The Stone-Čech compactification of a partially ordered set via bitopology Ralph Kopperman – January 3, 2003

Abstract: Each partially ordered set (X, \leq) has a *Čech order compactification* μX , constructed in the same fashion as Čech's construction of the Stone-Čech compactification, but using the order preserving functions from X into the unit interval [0, 1]. This is already known; our point is that its existence and properties can be seen using straightforward adaptations of textbook results and proofs.

1. Introduction

We shall be concerned with generalizing the Stone-Cech compactification of a discrete space to that obtained when (X, \leq) is a poset (partially ordered set), for use with discrete ordered semigroups. This is a natural compactification from the point of view of ordered sets, but note that this situation is asymmetric in the following sense: Above, we considered functions from the topological space (X, τ) to $([0, 1], \mathcal{E})$, \mathcal{E} the usual Euclidean topology on this closed interval; the map $x \to 1 - x$ is an automorphism of the latter. We will now be concerned with an ordered topological space (X, τ, \leq) and we must replace the closed interval with $([0, 1], \mathcal{E}, \leq)$; now there is no automorphism taking $1 \to 0$. This problem is as easy to overcome as it is to state: Now $x \to 1 - x$ is instead an isomorphism from $([0, 1], \mathcal{E}, \leq)$ to its *dual*, $([0, 1], \mathcal{E}, \geq)$. As we will see, the construction is almost identical, but the asymmetry (as evidenced by the need for a dual) remains.

The first to handle such asymmetry well was Nachbin (much of his research was done in the late 1940's; his most convenient reference is [Na]). Due to the simplicity of translating the conditions in the Stone-Čech theorem, we prefer to explain this in terms of bitopological spaces, and then specialize to ordered topological spaces. Bitopological spaces were developed by J. C. Kelly [Ky]. Prof. Kelly died in October, 2002, and we dedicate this paper to his memory, since it owes so much to his work. The first to construct a bitopological Stone-Čech compactification to our knowledge, was Salbany ([Sa]), a detailed such construction in our notation can be found in [Ko]. Our main point is that the reader already knows such a proof. Then we apply the result to discrete ordered spaces.

A major tool for us is the *specialization order*; for any topology τ , it is defined by $x \leq_{\tau} y \Leftrightarrow x \in c\ell_{\tau}(y)$ (here $c\ell(A)$ denotes the closure of $A \subseteq X$, and notation is abused for singletons by using $c\ell(x)$ in place of $c\ell(\{x\})$). Certainly, $x \leq_{\tau} y$ if and only if $c\ell_{\tau}(x) \subseteq c\ell_{\tau}(y)$, so \leq_{τ} is a *pre-order* (that is, a reflexive, transitive relation); it is antisymmetric, thus a partial order, if and only if τ is T_0 . It is equality if and only if τ is T_1 , so is rarely used. In general, a topology is considered asymmetric if \leq_{τ} is not a symmetric relation.

It is useful to notice how the specialization interacts with products and subspaces. Certainly if $x \in \prod_I X_i$ then $\prod_I c\ell(x_i)$ is closed, so $c\ell(x) \subseteq \prod_I c\ell(x_i)$. But if $y \notin c\ell(x)$ then for some open U in the product, $y \in U$ but $x \notin U$. By definition of the product topology, there is a finite $F \subseteq I$ and for each $i \in F$ an open U_i such that $y \in \bigcap_{i \in F} \pi_i^{-1}[U_i] \subseteq U$, where as usual, π_i denotes the *i*'th projection. Since $x \notin U$ there must be some $i \in F$ such that $x_i \notin U_i$; this shows some $y_i \notin c\ell(x_i)$, so $y \notin \prod_I c\ell(x_i)$. Thus $c\ell(x) = \prod_I c\ell(x_i)$, or in other words, $y \leq_{\prod_I \tau_i} x \Leftrightarrow (\forall i \in I)(y_i \leq_{\tau_i} x_i)$. Similarly, if $x, y \in Z$, Z a subspace of (X, τ) , then $y \leq_{\tau} x \Leftrightarrow y \leq_{\tau|Z} x$, where $\tau|Z$ denotes the relative topology on Z.

Below, in sections 2-4, we note that many textbook topological proofs extend without much effort to bitopological spaces and in particular, to the Stone-Čech compactification.

In section 5, we develop some useful techniques of bitopological spaces and spaces with topology and order.

2. The category of bitopological spaces

A bitopological space, is a space with two topologies, $X = (X, \tau_X, \tau_X^*)$. For such spaces, a pairwise continuous map (or simply map), $f: X \to Y$ is a function from X to Y that is continuous from τ_X to τ_Y and from τ_X^* to τ_Y^* ; such a function is pairwise open if open from τ_X to τ_Y and from τ_X^* to τ_Y^* . When we write $f: X \to Y$, and X, Y are bitopological spaces, we mean that f is a pairwise continuous map. As a result of these definitions, many topological ideas can be extended to bitopological spaces "topologywise". For example, given a space (Y, τ_Y, τ_Y^*) and $X \subseteq Y$, the bitopological subspace is found by individually restricting the two topologies: $(X, \tau_Y | X, \tau_Y^* | X)$ – then we already know, topology by topology, that the inclusion map, i_X , is pairwise continuous, and any pairwise continuous map into Y whose image is contained in X, factors uniquely through i_X ; the product of an indexed set of bitopological space is the product set with the product topologies, $(\prod_{i \in I} X_i, \prod_{i \in I} \tau_i, \prod_{i \in I} \tau_i^*)$; this is similarly seen to be the categorical product if the reader cares to do so. If we have a collection F of functions from a set X to a bitopological space (Y, τ_Y, τ_Y^*) , then the weak bitopological space is $(X, \omega_F, \omega_F^*)$, where ω_F is the weak topology for F and τ_Y , and ω_F^* that for F and τ_Y^* , giving us the weakest bitopological space such that each $f \in F$ becomes a pairwise continuous map (here $Z = (Z, \tau_Z, \tau_Z^*)$ is weaker than $Z' = (Z, \theta_Z, \theta_Z^*)$ if the "identity" $I: Z' \to Z$ is pairwise continuous).

In his construction of the Stone-Čech compactification [Če], E. Čech embedded a completely regular Hausdorff space X into the product $[0,1]^F$ where F = C(X, [0,1]), the set of all continuous functions from X into [0,1]. We now sketch a bitopological construction along lines suggested by Kelley's version of the Čech proof (4.5, p. 116 and 5.23-4, p. 153 of [Ke]): Kelley points out in 4.5 that for any family F of continuous functions such that for each $f \in F$, $f : (X, \tau) \to (Y_f, \tau_f)$, the evaluation map $e : X \to \prod_{f \in F} Y_f$, defined by e(x)(f) = f(x), is continuous. Further, it is open to the subspace determined by its image, if and only if, for each $x \in X$ and closed $C \subseteq X$, if $x \notin C$ then there is an $f \in F$ such that $f(x) \notin c\ell(f[C])$. This is already a bitopological lemma: e is pairwise continuous, because Kelley's result can be used topology by topology; similarly, pairwise openness is checked one topology at a time. Finally, Kelley points out that e is one-one if and only if, for each pair of distinct points $x, y \in X$, there is an $f \in F$ such that $f(x) \neq f(y)$; this does not refer to any topology, and surely holds in our situation.

To continue our proof, we must discuss bitopological separation and compactness.

3. Separation

1 Definitions. The *bitopological unit interval* is $II = ([0, 1], \mathcal{U}, \mathcal{L})$, where $\mathcal{U} = \{(a, 1] \mid a \in [0, 1]\} \cup \{[0, 1]\}$ and $\mathcal{L} = \{[0, a) \mid a \in [0, 1]\} \cup \{[0, 1]\}$.

Given a bitopological space, $X = (X, \tau, \tau^*)$, its symmetrization topology is $\tau^S = \tau \vee \tau^*$; also $X^S = (X, \tau^S)$. A bitopological space is:

normal if whenever C, D are disjoint, $C \tau$ -closed and $D \tau^*$ -closed, then there are disjoint $T \in \tau, U \in \tau^*$ such that $C \subseteq U$ and $D \subseteq T$,

completely regular if whenever $x \in T \in \tau$ then there is a map $f: X \to \mathbb{I}$ such that f(x) = 1 and $f[X \setminus T] = \{0\}$, Tychonoff $(T_{3.5})$ if completely regular and τ^S is T_0 ;

pseudoHausdorff (pH) if whenever $x \notin c\ell_{\tau}(y)$ then there are $T \in \tau$ and $U \in \tau^*$ which are disjoint and such that $x \in T$ and $y \in U$, Hausdorff (T_2) if pH and τ^S is T_0 ;

weakly symmetric (ws) if $y \in c\ell_{\tau^*}(x) \Rightarrow x \in cl_{\tau}(y), T_1$ if we and τ^S is T_0 .

The dual of X is $X^* = (X, \tau^*, \tau)$ (obtained by reversing the order in which the topologies are considered). Given a property **Q** of bitopological spaces, X is pairwise **Q** if both X and its dual X^* satisfy **Q**.

A major purpose of bitopology is to view separation as a relationship between two topologies, rather than a property of a single one. Though neither \mathcal{U} nor \mathcal{L} is completely regular, II is pairwise completely regular (if $x \in T = (z, 1] \in \mathcal{U}$, consider $f = (0 \lor \frac{y-z}{x-z}) \land 1$). So techniques used with separation can help to study topologies which lack it.

A separation axiom implies lower ones. For example, if X is completely regular, and $x \notin c\ell_{\tau}(y)$ then $x \in X \setminus c\ell_{\tau}(y)$, so there is a pairwise continuous $f: X \to \mathbb{I}$ such that f(x) = 1 and $f[c\ell_{\tau}(y)] = \{0\}$. But as a result, $x \in f^{-1}[(.5, 1]] \in \tau$ and $y \in f^{-1}[[0, .5)] \in \tau^*$, and these inverse images are disjoint, so X is pH. Also, products and subspaces of pH (resp., $T_1, T_2, T_{3.5}$, ws, completely regular) spaces are pH (resp., $T_1, T_2, T_{3.5}$, ws, completely regular) spaces are pH (resp., $T_1, T_2, T_{3.5}$, ws, completely regular) have essentially the usual proofs.

The symmetrization topology inherits by itself the symmetric (separation) properties shared by the two topologies. For example, we show that if X is pairwise completely regular, then (X, τ^S) is completely regular; in the proof we use the general observation that if $f: X \to Y$ is pairwise continuous, then f is continuous as a map from τ_X^S to τ_Y^S : Therefore, if X is pairwise completely regular and $x \in T \in \tau_X^S$, then for some $U \in \tau_X, V \in$ $\tau_X^*, x \in U \cap V \subseteq T$. Thus there are $f: X \to \mathbb{I}$, $g: X^* \to \mathbb{I}$ such that f(x) = g(x) = 1and $f[X \setminus U] = g[X \setminus V] = \{0\}$. But then f, g are both continuous from τ_X^S to $\mathcal{U} \vee \mathcal{L} = \mathcal{E}$, thus so is $f \wedge g$, and this map is 1 at x and 0 off $U \cap V \subseteq T$. Thus if τ_X^S is T_0 then τ_X^S is Tychonoff. A proof of similar difficulty shows that if X is pairwise pH and τ_X^S is T_0 then τ_X^S is Hausdorff. Further definitions and proofs of similar results for separation axioms between T_1 and $T_{3.5}$ are in [Ko], section 2, as is a counterexample for normality.

Given bitopological spaces X, Y, let P(X, Y) and Y^X (depending on convenience) denote the set of pairwise continuous maps from X to Y. If X is completely regular and $P = P(X, \mathbb{I})$, then e is open from (X, τ_X) to the subspace determined by its image in $([0, 1], \mathcal{U})^P$ (as in [Ke, 5.13]), because if $x \notin C$, C closed, then there is an $f \in F$ such that f(x) = 1 and $f[C] = \{0\}$; thus if X is pairwise completely regular, then e is also open from τ_X^* to its image in $([0, 1], \mathcal{L})^P$, so e is pairwise open. Finally, if X is pairwise Tychonoff, then e is one-one, since if $x \neq y$ then there is an open set in one of the topologies containing exactly one of these points, without loss of generality, let $x \in T \in \tau_X$, $y \notin T$. There is then by our definition, $f \in P$ such that f(x) = 1 and since $y \notin T$, f(y) = 0. Thus if X is pairwise Tychonoff, then e is a bitopological imbedding.

4. Compactness

A bitopological space, X, is *joincompact* if it is pairwise T_2 , and τ^S is compact.

Whenever X is joincompact, then clearly so is its dual, X^* , and we use this below. A straightforward adaptation of the usual proofs shows that joincompact spaces are (pairwise) normal; since they are pairwise T_1 , they are pairwise Tychonoff (the proof of the bitopological Urysohn lemma is like that of the topological version, and can be found in [Ko]). The product of the symmetrization topologies is the symmetrization of the product: certainly $\prod_{I} \tau_{i} \vee \prod_{I} \tau_{i}^{*} \subseteq \prod_{I} \tau_{i}^{S}$, and if $x \in U \in \prod_{I} \tau_{i}^{S}$, then there is a finite $F \subseteq I$ and for each $i \in F$ there are $V_{i} \in \tau_{i}$, $W_{i} \in \tau_{i}^{*}$, such that $y \in \bigcap_{i \in F} \pi_{i}^{-1}[V_{i} \cap W_{i}] = (\bigcap_{i \in F} \pi_{i}^{-1}[V_{i}]) \cap (\bigcap_{i \in F} \pi_{i}^{-1}[W_{i}]) \subseteq U$, showing by the arbitrary nature of x, that $U \in \prod_{I} \tau_{i} \vee \prod_{I} \tau_{i}^{*}$.

Since we already know that a product of pH bitopological space is pH, and have long known that a product of T_0 topological spaces is T_0 , and that a product of compact topological spaces is compact, the above shows that each product of joincompact spaces is joincompact. It is also clear that subspaces of pH bitopological spaces are pH, and therefore that symmetrically closed (that is, τ^S -closed) subspaces of joincompact spaces are joincompact. Notice that II is joincompact: Surely, $II^S = \mathcal{U} \lor \mathcal{L} = \mathcal{E}$, which is compact and T_0 . To see that II is pH, first note that, as the complement of the largest \mathcal{U} -open set not containing y, $c\ell_{\mathcal{U}}(y) = [0, y]$. This shows that $\leq_{\mathcal{U}} = \leq$; similarly $\leq_{\mathcal{L}} = \geq$. Thus $x \notin c\ell_{\mathcal{U}}(y)$ if and only if x > y, so for z between the two, $x \in (z, 1] \in \mathcal{U}, y \in [0, z) \in \mathcal{L}$, and $(z, 1] \cap [0, z) = \emptyset$.

By the above discussion, given a pairwise Tychonoff bitopological space X, the symmetric closure of its image in $\mathbb{I}^{P(X,\mathbb{I})}$ which we call $(\nu(X), \tau_{\nu}, \tau_{\nu}^{*})$ below, is a joincompact space in which X is symmetrically densely imbedded; further, if $f \in P(X,\mathbb{I})$ then the map $\tilde{f} = \pi_{f}|\nu(X)$ is pairwise continuous from $\nu(X)$ to \mathbb{I} , and for each $x \in X$, $\tilde{f}(e(x)) = e(x)(f) = f(x)$, showing that $\tilde{f}e = f$. This results in (a) of the following theorem:

2 Theorem. (a) Let X be a pairwise Tychonoff bitopological space. Then there is a joincompact space $\nu(X)$ and an imbedding e such that e[X] is symmetrically dense in $\nu(X)$ and for every pairwise continuous $f: X \to \mathbb{I}$, there is a unique $\tilde{f}: \nu(X) \to \mathbb{I}$ such that $f = \tilde{f}e$.

(b) If $f: X \to Y$ is a pairwise continuous map and Y is joincompact, then there is a unique $\tilde{f}: \nu(X) \to Y$ such that $f = \tilde{f}e$.

Part (b) is simply the bitopological version of the usual extension property of continuous functions to the Stone-Čech compactification, and again was essentially shown by Kelley, although in his (5.23, 152), we can't quite use the statement of the result, since it refers to the set of all continuous functions. But his proof holds without change for each topology, and the class of pairwise continuous functions. Finally, the extension property (5.24, 153) is shown by changing " $\beta(Y)$ because Y is compact Hausdorff" to " $\nu(Y)$ because Y is joincompact", and observing that this is true.

5. Stone-Čech order compactifications of discrete ordered spaces

Now, given a pre-ordered set (P, \leq) , and an $S \subseteq P$, we let $\uparrow S = \{x \mid s \leq x \text{ for some } s \in S\}$ ($\uparrow x$ abbreviates $\uparrow \{x\}$), and say that S is an upper set if $S = \uparrow S$; $\downarrow S$ and lower set are defined similarly. Its Alexandroff topology $\alpha(\leq)$ is that in which a set is open if and only if it is an upper set, and a topology is Alexandroff if it is of the form $\alpha(\leq)$ for some pre-order. The weak topology of this pre-ordered set, $W(\leq)$, is that whose closed sets have as a subbase, $\{\downarrow x : x \in X\}$ (this clashes with the usual use of weak topology generated by a set of functions; in fact, $W(\leq)$ is the weak topology generated by the characteristic functions $\chi_{X \setminus \downarrow x}$ for $x \in X$). Clearly, arbitrary intersections and unions of upper sets are

upper sets (so $\alpha(\leq)$ is a topology and also $W(\leq) \subseteq \alpha(\leq)$); also a set is upper if and only if its complement is a lower set, thus is closed in $\alpha(\leq)$ if and only if it is a lower set. Also, let $O(X, \leq)$ denote the set of order-preserving functions from (X, \leq) to [0, 1], with its usual order. On [0, 1], $W(\leq) = \mathcal{U}$ and $W(\geq) = \mathcal{L}$. Here are some other useful properties of these notations:

3 Theorem. (a) For any topology, $W(\leq_{\tau}) \subseteq \tau \subseteq \alpha(\leq_{\tau})$; the latter says that each open set is a specialization-upper set. Conversely, if $W(\leq) \subseteq \tau \subseteq \alpha(\leq)$, then \leq equals \leq_{τ} .

(b) Each continuous map preserves the specialization, and the converse holds if the topology on the domain is Alexandroff. If (X, τ, τ^*) is completely regular, then $x \leq_{\tau} y$ if and only if $f(x) \leq f(y)$ for each $f \in P(X, \mathbb{I})$.

(c) $O(X, \leq) = P((X, \alpha(\leq), \alpha(\geq)), \mathbb{I})$. Also for any poset, $\alpha(\leq) \lor \alpha(\geq) = \alpha(=)$, the discrete topology.

(d) For a poset (X, \leq) , $(X, \alpha(\leq), \alpha(\geq))$ is pairwise Tychonoff. If \leq is a linear order of X, then $W(\leq) \lor W(\geq)$ is the order topology on X (that is, the one generated by $\{(a, \infty) : a \in X\} \cup \{(-\infty, a) : a \in X\}$, where for any poset, we use the notations (a, ∞) for $\{x \in X : x > a\}$ and $(-\infty, a)$ for $\{x \in X : x < a\}$).

Proof: (a) By definition we have that for each $x \in X$, $\downarrow_{\leq_{\tau}} (x) = c\ell_{\tau}(x)$ so each such set is closed in τ , and our first inclusion results. For the second, if $T \in \tau$ then $X \setminus T$ is closed, so for each $x \in X \setminus T$, $\downarrow_{\leq_{\tau}} (x) \subseteq X \setminus T$. As a result, if $w \in T$ and $w \leq_{\tau} x$ then $x \in T$, showing that the arbitrary τ -open T is in $\alpha(\leq_{\tau})$.

For the converse, if $\tau \subseteq v$ then for each $x \in X$, $c\ell_{\tau}(x)$ is v-closed, so $c\ell_{v}(x) \subseteq c\ell_{\tau}(x)$; as a result, we have that in this situation, $\leq_{v} \subseteq \leq_{\tau}$, so in particular, $\leq_{\alpha(\leq)} \subseteq \leq_{W(\leq)}$. Further, $\downarrow x$ is closed by definition in $W(\leq)$, showing that $\leq_{W(\leq)} \subseteq \leq$. Also, any set closed in $\alpha(\leq)$ is a lower set, so $\downarrow x \subseteq c\ell_{\alpha(\leq)}(x)$, and $\leq \subseteq \leq_{\alpha(\leq)}$; thus these orders are equal.

(b) To see that continuous maps preserve the specialization: if $x \leq_{\tau} y$ and $f: (X, \tau) \to (Y, \tau')$ then $f(x) \in f(c\ell_{\tau}(y)) \subseteq c\ell_{\tau'}(f(y))$, so $f(x) \leq_{\tau'} f(y)$. For the converse, suppose f is specialization preserving, and consider a typical open set T; by (a), T is a $\leq_{\tau'}$ -upper set. Since f is specialization preserving, $f^{-1}[T]$ is a \leq_{τ} -upper set, so open in $\alpha(\leq_{\tau}) = \tau$.

In particular, if $x \leq_{\tau} y$ then for each $f \in P(X, \mathbb{I})$, $f(x) \leq f(y)$; but if not and our space is completely regular, then there is an $f \in P(X, \mathbb{I})$ such that f(x) = 1 and f is 0 off $X \setminus c\ell(y)$, so in particular, $f(y) = 0 \geq f(x)$.

(c) Then $O(X, \leq) = P((X, \alpha(\leq), \alpha(\geq)), \mathbb{I})$ by (a) since on $[0, 1], W(\leq) = \mathcal{U}$. The last assertion results from the fact that for each $\in X$, $\{x\} = \uparrow x \cap \downarrow x$, so is open in the join $\alpha(\leq) \lor \alpha(\geq)$ and in $\alpha(=)$.

(d) Suppose $x \in T \in \alpha(\leq)$, and define $f: X \to [0,1]$ by f(y) = 1 if $y \geq x$, f(y) = 0 otherwise. Then f is order preserving, for if $y \leq z$ then $f(y) = 0 \leq f(z)$, or f(y) = 1 thus $x \leq y \leq z$, whence $z \geq x$ so $f(z) = 1 \geq f(y)$. Thus f is pairwise continuous to II and f(x) = 1. Also, if $y \notin T$ then $y \not\geq x$ (since T is an upper set and $x \in T$), so f(y) = 0. This shows that $(X, \alpha(\leq), \alpha(\geq))$ is completely regular and similarly, $(X, \alpha(\geq), \alpha(\leq))$ is completely regular, so $(X, \alpha(\leq), \alpha(\geq))$ is pairwise completely regular. By (b), X^S is discrete (thus T_0).

If \leq is a linear order then each $X \setminus \downarrow x = (x, \infty)$, so the latter is a subbasic open set; dually this holds for \geq as well. Thus the sets of the forms $(x, \infty), (-\infty, x)$, are a subbase for $W(\leq) \lor W(\geq)$, as required.

The following corollary to Theorems 2 and 3, allows us to define $(\mu X, \mu(\leq), \mu(\geq)) = \nu(X, \alpha(\leq), \alpha(\geq))$ and call it the Stone-Čech order compactification of (X, \leq) .

4 Corollary. Let (X, \leq) be a poset and $f : X \to \mu X$ be an order preserving function. Then there is a unique pairwise continuous $\tilde{f} : \mu X \to \mu X$ such that $\tilde{f} \circ e = f$.

5 Lemma. Let (X, \leq) be a poset. Then for each $A \subseteq X$, $c\ell^{S}(\downarrow A)$ is $\mu(\geq)$ -open and $\mu(\leq)$ -closed in μX and $c\ell^{S}(\uparrow A)$ is $\mu(\leq)$ -open and $\mu(\geq)$ -closed there.

Proof: We define $f \in P(X, \mathbb{I})$ by f(y) = 0 if for some $x \in A$, $y \leq x$ and f(y) = 1 otherwise. Then $\downarrow A = f^{-1}[0] \subseteq \tilde{f}^{-1}[0]$, so $c\ell^{S}(\downarrow A) \subseteq \tilde{f}^{-1}[0]$ and similarly $c\ell^{S}(X \setminus \downarrow A) \subseteq \tilde{f}^{-1}[1]$. But $\mu X = c\ell^{S}(X) = c\ell^{S}(X \setminus \downarrow A) \cup c\ell^{S}(\downarrow A)$; thus in particular, $c\ell^{S}(\downarrow A) = \tilde{f}^{-1}[0]$, a $\mu(\leq)$ -closed, $\mu(\geq)$ -open set. The other assertions are the order duals of these. \Box

The evaluation map from a bitopological space to its joincompactification is a pairwise embedding, thus specialization preserving and reversing (by Theorem 3 (b)), so:

6 Lemma. Let (X, \leq) be a poset and let $x, y \in X$. Then $x \leq y \Leftrightarrow e(x) \leq e(y)$ in $\mu(X)$ – by earlier results, the order on the right can be taken to be the specialization $\leq_{\mu(\leq)}$, or equivalently the function order, which is the product of the usual order on [0, 1].

7 Lemma. Suppose that X is a pairwise T_1 bitopological space and the specialization \leq_{τ} is a linear order on an ^S-dense subspace of X. Then it is a linear order on X.

Proof: Let D be an ^S-dense subspace of X on which \leq_{τ} is a linear order, and let $x \in X$ be arbitrary. Suppose $x \in T \cap U$, $T \in \tau$, $U \in \tau^*$. Then there is a $d \in D \cap T \cap U$, and $D \subseteq \uparrow_{\leq_{\tau}} (d) \cup \downarrow_{\leq_{\tau}} (d) = c\ell_{\tau^*}(d) \cup c\ell_{\tau}(d)$, an ^S-closed set as the union of two such. Thus $X \subseteq c\ell_{\tau^*}(d) \cup c\ell_{\tau}(d) \subseteq T \cup U$. But this shows $X = (\bigcap\{T \mid x \in T \in \tau\}) \cup (\bigcap\{U \mid x \in U \in \tau^*\}) = \{y \mid x \in c\ell_{\tau}(y)\} \cup \{y \mid x \in c\ell_{\tau^*}(y)\} = \uparrow_{\leq_{\tau}} (x) \cup \downarrow_{\leq_{\tau}} (x)$, as required. \Box

8 Definition. Let (X, τ) be a topological space. A subset $S \subseteq X$ is saturated if it is a \leq_{τ} -upper set. The de Groot dual (cocompact dual) of τ is the topology τ^G , whose closed sets are generated by the compact saturated subsets of X. The space (X, τ) is skew compact if there is a second topology on X such that (X, τ, τ^*) is joincompact.

In our new terminology, the assertion in Theorem 3 (a) that for each topology, $\tau \subseteq \alpha(\leq_{\tau})$, says that each open set is saturated. Part (a) of the next theorem shows that τ^{G} is the only possible topology τ^{*} such that (X, τ, τ^{*}) can be joincompact. Part (b) applies this to our order-compactifications.

9 Theorem. (a) ([Ko]) If X is a pH bitopological space, then $\tau^G \subseteq \tau^*$; further, if X is joincompact, then $\tau^G = \tau^*$.

(b) If τ is skew compact, then $\tau \subseteq W(\geq_{\tau})^{G}$. If, further, \leq_{τ} is a linear order on X, then $\tau = W(\leq_{\tau}), \tau^{G} = W(\geq_{\tau})$, and τ^{S} is the order topology.

Proof: (a) The first assertion is shown by the asymmetric version of the proof that compact subsets of Hausdorff spaces are closed ([Ko], 3.1): suppose K is compact and $x \notin \uparrow_{\tau} K$. Then for each $y \in K$, we have $y \notin \downarrow_{\tau} x = c\ell_{\tau}(x)$, so there are $T_y \in \tau$, $U_y \in \tau^*$, such that $y \in T_y$, $x \in U_y$, and $T_y \cap U_y = \emptyset$. Thus $K \subseteq \bigcup_{y \in K} T_y$, so for some finite $F \subseteq K$, $K \subseteq \bigcup_{y \in F} T_y$. But then $U = \bigcap_{y \in F} U_y \in \tau^*$, $x \in U$, and U is disjoint from $\bigcup_{y \in F} T_y$ and therefore from K. This shows that $\uparrow_{\tau} K$ is τ^* -closed.

For the second assertion, let τ^S be compact and X^* be ws; that is, $\leq_{\tau^*} \subseteq \geq_{\tau}$. Then $\tau^* \subseteq \tau^G$: Each τ^* -closed set is τ^S -closed, thus is τ^S -compact, and so is τ -compact; it is a \leq_{τ^*} -lower set, therefore saturated, showing this observation.

(b) Our first result comes from (a) and several observations: (i) If $x \in K \subseteq \uparrow x$ then K is compact: for given any cover of K by open sets, x is in one of them, say T. Then $\uparrow x \subseteq T$, showing that T covers K. Thus in particular, each $\uparrow x \in \tau^G$, showing $W(\geq_{\tau}) \subseteq \tau^G$.

(ii) Since each subbasic closed set of τ^G is a \geq_{τ} -lower set, so are all its closed sets. That is, $\tau^G \subseteq \alpha(\geq_{\tau})$, so by (i) and Theorem 3 (a), \geq_{τ} is the specialization of τ^G .

(iii) If $\tau \subseteq v$ and the two have the same specialization, then $v^G \subseteq \tau^G$, since v-compact sets are clearly τ -compact sets, and sets have the same saturation in the two.

Applying (a) to (X, τ^*, τ) , we see $\tau = (\tau^*)^G \subseteq W(\geq_{\tau})^G$, the first assertion of (b).

For its second assertion, note: (iv) when \leq_{τ} is a total order, then $\tau^G \subseteq W(\geq_{\tau})$. For this, notice that $K \subseteq X$ is τ -compact if and only if it has a least element: for if x is the least element of K then by (i), K is compact; if K has no least element, then $\{\downarrow_{\leq_{\tau}} (x) : x \in K\}$ is a chain of τ -closed sets in X, each of which meets K, but whose intersection fails to meet K, so K is not compact. But if K is also saturated, then $K \subseteq \uparrow x \subseteq \uparrow K = K$, so $K = \uparrow x$, a $W(\geq_{\tau})$ -closed set. This shows that each subbasic τ^G -closed set is $W(\geq_{\tau})$ closed and so (iv) holds, and combining this with part (a) of this theorem and assertion (i) we have $\tau^* = W(\geq_{\tau})$. Applying this to the dual, which is joincompact, we have $\tau = W(\geq_{\tau^*}) = W(\leq_{\tau})$. This and Theorem 3 (c) show the second assertion. \Box

10 Lemma. (a) A bitopological space X is pH if and only if \leq_{τ} is closed in $(X, \tau) \times (X, \tau^*)$. (b) Let D be an ^S-dense subspace of a pH bitopological space, X. Then $\leq_{\tau} = c\ell_{\tau \times \tau^*}(\Delta_D)$, where Δ_D denotes the diagonal of $D = \{(x, x) \mid x \in D\}$.

(c) In the Stone-Čech order-compactification, $\leq_{\mu(\leq)} = c\ell_{(\mu(\leq)\vee\mu(\geq))^2}(\leq)$.

Proof: (a) Suppose first that \leq_{τ} is closed, and let $x, y \in X$, $x \not\leq_{\tau} y$. Then since $(x, y) \in X^2 \setminus \leq_{\tau}$, an open set in the product, there are open $T \in \tau, U \in \tau^*$ with $x \in T, y \in U$ so that $[T \times U] \cap \leq_{\tau} = \emptyset$. But this implies that if $t \in T$, $u \in U$, then $t \not\leq_{\tau} u$, whence $t \neq u$; in other words, $T \cap U = \emptyset$, so these are disjoint neighborhoods of the arbitrary x, y. Thus the space X is pH. For the converse, simply note that if X is pH and $x \not\leq_{\tau} y$, then there are $T \in \tau, U \in \tau^*$, disjoint, with $x \in T, y \in U$, and so $(x, y) \in T \times U$; $(T \times U) \cap \leq_{\tau} = \emptyset$ since if $t \in T, t \leq_{\tau} u$, then $u \in T$, so $u \notin U$. Thus \leq_{τ} is closed in $(X, \tau) \times (X, \tau^*)$.

(b) By (a), $c\ell_{\tau \times \tau^*}(\Delta_D) \subseteq \leq_{\tau}$, since the latter is a $\tau \times \tau^*$ -closed set containing Δ_D . To see the reverse inclusion, let $x, y \in X$, $x \leq_{\tau} y$, and suppose $x \in T \in \tau$, $y \in U \in \tau^*$. Then $y \in T$ (since $x \in c\ell(y)$), so $T \cap U$ is a nonempty open set in τ^S . Therefore, there is some $z \in T \cap U \cap D$. This says that if $(x, y) \in \leq_{\tau} \cap (T \times U)$ then, $\emptyset \neq \Delta_D \cap (T \times U)$, showing the denseness of Δ_D in \leq_{τ} , and thus the reverse inclusion.

(c) Again by (a), $c\ell_{(\mu(\leq)\vee\mu(\geq))^2}(\leq) \subseteq \leq_{\mu(\leq)}$. For the reverse inclusion, suppose $p \leq q$, and let $p \in U \in \mu^S$, $q \in V \in \mu^S$ (where we have abbreviated $\mu(\leq) \vee \mu(\geq)$ to μ^S), and let $U' = U \cap X$. But then $p \in c\ell^S(U') \subseteq c\ell^S(\uparrow U')$; since $p \leq q$ and by Lemma 5 $c\ell^S(\uparrow U')$ is $\mu(\leq)$ -open, thus $\mu(\leq)$ -saturated so in particular, a \leq -upper set, $q \in c\ell^S(\uparrow U')$. Thus

 $c\ell^{S}(\uparrow U') \cap V$ is a $\mu(\leq)$ -neighborhood of q, so it meets X. But if $x \in U'$, $y \in c\ell^{S}(\uparrow U') \cap V \cap X$ then $x \leq_{\mu(\leq)} y$, but in the subspace X this implies $x \leq y$. So in arbitrary μ^{S} -neighborhoods of p, q we have found $x \leq y$, thus $(p,q) \in c\ell_{(\mu^{S})^{2}}(\leq)$. \Box

As a topological compactification of the discrete space X, necessarily μX is a quotient of the Stone-Čech compactification βX of X. If there are strictly monotonic sequences in X, a significant amount of collapsing occurs when this quotient is formed, since:

11 Theorem. Let Y be joincompact and $D \subseteq Y$ be filtered (resp. directed) by \leq_{τ_Y} . Then D has a infimum (resp. supremum) in Y, and converges to it in τ_Y^S .

Proof: In any compact space, a $D \subseteq Y$ directed by \geq_{τ_Y} is bounded below, since $D' = \{\downarrow_{\tau_Y} (y) \mid y \in D\}$ is a collection of closed sets with fip (for any finite number, $\downarrow_{\tau_Y} (y_1), \ldots, \downarrow_{\tau_Y} (y_n)$, let $y \leq_{\tau_Y} y_1, \ldots, y_n$; then $y \in \downarrow_{\tau_Y} (y_1) \cap \ldots \cap \downarrow_{\tau_Y} (y_n)$), thus $\bigcap D' \neq \emptyset$, but any $z \in \bigcap D'$ is a lower bound for D.

Since $c\ell_{\tau_Y} {}^{S}D$ is closed in the compact τ_Y^S , it is ^S-compact, it inherits pH and T_0 from Y, so it is a joincompact subspace of Y, so a compact subspace of (Y, τ_Y) . In particular, D is bounded below by an element $z \in c\ell_{\tau_Y^S}D$. If w is any lower bound for D, we must have $D \subseteq \uparrow_{\tau_Y} w$ so $z \in c\ell_{\tau_Y^S}D \subseteq \uparrow_{\tau_Y} w$, whence z is the greatest lower bound for D.

To see that the net $D \ \tau_Y^S$ -converges to z, note that whenever $z \in T \in \tau_Y^S$, there are $U \in \tau_Y$, $V \in \tau_Y^*$, such that $z \in U \cap V \subseteq T$, and since $z \in c\ell_{\tau_Y^S}D$, there is a $d \in D \cap U \cap V$. Because U is τ_Y -saturated, $\uparrow_{\tau_Y} z \subseteq U$ and V is τ_Y^* -saturated, so $\downarrow_{\tau_Y} d \subseteq V$. But then if $d \geq d' \in D, \ d' \in \uparrow_{\tau_Y} z \cap \downarrow_{\tau_Y} d \subseteq U \cap V \subseteq T$. This shows that z is a limit for D. We get the theorem for sets directed by \leq_{τ_Y} by applying the above to the dual, Y^* . \Box

Nachbin's spaces are triples, (X, τ, \leq) where \leq is closed in $(X, \tau) \times (X, \tau)$. The topology τ^{\leq} of τ -open \leq -upper sets can then be defined, and similarly, τ^{\geq} , giving a bitopological space. If $\tau = \tau^{\leq} \vee \tau^{\geq}$, then the τ -continuous, order-preserving maps to [0, 1] are precisely the pairwise continuous functions from the bitopological space B(X) = $(X, \tau^{\leq}, \tau^{\geq})$ to II. Thus B(X) is Tychonoff if and only if, whenever $x \in T \in \tau^{\leq}$ then there is an order-preserving, continuous $f : (X, \tau, \leq) \to ([0, 1], \mathcal{E}, \leq)$ such that f(x) = 1 and fis 0 off T. Much of the above is more widely true than suggested in our title, and indeed the majority of our results were stated in wider generality (e.g., Theorems 2, 3, 9, 11, and Lemmas 7 and 10 (a) and (b)). But this wider theory is not required for our applications in [HK], and we have not tried to develop it where inconvenient.

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