of Hull
Hull HU6 7RX
United Kingdom