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Recent Results on the Algebraic Structure of βS

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Abstract. We survey recent results on the algebraic structure of the Stone-Čech compactification of a discrete semigroup. We include a complete proof of the result of Dona Strauss that $\beta\mathbb{N}\backslash\mathbb{N}$ does not contain a topological and algebraic copy of $\beta\mathbb{N}$.

1. Introduction. The "recent" in the title refers to results established since my last survey of this area [13] was written in 1989. Because of space limitations we limit our attention to results which are primarily algebraic in their statements, leaving results of a topological dynamical nature as well as applications to combinatorics for a later time.

With one exception we will follow the usual practice in survey articles, citing published results without proof and proving only those results which are appearing here for the first time. The one exception is the remarkable proof by D. Strauss [20] that $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ does not contain a topological and algebraic copy of $\beta\mathbb{N}$. This problem had remained open since it was first posed by van Douwen in 1978. We present this proof in its entirety in Section 2 because we feel it is sufficiently important to merit such special treatment and because we believe some people will find it helpful to view the proof from a different angle.

In Section 3 we present information about certain subsemigroups of $\beta\mathbb{N}$, both those known to exist there and those whose existence or nonexistence is conjectured.

In Section 4 we present results about cancellation in $(\beta\mathbb{N}, +)$ as well as results about βS for arbitrary semigroups S.

Section 5 deals with the effects of the choice of left or right continuity for βS where S itself is not commutative.

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Until Section 5 we take the operation $\cdot$ on the Stone-Čech compactification βS of S to be the extension of $\cdot$ on S which makes βS a left topological semigroup (i.e. λ_p is continuous for each $p \in \beta S$, where $\lambda_p(q) = p \cdot q$) with S contained in its topological center (*i.e.* ρ_x is continuous for each $x \in S$, where $\rho_x(p) = p \cdot x$). We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. The set $\{c\ell A : A \subseteq S\}$ is a basis for the open sets and a basis for the closed sets of βS. Observe that if $p, q \in \beta S$ and $A \subseteq S$, then A $\in p \cdot q$ if and only if $\{x \in S : A/x \in p\} \in q$ where $A/x = \{y \in S : y \cdot x \in A\}$.

We take $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\omega = \{0, 1, 2, \ldots\}$.

2. $\mathbb{N}^*$**does not contain an algebraic and topological copy of** $(\beta\mathbb{N}, +)$. We present here the proof from [20] that in fact if $\varphi : \beta\mathbb{N} \longrightarrow \mathbb{N}^*$ is a continuous homomorphism then $\varphi[\beta\mathbb{N}]$ is a finite group. All of the ideas are due to Strauss. The proofs will appear different for two reasons. The first, a trivial difference, is that she takes $(\beta\mathbb{N}, +)$ to be right topological. Secondly, I am deliberately taking a different tack on some of the proofs in order to provide an additional point of view.

We begin with a fundamental result about the structure of $\beta\mathbb{N}$ which Strauss utilizes in most of her beautiful results about $\beta\mathbb{N}$. The proof which I present is based on the proof of [7, Lemma 8.2].

2.1 **Lemma**. *Let X and Y be countable subsets of $\beta\mathbb{N}$. If $(c\ell X) \cap (c\ell Y) \neq \emptyset$, then either $X \cap (c\ell Y) \neq \emptyset$ or $(c\ell X) \cap Y \neq \emptyset$. In fact, if $p \in (c\ell X) \cap (c\ell Y)$, then $p \in c\ell(X \cap (c\ell Y)) \cup c\ell((c\ell X) \cap Y)$.*

Proof. Let $p \in (c\ell X) \cap (c\ell Y)$. Let $C = X \backslash c\ell Y$ and let $D = Y \backslash c\ell X$, and let $Z = C \cup D$. Then Z is a countable subspace of $\beta\mathbb{N}$ which is an F-space, and hence Z is C^*-embedded in $\beta\mathbb{N}$ [11, Section 14N(5)]. The function $f : Z \longrightarrow [0, 1]$ with $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in D$ is continuous since C is open and closed in Z. Then f extends continuously to $\beta\mathbb{N}$ so $c\ell C \cap c\ell D = \emptyset$. Since $c\ell X = c\ell C \cup c\ell(X \cap c\ell Y)$ and $c\ell Y = c\ell D \cup c\ell(Y \cap c\ell X)$ one must have $p \in c\ell(X \cap c\ell Y) \cup c\ell(Y \cap c\ell X)$ as claimed. []2.2

2.2 **Lemma**. *Let X be a topological space, let $f : \beta\mathbb{N} \longrightarrow X$ be continuous, and let $p \in \beta\mathbb{N}$. If U is a neighborhood of $f(p)$ then $\mathbb{N} \cap f^{-1}[U] \in p$.*

Proof. Pick $B \in p$ with $f[c\ell B] \subseteq U$. Then $B \subseteq \mathbb{N} \cap f^{-1}[U]$. []

2.3 **Lemma**. *Let $k \in \mathbb{N}$ and let p be an additive idempotent in $\beta\mathbb{N}$. For each $q \in \mathbb{N}^*$ there exists $r \in c\ell(\mathbb{N}k)$ such that $p + q + r \neq p + r + q$.*

Proof. We may presume $k > 1$ since $\mathbb{N}2k \subseteq \mathbb{N}k$. Define the function $f : \mathbb{N} \longrightarrow \{0, 1, \ldots, k-1\}^\omega$ so that $f(m)_i$ is the ith term of the base k expansion of m. That is $m = \sum_{i=0}^{\infty} f(m)_i \cdot k^i$. Let $\mu(m) = \min\{i \in \omega : f(m)_i \neq 0\}$. Define functions g and c from $\mathbb{N}$ to $\mathbb{N}$ by $g(m) = f(m)_{\mu(m)} \cdot k^{\mu(m)}$ and $c(m) = |\{i \in \omega : f(m)_i \neq 0\}|$. Let $f^\beta : \beta\mathbb{N} \longrightarrow \{0, 1, \ldots, k-1\}^\omega$, $g^\beta : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$, and $c^\beta : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$ be the continuous extensions of f, g, and c, respectively. We first show that for any $q \in \beta\mathbb{N}$ and any $\ell \in \mathbb{N}$, if $m = \sum_{i=0}^{\ell} f^\beta(q)_i \cdot k^i$, then $\mathbb{N}k^{\ell+1} + m \in q$. Indeed pick $n \in \{0, 1, 2, \ldots, k^{\ell+1} - 1\}$ such that $\mathbb{N}k^{\ell+1} + n \in q$. Then for $i \in \{0, 1, \ldots, \ell\}$ one has $\pi_i \circ f$ is constantly equal to $f(n)_i$ on $\mathbb{N}k^{\ell+1} + n$ and hence $f^\beta(q)_i = f(n)_i$. That is $m = n$.

We now claim that given any $r \in \beta\mathbb{N}$ and any $s \in \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}k^n)$ one has $g^\beta(s+r) = g^\beta(r)$ and $c^\beta(s+r) = c^\beta(s) + c^\beta(r)$. For the first equality it suffices to show that $g^\beta \circ \lambda_s$ and g^β agree on the dense subspace $\mathbb{N}$ of $\beta\mathbb{N}$, so let $m \in \mathbb{N}$ be given. Now for $x \in \mathbb{N}k^{\mu(m)+1}$, $g(x+m) = g(m)$. That is $g \circ \rho_m$ is constantly equal to $g(m)$ on $\mathbb{N}k^{\mu(m)+1}$. Since $s \in c\ell(\mathbb{N}k^{\mu(m)+1})$ this says $g(s+m) = g(m)$ as required. To see that $c^\beta(s+r) = c^\beta(s) + c^\beta(r)$ we show that $c^\beta \circ \lambda_s$ and $\lambda_{c^\beta(s)} \circ c^\beta$ agree on $\mathbb{N}$. So let $m \in \mathbb{N}$ be given and pick ℓ such that $m < k^\ell$ (so that $f(m)_i = 0$ for $i \geq \ell$). We want to show that $c^\beta(s+m) = c^\beta(s) + c(m)$. Now $c^\beta \circ \rho_m$ and $\rho_{c(m)} \circ c^\beta$ agree on $\mathbb{N}k^\ell$ and $s \in c\ell(\mathbb{N}k^\ell)$ so $c^\beta(s+m) = c^\beta(s) + c(m)$ as required.

Now let $q \in \mathbb{N}^*$ be given and suppose that for each $r \in c\ell(\mathbb{N}k)$ one has $p+q+r = p+r+q$. We claim first that $f^\beta(q)$ is not eventually 0. Suppose instead that it is and pick $\ell \in \mathbb{N}$ such that $f^\beta(q) = 0$ for $i \geq \ell$. Let $m = \sum_{i=0}^{\ell} f^\beta(q)_i k^i$. Then as we have seen, for $n > \ell$, $\mathbb{N}k^n + m \in q$. Thus if $s = q - m (= \{A - m : A \in q\})$ we have $s \in \bigcap_{n=\ell+1}^{\infty} c\ell(\mathbb{N}k^n) = \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}k^n)$. Now given any $r \in c\ell(\mathbb{N}k)$ we have $p+s+m+r = p+q+r = p+r+q = p+r+s+m$. Since the center of $(\beta\mathbb{N}, +)$ is $\mathbb{N}$ [8, Theorem 7.5] one has $p+s+r+m = p+r+s+m$ each $r \in c\ell(\mathbb{N}k)$. Since cancellation holds at each point of $\mathbb{N}$, we have $p+s+r = p+r+s$ for each $r \in c\ell(\mathbb{N}k)$. Now p is an idempotent so for each $n \in \mathbb{N}$, $\mathbb{N}n \in p$. (The homomorphism sending each element of $\mathbb{N}$ to its congruence class in $\mathbb{Z}_n$ extends to a homomorphism on $\beta\mathbb{N}$ which thus takes p to 0.) Then $p \in \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}k^n)$ and hence also $p+s \in \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}k^n)$. Now pick $r \in (c\ell\{k^n : n \in \mathbb{N}\}) \setminus (\mathbb{N} \cup \{g^\beta(s)\})$. Now observe that g is equal to the identity on $\{k^n : n \in \mathbb{N}\}$ and hence we have $g^\beta(r) = r$. Thus $r = g^\beta(r) = g^\beta(p+s+r) = g^\beta(p+r+s) = g^\beta(s)$, a contradiction.

We have thus established that $f^\beta(q)$ is not eventually equal to 0. We next show that it is not eventually $k-1$. Suppose instead we have $\ell \in \mathbb{N}$ such that $f^\beta(q)_i = k-1$ for all $i > \ell$. Let $m = k^{\ell+1} - \sum_{i=0}^{\ell} f^\beta(q)_i \cdot k^i$ and let $s = q + m$. Now given any

$n \geq \ell$ we have already seen that $\mathbb{N}k^{n+1} + \sum_{i=0}^{n} f^\beta(q)_i \cdot k^i \in q$, and hence $\mathbb{N}k^{n+1} + \sum_{i=0}^{n} f^\beta(q)_i \cdot k^i + m \in q + m = s$. Since $\sum_{i=0}^{n} f^\beta(q)_i \cdot k^i + m = \sum_{i=0}^{\ell} f^\beta(q)_i \cdot k^i + \sum_{i=\ell+1}^{n}(k-1) \cdot k^i + k^{\ell+1} - \sum_{i=0}^{\ell} f^\beta(q)_i \cdot k^i = k^{n+1}$ we have $\mathbb{N}k^{n+1} + k^{n+1} \in s$ so that $\mathbb{N}k^{n+1} \in s$. Thus $s \in \bigcap_{n=\ell+1}^{\infty} c\ell(\mathbb{N}k^n) = \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}k^n)$. Now given any $r \in c\ell(\mathbb{N}k)$ one has $p+s+r = p+q+m+r = p+q+r+m = p+r+q+m = p+r+s$ and hence we get the same contradiction as before.

Now let $L = \{n \in \mathbb{N} : 0 < f^\beta(q)_n < k-1\}$. We claim that L is finite. Suppose instead that L is infinite and pick $r \in (c\ell\{k^n : n \in L\})\backslash\mathbb{N}$. Then $r \in \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}k^n)$ so $c^\beta(p+r+q) = c^\beta(p) + c^\beta(r+q) = c^\beta(p) + c^\beta(r) + c^\beta(q) = c^\beta(p) + 1 + c^\beta(q)$ (since c is constantly equal to 1 on $\{k^n : n \in L\}$). On the other hand we will show that $c^\beta(q+r) = c^\beta(q)$ from which it follows that $c^\beta(p+q+r) = c^\beta(p) + c^\beta(q+r) = c^\beta(p) + c^\beta(q) \neq c^\beta(p) + c^\beta(q) + 1 = c^\beta(p+r+q)$, a contradiction. To see that $c^\beta(q+r) = c^\beta(q)$ suppose instead they are distinct and pick disjoint neighborhoods U and V of $c^\beta(q+r)$ and $c^\beta(q)$ respectively. Pick $A \in q+r$ and $B \in q$ such that $c^\beta[A] \subseteq U$ and $c^\beta[B] \subseteq V$. Let $C = \{x \in \mathbb{N} : A - x \in q\}$ so that $C \in r$. Since $\{k^n : n \in L\} \in r$, pick $n \in L$ with $k^n \in C$. Then $A - k^n \in q$, $B \in q$, and $\mathbb{N}k^{n+2} + \sum_{i=0}^{n+1} f^\beta(q)_i \cdot k^i \in q$ so pick $y \in (A - k^n) \cap B \cap (\mathbb{N}k^{n+2} + \sum_{i=0}^{n+1} f^\beta(q)_i \cdot k^i)$. (The exponents are chosen in order to be able to repeat the argument below.) Then $f(y)_n = f^\beta(q)_n \in \{1, 2, \ldots, k-2\}$ so $f(y+k^n)_n = f^\beta(q)_n + 1 \in \{2, 3, \ldots, k-1\}$, and for all $i \neq n$ $f(y+k^n)_i = f(y)_i$. Thus $c(y+k^n) = c(y)$ while $c(y+k^n) \in c[A] \subseteq U$ and $c(y) \in c[B] \subseteq V$, a contradiction.

We have thus established that $M = \{n \in \mathbb{N} : f^\beta(q)_n = k-1 \text{ and } f^\beta(q)_{n+1} = 0\}$ is infinite so let $r \in (c\ell\{k^n : n \in M\})\backslash\mathbb{N}$. Then as above we have $c^\beta(p+r+q) = c^\beta(p) + 1 + c^\beta(q)$ and it will suffice as above to show that $c^\beta(q+r) = c^\beta(q)$. Suppose otherwise and proceed verbatim as in the paragraph above obtaining $n \in M$ and $y \in (A - k^n) \cap B \cap (\mathbb{N}k^{n+2} + \sum_{i=0}^{n+1} f^\beta(q)_i \cdot k^i)$. Now one has $f(y)_n = f^\beta(q)_n = k-1$ and $f(y)_{n+1} = f^\beta(q)_{n+1} = 0$ so that $f(y+k^n)_n = 0$ and $f(y+k^n)_{n+1} = 1$ while for all $i \notin \{n, n+1\} f(y+k^n)_i = f(n)_i$. Consequently $c(y+k^n) = c(y)$, completing the contradiction and the proof. $\Box$

2.4 **Lemma**. *Let $\varphi : \beta\mathbb{N} \longrightarrow \mathbb{N}^*$ be a continuous homomorphism and let p be an idempotent in $\beta\mathbb{N}$. There exists $m \in \mathbb{N}$ such that for all $x \geq m$ in $\mathbb{N}$, $\varphi(x) = \varphi(p) + \varphi(x)$.*

Proof. Observe first that $\varphi(1) + \varphi(p) = \varphi(1+p) = \varphi(p+1) = \varphi(p) + \varphi(1)$ (recalling that the center of $\beta\mathbb{N}$ is $\mathbb{N}$). Since $\varphi(p) + \varphi(1) \in \varphi(p) + \beta\mathbb{N} = c\ell\{\varphi(p) + k : k \in \mathbb{N}\}$ and $\varphi(1) + \varphi(p) = \varphi(1+p) \in \varphi[\beta\mathbb{N}] = c\ell\varphi[\mathbb{N}]$ we have that $c\ell\{\varphi(p) + k : k \in \mathbb{N}\} \cap c\ell\varphi[\mathbb{N}] \neq \emptyset$. Thus by Lemma 2.1 we have either $\{\varphi(p) + k : k \in \mathbb{N}\} \cap c\ell\varphi[\mathbb{N}] \neq \emptyset$ or $\varphi[\mathbb{N}] \cap c\ell\{\varphi(p) + k : k \in \mathbb{N}\} \neq \emptyset$.

Assume first we have some $k \in \mathbb{N}$ with $\varphi(p) + k \in c\ell\varphi[\mathbb{N}] = \varphi[\beta\mathbb{N}]$. We show by induction on m that for all $m \in \mathbb{N}$, $\varphi(p) + mk \in \varphi[\beta\mathbb{N}]$. Indeed, let m be given and assume $\varphi(p) + mk \in \varphi[\beta\mathbb{N}]$. Then $\varphi(p) + (m+1)k = \varphi(p+p) + mk + k = \varphi(p) + \varphi(p) + mk + k = \varphi(p) + mk + \varphi(p) + k \in \varphi[\beta\mathbb{N}] + \varphi[\beta\mathbb{N}] \subseteq \varphi[\beta\mathbb{N}]$. Thus one has $\varphi(p) + \mathbb{N}k \subseteq \varphi[\beta\mathbb{N}]$ so $\varphi(p) + c\ell(\mathbb{N}k) \subseteq \varphi[\beta\mathbb{N}]$. By Lemma 2.3 pick $r \in c\ell(\mathbb{N}k)$ such that $\varphi(p) + \varphi(1) + r \neq \varphi(p) + r + \varphi(1)$. We have $\varphi(p) + r = \varphi(t)$ for some $t \in \beta\mathbb{N}$. Thus $\varphi(p) + r + \varphi(1) = \varphi(t) + \varphi(1) = \varphi(t+1) = \varphi(1+t) = \varphi(1) + \varphi(t) = \varphi(1) + \varphi(p) + r = \varphi(1+p) + r = \varphi(p+1) + r = \varphi(p) + \varphi(1) + r$, a contradiction.

Thus we must have some $m \in \mathbb{N}$ such that $\varphi(m) \in c\ell\{\varphi(p)+k : k \in \mathbb{N}\} = \varphi(p)+\beta\mathbb{N}$. Then given $x > m$, $\varphi(x) = \varphi(m+(x-m)) = \varphi(m)+\varphi(x-m) \in \varphi(p)+\beta\mathbb{N}+\varphi(x-m) \subseteq \varphi(p) + \beta\mathbb{N}$. So given $x \geq m$ we have some $t \in \beta\mathbb{N}$ such that $\varphi(x) = \varphi(p) + t$. So $\varphi(p) + \varphi(x) = \varphi(p) + \varphi(p) + t = \varphi(p+p) + t = \varphi(p) + t = \varphi(x)$ as required. $\square$

In the proof of Theorem 2.6 we will make use of the minimal ideal structure of compact left topological semigroups. We state next the portions of that structure theory which we will utilize.

2.5 **Theorem**. *Let $(S, \cdot)$ be a compact left topological semigroup. Then S has idempotents. Further S has a smallest two sided ideal which is the union of all minimal left ideals of S and is also the union of all minimal right ideals of S. Each minimal right ideal is compact. Given a minimal left ideal L of S and a minimal right ideal R of S, $L \cap R$ is a group.*

Proof. See [4, Theorem 3.11]. $\square$

2.6 **Theorem** (Strauss). *Let $\varphi : \beta\mathbb{N} \longrightarrow \mathbb{N}^*$ be a continuous homomorphism. Then $\varphi[\mathbb{N}^*]$ is a finite group and $\{x \in \mathbb{N} : \varphi(x) \notin \varphi[\mathbb{N}^*]\}$ is finite. (In particular φ is not one-to-one so $\mathbb{N}^*$ does not contain a topological and algebraic copy of $\beta\mathbb{N}$.)*

Proof. Let $S = \varphi[\mathbb{N}^*]$. We show first that S is right simple, that is, S is a minimal right ideal of itself. Now S is a compact left topological semigroup so it has a minimal right ideal R. As R is itself a compact left topological semigroup it has an idempotent t. Now $\varphi^{-1}[\{t\}]$ is a compact left topological semigroup so we may pick an idempotent $p \in \varphi^{-1}[\{t\}]$. Pick by Lemma 2.4 some $m \in \mathbb{N}$ such that for all $x \geq m$ in $\mathbb{N}$, $\varphi(x) = \varphi(p) + \varphi(x)$. Then for all $q \in \mathbb{N}^*$, $\varphi(q) = \varphi(p) + \varphi(q)$. Thus $S = \varphi[\mathbb{N}^*] = \varphi(p) + \varphi[\mathbb{N}^*] \subseteq R + S \subseteq R \subseteq S$. That is $S = R$ as claimed.

Next we claim that S is also left simple. Since S is right simple, it is its own smallest ideal so the only alternative is that there are two distinct minimal left ideals L_1 and L_2 of S, which are then disjoint. Let t_1 be the identity of $L_1 = L_1 \cap R$ and let t_2 be the identity of $L_2 = L_2 \cap R$. Choose as above idempotents p_1 and p_2 in $\beta\mathbb{N}$ with $\varphi(p_1) = t_1$ and $\varphi(p_2) = t_2$. Pick by Lemma 2.4 some m_1 and m_2 in $\mathbb{N}$ such that $\varphi(x) = \varphi(p_1)+\varphi(x)$ for all $x \geq m_1$ and $\varphi(x) = \varphi(p_2) + \varphi(x)$ for all $x \geq m_2$. Let $x = \max\{m_1, m_2\}$. Then $\varphi(x) = \varphi(p_1) + \varphi(x) = \varphi(p_1 + x) = \varphi(x + p_1) = \varphi(x) + \varphi(p_1) \in L_1$ and similarly $\varphi(x) \in L_2$ so $L_1 \cap L_2 \neq \emptyset$, a contradiction.

Thus S is both a minimal left ideal of S and a minimal right ideal of S so S is a group. Next observe that S is homogeneous. (Given q and r in S, if s is the inverse of r then $\lambda_{q+s}(r) = q$ and $\lambda_{q+s} : S \longrightarrow S$, being continuous, one-to-one, and onto, is a homeomorphism of S.) As no infinite compact F-space is homogeneous [7, Corollary 8.7], one must then have that $\varphi[\mathbb{N}^*]$ is finite.

Finally, pick any idempotent p of $\mathbb{N}$ and pick by Lemma 2.4 some $m \in \mathbb{N}$ such that $\varphi(x) = \varphi(p) + \varphi(x)$ for all $x \geq m$. Then given $x \geq m$, $\varphi(x) = \varphi(p + x) \in \varphi[\mathbb{N}^*]$. Therefore $\varphi[\beta\mathbb{N}] = \varphi[\mathbb{N}^*] \cup \{\varphi(x) : x \in \mathbb{N} \text{ and } x < m\}$. ▯

We conclude this section by stating a generalization of the fact from Theorem 2.6 that $\varphi[\beta\mathbb{N}]$ must be finite. Recall that the topological center $\Lambda(S)$ of a left topological semigroup S is $\{x \in S : \rho_x \text{ is continuous}\}$.

2.7 **Theorem** (Budak, Isik, and Pym). *Let S be a discrete countable semigroup which is isomorphic with a subsemigroup of a countable direct sum of copies of $\mathbb{Q}$, $\mathbb{Z}(p^\infty)$ (for various primes p), and finite groups. Let T be a compact subsemigroup of βS such that some countable subsemigroup T_0 of $\Lambda(T)$ is dense in T and assume $\Lambda(T)\backslash S \neq \emptyset$. Then*

(i) T_0 contains an idempotent which lies in the smallest ideal of T.

(ii) Each minimal left ideal of T is finite.

(iii) If T_0 is a group, then T is a finite group.

(iv) If T_0 is commutative, the smallest ideal of T is a finite commutative group.

(v) If T_0 is singly generated, T is finite.

Proof. [6, Theorem 8.3]. ▯

3. **Subsemigroups of** $\beta\mathbb{N}$. One of the most significant and stubborn problems about the algebraic structure of $(\beta\mathbb{N}, +)$ is determining whether it contains any nontrivial finite groups. The only remaining problem about continuous homomorphisms from $\beta\mathbb{N}$ to $\mathbb{N}^*$ is to determine whether there are any nontrivial ones. (By a trivial continuous homomorphism we mean a map which sends all of $\beta\mathbb{N}$ to a given idempotent.) We began this section by recording the easy observation that these problems are in fact almost the same problem.

3.1 **Theorem**. *There is a nontrivial continuous homomorphism from* $\beta\mathbb{N}$ *to* $\mathbb{N}^*$ *if and only if either there is a nontrivial finite group in* $\mathbb{N}^*$ *or there exist* $p \neq q$ *in* $\mathbb{N}^*$ *with* $p+p=q=q+q=q+p=p+q$.

Proof. Necessity. Let φ be a nontrivial homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$. By Theorem 2.6 $\varphi[\mathbb{N}^*]$ is a finite group and there is some $m \in \mathbb{N}$ with $\varphi(x) \in \varphi[\mathbb{N}^*]$ for all $x \geq m$. If $|\varphi[\mathbb{N}^*]| > 1$ we are done, so assume $\varphi[\mathbb{N}^*] = \{q\}$. Since φ is nontrivial, there is a largest x in $\mathbb{N}$ such that $\varphi(x) \neq q$. Let $p = \varphi(x)$. Then $p+p = \varphi(x+x) = q$ and $q+p = p+q = \varphi(x+x+1) = q$.

Sufficiency. If there is a nontrivial finite group in $\mathbb{N}^*$, then for some $n \in \mathbb{N}\backslash\{1\}$ there is a one-to-one homomorphism $\tau : \mathbb{Z}_n \longrightarrow \mathbb{N}^*$. Define $\varphi : \mathbb{N} \longrightarrow \mathbb{N}^*$ by $\varphi(k) = \tau(i)$ where $i \in \mathbb{Z}_n$ and $k \equiv i \bmod n$. Then the continuous extension $\varphi^\beta : \beta\mathbb{N} \longrightarrow \mathbb{N}^*$ is a nontrivial continuous homomorphism. If one has $p \neq q$ with $p+p=q=q+q=q+p=p+q$ define $\varphi : \beta\mathbb{N} \longrightarrow \mathbb{N}^*$ by $\varphi(1) = p$ and $\varphi(r) = q$ for all $r \in \beta\mathbb{N}\backslash\{1\}$. ▯

One way to distinguish between points of $\beta\mathbb{N}$ is to produce a homomorphism which takes on different values. For example if one were trying to find a copy of $\mathbb{Z}_2$ in $\beta\mathbb{N}$ one could take a homomorphism from $\mathbb{N}$ to the circle group $\mathbb{T}$ and look for a generator of $\mathbb{Z}_2$ in the inverse image of $e^{i\pi}$, the element of order 2 in $\mathbb{T}$. This approach cannot work.

3.2 **Theorem** (Baker, Hindman, Pym). *Let* G *be a compact topological group and let* $f : \mathbb{N} \longrightarrow G$ *be a homomorphism. If* H *is a finite subgroup of* $\mathbb{N}^*$ *then* $f^\beta[H]$ *consists only of the identity of* G.

Proof. [2, Corollary 2.3]. ▯

A particular semigroup of $\beta\mathbb{N}$ which has arisen in several contexts is the semigroup $H = \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}2^n)$. See for example [18]. It should be emphasized that by contrast with the entirety of $\beta\mathbb{N}$, topological and algebraic copies of H are plentiful in $\mathbb{N}^*$. In fact we have the following theorem which is well known among aficianados but I believe is unpublished. (See however [17, Theorem 2.3].) Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ write as usual $FS(\langle x_n \rangle_{n=m}^{\infty}) = \{\sum_{n\in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}$ and $\min F \geq m\}$. We say $\langle x_n \rangle_{n=1}^{\infty}$ "satisfies uniqueness of finite sums" provided $\sum_{n\in F} x_n = \sum_{n\in G} x_n$ implies $F = G$. (For instance, any sequence with each $x_{n+1} > \sum_{t=1}^{n} x_t$ satisfies uniqueness of finite sums.)

3.3 **Theorem**. *Let* $\langle x_n \rangle_{n=1}^{\infty}$ *be a sequence in* $\mathbb{N}$ *which satisfies uniqueness of finite sums. Then* $\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n \rangle_{n=m}^{\infty})$ *is homeomorphic and isomorphic to* H *via the same function.*

Proof. Define $f : \mathbb{N} \longrightarrow FS(\langle x_n \rangle_{n=1}^{\infty})$ by $f(\sum_{n\in F} 2^{n-1}) = \sum_{n\in F} x_n$. Let $f^\beta : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$ be the continuous extension of f and let $\varphi = f^\beta|_H$. Since f is one-to-one so is φ. For each n, $f[\mathbb{N}2^{n-1}] = FS(\langle x_m \rangle_{m=n}^{\infty})$ so $f^\beta[c\ell\mathbb{N}2^{n-1}] = c\ell FS(\langle x_m \rangle_{m=n}^{\infty})$ and hence $\varphi[H] = \bigcap_{n=1}^{\infty} c\ell FS(\langle x_m \rangle_{m=n}^{\infty})$. To see that φ is a homomorphism, let $p, q \in H$. We show that $\varphi(p+q) = \varphi(p) + \varphi(q)$ by showing that $\varphi \circ \lambda_p$ and $\lambda_{\varphi(p)} \circ \varphi$ agree on $\mathbb{N}$. Let $z \in \mathbb{N}$ be given, write $z = \sum_{m\in H} 2^{m-1}$, and let $n = \max H$. To see that $\varphi(p+z) = \varphi(p) + \varphi(z)$ we show that $\varphi \circ \rho_z$ and $\rho_{\varphi(z)} \circ \varphi$ agree on $\mathbb{N}2^n$. Indeed let $y \in \mathbb{N}2^n$ and write $y = \sum_{m\in G} 2^{m-1}$ where $\min G > n$. Then $z + y = \sum_{m\in H\cup G} 2^{m-1}$ so $\varphi(z+y) = \sum_{m\in H\cup G} x_m = \sum_{m\in H} x_m + \sum_{m\in G} x_m = \varphi(z) + \varphi(y)$. $\Box$

It is an old result [15] that H contains copies of the free group on 2^c generators. Additional light is shed by the following. By the free product of a set of pairwise disjoint semigroups is meant the set of all words $a_1a_2\ldots a_n$ with letters from the semigroups in question and with no a_i, a_{i+1} coming from the same semigroup. The product of $a_1a_2\ldots a_n$ and $b_1b_2\ldots b_m$ is $a_1a_2\ldots a_nb_1b_2\ldots b_m$ if a_n and b_1 come from different semigroups and is $a_1a_2\ldots a_{n-1}cb_2\ldots b_m$ if a_n and b_1 are from the same semigroup and $a_nb_1 = c$.

3.4. **Theorem** (El-Mabhouh, Pym, and Strauss). *The semigroup H contains the free product of 2^c copies of the semigroup* $(\mathbb{N}, \max)$.

Proof. [10, Theorem 4]. $\Box$

As we noted earlier all minimal right ideals in a compact left topological semigroup S are compact. In fact given $p \in S$, $p \cdot S = \lambda_p[S]$ is compact. Compact left ideals are not so easy to come by. In fact it is a consequence of Corollary 5.6 below that $(\beta\mathbb{N}, +)$ does not have two disjoint compact left ideals. We see however that compact left ideals are plentiful in H. (But note we are not claiming that the left ideals produced below are minimal.)

3.5 **Theorem**. *The semigroup H is the union of c pairwise disjoint compact left ideals of H.*

Proof. Pick by [22, Theorem 10] a collection $\{E_\sigma : 0 < \sigma < c\}$ of almost disjoint subsets of ω and for each σ with $0 < \sigma < c$, let $B_\sigma = \{\sum_{n\in F} 2^n : F$ is a finite nonempty subset of ω and $\min F \in E_\sigma\}$ and let $L_\sigma = H \cap c\ell B_\sigma$. Let $L_0 = H\backslash \bigcup\{L_\sigma : 0 < \sigma < c\}$. Then each L_σ is clopen in H for $0\langle\sigma < c$ so L_0 is compact. Also $\{\mathbb{N}\backslash B_\sigma : 0 < \sigma < c\} \cup \{\mathbb{N}2^n : n \in \mathbb{N}\}$ has the finite intersection property so $L_0 \neq \emptyset$. Given $0 < \sigma < \tau < c$ and $n \in \mathbb{N}$ such that $E_\sigma \cap E_\tau \subseteq \{0, 1, \ldots, n-1\}$, $B_\sigma \cap B_\tau \cap \mathbb{N}2^n = \emptyset$ so $L_\sigma \cap L_\tau = \emptyset$.

Now let $\sigma < c$ be given, let $p \in H$, and let $q \in L_\alpha$. We show $p+q \in L_\sigma$. Assume first $\sigma > 0$. We need to show that $B_\sigma \in p+q$ so we show that $B_\sigma \subseteq \{x \in \mathbb{N} : B_\sigma - x \in p\}$. So let $x \in B_\sigma$ and write $x = \sum_{n \in F} 2^n$ where $\min F \in E_\sigma$. Let $m = \max F + 1$. (Actually $m = \min F + 1$ would be large enough.) Then $\mathbb{N}2^m \in p$ and $\mathbb{N}2^m \subseteq B_\sigma - x$.

Next assume $\sigma = 0$ and suppose $p + q \notin L_0$. Pick $\tau < c$ such that $p + q \in L_\tau$. Now $B_\tau \in p+q$ and $B_\tau \notin q$ so that $\mathbb{N}\backslash B_\tau \in q$. Choose $x \in \mathbb{N}\backslash B_\tau$ such that $B_\tau - x \in p$. Write $x = \sum_{n \in F} 2^n$ so that $\min F \notin E_\tau$. Let $m = \min F + 1$ and pick $y \in \mathbb{N}2^m \cap (B_\tau - x)$. Then $y + x = \sum_{n \in H} 2^n$ where $\min H = \min F$ so $y + x \notin B_\tau$, a contradiction. ▯

By contrast with the highly structured semigroups known to exist in portions of $\beta\mathbb{N}$, we have the following result showing that the elements of $\mathbb{N}^*$ which are not sums of other elements of $\mathbb{N}^*$ almost generate a free semigroup.

3.6 **Theorem** (Strauss). *Define an equivalence relation $\equiv$ on $\mathbb{N}^*$ by agreeing that $p \equiv q$ if and only if $p = q$ or $p \in q + \mathbb{N}$ or $q \in p + \mathbb{N}$. Let $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_m$ be elements of $\mathbb{N}^* \backslash (\mathbb{N}^* + \mathbb{N}^*)$. If $p_1 + p_2 + \ldots + p_n = q_1 + q_2 + \ldots + q_m$, then $n = m$ and for each i, $p_i \equiv q_i$.*

Proof [21, Theorem 3]. ▯

Of course, $\mathbb{N}^* + \mathbb{N}^*$ is an ideal of $\beta\mathbb{N}$ so the smallest ideal of $\beta\mathbb{N}$ is contained in $\mathbb{N}^* + \mathbb{N}^*$. On the other hand, not only is the closure of the smallest ideal not contained in $\mathbb{N}^* + \mathbb{N}^*$, we in fact have the following.

3.7 **Theorem**. *Let $M = \{p \in \beta\mathbb{N} : p$ is in the smallest ideal of $\beta\mathbb{N}$ and $p = p + p\}$. Then $(c\ell M)\backslash(N^* + \mathbb{N}^*) \neq \emptyset$.*

Proof. By [3, Theorem 5.4], $c\ell M$ is a right ideal of $(\beta\mathbb{N}, \cdot)$. Using Theorem 2.5 pick $q \in c\ell M$ such that $q = q \cdot q$. Since $M \subseteq \bigcap_{n=1}^\infty c\ell(\mathbb{N}n)$ one has $c\ell M \subseteq \bigcap_{n=1}^\infty c\ell(\mathbb{N}n)$. Thus for each n, $\mathbb{N}n \in q$. Then by [12, Theorem 5.3] $q = q \cdot q \notin \mathbb{N}^* + \mathbb{N}^*$. ▯

4. **Cancellation**. Numerous characterizations are known of points at which left cancellation holds in $(\beta\mathbb{N}, +)$.

4.1 **Theorem** (Blass, Hindman, Strauss). *Let $p \in \mathbb{N}^*$. The following statements are equivalent.*

(a) λ_p is one-to-one on $\beta\mathbb{N}$ (i.e. left cancellation holds at p);

(b) λ_p is one-to-one on $\mathbb{N}^*$;

(c) λ_p is one-to-one on $\bigcap_{n=1}^\infty c\ell(\mathbb{N}n)$;

(d) there exist a first countable topological group $(G, +)$ and a continuous homomorphism $h : \beta\mathbb{N} \longrightarrow G$ such that λ_p is one-to-one on $\ker(h)$;

(e) for each $A \subseteq \mathbb{N}$, there exists $B \subseteq \mathbb{N}$ such that $A = \{x \in \mathbb{N} : B - x \in p\}$;

(f) for each $A \subseteq \mathbb{N}$, there exists $B \subseteq \mathbb{N}$ such that $A\Delta\{x \in \mathbb{N} : B - x \in p\}$ is finite;

(g) $\{p + n : n \in \mathbb{N}\}$ is discrete.

(h) for each $q \in \mathbb{N}^*$, p strictly precedes $p + q$ in the Rudin-Keisler order;

(i) for each $q \in \mathbb{N}^*$, p is not type equivalent to $p + q$;

(j) $p \notin p + \beta\mathbb{N}$;

(k) $p \notin p + \mathbb{N}^*$;

(l) there exists $A \in p$ such that for all $t \in \mathbb{N}$, $A - t \notin p$;

(m) $p + \mathbb{N}^*$ is not separable;

(n) there is an increasing sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that for each $k \in \mathbb{N}$, $\{x_n : x_{n+1} > x_n + k\} \in p$;

(o) there is a one-to-one function $f : \mathbb{N} \longrightarrow \beta\mathbb{N}$ such that $\{f(n) : n \in \mathbb{N}\}$ is discrete and for all $q \in \beta\mathbb{N}$, $f^{\beta}(q) = p + q$;

(p) there is a function $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $q \in \beta\mathbb{N}$, $g^{\beta}(p + q) = q$;

(q) there is a function $h : \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $q \in \beta\mathbb{N}$, $h^{\beta}(p + q) = p$.

Proof. [5, Theorem 2.1], [20, Theorem 2], and [16, Theorem 5.5]. $\Box$

Some of these characterizations hold in a more general setting.

4.2 **Theorem** (Hindman and Strauss). *Let $(S, \cdot)$ be a discrete semigroup and let $p \in \beta S$. Statement (a) implies statements (b), (c), and (d) which are equivalent. These statements imply statements (e) and (f) which are equivalent. If S is countable, all of these statements are equivalent.*

(a) λ_p is one-to-one on S and $\{p \cdot x : x \in S\}$ is strongly discrete;

(b) for each subset A of S there exists $B \subseteq S$ such that $A = \{x \in S : B/x \in p\}$;

(c) λ_p is one-to-one on βS (i.e. left cancellation holds at p);

(d) λ_p is one-to-one on βS and $\{p \cdot x : x \in S$ is discrete$\}$;

(e) λ_p is one-to-one on S and $\{p \cdot x : x \in S$ is discrete$\}$;

(f) for each $x \in S$ and each $q \in \beta S \backslash \{x\}$, $p \cdot x \neq p \cdot q$.

Proof. [16, Theorem 2.22]. $\Box$

No similar characterizations of right cancellation are known, but see [16, Section 4] for several sufficient conditions for right cancellation to hold at points of $(\beta\mathbb{N}, +)$.

In [14] it was shown by Pym and this writer that if $\langle x_n \rangle_{n=1}^{\infty}$ satisfies uniqueness of finite sums in $\mathbb{N}$ and $p \in (c\ell\{x_n : n \in \mathbb{N}\})\backslash\mathbb{N}$, then $c\ell\{p, p + p, p + p + p, \ldots\}$ is not a semigroup. This result was generalized by Strauss.

4.3 **Theorem** (Strauss). *Let $p \in \mathbb{N}^*$ such that left cancellation holds at p. Then $c\ell\{p, p+p, p+p+p, \ldots\}$ is not a semigroup.*

Proof. [20, Theorem 3]. $\Box$

Musing about this I wondered if $c\ell\{p, p+p, p+p+p, \ldots\}$ could ever be a semigroup when $p \in \mathbb{N}^*$ and $\{p, p+p, p+p+p, \ldots\}$ is infinite. Amir Maleki answered this question in a private communication.

4.4 **Theorem** (Maleki). *Let $q = q + q$ in $\beta\mathbb{N}$, let $k \in \mathbb{N}$, and let $p = q + k$. Then $c\ell\{p, p+p, p+p+p, \ldots\}$ is a semigroup.*

Proof. $\{p, p+p, p+p+p, \ldots\} = q + \mathbb{N}k$ so $c\ell\{p, p+p, p+p+p, \ldots\} = c\ell(q+\mathbb{N}k) = q + c\ell(\mathbb{N}k)$. Since $q \in c\ell(\mathbb{N}k)$, one has $q + c\ell(\mathbb{N}k)$ is a semigroup. $\Box$

5. **Commutativity**. The observant reader will have noted that commutativity played an important role in Strauss' proof that $\mathbb{N}^*$ does not contain a topological and algebraic copy of $\beta\mathbb{N}$. That is the center of $\beta\mathbb{N}$ is $\mathbb{N}$ while points p and q of $\mathbb{N}^*$ which commute with each other are rare.

We look in this section at a different aspect of commutativity. We have chosen to take the operation on βS which makes it a left topological semigroup with ρ_x continuous for each $x \in S$. Alternately one could choose (and many writers do) the operation on βS which makes it a right topological semigroup with λ_x continuous for each $x \in S$.

5.1 **Definition**. Let $(S, \cdot)$ be a semigroup. Denote also by $\cdot$ the operation on βS making βS a left topological semigroup with ρ_x continuous for $x \in S$ and denote by $*$ the operation on βS making βS a right topological semigroup with λ_x continuous for each $x \in S$. Let K_ℓ be the smallest ideal of $(\beta S, \cdot)$ and let K_r be the smallest ideal of $(\beta S, *)$.

Until recently it was tacitly assumed that it didn't make any substantive difference which operation was chosen. If S is commutative, this is true for then $p \cdot q = q * p$ for all $p, q \in \beta S$. Thus in particular a minimal left ideal of $(\beta S, \cdot)$ is a minimal right ideal of $(\beta S, *)$ so that $K_\ell = K_r$. By contrast we have:

5.2 **Theorem** (Anthony). *Let S be either the free semigroup on two generators or the group of permutations of $\mathbb{N}$ which move only finitely many elements. Then $K_r \backslash c\ell K_\ell \neq \emptyset$ and $K_\ell \backslash c\ell K_r \neq \emptyset$.*

Proof. [1, Corollaries 2.6 and 3.4]. $\Box$

On the other hand, the smallest ideals must remain close to each other.

5.3 **Theorem** (Anthony). *Let S be any semigroup. Then $K_\ell \cap c\ell K_r \neq \emptyset$ and $K_r \cap c\ell K_\ell \neq \emptyset$.*

Proof. [1, Theorem 4.1]. $\square$

It remains an open question whether it is possible to have $K_\ell \cap K_r = \emptyset$.

In a similar vein we see that the structures of $(\beta S, \cdot)$ and $(\beta S, *)$ can differ significantly at places far removed from the smallest ideals.

5.4 **Theorem** (El-Mabhouh, Pym, and Strauss). *Let S be the free semigroup on countably many generators. There is a subsemigroup H of $(\beta S, *)$ such that*

(1) $H \cap (\beta S \cdot \beta S) = \emptyset$ and

(2) for all $p, q \in \beta S$, $p * q \in H$ if and only if $p \in H$ and $q \in H$.

Proof. [9]. $\square$

We close with another simple contrast between commutative and noncommutative S provided by Theorems 5.5 and 5.7.

5.5 **Theorem** (Ruppert). *Let S be a discrete commutative semigroup, let K be the smallest ideal of βS, and let L be a minimal left ideal of βS. Then $c\ell L = c\ell K$.*

Proof. By the Proposition of [19], $c\ell L$ is an ideal so $K \subseteq c\ell L$ so $c\ell K \subseteq c\ell L \subseteq c\ell K$. $\square$

5.6 **Corollary**. *Let S be a commutative discrete semigroup. Then βS does not have disjoint closed left ideals.*

Proof. If $L$is a closed left ideal of βS then by Theorem 5.5 $c\ell K \subseteq L$. $\square$

5.7 **Theorem**. *Let S be the free semigroup on two generators. Then βS does have two disjoint compact left ideals. In fact there is a sequence $\langle L_n \rangle_{n=1}^\infty$ of pairwise disjoint clopen left ideals such that $S * \setminus \bigcup_{n=1}^\infty L_n$ is also a left ideal.*

Proof. Let the generators of S be the letters a and b. For each $n \in \mathbb{N}$ let $B_n = \{wab^n : w \in S\}$ and let $L_n = c\ell B_n$. To see that L_n is a left ideal of βS, let $p \in L_n$ and let $q \in \beta S$. Then given any $w \in S$, $S \subseteq B_n/(wab^n)$ so $B_n \in q \cdot p$. To see that $S * \setminus \bigcup_{n=1}^\infty L_n$ is a left ideal of βS, let $p \in S^* \setminus \bigcup_{n=1}^\infty L_n$ and let $q \in \beta S$. Then immediately $q \cdot p \in S^*$. Suppose we have $q \cdot p \in L_n$ for some $n \in \mathbb{N}$. Also $D = S \setminus (B_n \cup \{b^k : k \leq n\} \cup \{ab^n\}) \in p$ so pick $w \in D \cap \{v \in S : B_n/v \in q\}$. Pick $u \in B_n/w$. Then $uw \in B_n$ so for some $v \in S$, $uw = vab^n$. Since $w \notin \{b^k : k \leq n\} \cup \{ab^n\}$ one has that $w \in B_n$, a contradiction. $\square$

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