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QUASI-CENTRAL SETS AND THEIR DYNAMICAL CHARACTERIZATION

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Abstract. *Central* sets were originally defined for subsets of $\mathbb{N}$ by Furstenberg using notions from topological dynamics, and he proved the powerful *Central Sets Theorem* for such sets. Subsequently, central sets in any semigroup S were characterized as members of idempotents in the smallest ideal of βS, the Stone-Čech compactification of S. *Quasi-central* sets are members of idempotents in the closure of the smallest ideal of βS. They have a much simpler combinatorial characterization than do central sets. And they satisfy all known versions of the Central Sets Theorem. We provide a simple proof of this latter assertion for commutative semigroups and obtain a dynamical characterization of quasi-central sets which is similar to Furstenberg's original definition.

1. Introduction

We shall be concerned in this paper with two notions of largeness in an arbitrary semigroup $(S,\cdot)$, namely *central* and *quasi-central* sets. These notions in turn are related to two other notions which we define now. Given $s \in S$ and $A \subseteq S$,

$$s^{-1}A = \{t \in S : st \in A\}.$$

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(We are not assuming that s has an inverse, or that S has an identity.) For any set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X.

Definition 1.1. Let S be a semigroup and let $A \subseteq S$.

(a) The set A is *syndetic* if and only if there is some $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in G} t^{-1}A$.
(b) The set A is *piecewise syndetic* if and only if there is some $G \in \mathcal{P}_f(S)$ such that for all $F \in \mathcal{P}_f(S)$ there is some $x \in S$ with $Fx \subseteq \bigcup_{t \in G} t^{-1}A$.

In the semigroup $(\mathbb{N}, +)$, a set A is syndetic if and only if it has bounded gaps and is piecewise syndetic if and only if there are arbitrarily long intervals in which that gaps of A are bounded by a fixed bound.

In [3] H. Furstenberg defined a central subset of the set $\mathbb{N}$ of positive integers in terms of the notions of *proximality* and *uniform recurrence* in a dynamical system.

Definition 1.2.

(a) A *dynamical system* is a pair $(X, \langle T_s \rangle_{s \in S})$ such that
  (i) X is a compact Hausdorff space,
  (ii) S is a semigroup,
  (iii) for each $s \in S$, $T_s : X \to X$ and T_s is continuous, and
  (iv) for all $s, t \in S$, $T_s \circ T_t = T_{st}$.
(b) If $(X, \langle T_s \rangle_{s \in S})$ is a dynamical system, then a point $y \in X$ is *uniformly recurrent* if and only if, for every neighborhood U of y, $\{s \in S : T_s(y) \in U\}$ is syndetic.
(c) If $(X, \langle T_s \rangle_{s \in S})$ is a dynamical system, then points x and y of X are *proximal* if and only if for every neighborhood U of the diagonal in $X \times X$, there is some $s \in S$ such that $\big(T_s(x), T_s(y)\big) \in U$.

For Furstenberg the phase space X of a dynamical system was assumed to be metric. In this case, one easily sees that points x and y are proximal if and only if there is a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S such that $\lim_{n \to \infty} d\big(T_{s_n}(x), T_{s_n}(y)\big) = 0$.

Furstenberg's definition of central sets, generalized to apply to an arbitrary semigroup, was the following.

Definition 1.3. Let S be a semigroup and let $C \subseteq S$. Then C is *central* if and only if there exist a dynamical system $(X, \langle T_s \rangle_{s \in S})$, points x and y of X, and a neighborhood U of y such that y is uniformly recurrent, x and y are proximal, and $C = \{s \in S : T_s(x) \in U\}$.

The importance of this notion came from the following theorem.

Theorem 1.4 (Furstenberg)**.** *Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Let C be a central subset of $\mathbb{N}$. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) *for all n, $\max H_n < \min H_{n+1}$ and*
(2) *for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n \in F} (a_n + \sum_{t \in H_n} y_{i,t}) \in C$.*

Proof. [3, Proposition 8.21]. □

In [3] Furstenberg used the Central Sets Theorem to prove Rado's Theorem [7] by showing that any central subset of $\mathbb{N}$ contains solutions to all partition regular systems of homogeneous linear equations. Many other strong properties of central sets have been derived. See [6, Part III] for a number of these.

Subsequently, after looking at an early draft of the paper [4] by Furstenberg and Katznelson which derived Ramsey Theoretic results using idempotents in enveloping semigroups, Vitaly Bergelson had the idea that one might be able to derive the conclusion of the Central Sets Theorem for a set $C \subseteq \mathbb{N}$ which had an idempotent in the smallest ideal of $\beta\mathbb{N}$ in its closure. He was right. We shall consider this characterization after a brief review of the algebraic structure of βS.

Given a discrete semigroup $(S, \cdot)$ we take the points of the Stone-Čech compactification βS of S to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$ and the set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (and a basis for the closed sets) of βS. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$. In particular, the operation $\cdot$ on βS extends the operation $\cdot$ on S.

With this operation, $(\beta S, \cdot)$ is a compact Hausdorff right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$

is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. A subset I of a semigroup T is a *left ideal* provided $T \cdot I \subseteq I$, a *right ideal* provided $I \cdot T \subseteq I$, and a *two sided ideal* (or simply an *ideal*) provided it is both a left ideal and a right ideal.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T) = \bigcup\{L : L$ is a minimal left ideal of $T\} = \bigcup\{R : R$ is a minimal right ideal of $T\}$. Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is a *minimal* idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal. A point $p \in \beta S$ is in $c\ell K(\beta S)$ if and only if every member of p is piecewise syndetic. See [6] for additional information on the algebraic structure of βS.

Bergelson's characterization can now be presented.

Definition 1.5. Let S be a discrete semigroup and let C be a subset of S. Then C is *central* if and only if there is an idempotent p in $K(\beta S)$ such that $C \in p$.

In [1] it was shown, with the assistance of B. Weiss, that a subset C of $\mathbb{N}$ is central according to Definition 1.5 if and only if C is central according to Definition 1.3 and in [8] Hong-ting Shi and Hong-wei Yang showed that the two definitions are equivalent in general.

The Central Sets Theorem has been extended a few times. The most general version was recently obtained in [2]. We shall state here the version for commutative semigroups which is much simpler to state than the general version.

Theorem 1.6. *Let S be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$, the set of sequences in S. Let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and*
(2) *whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $\langle y_{i,n} \rangle_{n=1}^{\infty} \in G_i$, one has $\prod_{i=1}^{m} (\alpha(G_i) \cdot \prod_{t \in H(G_i)} y_{i,t}) \in C$.*

Proof. [2, Theorem 2.2]. □

The notion of *quasi-central* sets was introduced in [5].

Definition 1.7. Let S be a discrete semigroup and let C be a subset of S. Then C is *quasi-central* if and only if there is an idempotent p in $c\ell K(\beta S)$ such that $C \in p$.

While quasi-central sets have received much less attention than central sets, they have some significant virtues. In the first place, Theorem 1.6 is true for quasi-central sets, as is its noncommutative version as well. Secondly, in [2] combinatorial characterizations of both central and quasi-central sets were obtained. The characterization of quasi-central sets is much simpler than the characterization of central sets. In Section 2 we shall demostrate how easy it is to prove Theorem 1.6 as applied to quasicentral sets and discuss the combinatorial characterization of these sets.

In Section 3 we complete a cycle by providing a characterization of quasi-central sets which is similar to Definition 1.3.

2. Quasi-central sets

We shall begin by showing that quasi-central sets satisfy the commutative version of the Central Sets Theorem. It is a fact, which follows from results in [2], that the same is true for the general version as well. To illustrate precisely how easy it is, we present the details of the following theorem which is taken from [2].

Theorem 2.1. *Let S be a commutative semigroup and let $l \in \mathbb{N}$. For each $i \in \{1, 2, \ldots, l\}$, let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in S. Let C be a piecewise syndetic subset of S and let $m \in \mathbb{N}$. There exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > m$ and for each $i \in \{1, 2, \ldots, l\}$, $a \cdot \prod_{t \in H} y_{i,t} \in C$.*

Proof. Let $Y = \times_{t=1}^{l} \beta S$. Then by [6, Theorem 2.22] Y is a compact right topological semigroup and if $x \in \times_{t=1}^{l} S$, then λ_x is continuous. For $i \in \mathbb{N}$, let

$$I_i = \{(a \cdot \textstyle\prod_{t \in H} y_{1,t}, \ldots, a \cdot \prod_{t \in H} y_{l,t}) : \ a \in S\,,\ H \in \mathcal{P}_f(\mathbb{N})\,, \text{ and } \min H > i\}$$

and let $E_i = I_i \cup \{(a, a, \ldots, a) : a \in S\}$.

Let $E = \bigcap_{i=1}^{\infty} \overline{E_i}$ and let $I = \bigcap_{i=1}^{\infty} \overline{I_i}$. We claim that E is a subsemigroup of Y and I is an ideal of E. To this end, let $p, q \in E$. We show that $p \cdot q \in E$ and if either $p \in I$ or $q \in I$, then $p \cdot q \in I$.

Let U be an open neighborhood of $p \cdot q$ and let $i \in \mathbb{N}$. Since ρ_q is continuous, pick a neighborhood V of p such that $V \cdot q \subseteq U$. Pick $x \in E_i \cap U$ with $x \in I_i$ if $p \in I$. If $x \in I_i$ so that $x = (a \cdot \prod_{t \in H} y_{1,t}, \ldots, a \cdot \prod_{t \in H} y_{l,t})$ for some $a \in S$ and some $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H > i$, let $j = \max H$. Otherwise, let $j = i$. Since λ_x is continuous, pick a neighborhood W of q such that $x \cdot W \subseteq U$. Pick $y \in E_j \cap W$ with $y \in I_j$ if $q \in I$. Then $x \cdot y \in E_i \cap U$ and if either $p \in I$ or $q \in I$, then $x \cdot y \in I_i \cap U$.

By [6, Theorem 2.23] $K(Y) = \times_{t=1}^{l} K(\beta S)$. Pick $p \in K(\beta S) \cap \overline{C}$. Then $\overline{p} = (p, p, \ldots, p) \in K(Y)$. We claim that $\overline{p} \in E$. To see this, let U be a neighborhood of $\overline{p}$, let $i \in \mathbb{N}$, and pick $A_1, A_2, \ldots, A_l \in p$ such that $\times_{t=1}^{l} \overline{A_t} \subseteq U$. Pick $a \in \bigcap_{t=1}^{l} A_t$. Then

$$\overline{a} = (a, a, \ldots, a) \in U \cap E_i \,.$$

Thus $\overline{p} \in K(Y) \cap E$ so $K(E) = K(Y) \cap E$ by [6, Theorem 1.65] and so $\overline{p} \in K(E) \subseteq I$. Then $I_m \cap \times_{t=1}^{l} \overline{C} \neq \emptyset$ so pick $z \in I_m \cap \times_{t=1}^{l} \overline{C}$ and pick $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H > m$ such that $z = (a \cdot \prod_{t \in H} y_{1,t}, \ldots, a \cdot \prod_{t \in H} y_{l,t})$. □

Theorem 2.2. *Let S be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$, the set of sequences in S. Let C be a quasi-central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

1. *if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and*
2. *whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $\langle y_{i,n} \rangle_{n=1}^{\infty} \in G_i$, one has $\prod_{i=1}^{m} (\alpha(G_i) \cdot \prod_{t \in H(G_i)} y_{i,t}) \in C$.*

Proof. Pick an idempotent $p \in c\ell K(\beta S)$ such that $C \in p$. Let

$$C^\star = \{x \in C : -x \cdot C \in p\} \,.$$

Since $p \cdot p = p$, $C^\star \in p$. Also by [6, Lemma 4.14], if $x \in C^\star$, then $x^{-1} C^\star \in p$.

We define $\alpha(F) \in S$ and $H(F) \in \mathcal{P}_f(\mathbb{N})$ for $F \in \mathcal{P}_f(\mathcal{T})$ by induction on $|F|$ satisfying the following inductive hypotheses:

1. if $\emptyset \neq G \subsetneq F$, then $\max H(G) < \min H(F)$ and
2. if $n \in \mathbb{N}$, $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n = F$, and $\langle f_i \rangle_{i=1}^{n} \in \times_{i=1}^{n} G_i$, then $\prod_{i=1}^{n} \big(\alpha(G_i) \cdot \prod_{t \in H(G_i)} f_i(t)\big) \in C^\star$.

Assume first that $F = \{f\}$. Pick by Theorem 2.1 $a \in S$ and $L \in \mathcal{P}_f(\mathbb{N})$ such that $a \cdot \prod_{t\in L} f(t) \in C^\star$. Let $\alpha(\{f\}) = a$ and $H(\{f\}) = L$.

Now assume that $|F| > 1$ and $\alpha(G)$ and $H(G)$ have been defined for all proper subsets G of F. Let $K = \bigcup\{H(G) : \emptyset \neq G \subsetneq F\}$ and let $m = \max K$. Let $M = \{\prod_{i=1}^n \big(\alpha(G_i) \cdot \prod_{t\in H(G_i)} f_i(t)\big) : n \in \mathbb{N}, \emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n \subsetneq F, \text{ and } \langle f_i\rangle_{i=1}^n \in \times_{i=1}^n G_i\}$. Then M is finite and by hypothesis (2), $M \subseteq C^\star$. Let

$$B = C^\star \cap \bigcap_{x\in M} x^{-1}C^\star .$$

Then $B \in p$ so pick by Theorem 2.1 $a \in S$ and $L \in \mathcal{P}_f(\mathbb{N})$ such that $\min L > m$ and for each $f \in F$, $a \cdot \prod_{t\in L} f(t) \in B$. Let $\alpha(F) = a$ and $H(F) = L$.

Since $\min L \geq m$ we have that hypothesis (1) is satisfied. To verify hypothesis (2), let $n \in \mathbb{N}$, let $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n = F$, and let $\langle f_i\rangle_{i=1}^n \in \times_{i=1}^n G_i$ If $n = 1$, then

$$\prod_{i=1}^n \big(\alpha(G_i) \cdot \prod_{t\in H(G_i)} f_1(t)\big) = a \cdot \prod_{t\in L} f_1(t) \in B \subseteq C^\star .$$

So assume that $n > 1$ and let $y = \prod_{i=1}^{n-1} \big(\alpha(G_i) \cdot \prod_{t\in H(G_i)} f_i(t)\big)$. Then $y \in M$ so $a \cdot \prod_{t\in L} f_n(t) \in B \subseteq y^{-1}C^\star$ and thus

$$\prod_{i=1}^n \big(\alpha(G_i) \cdot \prod_{t\in H(G_i)} f_i(t)\big) = y \cdot a \cdot \prod_{t\in L} f_n(t) \in C^\star$$

as required. □

Consider now the following combinatorial characterization of quasi-central sets from [5].

Theorem 2.3. *Let S be an infinite semigroup and let $A \subseteq S$. Statements* (1) *and* (2) *are equivalent and are implied by statement* (3)*. If S is countable, then all three statements are equivalent.*

(1) *A is quasi-central.*
(2) *There is a downward directed family $\langle C_F\rangle_{F\in I}$ of subsets of A such that*
  (a) *for each $F \in I$ and each $x \in C_F$ there exists $G \in I$ with $C_G \subseteq x^{-1}C_F$ and*
  (b) *for each $F \in I$, C_F is piecewise syndetic.*
(3) *There is a decreasing sequence $\langle C_n\rangle_{n=1}^\infty$ of subsets of A such that*
  (a) *for each $n \in \mathbb{N}$ and each $x \in C_n$, there exists $m \in \mathbb{N}$ with $C_m \subseteq x^{-1}C_n$ and*

(b) *for each* $n \in \mathbb{N}$, C_n *is piecewise syndetic.*

Proof. [5, Theorem 3.7]. □

By contrast, the characterization of central requires that the family $\{C_F : F \in I\}$ (or the family $\{C_n : n \in \mathbb{N}\}$) be *collectionwise piecewise syndetic*, a very complicated notion which we are not going to inflict on the reader. (If she is curious, she may check [6, Definition 14.19].)

3. A dynamical characterization

It seemed natural to us to ask whether quasi-central sets could be characterized in term of notions defined in terms of dynamical systems since that had, after all, been the origin of central sets. It turns out that the answer is "yes", although the crucial notion is not one of the standard notions of topological dynamics.

Definition 3.1. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system and let $x, y \in X$. The pair (x, y) is *jointly intermittently uniformly recurrent* (abbreviated JIUR) if and only if for every neighborhood U of y, $\{s \in S : T_s(x) \in U \text{ and } T_s(y) \in U\}$ is piecewise syndetic.

Notice that trivially if the pair (x, y) is JIUR, then x and y are proximal. Further, one can show that if x and y are proximal and y is uniformly recurrent, then the pair (x, y) is JIUR.

Recall that given $p \in \beta S$ and $\langle x_s \rangle_{s \in S}$ in a topological space, $p\text{-}\lim_{s \in S} x_s = y$ if and only if for each neighborhood U of y,

$$\{s \in S : x_s \in U\} \in p\,.$$

Definition 3.2. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system, let $x \in X$, and let $p \in \beta S$. Then $T_p(x) = p\text{-}\lim_{s \in S} T_s(x)$.

Lemma 3.3. *Let* $(X, \langle T_s \rangle_{s \in S})$ *be a dynamical system and let* $x, y \in X$*. The following statements are equivalent.*

(a) *The pair* (x, y) *is JIUR.*
(b) *There exists* $r \in c\ell K(\beta S)$ *such that* $T_r(x) = T_r(y) = y$*.*
(c) *There exists* $r \in c\ell K(\beta S)$ *such that* $rr = r$ *and* $T_r(x) = T_r(y) = y$*.*

Proof. $(a) \Rightarrow (b)$. For each neighborhood U of y, let

$$B_U = \{s \in S : T_s(x) \in U \text{ and } T_s(y) \in U\}\,.$$

By assumption each B_U is piecewise syndetic. Further, the property of being piecewise syndetic is partition regular in the sense that if a finite union of sets is piecewise syndetic, one of them is. Also, if U and V are neighborhoods of y, then $B_{U\cap V} \subseteq B_U \cap B_V$. Thus by [6, Theorem 3.11] we may pick $r \in \beta S$ such that

$$\{B_U : U \text{ is a neighborhood of } y\} \subseteq r$$

and each member of r is piecewise syndetic. Then by [6, Corollary 4.41], $r \in c\ell K(\beta S)$. Then $T_r(x) = T_r(y) = y$.

(*b*) ⇒ (*c*). Let $M = \{r \in c\ell K(\beta S) : T_r(x) = T_r(y) = y\}$. By assumption $M \neq \emptyset$. Note also that M is closed. (If, say, $T_r(x) \neq y$, pick a neighborhood U of y such that $A = \{s \in S : T_s(x) \in U\} \notin r$, then $\overline{S \setminus A}$ is a neighborhood of r missing M.) We need to show that M is a subsemigroup of βS, since then, as a compact right topological semigroup, it will have an idempotent. So let $q, r \in M$. Then $qr \in c\ell K(\beta S)$ by [6, Theorem 4.44] and by [6, Remark 19.13] $T_{qr}(x) = T_q\big(T_r(x)\big) = T_q(y) = y$ and $T_{qr}(y) = T_q\big(T_r(y)\big) = T_q(y) = y$.

(*c*) ⇒ (*a*). Pick r as guaranteed. Let U be a neighborhood of y. Then $\{s \in S : T_s(x) \in U\} \in r$ and $\{s \in S : T_s(y) \in U\} \in r$, so $\{s \in S : T_s(x) \in U \text{ and } T_s(y) \in U\} \in r$ and is therefore piecewise syndetic. □

We should emphasize that we are not making any special assumptions about the semigroup S in the following theorem. The proof of the necessity in the following theorem is very similar to the corresponding part of the proof that Definitions 1.3 and 1.5 are equivalent.

Theorem 3.4. *Let S be a semigroup and let $C \subseteq S$. The set C is quasi-central if and only if there exist a dynamical system $(X, \langle T_s\rangle_{s\in S})$, points x and y of X such that x and y are JIUR, and a neighborhood U of y such that $C = \{s \in S : T_s(x) \in U\}$.*

Proof. Sufficiency. By Lemma 3.3 pick $r \in c\ell K(\beta S)$ such that $rr = r$ and $T_r(x) = T_r(y) = y$. Since U is a neighborhood of y and $T_r(x) = y$ we have that $C \in r$.

Necessity. Let $R = S \cup \{e\}$ where e is an identity adjoined to S (even if S already had an identity. Let $X = {}^{R}\{0, 1\}$, the set of functions from R to $\{0, 1\}$ with the product topology. For $s \in S$

define $T_s : X \to X$ by $T_s = f \circ \rho_s$. By [6, Lemma 19.14] $(X, \langle T_s \rangle_{s\in S})$ is a dynamical system.

Let $x = \chi_C$, the characteristic function of C. Pick an idempotent $r \in c\ell K(\beta S)$ such that $C \in r$ and let $y = T_r(x)$. Then by [6, Remark 19.13] we have that $T_r(y) = T_r\big(T_r(x)\big) = T_{rr}(x) = T_r(x) = y$ so by Lemma 3.3 the pair (x, y) is JIUR.

Let $U = \{z \in X : z(e) = y(e)\}$. Then U is a neighborhood of y in X. Notice that $y(e) = 1$. Indeed, $y = T_r(x)$ so $\{s \in S : T_s(x) \in U\} \in r$ so pick $s \in C$ such that $T_s(x) \in U$. Then $y(e) = T_s(x)(e) = \chi_C \circ \rho_s(e) = \chi_C(s) = 1$. Now given any $s \in S$,

$$\begin{aligned} s \in C &\Leftrightarrow x(s) = 1 \\ &\Leftrightarrow \chi_C(es) = 1 \\ &\Leftrightarrow T_s(x) \in U\,. \end{aligned}$$

□

In [5, Theorem 4.4] there is an example of a quasi-central set in $(\mathbb{N}, +)$ which is not central. If x and y are as produced in the proof of the necessity in Theorem 3.4 for that set, then the pair (x, y) is JIUR (and in particular, x and y are proximal) but y is not uniformly recurrent.

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