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Notions of Size and Combinatorial Properties of Quotient Sets in Semigroups

Vitaly Bergelson[1]

Neil Hindman[1]

and

Randall McCutcheon

Abstract An IP* set in a semigroup is one which must intersect the set of finite products from any specified sequence. (If the semigroup is noncommutative, one must specify the order of the products, resulting in "left" and "right" IP* sets.) If A is a subset of $\mathbb{N}$ with positive upper density, then the difference set $A - A = \{x \in \mathbb{N} :$ there exists $y \in A$ with $x + y \in A\}$ is an IP* set in $(\mathbb{N}, +)$. Defining analogously the quotient sets AA^{-1} and $A^{-1}A$, we analyze notions of largeness sufficient to guarantee that one or the other of these quotient sets are IP* sets. Among these notions are *thick*, *syndetic*, and *piecewise syndetic* sets, all of which come in both "left" and "right" versions. For example, we show that if A is any left syndetic subset of a semigroup S, then AA^{-1} is both a left IP* set and a right IP* set, while $A^{-1}A$ need be neither a left IP* set nor a right IP* set, even in a group. We also investigate the relationships among these notions of largeness.

1. Introduction.

In a commutative semigroup $(S, +)$, we write $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\Sigma_{n \in F}\, x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ where $\mathcal{P}_f(\mathbb{N}) = \{A : A$ is a finite nonempty subset of $\mathbb{N}\}$. Loosely following Furstenberg [5] we say that a set $A \subseteq S$ is an IP set if and only if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S with $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$. A set $C \subseteq S$ is then an IP* set if and only if $C \cap A \neq \emptyset$ for every IP set A (equivalently if and only if $C \cap FS(\langle x_n \rangle_{n=1}^{\infty}) \neq \emptyset$ for every sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S).

Recall that given a sequence of intervals $\langle (a_n, b_n] \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ with $\lim_{n\to\infty} (b_n - a_n) = \infty$, there are associated natural notions of upper density and density of a subset A of $\mathbb{N}$, namely $\overline{d}(A) = \limsup_{n\to\infty} \dfrac{|(a_n, b_n] \cap A|}{(b_n - a_n)}$ and $d(A) = \lim_{n\to\infty} \dfrac{|(a_n, b_n] \cap A|}{(b_n - a_n)}$ if the latter limit exists. (In $\mathbb{N}$, the expressions $\overline{d}(A)$ and $d(A)$ are typically used to refer to the upper density and density, respectively, of A with respect to the sequence of intervals

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$\langle(0,n]\rangle_{n=1}^\infty$. We are using them here somewhat more liberally.) Further, these notions are translation invariant. That is, given $t\in\mathbb{N}$, $\overline{d}(A-t)=\overline{d}(A)$. While upper density $\overline{d}$ is certainly not additive, density d is, in the sense that if $d(E)$ and $d(F)$ exist, where $E\cap F=\emptyset$, then $d(E\cup F)$ exists and equals $d(E)+d(F)$.

If $\overline{d}(A)>0$, then by passing to a subsequence of the intervals used to determine $\overline{d}(A)>0$, one can get the (positive) density of A to exist. This observation allows one to prove the following simple (and well known) fact. (By the *difference set* $A-A$, we mean $\{b\in\mathbb{N} : \text{there exists } c\in A \text{ such that } b+c\in A\}=\{x-y : x,y\in A \text{ and } x>y\}$.)

1.1 Theorem. *Let $A\subseteq\mathbb{N}$ and assume there is a sequence $\langle(a_n,b_n]\rangle_{n=1}^\infty$ of intervals in $\mathbb{N}$ (with $\lim_{n\to\infty}(b_n-a_n)=\infty$) with respect to which $\overline{d}(A)>0$. Then $A-A$ is an IP* set.*

Proof. Let $\alpha=\overline{d}(A)=\limsup_{n\to\infty}\frac{|(a_n,b_n]\cap A|}{(b_n-a_n)}$. Choose a subsequence $\langle(c_n,d_n]\rangle_{n=1}^\infty$ of $\langle(a_n,b_n]\rangle_{n=1}^\infty$ with respect to which $d(A)=\alpha$.

Let a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ be given. Then for each $m\in\mathbb{N}$, $d(A-\Sigma_{t=1}^m x_t)=\alpha$ (density relative to the sequence $\langle(c_n,d_n]\rangle_{n=1}^\infty$). Pick $k\in\mathbb{N}$ such that $1/k<\alpha$. Since d is additive one cannot have $\{A-x_1, A-(x_1+x_2),\ldots,A-(x_1+x_2+\ldots+x_k)\}$ pairwise disjoint, so pick $m<n$ such that $(A-\Sigma_{t=1}^m x_t)\cap(A-\Sigma_{t=1}^n x_t)\neq\emptyset$. Then $\Sigma_{t=m+1}^n x_t\in A-A$. $\square$

This simple result can be extended to a much wider class of semigroups. Let us recall the notion of a Følner sequence.

1.2 Definition. Let $(S,\cdot)$ be a countable semigroup. A sequence $\langle A_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(S)$ is said to be a left (respectively right) Følner sequence for S if for each $s\in S$, $\lim_{n\to\infty}\frac{|sA_n\bigtriangleup A_n|}{|A_n|}=0$ (respectively $\lim_{n\to\infty}\frac{|A_ns\bigtriangleup A_n|}{|A_n|}=0$).

Existence of Følner sequences in semigroups is related to the notion of *amenability.* A discrete semigroup S is said to be left amenable if there exists a left invariant mean μ (that is, positive linear functional satisfying $\mu(1)=1$) on $l_\infty(S)$, the space of bounded complex valued functions on S. By left invariance here we mean that for every $x\in S$ and every $\phi\in l_\infty(S)$ we have $\mu({}_x\phi)=\mu(\phi)$, where ${}_x\phi(t)=\phi(xt)$ for $t\in S$. (That is, ${}_x\phi=\phi\circ\lambda_x$ where $\lambda_x(t)=xt$.) Right amenability is similarly defined. We shall usually state our results for one side only, leaving the obvious left-right switches to the reader.

It is well known that the set of left invariant means μ is in one to one correspondence with the set of left invariant finitely additive probability measures m via the mapping $m\leftrightarrow\mu$, where $m(B)=\mu(\chi_B)$, μ extending continuously and linearly to $l_\infty(S)$. See, for example, [10, Section 0.1]. Therefore, we shall sometimes refer to such measures as "means", as well.

We shall be using the following relationships between amenability and existence of Følner sequences. If S admits a left Følner sequence, then S is left amenable. The converse does not hold in general, however any left amenable semigroup which is also left cancellative does admit a left Følner sequence. (See [10, Section 4.22] and [9, Corollary 3.6].)

For groups, left and right amenability are equivalent. Indeed, if S is a group and μ is a left invariant mean we put $\widetilde{f}(x) = f(x^{-1})$, $f \in l_\infty(S)$ and let $\nu(f) = \mu(\widetilde{f})$. ν is a right invariant mean. This equivalence does not hold for semigroups, which may be left but not right amenable. Indeed, for any set S with $|S| \geq 2$, letting $xy = y$ for $x, y \in S$ (so that S is a "right zero" semigroup) one may show that S is a left amenable semigroup that is not right amenable.

In this paper we shall be concerned with describing the "largeness" of certain subsets B of a semigroup S. Our philosophy is that the best notions of largeness should be closed under supersets, partition regular, and satisfy some sort of shift invariance.

A collection $\mathcal{L}$ of subsets of a set S is called partition regular if whenever $A \cup B \in \mathcal{L}$ one must have either $A \in \mathcal{L}$ or $B \in \mathcal{L}$.

In a non-commutative semigroup, there are four possible kinds of shift invariance. A set $\mathcal{L}$ of subsets of a set S is *left invariant* (respectively *left inverse invariant*) if and only if whenever $A \in \mathcal{L}$ and $s \in S$ one has $sA \in \mathcal{L}$ (respectively $s^{-1}A \in \mathcal{L}$), where $s^{-1}A = \{t \in S : st \in A\}$. If S is a group, so that $s^{-1}A = \{s^{-1}t : t \in A\}$, these notions coincide. Right invariance and right inverse invariance are defined analogously.

1.3 Lemma. *Let S be a left amenable semigroup, let m be a left invariant mean, let $s \in S$, and let $A \subseteq S$. Then $m(s^{-1}A) = m(A)$. If in addition, S is left cancellative, then $m(sA) = m(A)$.*

Proof. Let μ be the linear functional corresponding to m. Then $m(s^{-1}A) = \mu(\chi_{s^{-1}A}) = \mu(\chi_A \circ \lambda_s) = \mu(\chi_A) = m(A)$. If S is left cancellative, then $s^{-1}(sA) = A$ and so $m(sA) = m\big(s^{-1}(sA)\big) = m(A)$. □

If S is left amenable and m is a left invariant mean then $m(B)$ may be thought of as the "size" of the set B, relative to m, at least. We will usually be interested in distinguishing sets B for which there exists *some* left invariant mean m with $m(B) > 0$. Accordingly, we define $m_l^*(B)$ (respectively $m_r^*(B)$) to be the supremum of $m(B)$ over all left (respectively right) invariant means m. We call $m_l^*(B)$ the *left upper Banach* *mean density* of B, and remark that one can always find a left invariant mean μ for which $\mu(B) = m_l^*(B)$ (simply take a sequence of left invariant means $(\mu_i)_{i=1}^\infty$ such that $\mu_i(B)$ converges to $m_l^*(B)$, and let μ be any weak* limit point of this sequence). Then the condition $B \in \mathcal{L}$ if and only if $m_l^*(B) > 0$ serves as a notion of largeness which has the properties we desire, namely closure under supersets, partition regularity, and, by Lemma 1.3, left inverse invariance. Also by Lemma 1.3, if S is left cancellative, then $\mathcal{L}$ is left invariant.

A closely related notion of size for subsets B of semigroups S requires the existence of Følner sequences. Suppose that S is a countable, left cancellative, left amenable semigroup. For every left Følner sequence $\langle A_n \rangle_{n=1}^\infty$ one has naturally associated notions of upper density and density, namely $\overline{d}(B) = \limsup_{n\to\infty} \frac{|A_n \cap B|}{|A_n|}$ and $d(B) = \lim_{n\to\infty} \frac{|A_n \cap B|}{|A_n|}$ (provided the latter limit exists). Also d and $\overline{d}$ are left invariant and left inverse invariant: given $s \in S$ and $B \subseteq S$, one has $\overline{d}(sB) = \overline{d}(s^{-1}B) = \overline{d}(B)$. (It is routine to verify these assertions. One only needs to note that, while $s(s^{-1}B)$ need not equal B, it is true that $s(A \cap s^{-1}B) = sA \cap B$.) We also let $d_l^*(B)$ be the supremum of $\overline{d}(B)$ over all left Følner sequences $\langle A_n \rangle_{n=1}^\infty$. Then $d_l^*(B)$ will be called the *left upper Banach density* of B. Again, this supremum is achieved. In fact, for each $B \subseteq S$, there exists a left Følner sequence $\langle A_n \rangle_{n=1}^\infty$ (depending on B) such that $d(B)$ (with respect to this sequence) exists and equals $d_l^*(B)$. It follows that $d_l^*(sB) = d_l^*(s^{-1}B) = d_l^*(B)$ as well. Furthermore, as it is well known that a left Følner sequence $\langle A_n \rangle_{n=1}^\infty$ for which $d(B)$ exists may be used to define a left invariant mean m for which $m(B) = d(B)$, one easily obtains the inequality $m_l^*(B) \geq d_l^*(B)$.

We want to consider analogues to Theorem 1.1 for semigroups S which are possibly non-commutative. In such semigroups there are two reasonable interpretations for $\Pi_{n\in F}\, x_n$. That is, one may take the products in increasing or decreasing order of indices. The "left" and "right" terminology in the following definition comes from the choice of continuity in the Stone-Čech compactification βS of S, a topic that we shall discuss later in this introduction.

1.4 Definition. Let S be a semigroup and let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in S.

(a) Let $F = \{n_1, n_2, \ldots, n_k\} \in \mathcal{P}_f(\mathbb{N})$ with $n_1 < n_2 < \ldots < n_k$. Then $\prod^{\uparrow}_{n\in F} x_n = x_{n_1} \cdot x_{n_2} \cdot \ldots \cdot x_{n_k}$ and $\prod^{\downarrow}_{n\in F} x_n = x_{n_k} \cdot x_{n_{k-1}} \cdot \ldots \cdot x_{n_1}$.

(b) $FP_D(\langle x_n \rangle_{n=1}^\infty) = \{\prod^{\downarrow}_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$.

(c) $FP_I(\langle x_n \rangle_{n=1}^\infty) = \{\prod^{\uparrow}_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$.

(d) A subset A of S is a right (respectively left) IP set if and only if there is a sequence $\langle x_n\rangle_{n=1}^\infty$ in S with $FP_I(\langle x_n\rangle_{n=1}^\infty) \subseteq A$ (respectively $FP_D(\langle x_n\rangle_{n=1}^\infty) \subseteq A$).

(e) A subset A of S is a right (respectively left) IP* set if and only if for every right (respectively left) IP set B, $A \cap B \neq \emptyset$.

We shall see now that a natural analogue of Theorem 1.1 holds for any countable, left cancellative, left amenable semigroup. The "quotient set" $AA^{-1} = \{x \in S :$ there exists $y \in A$ such that $xy \in A\}$ is a natural analogue of the difference set $A - A$ used in Theorem 1.1. (Another natural analogue is $A^{-1}A = \{x \in S :$ there exists $y \in A$ such that $yx \in A\}$.) Note that if S is a group, then $AA^{-1} = \{xy^{-1} : x, y \in A\}$.

Theorem 1.5 will be seen to be a corollary to Theorem 3.1. We include its (short) proof now to illustrate how the proof of Theorem 1.1 is adapted.

1.5 Theorem. *Let S be a countable, left cancellative, left amenable semigroup and let $B \subseteq S$ with $d_l^*(B) > 0$. Then BB^{-1} is both a left IP* set and a right IP* set.*

Proof. There exists a left Følner sequence $\langle A_n\rangle_{n=1}^\infty$ with respect to which $d(B) = d_l^*(B) > 0$. Let a sequence $\langle x_n\rangle_{n=1}^\infty$ be given. For each $n \in \mathbb{N}$ we have $d\big((\prod_{t=1}^n x_t)B\big) = d(B)$ so by the additivity of d, $\{(\prod_{t=1}^n x_t)B : n \in \mathbb{N}\}$ cannot be a disjoint collection. Consequently one may pick $m < n$ such that $(\prod_{t=1}^m x_t)B \cap (\prod_{t=1}^n x_t)B \neq \emptyset$. Pick $a, b \in B$ such that $(\prod_{t=1}^m x_t)a = (\prod_{t=1}^n x_t)b$. Cancelling $\prod_{t=1}^m x_t$, one has that $a = (\prod_{t=m+1}^n)b$ so that $\prod_{t=m+1}^n \in BB^{-1}$.

Similarly we may pick $r < s$ such that $(\prod_{t=1}^r x_t)^{-1}B \cap (\prod_{t=1}^s x_t)^{-1}B \neq \emptyset$. Let $a \in (\prod_{t=1}^r x_t)^{-1}B \cap (\prod_{t=1}^s x_t)^{-1}B$. Then $(\prod_{t=1}^r x_t)a \in B$. Let $b = (\prod_{t=1}^r x_t)a$. Then $(\prod_{t=r+1}^s x_t)b = (\prod_{t=1}^s x_t)a \in B$ so $\prod_{t=r+1}^s x_t \in BB^{-1}$. □

In Theorem 3.1, this result is expanded, replacing the condition $d_l^*(B) > 0$ with the condition $m_l^*(B) > 0$, to include the case of left amenable semigroups S not admitting Følner sequences. The question naturally arises as to whether analogues of these results are available in non-amenable semigroups. Without invariant means or Følner sequences, none of our previous natural notions of largeness are applicable. We desire a different notion of largeness, one which has meaning in any semigroup. One class of sets which seem to be reasonable candidates to replace sets of positive upper Banach density (or positive upper Banach mean density) as our class of "large sets" are *piecewise syndetic* sets. In Section 2 we investigate basic information about these sets and the related notions of *thick* and *syndetic* sets.

In Section 3 of this paper we investigate the extent to which one can generalize Theorem 1.5 to the situation of non-amenable semigroups, using left or right piecewise syndetic or syndetic sets in place of sets having positive upper density. It turns out that the generalizations are surprisingly weak. We give examples showing that stronger versions are not possible.

In Section 4, we define a property for groups which is stronger than the IP* property, namely the Δ^* property, and investigate the extent to which the results of Section 2 carry over to this stronger property. Finally, we conclude with a few of the more natural questions which are suggested by the material we treat there.

Some of our proofs in Section 2 utilize the algebraic structure of the Stone-Čech compactification βS of a discrete semigroup S. We take βS to be the set of all ultrafilters on S, identifying the principal ultrafilters with the points of S. We denote also by $\cdot$ the operation on βS making $(\beta S, \cdot)$ a right topological semigroup with S contained in its topological center. That is, for all $p \in \beta S$, the function $\rho_p : \beta S \longrightarrow \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous and for all $x \in S$, the function $\lambda_x : \beta S \longrightarrow \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. The reader is referred to [6] for an elementary introduction to this operation. The basic fact characterizing the right continuous operation on βS is, given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$ where $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

The fact that one may extend the operation to βS so that $(\beta S, \cdot)$ is either right topological or left topological (but not both) is behind the "left" and "right" terminology introduced earlier. Thus if one takes $(\beta S, \cdot)$ to be right (respectively left) topological, then a subset A of S is a right (respectively left) IP set if and only if A is a member of some idempotent in βS and A is a right (respectively left) IP* set if and only if A is a member of every idempotent in βS. (See [6, Theorem 5.12].)

2. Thick, Syndetic, and Piecewise Syndetic Sets.

In this section we study the notions of right and left thick, syndetic, and piecewise syndetic sets and the relations among them. We state the definitions for the right versions, leaving the obvious left versions to the reader to formulate.

2.1 Definition. Let S be a semigroup and let $A \subseteq S$.

(a) A is *right thick* if and only if for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ such that $Fx \subseteq A$.

(b) A is *right syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} t^{-1}A$.

(c) A is *right piecewise syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} t^{-1}A$ is right thick.

We observe that right thickness is equivalent to a superficially stronger statement.

2.2 Lemma. *Let S be a semigroup and let $A \subseteq S$. Then A is right thick if and only if for every $F \in \mathcal{P}_f(S)$ there is some $x \in A$ such that $Fx \subseteq A$.*

Proof. Let $y \in S$ be arbitrary. By definition there exists $z \in S$ such that $(Fy \cup \{y\})z \subseteq A$. Now let $x = yz$. $\square$

Right thickness is a right invariant and left inverse invariant property. Indeed, if B is right thick, $F \in \mathcal{P}_f(S)$, and $g \in S$, then choosing $x \in S$ such that $Fx \subseteq B$ we have $F(xg) \subseteq Bg$, so that Bg is right thick. Similarly, one may show that if B is right thick then $g^{-1}B$ is right thick. Therefore, if S is a group then right thickness is a left invariant property as well.

Right thickness is easily seen not to be left invariant nor right inverse invariant in general for semigroups S, however, indeed not even for cancellative semigroups. For example, if S is the free semigroup on the letters a and b then clearly S is right thick in itself. aS, however, is not right thick since $bx \notin aS$ for all $x \in S$. Also, letting $H = Sa$, one easily sees that H is right thick, but $Hb^{-1} = \emptyset$ and hence is not right thick.

Right syndeticity is a left invariant and right inverse invariant property. Indeed, if E is right syndetic in S and $g \in S$, then $S = \bigcup_{t \in H} t^{-1}E$ for some $H \in \mathcal{P}_f(S)$. Then also $S = \bigcup_{t \in gH} t^{-1}gE = \bigcup_{t \in H} (gt)^{-1}gE$. To see this, simply note that for every $s \in S$, $ts \in E$ for some $t \in H$, so that $gts \in gE$ and $s \in (gt)^{-1}gE$. Also if E is right syndetic, then Eg^{-1} is right syndetic for all $g \in S$. To see this, let $H \in \mathcal{P}_f(S)$ have the property that $\bigcup_{t \in H} t^{-1}E = S$. We claim that $\bigcup_{t \in H} t^{-1}Eg^{-1} = S$. To see this, let $s \in S$. Since $sg \in \bigcup_{t \in H} t^{-1}E$, there exists $t \in H$ such that $tsg \in E$, that is $ts \in Eg^{-1}$ and $s \in t^{-1}Eg^{-1}$. It follows that right syndeticity is right invariant for groups.

Right syndeticity is neither right invariant nor left inverse invariant in general for cancellative semigroups. Again let S be the free semigroup on the letters a and b. S is right syndetic in itself, however Sa is not right syndetic, for $b \notin x^{-1}(Sa)$ for all $x \in S$. Also, if we let $J = aS$, then $a^{-1}J = S$, so that J is right syndetic. However, $b^{-1}J = \emptyset$, and hence is not right syndetic.

The right piecewise syndeticity property is both left and right invariant for semigroups.

2.3 Theorem. *Suppose that S is a semigroup, $a \in S$, and $E \subseteq S$ is right piecewise syndetic. Then aE and Ea are both right piecewise syndetic.*

Proof. There exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} t^{-1}E$ is right thick. Let $F \in \mathcal{P}_f(S)$. There exists $x \in S$ such that $Fx \subseteq \bigcup_{t \in H} t^{-1}E$. One easily checks that $F(xa) = (Fx)a \subseteq \big(\bigcup_{t \in H} t^{-1}E\big)a \subseteq \bigcup_{t \in H} t^{-1}(Ea)$. This shows that $\bigcup_{t \in H} t^{-1}(Ea)$ is right thick and hence that Ea is right piecewise syndetic. On the other hand, one easily checks that $\bigcup_{t\in H} t^{-1}E \subseteq \bigcup_{t\in H}(at)^{-1}(aE)$ and hence $\bigcup_{t\in aH} t^{-1}(aE)$ is right thick. This shows that aE is right piecewise syndetic. $\square$

The following theorem indicates some of the interrelationships among the various notions we are dealing with.

2.4 Theorem. *Let S be a semigroup and suppose that $E \subseteq S$.*

(a) E is right syndetic if and only if E intersects every right thick set non-trivially.

(b) If E is right thick then E contains a right IP set.

(c) If E is a right IP set then E is right syndetic.*

(d) E is right piecewise syndetic if and only if there exist a right syndetic set B and a right thick set C such that $E = B \cap C$.

Proof. (a). If E is right syndetic then there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} t^{-1}E = S$. Let B be a right thick set. There exists $x \in S$ such that $Hx \subseteq B$. Furthermore, there exists $t \in H$ such that $tx \in E$. Therefore $tx \in E \cap B$ so E intersects B non-trivially. Conversely, if E intersects every right thick set non-trivially then $S\backslash E$ fails to be right thick. In other words, there exists $H \in \mathcal{P}_f(S)$ having the property that for every $x \in S$, $Hx \cap E \neq \emptyset$. This means that $S = \bigcup_{t\in H} t^{-1}E$.

(b). Suppose E is right thick. Choose $x_1 \in E$. Now, by Lemma 2.2, choose $x_2 \in E$ such that $x_1x_2 \in E$. Choose $x_3 \in E$ such that $\{x_1, x_2, x_1x_2\}x_3 \subseteq E$. Continuing in this fashion we obtain a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $FP_I(\langle x_n\rangle_{n=1}^{\infty}) \subseteq E$.

(c). Suppose that E is a right IP* set. By (b), E must intersect every right thick set non-trivially. By (a), E is therefore right syndetic.

(d). Suppose first that E is right piecewise syndetic and pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} t^{-1}E$ is right thick. Let $C = E \cup \bigcup_{t\in H} t^{-1}E$ and let $B = E \cup (S\backslash C)$. Then trivially C is right thick and $E = B \cap C$. Thus it suffices to show that B is right syndetic. Suppose not. Then by (a), $S\backslash B$ is right thick and

$$S\backslash B = C\backslash E \subseteq \bigcup_{t\in H} t^{-1}E\,.$$

Pick by Lemma 2.2 some $x \in S\backslash B$ such that $Hx \subseteq S\backslash B$. Then for some $t \in H$, $tx \in E$ so $tx \in B$, a contradiction.

Now assume that $E = B \cap C$ where B is right syndetic and C is right thick. Pick $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t\in H} t^{-1}B$. Let $F \in \mathcal{P}_f(S)$ be given and pick x such that $HFx \subseteq C$. We claim that $Fx \subseteq \bigcup_{t\in H} t^{-1}(B \cap C)$. To see this, let $y \in F$ and pick $t \in H$ such that $yx \in t^{-1}B$. Then $tyx \in B \cap C$. $\square$

The following theorem, together with Theorem 2.3 and the obvious fact that they are closed under supersets, shows that the piecewise syndeticity properties give a satisfactory notion of largeness.

2.5 Theorem. *In a semigroup S, the right piecewise syndeticity property is partition regular.*

Proof. Assume that $A \cup B$ is right piecewise syndetic and pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} t^{-1}(A\cup B)$ is right thick. Suppose that neither A nor B is right piecewise syndetic. Then $\bigcup_{t\in H} t^{-1}A$ is not right thick so pick $F \in \mathcal{P}_f(S)$ such that for all $x \in S$ there exists $y \in F$ such that $yx \notin \bigcup_{t\in H} t^{-1}A$. That is

(∗) for all $x \in S$ there exists $y \in F$ such that $Hyx \cap A = \emptyset$.

Also $\bigcup_{t\in HF} t^{-1}B$ is not right thick so pick $L \in \mathcal{P}_f(S)$ such that for all $x \in S$, there exists $y \in L$ such that $HFyx \cap B = \emptyset$. Pick $x \in S$ such that $FLx \subseteq \bigcup_{t\in H} t^{-1}(A \cup B)$. Pick $y \in L$ such that $HFyx \cap B = \emptyset$. Then, replacing x by yx in (∗), pick $z \in F$ such that $Hzyx \cap A = \emptyset$. Then $zyx \notin \bigcup_{t\in H} t^{-1}(A \cup B)$, a contradiction. □

Alternatively, one may establish partition regularity using the fact (see Theorem 2.9 below) that a subset A of S is right piecewise syndetic if and only if A is a member of some ultrafilter in the smallest ideal of $(\beta S, \cdot)$ with its right topological structure.

Our next two theorems indicate how the properties of thickness, syndeticity, and piecewise syndeticity relate to our previous notions of size in amenable semigroups.

2.6 Theorem. *Suppose that S is a left amenable semigroup and $E \subseteq S$. Statements (a) and (b) are equivalent and statements (c) and (d) are equivalent. If in addition S is left cancellative, then all four statements are equivalent.*

(a) E is right thick.

(b) $m_l^*(E) = 1$.

(c) There exists a left Følner sequence for S whose members are contained in E.

(d) $d_l^*(E) = 1$.

Proof. $(a) \leftrightarrow (b)$. See [10, Proposition 1.21].

$(c) \to (d)$. Obvious.

$(d) \to (c)$. Pick a left Følner sequence $\langle A_n\rangle_{n=1}^{\infty}$ with respect to which $d(E) = 1$. We claim that $\langle A_n \cap E\rangle_{n=1}^{\infty}$ is a left Følner sequence. To see this, let $s \in S$ be given. Then for any n,

$$s(A_n \cap E)\backslash\big((A_n \cap E) \cup (sA_n\Delta A_n)\big) \subseteq A_n\backslash E$$

and

$$(A_n \cap E)\backslash\big(s(A_n \cap E) \cup (sA_n \Delta A_n)\big) \subseteq s(A_n \backslash E)$$

so that $|\big(s(A_n \cap E)\Delta(A_n \cap E)\big)\backslash(sA_n\Delta A_n)| \leq 2|A_n\backslash E|$. Thus

$$\begin{aligned}\frac{|s(A_n \cap E)\Delta(A_n \cap E)|}{|A_n \cap E|} &\leq \frac{|\big(s(A_n \cap E)\Delta(A_n \cap E)\big)\backslash(sA_n\Delta A_n)| + |sA_n\Delta A_n|}{|A_n \cap E|} \\ &\leq \frac{2|A_n\backslash E| + |sA_n\Delta A_n|}{|A_n|} \cdot \frac{|A_n|}{|A_n \cap E|}\end{aligned}$$

so that $\lim\limits_{n\to\infty} \dfrac{|s(A_n \cap E)\Delta(A_n \cap E)|}{|A_n \cap E|} = 0$ as required.

Now assume that S is left cancellative.

$(a) \to (c)$. Since S is left amenable and left cancellative, there exists a left Følner sequence $\langle A_n\rangle_{n=1}^{\infty}$. Since E is left thick, for every $n \in \mathbb{N}$ there exists $x_n \in S$ such that $A_n x_n \subseteq E$. Using the fact that for any $s \in S$ and any n, $sA_n x_n \Delta A_n x_n \subseteq (sA_n\Delta A_n)x_n$, one easily checks that $\langle A_n x_n\rangle_{n=1}^{\infty}$ is a left Følner sequence.

$(c) \to (a)$. Let $F = \{x_1, \cdots, x_k\} \subseteq S$ be any finite set. There exists a left Følner sequence $\langle A_n\rangle_{n=1}^{\infty}$ such that $A_n \subseteq E$ for all n. Let n be so large that for each $i \in \{1, 2, \ldots, k\}$,

$$\frac{|A_n\backslash {x_i}^{-1}A_n|}{|A_n|} \leq \frac{|{x_i}^{-1}A_n \triangle A_n|}{|A_n|} = \frac{|x_i({x_i}^{-1}A_n \triangle A_n)|}{|A_n|} \leq \frac{|A_n \triangle xA_n|}{|A_n|} < \frac{1}{k}\,.$$

(Left cancellation is used for the second to last inequality.) Then $\bigcap_{i=1}^{k} {x_i}^{-1}A_n \neq \emptyset$ Let $s \in \bigcap_{i=1}^{k} {x_i}^{-1}A_n$. Then $Fs \subseteq A_n \subseteq E$. □

2.7 Theorem. *Suppose that S is a left amenable semigroup and $E \subseteq S$.*

(a) E is right syndetic if and only if there exists $\alpha > 0$ such that $m(E) > \alpha$ for every left invariant mean m.

(b) If E is right piecewise syndetic then $m_l^*(E) > 0$. If S is also left cancellative, then $d_l^*(E) > 0$.

Proof. (a). Suppose E is right syndetic. By Theorem 2.4(a), $S\backslash E$ fails to be right thick. By Theorem 2.6, $\alpha = 1 - m_l^*(S\backslash E) > 0$. One easily checks that $m(E) \geq \alpha$ for every left invariant mean m. Conversely, if there exists $\alpha > 0$ such that $m(E) \geq \alpha$ for every left invariant mean m, then $m_l^*(S\backslash E) \leq 1 - \alpha < 1$ and $S\backslash E$ fails to be right thick. That is, E is right syndetic.

(b). Pick $H \in \mathcal{P}_f(S)$ such that $A = \bigcup_{t\in H}\ t^{-1}E$ is right thick. By Theorem 2.6, $m_l^*(A) = 1$, hence there is some left invariant mean m for S such that $m(A) = 1$.

Consequently, there is some $t \in H$ such that $\mu(\chi_{t^{-1}E}) > 0$. But $\mu(\chi_A) = \mu(\chi_A \circ \lambda_t) = \mu(\chi_{t^{-1}A})$. The second assertion is proved similarly. $\square$

One can easily guess that the converse to Theorem 2.7(b) is false. We see now that this is in fact the case.

2.8 Theorem. *Let* $E = \mathbb{Z} \setminus \left(\bigcup_{n=2}^{\infty} \bigcup_{k\in\mathbb{Z}} \{kn^3+1, kn^3+2, \ldots, kn^3+n\}\right)$. *Then* $d^*(E) > 0$, *but* E *fails to be piecewise syndetic.*

Proof. To see that E is not piecewise syndetic suppose instead that we have $H \in \mathcal{P}_f(\mathbb{Z})$ such that $\bigcup_{t\in H}(-t+E)$ is thick. Pick even $n \in \mathbb{N}$ such that $|t| < \frac{n}{2}$ for all $t \in H$. Let $F = \{1, 2, \ldots, n^3 + \frac{n}{2}\}$ and pick $x \in \mathbb{Z}$ such that $F + x \subseteq \bigcup_{t\in H}(-t+E)$. Pick $k \in \mathbb{Z}$ such that $(k-1)n^3 < x \leq kn^3$. Let $y = kn^3 + \frac{n}{2} - x$ and note that $y \in F$. Pick $t \in H$ such that $t+y+x \in E$. But $t+y+x \in \{kn^3+1, kn^3+2, \ldots, kn^3+n\}$, a contradiction.

Next note that for fixed $n \geq 2$, and $L \in \mathbb{N}$,

$$|\{-L^3+1, -L^3+2, \ldots, 0\} \cap \bigcup_{k\in\mathbb{Z}}\{kn^3+1, kn^3+2, \ldots, kn^3+n\}| =$$
$$|\{-L^3+1, -L^3+2, \ldots, 0\} \cap \bigcup\big\{\{kn^3+1, kn^3+2, \ldots, kn^3+n\} : \tfrac{-L^3}{n^3} \leq k \leq -1\big\}|$$
$$\leq \tfrac{L^3}{n^2}\,.$$

Moreover $\{-L^3+1, -L^3+2, \ldots, 0\} \cap \bigcup_{k\in\mathbb{Z}}\{kn^3+1, kn^3+2, \ldots, kn^3+n\} = \emptyset$ for $n > L$. It follows that

$$|\frac{\{-L^3+1, -L^3+2, \ldots, 0\}\setminus E|}{L^3} \leq \textstyle\sum_{n=2}^{L} \frac{1}{n^2} \leq \displaystyle\frac{\pi^2}{6} - 1\,.$$

This implies that $\dfrac{|\{-L^3+1, -L^3+2, \ldots, 0\} \cap E|}{L^3} \geq 2 - \dfrac{\pi^2}{6}$. Since L is arbitrary, $d^*(E) \geq 2 - \frac{\pi^2}{6} > 0$. $\square$

Any compact right (or left) topological semigroup S has a (unique) smallest two sided ideal $K(S)$, which is the union of all minimal left ideals and is also the union of all minimal right ideals. (See [3] or [6] for these and other unfamiliar algebraic facts.) The smallest ideal of βS and its closure may be characterized in terms of piecewise syndetic and syndetic sets. We thank the referee for providing the characterizations in (c) and (d) below.

2.9 Theorem. *Let* S *be a discrete semigroup and assume that the operation has been extended to* βS *making* $(\beta S, \cdot)$ *a right topological semigroup with* S *contained in its topological center.*

(a) Let $p \in \beta S$. *Then* $p \in K(\beta S)$ *if and only if for every* $A \in p$, $\{x \in S : x^{-1}A \in p\}$ *is right syndetic.*

(b) Let $A \subseteq S$. *Then* $c\ell A \cap K(\beta S) \neq \emptyset$ *if and only if* A *is right piecewise syndetic.*

(c) Let $A \subseteq S$. Then A is right thick if and only if $c\ell A$ contains a left ideal of βS.

(d) Let $A \subseteq S$. Then A is right syndetic if and only if for every left ideal L of βS, $c\ell A \cap L \neq \emptyset$.

Proof. (a) [6, Theorem 4.39].

(b) [6, Theorem 4.40].

(c) Necessity. Since A is right thick, $\{t^{-1}A : t \in S\}$ has the finite intersection property. Pick $p \in \beta S$ such that $\{t^{-1}A : t \in S\} \subseteq p$. Then $Sp \subseteq c\ell A$ and thus $\beta Sp \subseteq c\ell A$.

Sufficiency. Pick a left ideal L of βS such that $L \subseteq c\ell A$ and pick $p \in L$. Then for each $t \in S$, $tp \in c\ell A$ and so $t^{-1}A \in p$. Given $F \in \mathcal{P}_f(S)$, pick $x \in \bigcap_{t\in F} t^{-1}A$.

(d) Necessity. Pick $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t\in H} t^{-1}A$ and let L be a left ideal of βS. Pick $p \in L$ and pick $t \in H$ such that $t^{-1}A \in p$. Then $tp \in L \cap c\ell A$.

Sufficiency. Suppose that for each $H \in \mathcal{P}_f(S)$, $S\backslash \bigcup_{t\in H} t^{-1}A \neq \emptyset$. Then $\{S\backslash t^{-1}A : t \in S\}$ has the finite intersection property so pick $p \in \beta S$ such that $\{S\backslash t^{-1}A : t \in S\} \subseteq p$. Then $c\ell A \cap \beta Sp \neq \emptyset$ so pick $q \in \beta S$ such that $A \in qp$. Then $\{t \in S : t^{-1}A \in p\} \in q$ so for some $t \in S$, $t^{-1}A \in p$, a contradiction. □

It is worth noting that one is guaranteed (as an easy consequence of a result of P. Anthony) a certain minimal connection between right piecewise syndetic sets and left piecewise syndetic sets.

2.10 Theorem. *Let S be a semigroup, let $r \in \mathbb{N}$, and let $S = \bigcup_{i=1}^{r} A_i$. Then some A_i is both left piecewise syndetic and right piecewise syndetic.*

Proof. Let K_ℓ be the smallest ideal of $(\beta S, \cdot)$ with its left topological structure and let K_r be the smallest ideal of $(\beta S, \cdot)$ with its right topological structure. By [1, Theorem 4.1] $K_\ell \cap c\ell K_r \neq \emptyset$ so pick $p \in K_\ell \cap c\ell K_r$. Pick $r \in \{1, 2, \ldots, r\}$ such that $A_i \in p$. Then by Theorem 2.9(b) A_i is both left and right piecewise syndetic. □

We now introduce one more class of sets. One will notice that this class is a hybrid of what have been for us left and right notions.

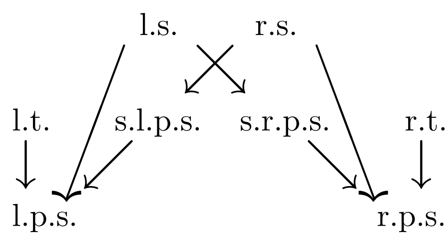
2.11 Definition. Let S be a semigroup and let $A \subseteq S$. Then A is *strongly right piecewise syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} At^{-1}$ is right thick.

Justification for our choice of terminology in the previous definition is given by the following result.

2.12 Theorem. *Let S be a semigroup. Then any strongly right piecewise syndetic subset of S is right piecewise syndetic.*

Proof. Let A be a strongly right piecewise syndetic subset of S and pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} At^{-1}$ is right thick. Then $\bigcup_{t\in H} At^{-1}$ is right piecewise syndetic so, by Theorem 2.5 there is some $t \in H$ such that At^{-1} is right piecewise syndetic. Pick $K \in \mathcal{P}_f(S)$ such that for each $F \in \mathcal{P}_f(S)$, $\bigcup_{s\in K} s^{-1}At^{-1}$ is right thick. Given $F \in \mathcal{P}_f(S)$, pick $x \in S$ such that $Fx \subseteq \bigcup_{s\in K} s^{-1}At^{-1}$. Then $Fxt \subseteq \bigcup_{s\in K} s^{-1}A$ so that $\bigcup_{s\in K} s^{-1}A$ is right thick. □

We then have that all of the implications in the following diagram hold.



We set out now to show that none of the missing implications is valid in general. Wherever possible, we shall present counterexamples in a group, specifically the free group on two generators. The one case in which this is not possible is the proof that not every right thick set is strongly right piecewise syndetic.

2.13 Theorem. *If S is a semigroup with nonempty center, then every right thick subset of S is strongly right piecewise syndetic. In particular, every right thick subset of a group is strongly right piecewise syndetic.*

Proof. Let A be a right thick subset of S. Pick y in the center of S and let $H = \{y\}$. To see that Ay^{-1} is right thick, let $F \in \mathcal{P}_f(S)$ be given and pick $x \in S$ such that $(Fy)x \subseteq A$. Then $(Fx)y \subseteq A$ so that $Fx \subseteq Ay^{-1}$ as required.

2.14 Theorem. *There is a subset of the free semigroup on countably many generators which is right thick but not strongly right piecewise syndetic.*

Proof. Let S be the free semigroup on the letters $\{y_n : n \in \mathbb{N}\}$ (without identity). For $w \in S$, let $\ell(w)$ be the length of w, that is the number of occurrences of letters in w. For $n \in \mathbb{N}$, let $T_n = \{w \in S : \ell(w) \leq n\}$ and let $A = \bigcup_{n=1}^{\infty} T_n y_n$.

Then A is trivially right thick. Now suppose we have some $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} At^{-1}$ is right thick. Let $m = \max\{n :$ there is some $t \in H$ such that y_n occurs in $t\}$. Let $F = \{{y_1}^m\}$ and pick $x \in S$ such that $Fx \subseteq \bigcup_{t\in H} At^{-1}$. Pick $t \in H$ such that ${y_1}^m xt \in A$ and pick $n \in \mathbb{N}$ and $z \in T_n$ such that ${y_1}^m xt = zy_n$. Then y_n occurs in t so

$n \leq m$ and consequently $\ell(z) \leq m$. But then $\ell(zy_n) \leq m+1$ while $\ell(y_1{}^m xt) \geq m+2$, a contradiction. □

Since a free semigroup on two generators contains a copy of a free semigroup on countably many generators, we find the following contrast to Theorem 2.14 interesting.

2.15 Theorem. *Any right thick subset of a free semigroup on two generators is strongly right piecewise syndetic.*

Proof. Let S be the free semigroup on the letters a and b and let A be a right thick subset of S. Let $H = \{a, b\}$. To see that $\bigcup_{t \in H} At^{-1}$ is right thick, let $F \in \mathcal{P}_f(S)$ be given. Then $Fa \in \mathcal{P}_f(S)$ so pick $x \in S$ such that $Fax \subseteq A$. Let t be the rightmost letter of x. Then $x = zt$ for some $z \in S \cup \{\emptyset\}$. Thus $az \in S$ and $t \in H$ and $Faz \subseteq At^{-1}$. □

2.16 Theorem. *There is a subset of the free group G on the letters a and b which is right syndetic and left thick but is not right thick and is not strongly right piecewise syndetic.*

Proof. Let $A = \{w \in G \backslash \{e\} :$ the leftmost letter of w is a or $a^{-1}\}$. Let $H = \{a, a^{-1}\}$. Then $G = \bigcup_{t \in H} t^{-1}A$ so that A is right syndetic. Also given $F \in \mathcal{P}_f(G)$, if $m = \max\{\ell(w) : w \in F\}$ (where $\ell(w)$ is the length of w), then $a^{m+1}F \subseteq A$ so A is left thick.

To see that A is not right thick, let $F = \{b, b^{-1}\}$. Then for any $x \in G$, either $bx \notin A$ or $b^{-1}x \notin A$.

To see that A is not strongly right piecewise syndetic, suppose that one has $H \in \mathcal{P}_f(G)$ such that $\bigcup_{t \in H} At^{-1}$ is right thick. Let $m = \max\{\ell(w) : w \in H\}$, let $F = \{b^{2m+1}, b^{-2m-1}\}$, and pick $x \in G$ such that $Fx \subseteq \bigcup_{t \in H} At^{-1}$. Pick $t \in H$ such that $b^{2m+1}xt \in A$. Then the leftmost $2m+1$ letters of xt must all be b^{-1} so $\ell(xt) \geq 2m+1$, while $\ell(xt) \leq \ell(x) + \ell(t) \leq \ell(x) + m$. Consequently, $\ell(x) \geq m+1$ and thus the leftmost letter of x is also the leftmost letter of xt, and thus the leftmost letter of x must be b^{-1}. Similarly, choosing $s \in H$ such that $b^{-2m-1}xs \in A$, one concludes that the leftmost letter of x is b. □

2.17 Theorem. *There is a subset of the free group G on the letters a and b which is right and left syndetic but is neither right nor left thick.*

Proof. Let $A = \{w \in G : \ell(w)$ is even$\}$. Letting $H = \{e, a\}$ one easily sees that $G = \bigcup_{t \in H} t^{-1}A = \bigcup_{t \in H} At^{-1}$, so that A is both right and left syndetic. (If $\ell(w)$ is odd, then $\ell(aw) = \ell(w) \pm 1$.)

Similarly, if $F = \{e, a\}$, one easily sees that there is no $x \in G$ with either $Fx \subseteq A$ or $xF \subseteq A$. □

2.18 Theorem. *There is a subset of the free group G on the letters a and b which is right thick and strongly right piecewise syndetic but is not left piecewise syndetic.*

Proof. Let $A = \{wa^n : w \in G, n \in \mathbb{N}, n \geq \ell(w)$, and either $w = e$ or the rightmost letter of w is not $a^{-1}\}$. To see that A is right thick (and consequently, by Theorem 2.13, is strongly right piecewise syndetic), let $F \in \mathcal{P}_f(G)$ be given. Let $m = \max\{\ell(w) : w \in F\}$. Then $Fa^{2m} \subseteq A$.

To see that A is not left piecewise syndetic, suppose instead that we have $H \in \mathcal{P}_f(G)$ such that $\bigcup_{t \in H} At^{-1}$ is left thick. Let $m = \max\{\ell(w) : w \in H\}$, let $F = \{b^{m+1}, b^{-m-1}\}$, and pick $x \in G$ such that $xF \subseteq \bigcup_{t \in H} At^{-1}$. Without loss of generality assume that either $x = e$ or the rightmost letter of x is not b^{-1}, and pick $t \in H$ such that $xb^{m+1}t \in A$. This is clearly impossible. □

Theorems 2.16, 2.17, and 2.18 (and their left-right switched versions) establish that none of the missing implications in the diagram appearing before Theorem 2.13 is valid in the free group on 2 generators, except that in any group right thick sets are strongly right piecewise syndetic (and left thick sets are strongly left piecewise syndetic).

We saw in Theorem 2.5 that the property of being right piecewise syndetic is partition regular. On the other hand, we saw in the proof of Theorem 2.16 that in the free group G on the letters a and b, the set $A = \{w \in G \backslash \{e\}$: the leftmost letter of w is a or $a^{-1}\}$ is not strongly right piecewise syndetic and similarly $B = \{w \in G \backslash \{e\}$: the leftmost letter of w is b or $b^{-1}\}$ is not strongly right piecewise syndetic. Since $A \cup B = G \backslash \{e\}$ is trivially left thick, and hence strongly right piecewise syndetic, one sees that the property of being strongly right piecewise syndetic is not partition regular.

We shall see in Theorem 3.6 that in any semigroup, the quotient set AA^{-1} of a strongly right piecewise syndetic A is both a left IP* set and a right IP* set. Consequently, if S is a semigroup in which the notions of strongly right piecewise syndetic and right piecewise syndetic coincide, one has the corollary that whenever $r \in \mathbb{N}$ and $S = \bigcup_{i=1}^{r} A_i$, one has for some $i \in \{1, 2, \ldots, r\}$ that $A_i {A_i}^{-1}$ is both a left IP* set and a right IP* set.

Other strong combinatorial consequences are also obtainable in semigroups for which these notions coincide. They obviously coincide in commutative semigroups, so it is of interest to determine how much noncommutativity is needed to separate the notions of strongly right piecewise syndetic and right piecewise syndetic. In our final result of this section, we shall show that the answer is “not much”.

Let G be the Heisenberg group. That is, $G = \mathbb{Z}^3$ with the operation defined by

$$(x, y, z) \cdot (u, v, w) = (x + u, y + v + xw, z + w)\,.$$

(To see that G is a group note that it is isomorphic to the upper triangular 3×3 matrices with integer entries and 1's on the main diagonal.) It is well known that G is a nilpotent group of rank 2. The center of G is rather large, namely $\{0\} \times \mathbb{Z} \times \{0\}$. Also, the centralizer of any element of G is always strictly larger than the center.

We thank the referee for providing a significant simplification of the following proof.

2.19 Theorem. *Let $G = \mathbb{Z}^3$ with the operation defined above. There is a right piecewise syndetic subset of G which is not strongly right piecewise syndetic.*

Proof. Let

$$A = \{(d, 2^{2l} \cdot d + e, 2^{2l} + f) : l \in \mathbb{N}\,, l \geq 9\,, d, e, f \in \{0, 1, \ldots, 2l\}\,,\ \text{and}\ d \in 2\mathbb{N}\}\,.$$

To see that A is right piecewise syndetic, let $H = \{(0, 0, 0), (1, 0, 0)\}$. Let $F \in \mathcal{P}_f(G)$ be given and pick $n \in \mathbb{N}\backslash\{1\}$ such that $F \subseteq \{m \in \mathbb{Z} : |m| \leq n\}^3$. Let $l = n^2 + 3n$ and let $x = (l, 2^{2l} \cdot l + l, 2^{2l} + n)$. To see that $Fx \subseteq \bigcup_{t\in H} t^{-1}A$, let $y = (u, v, w) \in F$. Pick $c \in \{0, 1\}$ such that $l + u + c$ is even and let $t = (c, 0, 0)$. Then

$$tyx = (l + u + c, 2^{2l}(l + u + c) + l + v + nu + cw + cn, 2^{2l} + w + n)\,.$$

Let $d = u+l+c$, $e = l+v+nu+cw+cn$, and $f = w+n$. Then $tyx = (d, 2^{2l} \cdot d+e, 2^{2l}+f)$. Now $d = l + u + c \leq l + n + 1 < 2l$, $d = l + u + c \geq l - n > 0$, and c was chosen to make d even. Also $0 = n - n \leq n + w \leq n + n < 2l$ so $f \in \{0, 1, \ldots, 2l\}$. Finally, $e = l + v + nu + (w + n)c \geq l - n - n^2 + 0 > 0$ and $e = l + v + nu + (w + n)c \leq l + n + n^2 + 2n = 2l$. Thus $tyx \in A$.

To see that A is not strongly right piecewise syndetic, suppose instead that one has $H \in \mathcal{P}_f(G)$ such that $\bigcup_{t\in H} At^{-1}$ is right thick. Let $m = \max(\{\max\{|u|, |v|, |w|\} : (u, v, w) \in H\} \cup \{9\})$.

Pick by Theorem 2.9(c) a left ideal L of βG such that $L \subseteq c\ell(\bigcup_{t\in H} At^{-1}$. By [6, Corollary 4.33], $G^* = \beta G\backslash G$ is a right ideal of βG and so $G^* \cap L \neq \emptyset$. Pick $q \in L \cap G^*$. Then $Gq \subseteq c\ell(\bigcup_{t\in H} At^{-1})$. Choose t and s in H such that $At^{-1} \in q$ and $As^{-1} \in (1, 0, 0)q$. Let $t^{-1} = (u, v, w)$ and let $s^{-1} = (u', v', w')$. Let

$$B = \{(d, 2^{2l} \cdot d + e, 2^{2l} + f) : l \in \mathbb{N}\,, l \geq m\,, d, e, f \in \{0, 1, \ldots, 2l\}\,,\ \text{and}\ d \in 2\mathbb{N}\}$$

and notice that $A\backslash B$ is finite. Consequently $Bs^{-1} \in (1, 0, 0)q$ and, because $(1, 0, 0)At^{-1} \in (1, 0, 0)q$, $(1, 0, 0)Bt^{-1} \in (1, 0, 0)q$. Choose $x \in Bs^{-1} \cap (1, 0, 0)Bt^{-1}$ and pick $l, d, e, f, l', d', e', f'$ such that $l, l' \in \mathbb{N}\,, l, l' \geq m\,, d, e, f \in \{0, 1, \ldots, 2l\}\,, d', e', f' \in \{0, 1, \ldots, 2l'\}\,, d, d' \in 2\mathbb{N}$, and

$$\begin{aligned} x &= (1 + d + u, 2^{2l}(1 + d) + e + v + dw + f + w, 2^{2l} + f + w) \\ &= (d' + u', 2^{2l'}d' + e' + v' + d'w', 2^{2l'} + f' + w')\,. \end{aligned}$$

Now the largest power of 2 less than or equal to $2^{2l} + f + w$ is either 2^{2l} or 2^{2l-1} and the largest power of 2 less than or equal to $2^{2l'} + f' + w'$ is either $2^{2l'}$ or $2^{2l'-1}$ and consequently, $l = l'$.

Also $|e + v + dw + f + w - e' - v' - d'w'| < 13l^2 < 2^{2l}$ and thus, since 2^{2l} divides $e + v + dw + f + w - e' - v' - d'w'$, $e + v + dw + f + w = e' + v' + d'w'$. Consequently $2^{2l}(1 + d) = 2^{2l}d'$, which is a contradiction because $1 + d$ is odd, while d' is even. □

3. Quotient Sets and the IP* Property.

We show that we can get a generalization of Theorem 1.5 in any left amenable semigroup. Making a left-right switch in the semigroup multiplication, it is clearly equivalent to a version for right amenable semigroups which may be formulated as easily. As before, we shall leave left-right switches to the reader. Recall that we have defined $AA^{-1} = \{x \in S :$ there exists $y \in A$ such that $xy \in A\}$ and $A^{-1}A = \{x \in S :$ there exists $y \in A$ such that $yx \in A\}$.

3.1 Theorem. *Let S be a left amenable semigroup and let $A \subseteq S$. If $m_l^*(A) > 0$, then AA^{-1} is a left IP* set. If also S is left cancellative, then AA^{-1} is a right IP* set.*

Proof. Pick a left invariant mean m such that $m(A) > 0$. Let a sequence $\langle x_n \rangle_{n=1}^{\infty}$ be given. Then for each $n \in \mathbb{N}$ we have by Lemma 1.3, $m\big((\prod_{t=1}^{n} x_t)^{-1}A\big) = m(A)$ so by the additivity of m, $\{(\prod_{t=1}^{n} x_t)^{-1}A : n \in \mathbb{N}\}$ cannot be a disjoint collection. Consequently we may pick $r < s$ such that $(\prod_{t=1}^{r} x_t)^{-1}A \cap (\prod_{t=1}^{s} x_t)^{-1}A \neq \emptyset$. Let $a \in (\prod_{t=1}^{r} x_t)^{-1}A \cap (\prod_{t=1}^{s} x_t)^{-1}A$. Then $(\prod_{t=1}^{r} x_t)a \in A$. Let $b = (\prod_{t=1}^{r} x_t)a$. Then $(\prod_{t=r+1}^{s} x_t)b = (\prod_{t=1}^{s} x_t)a \in A$ so $\prod_{t=r+1}^{s} x_t \in AA^{-1}$.

Now assume that S is left cancellative. Then one has by Lemma 1.3 for each $n \in \mathbb{N}$ that $m\big((\prod_{t=1}^{m} x_t)A\big) = m(A)$. Consequently one may pick $m < n$ such that $(\prod_{t=1}^{m} x_t)A \cap (\prod_{t=1}^{n} x_t)A \neq \emptyset$. So pick $a, b \in A$ such that $(\prod_{t=1}^{m} x_t)a = (\prod_{t=1}^{n} x_t)b$. Then cancelling $\prod_{t=1}^{m} x_t$ on the left, one has that $a = (\prod_{t=m+1}^{n})b$ so that $\prod_{t=m+1}^{n} \in AA^{-1}$.
□

In light of the inequality $d_l^*(A) \leq m_l^*(A)$ for left cancellative left amenable semigroups, one sees that we have in fact generalized Theorem 1.5. Another corollary is the following, which follows from Theorem 2.7.

3.2 Corollary. *Let S be a left amenable semigroup and let A be a right piecewise syndetic subset of S. Then AA^{-1} is a left IP* set. If S is left cancellative, then AA^{-1} is a right IP* set.*

A natural question is whether one can use $A^{-1}A$ instead of AA^{-1} without changing the other conditions. The answer, as we now see, is "no".

3.3 Theorem. *There exist a left cancellative, left amenable semigroup S and a subset A of S which is both right piecewise syndetic (in fact right thick) and left piecewise syndetic such that $A^{-1}A$ is neither a left IP* set nor a right IP* set.*

Proof. Let S be a finite right zero (i.e. $x \cdot y = y$ for all $x, y \in S$) semigroup with at least two members. As is well known, and easy to see, the function μ defined by $\mu(f) = \dfrac{\Sigma_{s\in S}\ f(s)}{|S|}$ is a left invariant mean for S. (See for example [3, p. 80].)

Now pick $x \in S$ and let $A = \{x\}$. Then $S = Ax^{-1}$ and $Sx = A$ so A is both left and right piecewise syndetic. Also $A^{-1}A = A$. Pick $z \in S\backslash A$ and for each $n \in \mathbb{N}$ let $y_n = z$. Then $FP_D(\langle y_n\rangle_{n=1}^{\infty}) = FP_I(\langle y_n\rangle_{n=1}^{\infty}) = \{z\}$. □

The reader may wish to verify that if one replaces the condition of Theorem 3.3 that S be left amenable and left cancellative by the condition that S be an amenable group, then as a consequence of the left-right switched version of Corollary 3.2, $A^{-1}A$ must be both left IP* and right IP*.

We have seen that in one sense, given a left amenable semigroup, AA^{-1} is a better analog than $A^{-1}A$ to the difference set $A - A$ obtained when S is commutative.

In the left-right switched version of Corollary 3.2, A is left piecewise syndetic. This prompts the question of whether we may replace right piecewise syndeticity of A with left piecewise syndeticity in Corollary 3.2. Again, the answer is "no", as we now show via an example which is almost exactly that given in [1, Theorem 3.3].

3.4 Theorem. *There exist an amenable group $(T, \cdot)$ and a left piecewise syndetic (in fact left thick) subset A of T such that AA^{-1} is neither a left IP* set nor a right IP* set.*

Proof. Let T be the group of all permutations of $\mathbb{N}$ that move only finitely many points. It is well known that $(T, \circ)$ is amenable. Indeed, if for each $n \in \mathbb{N}$ we let $S_n = \big\{g \in T : \{x \in \mathbb{N} : g(x) \neq x\} \subseteq \{1, 2, \ldots, n\}\big\}$, then $\langle S_n\rangle_{n=1}^{\infty}$ is a two-sided Følner sequence. Let $f_n = (1, n+1)(2, n+2)\ldots(n, 2n)$. That is,

$$f_n(k) = \begin{cases} k+n & \text{if } k \leq n \\ k-n & \text{if } n < k \leq 2n \\ k & \text{if } k > 2n\ . \end{cases}$$

Let $A = \bigcup_{n=2}^{\infty} f_n S_n$. One easily checks that A is left thick and hence left piecewise syndetic.

We now claim that $AA^{-1} \subseteq \{g \in T : g(1) \neq 1 \text{ or } g(2) = 2\}$. To see this let $g \in AA^{-1}$ and pick $n, m \in \mathbb{N}\backslash\{1\}$ and $h \in S_n$ and $k \in S_m$ such that $g \circ f_m \circ k = f_n \circ h$. Then $g = f_n \circ h \circ k^{-1} \circ f_m$, since $f_m = {f_m}^{-1}$. If $n = m$, then $g(2) = 2$. If $n > m$, then $g(1) = n + h(m+1) \neq 1$. If $n < m$, then $g(1) = f_n(m+1) \neq 1$.

Now for each $n \in \mathbb{N}$, let $g_n = (2, n+2)$. Given $F \in \mathcal{P}_f(\mathbb{N})$, let $k = \min F$ and let $\ell = \max F$. Then $(\prod^{\downarrow}_{n\in F} g_n)(1) = (\prod^{\uparrow}_{n\in F} g_n)(1) = (1)$ and $(\prod^{\downarrow}_{n\in F} g_n)(2) = k+2$ and $(\prod^{\uparrow}_{n\in F} g_n)(2) = \ell + 2$ so $FP_D(\langle g_n\rangle_{n=1}^{\infty}) \cap AA^{-1} = FP_I(\langle g_n\rangle_{n=1}^{\infty}) \cap AA^{-1} = \emptyset$. □

Notice that $d_r^*(A) = 1$ and (due to the fact that AA^{-1} is not IP*) $d_l^*(A) = 0$ for the set A constructed in the previous theorem.

If a semigroup S is partitioned into finitely many cells, then by Theorem 2.10, some cell is both left and right piecewise syndetic. Consequently, if a left and right amenable semigroup S is partitioned into finitely many cells then some cell A of the partition will have the property that AA^{-1} is a left IP* set and $A^{-1}A$ is a right IP* set. We see now that this can fail badly if the amenability assumption is deleted.

3.5 Theorem. *Let G be the free group with identity e on the letters a and b. There is a partition $\mathcal{F}$ of G into four sets such that*
(1) each $A \in \mathcal{F}$ is both left and right piecewise syndetic,
(2) for each $A \in \mathcal{F}$, neither $A^{-1}A$ nor AA^{-1} is either a left IP set nor a right IP* set.*

Proof. Let

$$\begin{aligned} A_1 = &\{e\} \cup \{w \in G\backslash\{e\} : \text{ the leftmost letter of } w \text{ is } a \text{ or } a^{-1} \\ &\quad \text{and the rightmost letter of } w \text{ is } a \text{ or } a^{-1}\} \\ A_2 = &\{w \in G\backslash\{e\} : \text{ the leftmost letter of } w \text{ is } b \text{ or } b^{-1} \\ &\quad \text{and the rightmost letter of } w \text{ is } b \text{ or } b^{-1}\} \\ A_3 = &\{w \in G\backslash\{e\} : \text{ the leftmost letter of } w \text{ is } a \text{ or } a^{-1} \\ &\quad \text{and the rightmost letter of } w \text{ is } b \text{ or } b^{-1}\} \\ A_4 = &\{w \in G\backslash\{e\} : \text{ the leftmost letter of } w \text{ is } b \text{ or } b^{-1} \\ &\quad \text{and the rightmost letter of } w \text{ is } a \text{ or } a^{-1}\} \end{aligned}$$

and let $\mathcal{F} = \{A_1, A_2, A_3, A_4\}$.

To see that A_2 is left piecewise syndetic, let $H = \{b, b^{-1}\}$ and let $F \in \mathcal{P}_f(G)$ be given. Let m be the maximum length of a word in F. Then given any $w \in F$ one has either $bwb^{m+2} \in A_2$ or $b^{-1}wb^{m+2} \in A_2$ so $Fb^{m+2} \subseteq \bigcup_{t \in H} t^{-1}A_2$. Similarly, A_2 is right piecewise syndetic. Proofs for left and right piecewise syndeticity of the other cells in the partition are nearly identical.

It is easy to see that

$$\begin{aligned} {A_2}^{-1}A_2 = A_2{A_2}^{-1} = &\{e\} \cup \{w \in G\backslash\{e\} : \text{ the leftmost letter of } w \text{ is } b \text{ or } b^{-1} \\ &\text{or the rightmost letter of } w \text{ is } b \text{ or } b^{-1}\}\ . \end{aligned}$$

Similarly one sees that

$$\begin{aligned} {A_1}^{-1}A_1 = A_1{A_1}^{-1} = &\{e\} \cup \{w \in G\backslash\{e\} : \text{ the leftmost letter of } w \text{ is } a \text{ or } a^{-1} \\ &\text{or the rightmost letter of } w \text{ is } a \text{ or } a^{-1}\} \end{aligned}$$

and that ${A_3}^{-1}A_3 = A_4{A_4}^{-1} = A_2{A_2}^{-1}$ and ${A_4}^{-1}A_4 = A_3{A_3}^{-1} = A_1{A_1}^{-1}$.

Consequently, given any $A \in \mathcal{F}$ one has that AA^{-1} and $A^{-1}A$ each miss either $FP(\langle a^n\rangle_{n=1}^{\infty})$ or $FP(\langle b^n\rangle_{n=1}^{\infty})$ (where we omit the subscripts $_I$ and $_D$ because the products in any order are the same). □

Theorem 3.5 demonstrates the following circumstance for sufficiently badly non-commutative semigroups S: any family $\mathcal{L}$ of subsets of S having the property that AA^{-1} is left or right IP* for any $A \in \mathcal{L}$, must fail partition regularity. In other words, it must be possible to partition some set from $\mathcal{L}$ into two sets, neither of which is an element of $\mathcal{L}$. As a result, membership in $\mathcal{L}$ will not constitute a good notion of largeness according to our established criteria. For instance in Theorem 3.5, if $B = \{w \in G\backslash\{e\}$: the rightmost letter of w is b or $b^{-1}\}$, then $BB^{-1} = G$ and $B = A_2 \cup A_4$.

In spite of this considerable drawback, we nevertheless proceed now to show that the strongly right piecewise syndetic sets form a class of sets sufficient to guarantee that AA^{-1} is left (and right) IP* for all members A of that class.

3.6 Theorem. *Let S be a semigroup and let A be a strongly right piecewise syndetic subset of S. Then AA^{-1} is both a left IP* set and a right IP* set.*

Proof. Pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} At^{-1}$ is right thick and let $l = |H|$. Let a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S be given.

Choose $a \in S$ such that $\{\prod_{k=1}^{n} x_k : 1 \le n \le l+1\}a \subseteq \bigcup_{t\in H} At^{-1}$. Pick $t \in H$ and m, n with $1 \le m < n \le l+1$ such that $\big(\prod_{k=1}^{n} x_k\big)a$ and $\big(\prod_{k=1}^{m} x_k\big)a$ both lie in At^{-1}. Then $(\prod_{k=1}^{m} x_k)at \in A$ and $(\prod_{k=1}^{n} x_k)at \in A$ so $(\prod_{k=m+1}^{n} x_k) \in AA^{-1}$. This shows that AA^{-1} is a left IP* set.

To see that AA^{-1} is right IP*, choose $a \in S$ such that $\{\prod_{k=n}^{l+1} x_k : 1 \le n \le l+1\}a \subseteq \bigcup_{t\in H} At^{-1}$. Pick $t \in H$ and m, n with $1 \le m < n \le l+1$ such that $\big(\prod_{k=n}^{l+1} x_k\big)a$ and

$(\prod_{k=m}^{l+1} x_k)a$ both lie in At^{-1}. Then $(\prod_{k=m}^{l+1} x_k)at \in A$ and $(\prod_{k=n}^{l+1} x_k)at \in A$ so $(\prod_{k=m}^{n-1} x_k) \in AA^{-1}$. □

We saw in Section 2 that not every right syndetic set is strongly right piecewise syndetic but that every *left* syndetic set *is*. Thus we come to the following, which, paradoxically, we have arrived at via a very "rightward" train of thought.

3.7 Corollary. *Let S be a semigroup and let A be a left syndetic subset of S. Then AA^{-1} is both a left IP* set and a right IP* set.*

Notice that the left-right switch of the previous corollary yields the result that if A is right syndetic then $A^{-1}A$ is both a left IP* set and a right IP* set. This prompts the question of whether a hybrid of these two versions is true. The answer, as we now see most strongly, is "no".

3.8 Theorem. *Let G be the free group with identity e on the letters a and b. There is a partition of G into two sets A and B such that*
(1) A and B are each right syndetic and
(2) AA^{-1} and BB^{-1} are each neither left nor right IP.*

Proof. Let

$$A = \{e\} \cup \{w \in G \setminus \{e\} : \text{ the leftmost letter of } w \text{ is } a \text{ or } a^{-1}\} \text{ and}$$
$$B = \{w \in G \setminus \{e\} : \text{ the leftmost letter of } w \text{ is } b \text{ or } b^{-1}\}.$$

Let $H = \{a, a^{-1}\}$ and let $K = \{b, b^{-1}\}$. Then $S = \bigcup_{t \in H} t^{-1}A = \bigcup_{t \in K} t^{-1}B$ so that A and B are right syndetic.

It is easy to check that AA^{-1} then consist of e and all words either beginning or ending with a or a^{-1} and that BB^{-1} consists of e and all words either beginning or ending with b or b^{-1}. Hence $AA^{-1} \cap FP(\langle b^n \rangle_{n=1}^{\infty}) = \emptyset$, and $BB^{-1} \cap FP(\langle a^n \rangle_{n=1}^{\infty}) = \emptyset$. Consequently neither AA^{-1} nor BB^{-1} is either left or right IP*. □

It is a routine fact that in $(\mathbb{N}, +)$, if A is a piecewise syndetic set then there is a syndetic set C with $C - C \subseteq A - A$, so that, as far as difference sets are concerned, it does not matter whether one is talking about syndetic or piecewise syndetic sets. (We remark that there exist sets A of positive density in $\mathbb{N}$ such that for no syndetic set C do we have $C - C \subseteq A - A$. This observation was made by Forrest in [4], and is a corollary of a result of Kriz ([7], see also [8]).) With the choice of AA^{-1} to replace the notion of difference sets one has the corresponding result in any semigroup. We are grateful to Imre Leader for providing a simple proof of this result, showing in fact that the same set H which establishes that A is right piecewise syndetic also establishes that C is right syndetic.

3.9 Theorem. *Let S be a semigroup, let A be a right piecewise syndetic subset of S, and pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} t^{-1}A$ is right thick. Then there exists a right syndetic subset C of S (in fact $S = \bigcup_{t\in H} t^{-1}C$) such that whenever $F \in \mathcal{P}_f(S)$ and $F \subseteq C$, there exists $x \in S$ such that $Fx \subseteq A$. In particular, $CC^{-1} \subseteq AA^{-1}$.*

Proof. For each $F \in \mathcal{P}_f(S)$, let $B_F = \{\tau \in {}^{S}\!H : (\exists x \in S)(\forall s \in F)(\tau(s) \cdot s \cdot x \in A)\}$, where ${}^{S}\!H$ is the set of functions from S to H. Given $F \in \mathcal{P}_f(S)$, pick $x \in S$ such that $Fx \subseteq \bigcup_{t\in H} t^{-1}A$ and for each $s \in F$, pick $\tau(s) \in H$ such that $\tau(s)\cdot s\cdot x \in A$. Defining τ at will on the rest of S, one has $\tau \in B_F$. Given $F, G \in \mathcal{P}_f(S)$ one has $B_{F\cup G} \subseteq B_F \cap B_G$. Thus $\{B_F : F \in \mathcal{P}_f(S)\}$ has the finite intersection property.

Also, given $F \in \mathcal{P}_f(S)$ and $\tau \in {}^{S}\!H \backslash B_F$ one has that $\{\sigma \in {}^{S}\!H : \sigma_{|F} = \tau_{|F}\}$ is a neighborhood of τ missing B_F. Thus $\{B_F : F \in \mathcal{P}_f(S)\}$ is a collection of closed subsets of the compact space ${}^{S}\!H$ with the finite intersection property. Pick $\tau \in \bigcap_{F\in\mathcal{P}_f(S)} B_F$ and let $C = \{\tau(s) \cdot s : s \in S\}$. Then trivially $S = \bigcup_{t\in H} t^{-1}C$.

Let $F \in \mathcal{P}_f(S)$ such that $F \subseteq C$. Pick $G \in \mathcal{P}_f(S)$ such that $F = \{\tau(s) \cdot s : s \in G\}$. Since $\tau \in B_G$, pick $x \in S$ such that for all $s \in G$, $\tau(s) \cdot s \cdot x \in A$.

To see the "in particular" conclusion, let $a \in CC^{-1}$ and pick $b \in C$ such that $ab \in C$. Let $F = \{b, ab\}$ and pick $x \in S$ such that $Fx \subseteq A$. Then $a \in AA^{-1}$. □

Our original proof of (a slightly weaker version of) Theorem 3.9 used the algebraic structure of βS. It is so short (given that one knows the characterizations of the smallest ideal of βS and its closure) that we present it now for comparison. (The slightly weaker version does not guarantee that the same H establishes the right piecewise syndeticity of A and the right syndeticity of C.)

Alternate proof. Let $(\beta S, \cdot)$ have the right continuous operation. By Theorem 2.9(b), $c\ell A \cap K(\beta S) \neq \emptyset$ so pick $p \in c\ell A \cap K(\beta S)$. Let $C = \{x \in S : x^{-1}A \in p\}$. Then by Theorem 2.9(a), C is right syndetic. Let $F \in \mathcal{P}_f(S)$ such that $F \subseteq C$. Pick $x \in \bigcap_{t\in F} t^{-1}A$. Then $Fx \subseteq A$. □

On the other hand one does not get the corresponding result for $A^{-1}A$, even in an amenable group, or even if one assumes A to be both left and right piecewise syndetic.

3.10 Theorem.

(a) There exists a group G and a set $A \subseteq G$ such that A is both left and right piecewise syndetic but there does not exist a right syndetic set C with $C^{-1}C \subseteq A^{-1}A$.

(b) There exists an amenable group T and a right piecewise syndetic (in fact right thick) set B such that there does not exist a right syndetic set C such that $C^{-1}C \subseteq B^{-1}B$.

Proof. (a). Let G be the free group with identity e on the letters a and b and let $A = \{w \in G\backslash\{e\}$: the leftmost letter of w is b or b^{-1} and the rightmost letter of w is b or $b^{-1}\}$. We saw in the proof of Theorem 3.5 that A is both left and right piecewise syndetic and that $A^{-1}A = \{e\} \cup \{w \in G\backslash\{e\}$: the leftmost letter of w is b or b^{-1} or the rightmost letter of w is b or $b^{-1}\}$.

Now suppose that we have a right syndetic set C with $C^{-1}C \subseteq A^{-1}A$. Pick $H \in \mathcal{P}_f(G)$ such that $G = \bigcup_{t\in H} t^{-1}C$. By the pigeon hole principle pick $m < n$ in $\mathbb{N}$ and $t \in H$ such that $ta^n \in C$ and $ta^m \in C$. Then $a^{n-m} \in C^{-1}C\backslash A^{-1}A$.

(b). Let T be the group of Theorem 3.4, and $B = A^{-1}$, where A is the set appearing there. Then B is right thick, but $B^{-1}B$ is neither left nor right IP*. Since the left-right switched version of Theorem 3.7 guarantees that $C^{-1}C$ is left and right IP* for every right syndetic C, we cannot have $C^{-1}C \subseteq B^{-1}B$. □

We now establish a dynamical equivalence pertaining to the kinds of questions we have been considering.

3.11 Definition. A *dynamical system* is a pair $(X, \langle T_s\rangle_{s\in S})$ where X is a compact Hausdorff space, S is a semigroup, for each $s \in S$, T_s is a continuous function from X to X (with T_e as the identity on X if e is an identity for S), and for each $s, t \in S$, $T_s \circ T_t = T_{st}$. The dynamical system $(X, \langle T_s\rangle_{s\in S})$ is *minimal* if and only if no proper closed subset of X is invariant under T_s for each $s \in S$.

We state the following theorem for general classes that are closed under supersets. In our previous results in this section we have been taking $\mathcal{G} = \{A \subseteq S : A$ is an IP* set$\}$.

3.12 Theorem. *Let S be a semigroup and let $\mathcal{G} \subseteq \mathcal{P}(S)$ such that $\mathcal{G}$ is closed under supersets. The following statements are equivalent.*

(a) For every right piecewise syndetic subset B of S, $BB^{-1} \in \mathcal{G}$.

(b) For every right syndetic subset B of S, $BB^{-1} \in \mathcal{G}$.

(c) For every minimal dynamic system $(X, \langle T_s\rangle_{s\in S})$ and every nonempty open subset U of X, $\{s \in S : U \cap {T_s}^{-1}U \neq \emptyset\} \in \mathcal{G}$.

Proof. That (a) implies (b) is trivial. To see that (b) implies (c) let $(X, \langle T_s\rangle_{s\in S})$ be a minimal dynamic system and let U be a nonempty open subset of X. Pick any $x \in S$ and let $B = \{s \in S : T_s(x) \in U\}$.

We claim that B is right syndetic. By the minimality of $(X, \langle T_s \rangle_{s \in S})$, pick $H \in \mathcal{P}_f(S)$ such that $X = \bigcup_{t \in H} \ {T_t}^{-1}U$. (See for example [5, Lemma 1.14].) Then, as can be routinely verified, $S = \bigcup_{t \in H} \ t^{-1}B$.

It is then easy to see that $BB^{-1} \subseteq \{s \in S : U \cap {T_s}^{-1}U \neq \emptyset\}$ so that $\{s \in S : U \cap {T_s}^{-1}U \neq \emptyset\} \in \mathcal{G}$.

To see that (c) implies (a), let B be right piecewise syndetic. Then by Theorem 2.9(b), $c\ell B \cap K(\beta S) \neq \emptyset$. Since $K(\beta S)$ is the union of all of the minimal left ideals of βS, pick a minimal left ideal L of βS such that $c\ell B \cap L \neq \emptyset$.

We claim that $(L, \langle \lambda_s \rangle_{s \in S})$ is a minimal dynamical system. Each λ_s is continuous since $s \in S$, and since L is a left ideal, $\lambda_s : L \longrightarrow L$. Trivially $\lambda_s \circ \lambda_t = \lambda_{st}$. To see that $(L, \langle \lambda_s \rangle_{s \in S})$ is minimal let Y be a closed nonempty subset of L which is invariant under each λ_s and pick $p \in Y$. Then $\beta S \cdot p$ is a left ideal which is contained in the minimal left ideal L so $\beta S \cdot p = L$ and since Y is invariant, $S \cdot p \subseteq Y$. Thus $L = \beta S \cdot p = c\ell(S \cdot p) \subseteq Y$.

Let $U = c\ell B \cap L$. Then U is a nonempty open subset of L. We claim that $\{s \in S : U \cap {\lambda_s}^{-1}U \neq \emptyset\} \subseteq BB^{-1}$. Let $s \in S$ such that $U \cap {\lambda_s}^{-1}U \neq \emptyset$ and pick $p \in U \cap {\lambda_s}^{-1}U$. Then $B \in p$ and $B \in s \cdot p$ and hence $s^{-1}B \in p$. Pick $t \in B \cap s^{-1}B$. Then $st \in B$ so $s \in BB^{-1}$. □

To summarize, right syndeticity and right piecewise syndeticity of A each guarantee by Corollary 3.2 that AA^{-1} will be both left and right IP* in left amenable, left cancellative semigroups, but not for free groups (Theorem 3.8). Left syndeticity of A, on the other hand, guarantees that AA^{-1} will be left and right IP* in general semigroups (Corollary 3.7), but left piecewise syndeticity does not, even for amenable groups (Theorem 3.4).

4. Quotient Sets and the Δ^* Property.

According to Theorem 1.1, if $A \subseteq \mathbb{N}$ with $d^*(A) > 0$, then $A - A$ is an IP* set. Upon examination of the proof, one sees in fact that $A - A$ intersects non-trivially the set of differences of any infinite set. Namely, if B is an infinite subset of $\mathbb{N}$, then $(A - A) \cap (B - B) \neq \emptyset$. This is a stronger property, for every IP-set in $\mathbb{N}$, say $FS(\langle x_n \rangle_{n=1}^{\infty})$, contains the difference set of some infinite set (for example, the set $B = \{\Sigma_{t=1}^{n} x_t : n \in \mathbb{N}\}$). We call a set of the form $B - B$, where B is an infinite subset of $\mathbb{N}$, a Δ *set*, and accordingly a subset $E \subseteq \mathbb{N}$ which intersects every Δ set non-trivially a Δ^* *set*.

Hence in $\mathbb{N}$, the Δ^* property is stronger than the IP* property. We wish to fashion a definition of Δ sets in more general semigroups or groups in such a way that the resulting Δ^* property remains stronger than the corresponding IP* property. One immediate concern is that we may need separate notions of "left" and "right" Δ^* sets in the non-commutative situation to correspond to the separate notions of left IP* and right IP*. There is another consideration, however, even more basic.

Consider the group $\mathbb{Z}$. Clearly the set E of negative integers will intersect $B - B$ non-trivially for any set B of infinite cardinality, but E is obviously not IP*. This motivates the following definition.

4.1 Definition. Let S be a semigroup and let $E \subseteq S$.

(a) Given a sequence $\langle b_n \rangle_{n=1}^{\infty}$ in S, let $\Delta_I(\langle b_n \rangle_{n=1}^{\infty}) = \{x \in S :$ there exist $m < n$ in $\mathbb{N}$ such that $b_m x = b_n\}$ and let $\Delta_D(\langle b_n \rangle_{n=1}^{\infty}) = \{x \in S :$ there exist $m < n$ in $\mathbb{N}$ such that $x b_m = b_n\}$.

(b) E is a *right* (respectively *left*) Δ set if and only if there exists a one-to-one sequence $\langle b_n \rangle_{n=1}^{\infty}$ in S such that $\Delta_I(\langle b_n \rangle_{n=1}^{\infty}) \subseteq E$ (respectively $\Delta_D(\langle b_n \rangle_{n=1}^{\infty}) \subseteq E$).

(c) E is a *right* (respectively *right*) Δ^* set if and only if E intersects every right (respectively left) Δ set in G non-trivially.

Notice that, if $B = \{b_n : n \in \mathbb{N}\}$, then $\Delta_I(\langle b_n \rangle_{n=1}^{\infty}) \subseteq B^{-1}B$ and $\Delta_D(\langle b_n \rangle_{n=1}^{\infty}) \subseteq BB^{-1}$.

In some semigroups, even cancellative semigroups, there may be no Δ^* sets. For example, in the free semigroup on the letters a and b, if $c_n = ab^n a$, then $\Delta_I(\langle c_n \rangle_{n=1}^{\infty}) = \Delta_D(\langle c_n \rangle_{n=1}^{\infty}) = \emptyset$. On the other hand, in a group any Δ set must be infinite and so all cofinite sets are Δ^* sets.

4.2 Theorem. *Let S be a left cancellative semigroup. Every right Δ^* set in S is a right IP* set.*

Proof. Let E be a right Δ^* set, and let a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S be given. For each $n \in \mathbb{N}$ let $b_n = \overrightarrow{\prod}_{t=1}^{n} x_t$. Pick $y \in E$ and $m < n$ in $\mathbb{N}$ such that $b_m y = b_n$. Since also $b_m \overrightarrow{\prod}_{t=m+1}^{n} x_t = b_n$ we have by left cancellation that $y = \overrightarrow{\prod}_{t=m+1}^{n} x_t$. □

We continue our practice of leaving the obvious left-right switches to the reader.

The converse to the previous result is false, even for Δ sets, as we shall see in Theorem 4.4. For this we need the following lemma.

4.3 Lemma. *Let $(G, +)$ be a countable abelian group and let T be a thick subset of G. There is a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in G such that $T = \Delta_I(\langle a_n \rangle_{n=1}^{\infty})$.*

Proof. Enumerate T as $\{t_n : n \in \mathbb{N}\}$. Pick any $a_1 \in G$ and let $a_2 = a_1 + t_1$. Inductively let $k \in \mathbb{N}$ and assume that $a_1, a_2, \ldots, a_{2k}$ have been chosen so that $a_{2k} - a_{2k-1} = t_k$ and whenever $i < j$ in $\{1, 2, \ldots, 2k\}$, $a_j - a_i \in T$. Let $F = \{-a_1, -a_2, \ldots, -a_{2k}\} \cup$

$\{t_{k+1} - a_1, t_{k+1} - a_2, \ldots, t_{k+1} - a_{2k}\}$ and pick a_{2k+1} such that $F + a_{2k+1} \subseteq T$. Let $a_{2k+2} = a_{2k+1} + t_{k+1}$. □

4.4 Theorem. *There is a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{Z}$ such that $\Delta_I(\langle a_n \rangle_{n=1}^{\infty})$ is an IP* set but not a Δ^* set.*

Proof. It is easy to see that $\mathbb{Z} \backslash \Delta_I(\langle 2^n \rangle_{n=1}^{\infty})$ is thick and an IP* set, and it is trivially not a Δ^* set. Pick by Lemma 4.3 a sequence $\langle a_n \rangle_{n=1}^{\infty}$ such that $\mathbb{Z} \backslash \Delta_I(\langle 2^n \rangle_{n=1}^{\infty}) = \Delta_I(\langle a_n \rangle_{n=1}^{\infty})$. □

As is well known to aficionados, the right IP* property in any semigroup S is preserved by finite intersections. That is, if A and B are both right IP* sets in S, then $A \cap B$ is right IP* as well [6, Remark 16.7]. We show now that the same result holds for right Δ^* sets in any left cancellative semigroup. (But recall that there may be no right Δ^* sets.)

4.5 Theorem. *Let S be a left cancellative semigroup. If E and F are right Δ^* sets in S, then $E \cap F$ is a right Δ^* set.*

Proof. Let $\langle b_n \rangle_{n=1}^{\infty}$ be a sequence in S. For a set X, let $[X]^2$ denote the set of two element subsets of X. Let $A_0 = \big\{\{m, n\} \in [\mathbb{N}]^2 : m < n$ and there exists $x \in E$ such that $b_m x = b_n\big\}$ and let $A_1 = [\mathbb{N}]^2 \backslash A_0$. Pick by Ramsey's Theorem [11] an infinite subset C of $\mathbb{N}$ and $i \in \{0, 1\}$ such that $[C]^2 \subseteq A_i$. Enumerate C in increasing order as $\langle k(n) \rangle_{n=1}^{\infty}$. Since $E \cap \Delta_I(\langle b_{k(n)} \rangle_{n=1}^{\infty}) \neq \emptyset$ and thus $i = 0$. Since F is a right Δ^* set, pick $y \in F \cap \Delta_I(\langle b_{k(n)} \rangle_{n=1}^{\infty})$ and pick $m < n$ in $\mathbb{N}$ such that $b_{k(m)} y = b_{k(n)}$. Since $\{k(m), k(n)\} \in [C]^2 \subseteq A_0$, pick $x \in E$ such that $b_{k(m)} x = b_{k(n)}$. By left cancellation, $x = y$ and so $E \cap F \cap \Delta_I(\langle b_{k(n)} \rangle_{n=1}^{\infty}) \neq \emptyset$ as required. □

One may easily check that in a group, AA^{-1} is left Δ^* if and only if it is right Δ^* . This is a consequence of the fact that AA^{-1} is closed under inverses, so that $x_i x_j^{-1} \in AA^{-1}$ if and only if $x_j x_i^{-1} \in AA^{-1}$. Trivially, if a semigroup is commutative then the left Δ^* and right Δ^* properties are equivalent. This is not true in general. Indeed, if G is the free group on the letters $\langle x_n \rangle_{n=1}^{\infty}$, one may readily check that the set $B = G \setminus \{x_i x_j^{-1} : i < j\}$ is a left Δ^* set (it is obviously not right Δ^*). However, we do have the general fact that in any group B is left Δ^* if and only if B^{-1} is right Δ^* .

The following theorem is a strengthening (for groups) of Theorem 3.1 and Corollary 3.2.

4.6 Theorem. *Let G be a countable, left amenable group.*

(a) If $B \subseteq G$ with $m_l^*(B) > 0$, then BB^{-1} is both a left Δ^* set and a right Δ^* set.

(b) If $B \subseteq G$ is left piecewise syndetic, then BB^{-1} is both a left Δ^* set and a right Δ^* set.

Proof. (a). Let m be a left invariant mean with $m(B) = m_l^*(B)$. Let a sequence $A = \langle x_n \rangle_{n=1}^{\infty}$ be given. For each $n \in \mathbb{N}$ we have $m(x_n^{-1}B) = m(B)$, so $\{x_n^{-1}B : n \in \mathbb{N}\}$ cannot be a disjoint collection. Consequently one may pick $n < m$ such that $x_m^{-1}B \cap x_n^{-1}B \neq \emptyset$. Let b_1 and b_2 be elements of B such that $x_m^{-1}b_1 = x_n^{-1}b_2$. Then $x_n x_m^{-1} = b_2 b_1^{-1} \in BB^{-1}$. Hence $BB^{-1} \cap \Delta_I A \neq \emptyset$ and BB^{-1} is a right Δ^* set. Since the left and right Δ^* properties are equivalent for quotient sets, BB^{-1} is a right Δ^* set as well. Statement (b) follows from (a) by Theorem 2.7. □

Notice that the proof of Theorem 4.6 is essentially the same as the proof of Theorem 3.1. Both proofs rely on the fact that if $m(B) > 0$ for some left invariant mean m then one cannot have pairwise disjointness of infinitely many shifts of B. Things are not so nice in the non-amenable situation, where one may have pairwise disjointness of infinitely many shifts of a right syndetic set. (Let G be the free group on letters a and b and let B be the set of words beginning with b or b^{-1}. Then $G = bB \cup b^{-1}B$, so B is right syndetic, but the shifts $\{a^n B : n \in \mathbb{N}\}$ are pairwise disjoint.)

As was the case for the IP* property, strong right piecewise syndeticity, in particular left syndeticity, of A is enough to guarantee that AA^{-1} is both left and right Δ^* .

4.7 Theorem. *Let G be a group. If B is a strongly right piecewise syndetic subset of G then BB^{-1} is both a left Δ^* set and a right Δ^* set.*

Proof. Pick $H = \{h_1, \cdots, h_l\} \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} Bt^{-1}$ is left thick. Let a sequence $A = \langle x_n \rangle_{n=1}^{\infty}$ in G be given. Choose $a \in G$ such that $\{x_n : 1 \leq n \leq l+1\}a \subseteq \bigcup_{t \in H} Bt^{-1}$. Pick $t \in H$ and m, n with $1 \leq m < n \leq l+1$ such that $x_n a \in Bt^{-1}$ and $x_m a \in Bt^{-1}$. Then $x_m x_n^{-1} \in BB^{-1}$. Since $x_m x_n^{-1} \in \Delta_I A$, this shows that BB^{-1} is a right Δ^* set, hence also a left Δ^* set. □

The following interesting finite intersection property is now readily obtained.

4.8 Corollary. *Suppose that G is a group and $B_1, \cdots, B_k$ are subsets of G such that either:*
(a) B_i is strongly right piecewise syndetic for $i \in \{1, 2, \ldots, k\}$, or
(b) G is amenable and $m_l^*(B_i) > 0$ for $i \in \{1, 2, \ldots, k\}$.
Then $\bigcap_{i=1}^{k} B_i B_i^{-1}$ is both left and right Δ^* .

Proof. This follows from Theorem 4.6 and 4.7, using the fact that the Δ^* property is preserved by finite intersections. □

Although Theorem 3.5 shows that AA^{-1} need not be IP* (much less Δ^*) for right piecewise syndetic sets A in non-amenable, groups, one still might wonder from Corollary 4.8 whether such difference sets AA^{-1} might nevertheless have some non-trivial finite intersection property. We now show by example that this is not the case for intersections of three such difference sets. (We do not know about intersections of two such sets. See Question 1 at the end.)

4.9 Theorem. *There exists a group G of infinite cardinality and a partition of G into right syndetic sets A, B, and C such that $AA^{-1} \cap BB^{-1} \cap CC^{-1} = \{e\}$.*

Proof. Let G be the free group on the letters a, b, and c. Let A consist of e and all words starting with a or a^{-1}. Let B consist of all words starting with b or b^{-1}, and let C consist of all words starting with c or c^{-1}. It is easily seen that A, B, and C are each right syndetic. Moreover, AA^{-1} consists of e and those words either starting or ending in a or a^{-1}, BB^{-1} consists of e and those words either starting or ending in b or b^{-1}, and CC^{-1} consists of e and those words either starting or ending in c or c^{-1}. Clearly $AA^{-1} \cap BB^{-1} \cap CC^{-1} = \{e\}$. □

Questions.

1. In a group, if A and B are both right syndetic, does it follow that $AA^{-1} \cap BB^{-1}$ necessarily contains more than the identity?

2. If $m_l(B) > 0$ for B in a left amenable semigroup, and A is infinite, does $BB^{-1} \cap AA^{-1}$ necessarily contain elements different from the identity?

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