

This paper was published in *Indagationes Mathematicae* **29** (2018), 293-312.

To the best of my knowledge, this is the final version as it was submitted to
the publisher. –NH

Recurrence in the dynamical system $(X, \langle T_s \rangle_{s \in S})$ and ideals of βS

Neil Hindman ^{*†} Dona Strauss [‡] Luca Q. Zamboni [§]

Abstract

A *dynamical system* is a pair $(X, \langle T_s \rangle_{s \in S})$, where X is a compact Hausdorff space, S is a semigroup, for each $s \in S$, T_s is a continuous function from X to X , and for all $s, t \in S$, $T_s \circ T_t = T_{st}$. Given a point $p \in \beta S$, the Stone-Čech compactification of the discrete space S , $T_p : X \rightarrow X$ is defined by, for $x \in X$, $T_p(x) = p\text{-}\lim_{s \in S} T_s(x)$. We let βS have the operation extending the operation of S such that βS is a right topological semigroup and multiplication on the left by any point of S is continuous. Given $p, q \in \beta S$, $T_p \circ T_q = T_{pq}$, but T_p is usually not continuous. Given a dynamical system $(X, \langle T_s \rangle_{s \in S})$, and a point $x \in X$, we let $U(x) = \{p \in \beta S : T_p(x) \text{ is uniformly recurrent}\}$. We show that each $U(x)$ is a left ideal of βS and for any semigroup we can get a dynamical system with respect to which $K(\beta S) = \bigcap_{x \in X} U(x)$ and $\text{cl}K(\beta S) = \bigcap \{U(x) : x \in X \text{ and } U(x) \text{ is closed}\}$. And we show that weak cancellation assumptions guarantee that each such $U(x)$ properly contains $K(\beta S)$ and has $U(x) \setminus \text{cl}K(\beta S) \neq \emptyset$.

*Department of Mathematics, Howard University, Washington, DC 20059, USA.
nhindman@aol.com

†This author acknowledges support received from the National Science Foundation (USA) via Grant DMS-1460023.

‡Department of Pure Mathematics, University of Leeds, Leeds LS2 9J2, UK.
d.strauss@hull.ac.uk

§Université de Lyon, Université Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43
boulevard du 11 novembre 1918, F69622 Villeurbanne Cedex, France
zamboni@math.univ-lyon1.fr

1 Introduction

We take the Stone-Čech compactification of a discrete semigroup (S, \cdot) to be the set of ultrafilters on S , identifying the points of S with the principal ultrafilters. Given $A \subseteq S$, we set $\bar{A} = \{p \in \beta S : A \in p\}$. The set $\{\bar{A} : A \subseteq S\}$ is a basis for the open sets and a basis for the closed sets of βS . The operation on S extends uniquely to βS so that $(\beta S, \cdot)$ is a right topological semigroup with S contained in its topological center, meaning that ρ_p is continuous for each $p \in \beta S$ and λ_x is continuous for each $x \in S$, where for $q \in \beta S$, $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$. So, for every $p, q \in \beta S$, $pq = \lim_{s \rightarrow p} \lim_{t \rightarrow q} st$, where s and t denote elements of S . If $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : xy \in A\}$. (We are following the custom of frequently writing xy for $x \cdot y$.)

The algebraic structure of βS is interesting in its own right, and has had substantial applications, especially to that part of combinatorics known as *Ramsey Theory*. See the book [4] for an elementary introduction to the structure of βS and its applications.

We are concerned in this paper with the relationship between the algebraic structure of βS and recurrence in *dynamical systems*.

Definition 1.1. A *dynamical system* is a pair $(X, \langle T_s \rangle_{s \in S})$ such that

- (1) X is a compact Hausdorff topological space (called the *phase space* of the system);
- (2) S is a semigroup;
- (3) for each $s \in S$, T_s is a continuous function from X to X ; and
- (4) for all $s, t \in S$, $T_s \circ T_t = T_{st}$.

Associated with any semigroup S are at least two interesting dynamical systems, namely $(\beta S, \langle \lambda_s \rangle_{s \in S})$ and $({}^S\{0, 1\}, \langle T_s \rangle_{s \in S})$ where ${}^S\{0, 1\}$ is the set of all functions from S to $\{0, 1\}$ with the product topology and $T_s(x) = x \circ \rho_s$. (We shall verify that this latter example is a dynamical system shortly.)

It is common to assume that the phase space of a dynamical system is a metric space, but we make no such assumption. If S is infinite, then βS is not a metric space. Everything we do here is boring if S is finite so whenever we write “let S be a semigroup” we shall assume that S is infinite. The interested reader can amuse herself by determining which of our results remain valid if that assumption is dropped.

The system $(\beta S, \langle \lambda_s \rangle_{s \in S})$ has significant general properties as can be seen in [4, Section 19.1], but will not be used much in this paper.

Given a product space $S\{0, 1\}$, recall that the product topology has a subbasis consisting of sets of the form $\pi_t^{-1}[\{a\}]$ for $t \in S$ and $a \in \{0, 1\}$, where, for $x \in S\{0, 1\}$, $\pi_t(x) = x(t)$.

Lemma 1.2. *Let R be a semigroup and let S be a subsemigroup of R . Let $Z = R\{0, 1\}$, the set of all functions from R to $\{0, 1\}$ with the product topology. For $x \in Z$ and $s \in S$, define $T_s(x) = x \circ \rho_s$. Then $(Z, \langle T_s \rangle_{s \in S})$ is a dynamical system.*

Proof. It is routine to verify that for $s, t \in S$, $T_s \circ T_t = T_{st}$. To see that T_s is continuous for each $s \in S$, let $s \in S$ be given. It suffices to show that the inverse image of each subbasic open set is open, so let $t \in R$ and $a \in \{0, 1\}$ be given. Then $T_s^{-1}[\pi_t^{-1}[\{a\}]] = \pi_{ts}^{-1}[\{a\}]$. \square

Recall that, if T is any discrete space, $p \in \beta T$, $\langle x_t \rangle_{t \in T}$ is any indexed family in a Hausdorff topological space X , and $y \in X$, then $p\text{-}\lim_{t \in T} x_t = y$ if and only if for every neighborhood U of y , $\{t \in T : x_t \in U\} \in p$. In compact spaces p -limits always exist.

Definition 1.3. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system and let $p \in \beta S$. Then $T_p : X \rightarrow X$ is defined by, for $x \in X$, $T_p(x) = p\text{-}\lim_{s \in S} T_s(x)$. So $T_p(x) = \lim_{s \rightarrow p} T_s(x)$ where s denotes an element of S .

Using [4, Theorem 4.5] one easily sees that for $p, q \in \beta S$, $T_p \circ T_q = T_{pq}$. However, $(X, \langle T_s \rangle_{s \in \beta S})$ is not in general a dynamical system, since T_p is not likely to be continuous when $p \in \beta S \setminus S$. However, for each $x \in X$, the map $p \mapsto T_p(x) : \beta S \rightarrow X$ is continuous. To see this, define $f_x(p) = T_p(x)$. If U is a neighborhood of $f_x(p)$ and $A = \{s \in S : T_s(x) \in U\}$, then $U \in p$ and $f_x[\bar{A}] \subseteq U$. Alternatively, one may note that $p \mapsto T_p(x)$ is the continuous extension to βS of the function $s \mapsto T_s(x) : S \rightarrow X$.

As a compact Hausdorff right topological semigroup, βS has a number of important algebraic properties, and we list some of those that we shall use. (Proofs can be found in [4, Chapters 1 and 2]. Assume that T is a compact Hausdorff right topological semigroup. A non-empty subset V of T is a *left ideal* if $tV \subseteq V$ for every $t \in T$, a *right ideal* if $Vt \subseteq V$ for every $t \in T$, and an *ideal* if it is both a left and a right ideal.

- (1) T contains an idempotent.
- (2) T has a smallest ideal $K(T)$, which is the union of the minimal left ideals of T and the union of the minimal right ideals of T .
- (3) For every $t \in K(T)$, Tt is a minimal left ideal of T and tT is a minimal right ideal of T .

- (4) The intersection of any minimal left ideal and any minimal right ideal of T is a group.
- (5) Every left ideal of T contains a minimal left ideal, and every right ideal of T contains a minimal right ideal.
- (5) Every minimal left ideal of T is compact.
- (6) If $\{t \in T : \lambda_t \text{ is continuous}\}$ is dense in T , then the closure of every ideal in T is also an ideal.

We introduce the main objects of study in this paper now. Given a set X , we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X .

Definition 1.4. Let S be a semigroup and let $A \subseteq S$. We say the set A is *syndetic* if and only if there exists $F \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in F} t^{-1}A$.

In the semigroup $(\mathbb{N}, +)$ a set is syndetic if and only if it has bounded gaps.

Definition 1.5. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system and let $x \in X$.

- (a) The point x is *uniformly recurrent* if and only if for every neighborhood V of x , $\{s \in S : T_s(x) \in V\}$ is syndetic.
- (b) $U(x) = U_X(x) = \{p \in \beta S : T_p(x) \text{ is uniformly recurrent}\}$.

In Section 2 of this paper we present well known results about $U(x)$ that are valid in arbitrary dynamical systems as well as the few simple results that we have in the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$.

In Section 3 we present results about the dynamical systems described in Lemma 1.2.

In Section 4 we consider the effect of slightly modifying the phase space in the dynamical systems described in Lemma 1.2.

In Section 5 we consider surjectivity of T_p and the set $NS = NS_X = \{p \in \beta S : T_p : X \rightarrow X \text{ is not surjective}\}$ which is a right ideal of βS whenever it is nonempty.

2 General results

We begin with some well known basic facts.

Lemma 2.1. *Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system, let L be a minimal left ideal of βS , and let $x \in X$. The following are equivalent:*

- (a) x is uniformly recurrent.

- (b) There exists $q \in L$ such that $T_q(x) = x$.
- (c) There exists an idempotent $q \in L$ such that $T_q(x) = x$.
- (d) There exists $y \in X$ and $q \in L$ such that $T_q(y) = x$.
- (e) There exists $q \in K(\beta S)$ such that $T_q(x) = x$.
- (f) There exists $y \in X$ and $q \in K(\beta S)$ such that $T_q(y) = x$.

Proof. The equivalence of (a)-(d) is shown in [4, Theorem 19.23]. Since (c) implies (e), and (e) implies (f), we shall show (f) implies (c) and this will establish the equivalence of all six statements. So assume that (f) holds. Let u denote the identity of the group $L \cap q\beta S$. Since $uq = q$, it follows that $T_u(x) = T_u T_q(y) = T_q(y) = x$. \square

Corollary 2.2. *Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system and let $x \in X$.*

- (1) If x is uniformly recurrent, $U(x) = \beta S$.
- (2) For each $x \in X$, $U(x)$ is a left ideal of βS .
- (3) For every $x \in X$, $K(\beta S) \subseteq U(x)$.
- (4) $\bigcap_{x \in X} U(x)$ is a two sided ideal of βS .

Proof. (1) Suppose that x is uniformly recurrent. Then $T_u(x) = x$ for some $u \in K(\beta S)$. Thus for every $v \in \beta S$, $T_v(x) = T_v T_u(x) = T_{vu}(x)$; since $vu \in K(\beta S)$, by Lemma 2.1(f), $T_v(x)$ is uniformly recurrent.

(2) Let $x \in X$, let $p \in U(x)$, and let $r \in \beta S$. By Lemma 2.1(e), pick $q \in K(\beta S)$ such that $T_q(T_p(x)) = T_p(x)$. Then $T_{rp}(x) = T_r(T_q(T_p(x))) = T_{rqp}(x)$. Now $rqp \in K(\beta S)$, so by Lemma 2.1(f), $T_{rp}(x)$ is uniformly recurrent.

(3) This is immediate from Lemma 2.1(f).

(4) By (3), $\bigcap_{x \in X} U(x)$ is nonempty, so by (2) $\bigcap_{x \in X} U(x)$ is a left ideal of βS , so it is enough to show that $\bigcap_{x \in X} U(x)$ is a right ideal of βS . So suppose that $x \in X$, $p \in \bigcap_{x \in X} U(x)$ and $q \in \beta S$. Since $p \in U(T_q(x))$, $T_{pq}(x)$ is uniformly recurrent and so $pq \in U(x)$. \square

The statements of Lemma 2.3 below are modifications of basic well known facts that are proved in [2]. (Furstenberg assumes that the phase space is metric, but the proofs given do not use this assumption.) We shall say that a subspace Z of X is *invariant* if $T_s[Z] \subseteq Z$ for every $s \in S$. Of course, if Z is closed and invariant, then $T_p[Z] \subseteq Z$ for every $p \in \beta S$. (Let $x \in Z$. Then $T_s(x) \in Z$ for each $s \in S$ so $p\text{-}\lim_{s \in S} T_s(x) \in Z$.)

Lemma 2.3. *Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. Let L be a minimal left ideal of βS .*

- (1) A subspace Y of X is minimal among all closed and invariant subsets of X if and only if there is some $x \in X$ such that $Y = \{T_p(x) : p \in L\}$.
- (2) Let Y be a subspace of X which is minimal among all closed and invariant subsets of X . Then every element of Y is uniformly recurrent.
- (3) If $x \in X$ is uniformly recurrent and $Y = \{T_p(x) : p \in \beta S\}$, then Y is minimal among all closed and invariant subsets of X .
- (4) If $x \in X$ is uniformly recurrent, then $T_p(x)$ is uniformly recurrent for every $p \in \beta S$.

Proof. (1) Suppose that Y is a subspace of X which is minimal among all closed and invariant subsets of X . Pick $x \in Y$ and let $Z = \{T_p(x) : p \in L\}$. We claim that Z is a closed and invariant subspace of Y and is therefore equal to Y . If $p \in L$ and $s \in S$, then $T_s(T_p(x)) = T_{sp}(x)$ and $sp \in L$, so Z is invariant and obviously $Z \subseteq Y$. To see that Z is closed, it suffices to show that any net in Z has a cluster point in Z . To this end, let $\langle p_\alpha \rangle_{\alpha \in D}$ be a net in L and pick a cluster point p in L of $\langle p_\alpha \rangle_{\alpha \in D}$. Then $T_p(x)$ is a cluster point of $\langle T_{p_\alpha}(x) \rangle_{\alpha \in D}$.

Conversely, let $x \in X$ and let $Y = \{T_p(x) : p \in L\}$. Then Y is invariant and one sees as above that Y is closed. We shall show that Y is minimal among all closed and invariant subsets of X . To see this, suppose that Z is a subset of Y which is closed and invariant. We shall show that $Y \subseteq Z$, so let $y \in Y$ be given. Pick $z \in Z$. Then $y = T_p(x)$ and $z = T_q(x)$ for some p and q in L . Since $Lq = L$, there exists $r \in L$ such that $rq = p$. It follows that $T_r(z) = y$ and hence that $y \in Z$ as required.

(2) Let Y be a subspace of X which is minimal among all closed and invariant subsets of X and let $x \in Y$. Pick $y \in X$ such that $Y = \{T_p(y) : p \in L\}$. Pick $p \in L$ such that $x = T_p(y)$. By Lemma 2.1(f), x is uniformly recurrent.

(3) Let x be a uniformly recurrent point of X and let $Y = \{T_p(x) : p \in \beta S\}$. By Lemma 2.1(b), pick $q \in L$ such that $T_q(x) = x$. By (1) it suffices that $Y = \{T_p(x) : p \in L\}$. To see this, let $y \in Y$ and pick $p \in \beta S$ such that $y = T_p(x)$. Then $y = T_p(T_q(x)) = T_{pq}(x)$ and $pq \in L$.

(4) Let x be a uniformly recurrent point of X and let $Y = \{T_p(x) : p \in \beta S\}$. By (3) Y is minimal among all closed and invariant subsets of X so (2) applies. \square

We conclude this section with a few results about the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$. We observe that, if we define $\lambda_p : \beta S \rightarrow \beta S$ in this system by $\lambda_p(q) = \lim_{s \rightarrow p} \lambda_s(q)$, where s denotes an element of S , then $\lambda_p(q) = pq$ for every p and q in βS . So this does not conflict with the previous definition of λ_p given in the introduction.

Theorem 2.4. *Let S be a semigroup and let $x \in \beta S$. Statements (a) and (b) are equivalent and imply (c). If βS has a left cancelable element, all three are equivalent.*

- (a) $x \in K(\beta S)$.
- (b) x is uniformly recurrent in the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$.
- (c) βSx is a minimal left ideal of βS .

Proof. To see that (a) implies (b), let $x \in K(\beta S)$ and let u be the identity of the group in $K(\beta S)$ to which x belongs. Then $x = \lambda_u(x)$ so by Lemma uniformrecurrence(e), x is uniformly recurrent.

To see that (b) implies (a), assume that x is uniformly recurrent. By Lemma 2.1(f) pick $y \in \beta S$ and $q \in K(\beta S)$ such that $\lambda_q(y) = x$. Then $x = qy \in K(\beta S)$.

To see that (a) implies (c), assume that $x \in K(\beta S)$ and pick the minimal left ideal L of βS such that $x \in L$. Then βSx is a left ideal of βS contained in L and so $\beta Sx = L$.

Now assume that βS has a left cancelable element z and that βSx is a minimal left ideal of βS . Pick an idempotent $u \in \beta Sx$. Then $zx \in \beta Sx$ so by [4, Lemma 1.30], $zx = zxu$ and therefore $x = xu \in \beta Sx \subseteq K(\beta S)$. \square

Corollary 2.5. *Let S be an infinite semigroup and let $x \in K(\beta S)$. Then $U(x) = \beta S$ with respect to the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$.*

Proof. By Theorem 2.4, x is uniformly recurrent, so by Lemma 2.3(4), $U(x) = \beta S$. \square

Corollary 2.6. *Let S be a semigroup and let $p, q \in \beta S$. Statements (a) and (b) are equivalent and imply statement (c). If βS has a left cancelable element, then all three statements are equivalent.*

- (a) $qp \in K(\beta S)$.
- (b) $q \in U(p)$ with respect to the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$.
- (c) βSqp is a minimal left ideal of βS .

Proof. We have that $q \in U(p)$ if and only if $\lambda_q(p)$ is uniformly recurrent and $\lambda_q(p) = qp$ so Theorem 2.4 applies. \square

It is an old and difficult problem to characterize when $K(\beta S)$ is prime or when $\text{cl}K(\beta S)$ is prime. There are trivial situations where the answer is known. For example if S is left zero or right zero, then so is βS and thus $K(\beta S) = \beta S$, and is necessarily prime. It is not known whether $K(\beta \mathbb{N}, +)$ is prime or $\text{cl}K(\beta \mathbb{N}, +)$ is prime. (Some partial results were obtained in [3].)

Corollary 2.7. *Let S be a semigroup. The following statements are equivalent.*

- (a) *There exists $p \in \beta S \setminus K(\beta S)$ such that, with respect to the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$, $K(\beta S) \subsetneq U(p)$.*

(b) $K(\beta S)$ is not prime.

Proof. This is an immediate consequence of Corollary 2.6. \square

3 Dynamical systems with phase space $R_{\{0,1\}}$

Throughout this section we assume that R is a semigroup, S a subsemigroup of R , and $(Z, \langle T_s \rangle_{s \in S})$ is the dynamical system of Lemma 1.2. While our results are valid in this generality, in practice we are interested in just two situations, one in which $R = S$ and the other in which $R = S \cup \{e\}$ where e is a two sided identity adjoined to S .

Our first results in this section are aimed at showing that for any semigroup S , there is a dynamical system such that both $K(\beta S)$ and $\text{cl}K(\beta S)$ are intersections of sets of the form $U(x)$.

Definition 3.1. Given $x \in Z$ we denote the continuous extension of x from βR to $\{0,1\}$ by \tilde{x} .

Of course, for each $x \in Z$, each $p \in \beta S$ and each $t \in R$, $T_p(x)(t) = p\text{-}\lim_{s \in S} T_s(x)(t) = p\text{-}\lim_{s \in S} x(ts)$ and so $T_p(x)(t) = \tilde{x}(tp)$.

Lemma 3.2. Let $x \in Z$, let $p \in \beta S$, and let L be a minimal left ideal of βS . The following statements are equivalent:

- (a) $p \in U(x)$.
- (b) There exists $q \in L$ such that $\tilde{x}(tp) = \tilde{x}(tqp)$ for all $t \in R$.
- (c) There exists an idempotent $q \in L$ such that $\tilde{x}(tp) = \tilde{x}(tqp)$ for all $t \in R$.

Proof. To see that (a) implies (c), assume that $T_p(x)$ is uniformly recurrent. By Lemma 2.1(c), pick an idempotent $q \in L$ such that $T_q(T_p(x)) = T(p)(x)$. Then $T_{qp}(x) = T_p(x)$ so as noted above, for all $t \in R$, $\tilde{x}(tqp) = \tilde{x}(tp)$.

Trivially (c) implies (b). To see that (b) implies (a), pick $q \in L$ such that $\tilde{x}(tp) = \tilde{x}(tqp)$ for all $t \in R$. Then $T_p(x) = T_{qp}(x) = T_q(T_p(x))$, so by Lemma 2.1(b), $T_p(x)$ is uniformly recurrent. \square

Lemma 3.3. Let $x \in Z$ and let $p \in \beta S$. Then $p \in U(x)$ if and only if for every minimal left ideal L of βS and every $F \in \mathcal{P}_f(R)$, there exists $q_F \in L$ such that for all $t \in F$, $\tilde{x}(tp) = \tilde{x}(tq_F p)$.

Proof. The necessity is an immediate consequence of Lemma 3.2(b).

For the sufficiency, let L be a minimal left ideal of βS . For each $F \in \mathcal{P}_f(R)$, pick $q_F \in L$ as guaranteed. Direct $\mathcal{P}_f(R)$ by agreeing that $F < G$ if and only if

$F \subseteq G$. Pick a cluster point $q \in L$ of the net $\langle q_F \rangle_{F \in \mathcal{P}_f(R)}$. It is then routine to show that for all $t \in R$, $\tilde{x}(tqp) = \tilde{x}(tp)$ so that by Lemma 3.2(b), $p \in U(x)$. \square

Theorem 3.4. (1) $K(\beta S) \subseteq \bigcap_{x \in Z} U(x)$.

(2) If $p \in \bigcap_{x \in Z} U(x)$, then, for every minimal left ideal L of βS , $\beta Sp = Lp$ and so βSp is a minimal left ideal of βS .

(3) If R contains a left cancelable element, then $K(\beta S) = \bigcap_{x \in Z} U(x)$. In particular, if R has a left identity, then $K(\beta S) = \bigcap_{x \in Z} U(x)$.

Proof. (1) $K(\beta S) \subseteq \bigcap_{x \in Z} U(x)$ by Corollary 2.2(3).

(2) Assume that $p \in \bigcap_{x \in Z} U(x)$. Let L be a minimal left ideal of βS . We shall show that, for every $t \in R$, $tp \in tLp$. To see this, assume the contrary. Then for some $t \in R$, there exists $A \subseteq R$ such that $A \in tp$ and $\overline{A} \cap tLp = \emptyset$. Let $x = \chi_A$. So \tilde{x} is the characteristic function of \overline{A} . Since $p \in U(x)$, it follows from Lemma 3.2 that $\tilde{x}(tp) = \tilde{x}(tqp)$ for some $q \in L$. However, $\tilde{x}(tp) = 1$ and $\tilde{x}(tqp) = 0$. This contradiction establishes that $tp \in tLp$ for every $t \in R$. In particular, $\beta Sp = c\ell_{\beta S} Sp \subseteq Lp$. So $\beta Sp \subseteq Lp$. By [4, Theorem 1.46], Lp is a minimal left ideal of βS , and so $\beta Sp = Lp$.

(3) Now suppose that R contains a left cancelable element t and let $p \in \bigcap_{x \in Z} U(x)$. Since t is left cancelable in βR by [4, Lemma 8.1] and $tp = tqp$ for some $q \in L$, it follows that $p = qp \in K(\beta S)$. \square

Recall that a subset A of a semigroup S is *piecewise syndetic* if and only if there is some $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$, there is some $x \in S$ with $Fx \subseteq \bigcup_{t \in G} t^{-1}A$. The important fact about piecewise syndetic sets is that they are the subsets of S whose closure meets $K(\beta S)$, [4, Theorem 4.40].

Definition 3.5. $\Omega = \Omega_Z = \{x \in Z : \overline{x^{-1}[\{1\}] \cap S} \cap K(\beta S) = \emptyset\}$.

Thus $\Omega = \{x \in Z : x^{-1}[\{1\}] \cap S \text{ is not piecewise syndetic in } S\}$. Note that, since $K(\beta S)$ is usually not topologically closed, we have by Theorem 3.4 that not all sets of the form $U(x)$ are closed.

Definition 3.6. Let $x \in Z$. $N(x) = \{p \in \beta S : (\forall t \in R)(T_p(x)(t) = 0)\}$.

Lemma 3.7. Let $x \in Z$. Then $N(x)$ is closed and $N(x) \subseteq U(x)$. If $N(x) = U(x)$, then $x \in \Omega$. If S is a left ideal of R , then $N(x) = U(x)$ if and only if $x \in \Omega$.

Proof. To see that $N(x)$ is closed, let $p \in \beta S \setminus N(x)$, pick $t \in R$ such that $T_p(x)(t) = 1$, and let $A = \{s \in S : T_s(x)(t) = 1\}$. Then $A \in p$ and $\overline{A} \cap N(x) = \emptyset$.

If $T_p(x)$ is constantly equal to 0 on R , then $T_p(x)$ is uniformly recurrent and thus $p \in U(x)$.

Let $A = x^{-1}[\{1\}] \cap S$.

First assume that $N(x) = U(x)$ and suppose that $x \notin \Omega$. Since $\overline{A} \cap K(\beta S) \neq \emptyset$, pick $p \in \overline{A} \cap K(\beta S)$. By Corollary 2.2(3), $p \in U(x)$ and so for all $t \in R$, $T_p(x)(t) = 0$. Since $K(\beta S)$ is a union of groups, there exists $q \in K(\beta S)$ such that $qp = p$. Pick $t \in S$ such that $t^{-1}A \in p$. Also $T_p(x)(t) = 0$ so $\{s \in S : x(ts) = 0\} \in p$. Pick $s \in t^{-1}A$ such that $x(ts) = 0$, a contradiction.

Now assume that S is a left ideal in R . Let $x \in \Omega$ and let $p \in U(x)$. We claim that $p \in N(x)$. To see this, suppose we have some $t \in R$ such that $T_p(x)(t) = 1$. By Lemma 3.2, there exists an idempotent $q \in K(\beta S)$ such that $\tilde{x}(tqp) = 1$. By [4, Theorem 2.17], βS is a left ideal of βR so $tqp \in \beta S$ and so $A \in tqp = tqqp$. Thus there is some $s \in S$ such that $tsqp \in \overline{A}$. Since $ts \in S$, $tsqp \in K(\beta S)$, a contradiction. \square

Lemma 3.8. *Let $p \in \bigcap_{x \in \Omega} U(x)$ and let $t \in R$. If $tp \in \beta S$, then $tp \in \text{cl}K(\beta S)$. In particular, $\beta Sp \subseteq \text{cl}K(\beta S)$.*

Proof. Assume that $tp \in \beta S \setminus \text{cl}(K\beta S)$. We can choose $A \in tp$ such that $A \subseteq S$ and $\overline{A} \cap K(\beta S) = \emptyset$. Let x be the characteristic function of A in R , so that $x \in \Omega$ and hence $p \in U(x)$. Observe that \tilde{x} is the characteristic function of $\text{cl}_{\beta R}(A)$ in βR and that $\text{cl}_{\beta R}(A) \subseteq \beta S$, because βS is clopen in βR . Since $\tilde{x}(tp) = 1$, it follows from Lemma 3.2(b) that there exists $q \in K(\beta S)$ such that $\tilde{x}(tqp) = 1$, and so $A \in tqp$. Now $\{r \in \beta S : tqr \in \beta S\}$ is non-empty and is a right ideal of βS . There exists an idempotent u in the intersection of this right ideal with the left ideal βSq of βS . Since $q \in \beta Su$, $qu = q$. So $tqp = tqqup \in K(\beta S)$, because $tqu \in \beta S$ and $u \in \beta Sq \subseteq K(\beta S)$. This contradicts the assumption that $\overline{A} \cap K(\beta S) = \emptyset$. \square

Corollary 3.9. *Each of the following statements implies that $\bigcap_{x \in \Omega} U(x) \subseteq \text{cl}K(\beta S)$.*

- (a) *There exists $e \in R$ such that $es = s$ for every $s \in S$.*
- (b) *S contains a left cancelable element.*

Proof. It follows from Lemma 3.8 that (a) implies that $\bigcap_{x \in \Omega} U(x) \subseteq \text{cl}K(\beta S)$. So assume that s is a left cancelable element in S and let $p \in \bigcap_{x \in \Omega} U(x)$. By [4, Lemma 8.1], s is left cancelable in βS . By Lemma 3.8, $sp \in \text{cl}K(\beta S)$. Now $s\beta S = \overline{sS}$ is clopen in βS . So $sp \in \text{cl}(K(\beta S) \cap s\beta S)$. We claim that, if $q \in K(\beta S) \cap s\beta S$, then $q \in sK(\beta S)$. To see this, suppose that $q \in K(\beta S)$ and that $q = sv$ for some $v \in \beta S$. There is an idempotent $u \in K(\beta S)$ for which $qu = q$. This implies that $sv = svu$ and hence that $v = vu \in K(\beta S)$. So $sp \in \text{cl}(sK(\beta S)) = s\text{cl}K(\beta S)$ and hence $p \in \text{cl}K(\beta S)$. \square

Corollary 3.10. *Assume that S is a left ideal of R . Then each of the hypotheses (a) and (b) of Corollary 3.9 implies that $\bigcap_{x \in \Omega} U(x) = \text{cl}K(\beta S)$.*

Proof. Assume that one of the hypotheses of Corollary 3.9 holds. Then

$$\bigcap_{x \in \Omega} U(x) \subseteq \text{cl}K(\beta S).$$

To see that $\text{cl}K(\beta S) \subseteq \bigcap_{x \in \Omega} U(x)$, let $x \in \Omega$ be given. By Lemma 3.7, $U(x) = N(x)$ and so $U(x)$ is closed. By Corollary 2.2(3), $K(\beta S) \subseteq U(x)$ and hence $\text{cl}K(\beta S) \subseteq U(x)$. \square

For the statement of the following corollary we depart from our standing assumptions about R , S , and $(Z, \langle T_s \rangle_{s \in S})$.

Corollary 3.11. *Let S be a semigroup. There exist a dynamical system $(X, \langle T_s \rangle_{s \in S})$ and a subset M of X such that $K(\beta S) = \bigcap_{x \in X} U(x)$ and $\text{cl}K(\beta S) = \bigcap_{x \in M} U(x)$.*

Proof. If S has a left identity, let $R = S$. Otherwise, let $R = S \cup \{e\}$ where e is an identity adjoined to S . The conclusion then follows from Theorem 3.4 and Corollary 3.10. \square

In the proof of the above corollary, we could have simply let $R = S \cup \{e\}$ where e is an identity adjoined to S , regardless of whether S has a left identity, as was done in [4, Theorem 19.27] to produce a dynamical system for any semigroup S establishing the equivalence of the notions of *central* and *dynamically central*. We shall investigate the relationship between the systems with phase space $X = \mathbb{R}\{0, 1\}$ and $Y = \mathbb{S}\{0, 1\}$ in the next section.

We note that it is possible that $\bigcap_{x \in Z} U(x) \neq K(\beta S)$ and there is no subset M of Z such that $\bigcap_{x \in M} U(x) = \text{cl}K(\beta S)$. To see this, let S be an infinite zero semigroup. That is, there is an element $0 \in S$ such that $st = 0$ for all s and t in S . Then $pq = 0$ for all p and q in βS and so $\text{cl}K(\beta S) = K(\beta S) = \{0\}$. Let $R = S$. Given $x \in T$, if $a = x(0)$, then for all $p \in \beta S$, $T_p(x)$ is constantly equal to a and so $T_p(x)$ is uniformly recurrent. That is, for any $x \in Z$, $U(x) = \beta S$.

In [1] it was shown that $\text{cl}K(\beta \mathbb{N})$ is the intersection of all of the closed two sided ideals that strictly contain it. In a similar vein, we would like to show that each $U(x)$ properly contains $K(\beta S)$. One cannot hope for this to hold in general. For example, as we have already noted, if S is either left zero or right zero then so is βS and then $K(\beta S) = \beta S$. Results establishing that $U(x)$ properly contains $K(\beta S)$ require some weak cancellation assumptions.

Definition 3.12. Let S be a semigroup and let $A \subseteq S$.

- (a) A is a *left solution set* if and only if there exist u and v in S such that $A = \{x \in S : ux = v\}$.
- (b) A is a *right solution set* if and only if there exist u and v in S such that $A = \{x \in S : xv = u\}$.

As is standard, we denote by ω the first infinite ordinal, which is also the first infinite cardinal. That is, $\omega = \aleph_0$.

Definition 3.13. Let S be a semigroup with $|S| = \kappa \geq \omega$.

- (a) S is *weakly left cancellative* if and only if every left solution set in S is finite.
- (b) S is *weakly right cancellative* if and only if every right solution set in S is finite.
- (c) S is *weakly cancellative* if and only if S is both weakly left cancellative and weakly right cancellative.
- (d) S is *very weakly left cancellative* if and only if the union of any set of fewer than κ left solution sets has cardinality less than κ .
- (e) S is *very weakly right cancellative* if and only if the union of any set of fewer than κ right solution sets has cardinality less than κ .
- (f) S is *very weakly cancellative* if and only if S is both very weakly left cancellative and very weakly right cancellative.

Given a set X and a cardinal κ , we let $U_\kappa(X)$ be the set of κ -uniform ultrafilters on X . That is, $U_\kappa(X) = \{p \in \beta X : (\forall A \in p)(|A| \geq \kappa)\}$.

Theorem 3.14. *Assume that $|R| = |S| = \kappa \geq \omega$, S is very weakly cancellative, and has the property that $|\{e \in S : (\exists s \in S)(es = s)\}| < \kappa$. Then for all $x \in Z$, $U(x) \cap U_\kappa(S) \setminus \text{cl}K(\beta S) \neq \emptyset$.*

Proof. Let $E = \{e \in S : (\exists s \in S)(es = s)\}$. Let $x \in Z$ and pick $q \in K(\beta S)$. Let $y = T_q(x)$. By Corollary 2.2(3), y is uniformly recurrent. For each $F \in \mathcal{P}_f(R)$, let $B_F = \{s \in S : (\forall t \in F)(x(ts) = y(t))\}$. Since

$$B_F = \{s \in S : T_s(x) \in \bigcap_{t \in F} \pi_t^{-1}[\{y(t)\}]\},$$

we have $B_F \in q$. By [4, Lemma 6.34.3], $K(\beta S) \subseteq U_\kappa(S)$ and so $|B_F| = \kappa$. Note that if $F \subseteq H$, then $B_H \subseteq B_F$.

Enumerate $\mathcal{P}_f(R)$ as $\langle F_\alpha \rangle_{\alpha < \kappa}$. Choose $t_0 \in B_{F_0} \setminus E$. Let $0 < \alpha < \kappa$ and assume that we have chosen $\langle t_\delta \rangle_{\delta < \alpha}$ satisfying the following inductive hypotheses.

- (1) For each $\delta < \alpha$, $t_\delta \in B_{F_\delta}$.
- (2) For each $\delta < \alpha$, $FP(\langle t_\beta \rangle_{\beta \leq \delta}) \cap E = \emptyset$.
- (3) For each $\delta < \alpha$, if $\delta > 0$, then $t_\delta \notin FP(\langle t_\beta \rangle_{\beta < \delta})$.
- (4) For each $\delta < \alpha$, if $\delta > 0$, $s \in FP(\langle t_\beta \rangle_{\beta < \delta})$, and $\gamma < \delta$, then $st_\delta \neq t_\gamma$.

The hypotheses are satisfied for $\delta = 0$. Let

$$\begin{aligned} M_0 &= \{t \in S : (\exists H \in \mathcal{P}_f(\alpha))((\prod_{\beta \in H} t_\beta)t \in E)\} \text{ and let} \\ M_1 &= \{t \in S : (\exists s \in FP(\langle t_\beta \rangle_{\beta < \alpha}))(\exists \gamma < \alpha)(st = t_\gamma)\}. \end{aligned}$$

Note that $|FP(\langle t_\beta \rangle_{\beta < \alpha})| \leq |\mathcal{P}_f(\alpha)| < \kappa$. Also, given $H \in \mathcal{P}_f(\alpha)$ and $s \in E$, $\{t \in S : (\prod_{\beta \in H} t_\beta)t = s\}$ is a left solution set so $|M_0| < \kappa$. Note also that, given $s \in FP(\langle t_\beta \rangle_{\beta < \alpha})$ and $\gamma < \alpha$, $\{t \in S : st = t_\gamma\}$ is a left solution set so $|M_1| < \kappa$. Thus we may choose $t_\alpha \in B_{F_\alpha} \setminus (E \cup FP(\langle t_\beta \rangle_{\beta < \alpha}) \cup M_0 \cup M_1)$. The induction hypotheses are satisfied for α .

Let $B = \{t_\alpha : \alpha < \kappa\}$ and let $C = \bigcap_{\alpha < \kappa} \text{cl}FP(\langle t_\beta \rangle_{\alpha < \beta < \kappa})$. By [4, Theorem 4.20], C is a compact subsemigroup of βS . We claim that $\overline{B} \cap K(C) = \emptyset$. Suppose instead that we have $p \in \overline{B} \cap K(C)$. Pick $r \in K(C)$ such that $p = pr$. (By [4, Lemma 1.30], an idempotent in the minimal left ideal L of C in which p lies will do.) Let $D = \{s \in S : s^{-1}B \in r\}$. Then $D \in p$ so $D \cap B \neq \emptyset$ so pick $\alpha < \kappa$ such that $t_\alpha^{-1}B \in r$. Then $t_\alpha^{-1}B \cap FP(\langle t_\beta \rangle_{\alpha < \beta < \kappa}) \neq \emptyset$ so pick finite $H \subseteq \{\beta : \alpha < \beta < \kappa\}$ such that $\prod_{\beta \in H} t_\beta \in t_\alpha^{-1}B$. Pick $\gamma < \kappa$ such that $t_\alpha \prod_{\beta \in H} t_\beta = t_\gamma$. Let $\max H = \mu$ and let $K = H \setminus \{\mu\}$. If $K = \emptyset$, then $t_\alpha t_\mu = t_\gamma$. If $K \neq \emptyset$, then $t_\alpha (\prod_{\beta \in K} t_\beta) t_\mu = t_\gamma$. If $\gamma > \mu$ we get a contradiction to hypothesis (3). If $\mu = \gamma$ we either get $t_\alpha \in E$ or $t_\alpha \prod_{\beta \in K} t_\beta \in E$, contradicting hypothesis (2). If $\gamma < \mu$ we get a contradiction to hypothesis (4). Thus $\overline{B} \cap K(C) = \emptyset$ as claimed.

Now we claim that $\overline{B} \cap K(\beta S) = \emptyset$. Suppose instead we have $p \in \overline{B} \cap K(\beta S)$. By [4, Lemma 6.34.3] we have that $p \in U_\kappa(S)$ and consequently, $p \in C$. Thus $K(\beta S) \cap C \neq \emptyset$ and so, by [4, Theorem 1.65], $K(C) = K(\beta S) \cap C$, contradicting the fact that $\overline{B} \cap K(C) = \emptyset$. Since \overline{B} is clopen, we thus have $\overline{B} \cap \text{cl}K(\beta S) = \emptyset$.

Now let $\mathcal{C} = \{B_F : F \in \mathcal{P}_f(S)\} \cup \{B\}$. We claim that \mathcal{C} has the κ -uniform finite intersection property. To see this, let $\mathcal{F} \in \mathcal{P}_f(\mathcal{P}_f(S))$ and let $H = \bigcup \mathcal{F}$. If $\delta < \kappa$ and $H \subseteq F_\delta$, then $t_\delta \in B \cap \bigcap_{F \in \mathcal{F}} B_F$. Since $|\{\delta < \kappa : H \subseteq F_\delta\}| = |\{F \in \mathcal{P}_f(S) : H \subseteq F\}| = \kappa$, we have that $|\bigcap \mathcal{C}| = \kappa$ as required. Pick by [4, Corollary 3.14] $p \in U_\kappa(S)$ such that $\mathcal{C} \subseteq p$.

Since $B_F \in p$ for each $F \in \mathcal{P}_f(R)$, we have $T_p(x) = y$ so $p \in U(x)$. Since $B \in p$, $p \notin \text{cl}K(\beta S)$. \square

Corollary 3.15. *Assume that $|R| = |S| = \kappa \geq \omega$ and that S is right cancellative and very weakly left cancellative. Then for all $x \in Z$, $U(x) \cap U_\kappa(S) \setminus \text{cl}K(\beta S) \neq \emptyset$.*

Proof. Let $E = \{e \in S : (\exists s \in S)(es = s)\}$. It suffices to show that $|E| < \kappa$. Pick $x \in S$. Given $e \in E$ and $s \in S$ such that $es = s$, we have that $res = xs$ so $xe = x$. Thus E is contained in the left solution set $\{t \in S : xt = x\}$. \square

Corollary 3.16. *Assume that $|R| = |S| = \kappa \geq \omega$, that S is very weakly cancellative, that S has a member e such that $es = s$ for all $s \in S$, and*

$|\{e \in S : (\exists s \in S)(es = s)\}| < \kappa$. Then $K(\beta S) = \bigcap_{x \in Z} U(x)$ and for each $x \in Z$, $U(x)$ properly contains $K(\beta S)$.

Proof. This is an immediate consequence of Theorems 3.4 and 3.14. \square

Corollary 3.17. *Assume that S is a left ideal of R , $|R| = |S| = \kappa \geq \omega$, S is very weakly cancellative, S has a member e such that $es = s$ for all $s \in S$, and $|\{e \in S : (\exists s \in S)(es = s)\}| < \kappa$. Then $clK(\beta S) = \bigcap_{x \in \Omega} U(x)$ and for each $x \in \Omega$, $U(x)$ properly contains $clK(\beta S)$.*

Proof. By Corollary 3.10 $clK(\beta S) = \bigcap_{x \in \Omega} U(x)$. By Theorem 3.14, for each $x \in \Omega$, $U(x)$ properly contains $clK(\beta S)$. \square

4 Relations between systems with phase spaces X and Y

Throughout this section we will let S be an arbitrary semigroup and let $Q = S \cup \{e\}$, where e is an identity adjoined to S , even if S already has an identity. We will let $(X, \langle T_{X,s} \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = Q$ let $(Y, \langle T_{Y,s} \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. For $x \in X$ we will let $U_X(x) = \{p \in \beta S : T_{X,p}(x) \text{ is uniformly recurrent}\}$ and let $U_Y(x) = \{p \in \beta S : T_{Y,p}(x) \text{ is uniformly recurrent}\}$.

We have from the results of the previous section that for any semigroup S , $K(\beta S) = \bigcap_{x \in X} U_X(x)$ and $clK(\beta S) = \bigcap_{x \in \Omega_X} U_X(x)$. We are interested in determining when the corresponding assertions hold with respect to Y . Of course, the simplest situation in which they do is when for each $x \in X$, $U_X(x) = U_Y(x|_S)$ so we address this problem first, beginning with the following simple observation.

Lemma 4.1. *Let $x \in X$. Then $U_X(x) \subseteq U_Y(x|_S)$.*

Proof. Let $y = x|_S$ and note that \tilde{y} is the restriction of \tilde{x} to βS . Let L be a minimal left ideal of βS . By Lemma 3.2, $p \in U_X(x)$ if and only if there exists $q \in L$, such that $\tilde{x}(tp) = \tilde{x}(tqp)$ for all $t \in Q$. And $p \in U_Y(x|_S)$ if and only if there exists $q \in L$ such that $\tilde{y}(tp) = \tilde{y}(tqp)$ for all $t \in S$. \square

Theorem 4.2. *The following statements are equivalent.*

- (a) *For all $x \in X$, $U_X(x) = U_Y(x|_S)$.*
- (b) *There do not exist $p \in \beta S$ and $x \in X$ such that $T_{X,p}(x)$ is the characteristic function of $\{e\}$ in X .*
- (c) *For every $p \in \beta S$, $p \in \beta Sp$.*

Proof. Assume that (a) holds and suppose we have $p \in \beta S$ and $x \in X$ such that $T_{X,p}(x)$ is the characteristic function of $\{e\}$ in X . Then $T_{Y,p}(x|_S)$ is constantly 0 so $p \in U_Y(x|_S)$. But $V = \{u \in X : w(e) = 1\}$ is a neighborhood of $w = T_{X,p}(x)$ in X , while $\{s \in S : T_{X,s}(w) \in V\} = \emptyset$, so $p \notin U_X(x)$.

To see that (b) implies (c), assume that (b) holds and suppose that we have some $p \in \beta S$ such that $p \notin \beta Sp$. Since $\beta Sp = \rho_p[\beta S]$, βSp is closed. Pick $A \in p$ such that $\bar{A} \cap \beta Sp = \emptyset$. Let x be the characteristic function of A in X . First let $s \in S$. Then $sp \notin \bar{A}$ so $s^{-1}(S \setminus A) \in p$ so to see that $T_{X,p}(s) = 0$, it suffices to observe that $s^{-1}(S \setminus A) \subseteq \{t \in S : T_{X,t}(x)(s) = 0\}$. Since $A \in p$ and for $t \in A$, $T_{X,t}(x)(e) = x(t) = 1$, we have that $T_{X,p}(x)(e) = 1$.

By Lemma 4.1, we have $U_X(x) \subseteq U_Y(x|_S)$ for all $x \in X$, so to show that (c) implies (a), it suffices to let $x \in X$, let $p \in U_Y(x|_S)$, assume that $p \in \beta Sp$, and show that $p \in U_X(x)$. By Lemma 3.3, it suffices to let L be a minimal left ideal of βS and let $F \in \mathcal{P}_f(Q)$ and show that there is some $q \in L$ such that $\tilde{x}(tp) = \tilde{x}(tqp)$ for every $t \in F$. For $t \in F$, let $B_t = \{s \in S : x(ts) = \tilde{x}(tp)\}$. Then $\bigcap_{t \in F} B_t \in p$ and $p \in \beta Sp = \text{cl}(Sp)$ so pick $v \in S$ such that $\bigcap_{t \in F} B_t \in vp$. Let $y = x|_S$. Now $Fv \in \mathcal{P}_f(S)$ so pick by Lemma 3.3 $q \in L$ such that for all $t \in F$, $\tilde{y}(tvp) = \tilde{y}(tvqp)$. Let $q' = vq$ and note that $q' \in L$. Let $t \in F$ be given. Then $B_t \in vp$ so $\tilde{x}(tvp) = \tilde{x}(tp)$ and thus $\tilde{x}(tp) = \tilde{y}(tvp) = \tilde{y}(tvqp) = \tilde{x}(tq'p)$. \square

Corollary 4.3. *If for all $p \in \beta S$, $p \in \beta Sp$, then $K(\beta S) = \bigcap_{x \in Y} U_Y(x)$ and $\text{cl}K(\beta S) = \bigcap_{x \in \Omega_Y} U_Y(x)$.*

Proof. The first assertion is an immediate consequence of Theorems 3.4 and 4.2. The second assertion follows from Corollary 3.10 and Theorem 4.2. \square

We have already mentioned the problem of determining whether $K(\beta S)$ or $\text{cl}K(\beta S)$ is prime. Recall that an ideal I in a semigroup is *semiprime* if and only if whenever $ss \in I$, one must have $s \in I$.

Corollary 4.4. (1) *If $K(\beta S) \neq \bigcap_{x \in Y} U_Y(x)$, then $K(\beta S)$ is not semiprime.*

(2) *If $\text{cl}K(\beta S) \neq \bigcap_{x \in \Omega_Y} U_Y(x)$, then $\text{cl}K(\beta S)$ is not semiprime.*

Proof. (1) If $p \in \bigcap_{x \in Y} U_Y(x) \setminus K(\beta S)$, then $pp \in \beta Sp$ and by Theorem 3.4, $\beta Sp \subseteq K(\beta S)$.

(2) If $p \in \bigcap_{x \in \Omega_Y} U_Y(x) \setminus \text{cl}K(\beta S)$, then $pp \in \beta Sp$ and by Lemma 3.8, $\beta Sp \subseteq \text{cl}K(\beta S)$. \square

By virtue of Theorem 4.2 we are interested in knowing when there is some $p \in \beta S$ such that $p \notin \beta Sp$.

Lemma 4.5. *Let $p \in \beta S$. Then $p \notin \beta Sp$ if and only if there exists $A \subseteq S$ such that for all $x \in S$, $x^{-1}A \in p$ and $A \notin p$.*

Proof. Let $C(p) = \{A \subseteq S : (\forall x \in S)(x^{-1}A \in p)\}$. By [4, Theorem 6.18], $p \in \beta Sp$ if and only if $C(p) \subseteq p$. \square

Theorem 4.6. *Assume that $|S| = \kappa \geq \omega$. There exists $p \in \beta S$ such that $p \notin \beta Sp$ if and only if there exists $\langle y_F \rangle_{F \in \mathcal{P}_f(S)}$ in S such that $\{y_F : F \in \mathcal{P}_f(S)\} \cap \bigcup \{Fy_F : F \in \mathcal{P}_f(S)\} = \emptyset$.*

Proof. Necessity. Pick $p \in \beta S$ such that $p \notin \beta Sp$. By Lemma 4.5, pick $A \subseteq S$ such that for all $x \in S$, $x^{-1}A \in p$ and $A \notin p$. For $F \in \mathcal{P}_f(S)$ pick $y_F \in (S \setminus A) \cap \bigcap_{x \in F} x^{-1}A$.

Sufficiency. Let $A = \bigcup \{Fy_F : F \in \mathcal{P}_f(S)\}$. Then $\{S \setminus A\} \cup \{x^{-1}A : x \in S\}$ has the finite intersection property so pick $p \in \beta S$ such that $\{S \setminus A\} \cup \{x^{-1}A : x \in S\} \subseteq p$. By Lemma 4.5, $p \notin \beta Sp$. \square

One of the assumptions in the following corollary is that $S^* = \beta S \setminus S$ is a right ideal of βS . A (not very simple) characterization of when S^* is a right ideal of βS is given in [4, Theorem 4.32]. By [4, Corollary 4.33 and Theorem 4.36] it is sufficient that S be either right cancellative or weakly cancellative.

Corollary 4.7. *Assume that $|S| = \kappa \geq \omega$ and assume that*

$$|S \setminus \{t \in S : (\exists s \in S)(st = t)\}| = \kappa.$$

If either S^ is a right ideal of βS or S is very weakly left cancellative, then there exists p in βS such that $p \notin \beta Sp$.*

Proof. Assume first that S^* is a right ideal of βS , and pick $t \in S$ such that there is no $s \in S$ with $st = t$. Then $t \notin St$ and $t \notin S^*t$.

Now assume that S is very weakly left cancellative. Enumerate $\mathcal{P}_f(S)$ as $\langle F_\alpha \rangle_{\alpha < \kappa}$. By Theorem 4.6, it suffices to produce $\langle t_\alpha \rangle_{\alpha < \kappa}$ in S such that $\{t_\alpha : \alpha < \kappa\} \cap \bigcup \{F_\alpha t_\alpha : \alpha < \kappa\} = \emptyset$.

Let $E = \{t \in S : (\exists s \in S)(st = t)\}$. Pick $t_0 \in S \setminus E$. Let $0 < \alpha < \kappa$ and assume we have chosen $\langle t_\delta \rangle_{\delta < \alpha}$ in $S \setminus E$ such that if $\delta > 0$, then $t_\delta \notin \bigcup_{\mu < \delta} F_\mu t_\mu$ and for each $x \in F_\delta$, $xt_\delta \notin \{t_\mu : \mu < \delta\}$.

For $x \in S$ and $\mu < \alpha$, let $H_{x,\mu} = \{t \in S : xt = t_\mu\}$. Then each $H_{x,\mu}$ is a left solution set so $|\bigcup \{H_{x,\mu} : x \in F_\alpha \text{ and } \mu < \alpha\}| < \kappa$. Pick

$$t_\alpha \in S \setminus (E \cup \bigcup \{H_{x,\mu} : x \in F_\alpha \text{ and } \mu < \alpha\} \cup \bigcup_{\mu < \alpha} F_\mu t_\mu).$$

Suppose we have some $\mu < \kappa$ such that $t_\mu \in \bigcup \{F_\alpha t_\alpha : \alpha < \kappa\}$ and pick $\alpha < \kappa$ and $x \in F_\alpha$ such that $t_\mu = xt_\alpha$. Then $\alpha \neq \mu$ because $t_\alpha \notin E$. If $\alpha < \mu$, we would have $t_\mu \in F_\alpha t_\alpha$. So we must have $\mu < \alpha$. But then $t_\alpha \in H_{x,\mu}$, a contradiction. \square

We conclude this section by exhibiting a sufficient condition which guarantees $K(\beta S) = \bigcap_{x \in Y} U_Y(x)$. We shall see that this does not require equality between $U_X(x)$ and $U_Y(x|_S)$ for all $x \in X$.

Theorem 4.8. *Assume that for all $p \in \bigcap_{x \in Y} U_Y(x)$ and all $A \in p$ the assumption that $\{t \in S : t^{-1}sA \in p\}$ is syndetic for every $s \in S$, implies that $\{t \in S : t^{-1}A \in p\} \neq \emptyset$. Then $K(\beta S) = \bigcap_{x \in Y} U_Y(x)$.*

Proof. Assume that $p \in \bigcap_{x \in Y} U_Y(x) \setminus K(\beta S)$. By Theorem 3.4(2), $\beta Sp \subseteq K(\beta S)$ so $p \notin \beta Sp$. Pick $A \in p$ such that $\overline{A} \cap \beta Sp = \emptyset$. Thus $\{t \in S : t^{-1}A \in p\} = \emptyset$. We claim that for all $s \in S$, $\{t \in S : t^{-1}sA \in p\}$ is syndetic. So let $s \in S$. By [4, Theorem 4.48] it suffices to let L be a minimal left ideal of βS and show that there is some $q \in L$ such that $\{t \in S : t^{-1}sA \in p\} \in q$. By Theorem 3.4(2), $sp \in Lp$ so pick $q \in L$ such that $sp = qp$. Then $sA \in qp$ so $\{t \in S : t^{-1}sA \in p\} \in q$ as required. \square

Note that by Theorem 3.4(3), $K(\beta\mathbb{N}, +) = \bigcap_{x \in Y} U_Y(x)$ while $1 \notin \beta\mathbb{N} + 1$ so by Theorem 4.2, it is not the case that for all $x \in X$, $U_X(x) = U_Y(x|_S)$. On the other hand, given $p \in K(\beta\mathbb{N}, +)$ one has $p = q + p$ for some $p \in K(\beta\mathbb{N}, +)$ so automatically for any $A \in p$, $\{t \in \mathbb{N} : -t + A \in p\} \neq \emptyset$ so the hypotheses of Theorem 4.8 are valid.

5 Recurrence and surjectivity of T_p

So far in this paper we have been considering the notion of uniform recurrence. We now introduce a notion which is usually weaker.

Definition 5.1. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. The point $x \in X$ is *recurrent* if and only if for each neighborhood V of x in X , $\{s \in S : T_s(x) \in V\}$ is infinite.

If all syndetic subsets of a semigroup S are infinite, then any uniformly recurrent point of X is recurrent. This is not always the case. For example, if S is a left zero semigroup and $x \in S$, then x is uniformly recurrent in the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$ but is not recurrent. (We have that $\{x\}$ is a neighborhood of x and $\{s \in S : \lambda_s(x) \in \{x\}\} = \{x\}$, which is syndetic, but finite.)

The following characterization of recurrence is very similar to the characterization of uniform recurrence in [4, Theorem 19.23]. Part of the results depend on the assumption that S^* is a subsemigroup of βS . There is a characterization of S^* as a subsemigroup in [4, Theorem 4.28]. By [4, Corollary 4.29 and Theorem 4.31] it is sufficient that S be right cancellative or weakly left cancellative.

Theorem 5.2. *Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. Statements (a) and (b) are equivalent and imply statements (c) and (d), which are equivalent. If S^* is a subsemigroup of βS , then all four statements are equivalent.*

- (a) There exists an idempotent $p \in S^*$ such that $T_p(x) = x$.
- (b) There exist $y \in X$ and an idempotent $p \in S^*$ such that $T_p(y) = x$.
- (c) There exists $p \in S^*$ such that $T_p(x) = x$.
- (d) x is recurrent.

Proof. Trivially (a) implies (b) and (a) implies (c). To see that (b) implies (a), pick $y \in X$ and an idempotent $p \in S^*$ such that $T_p(y) = x$. Then $x = T_p(y) = T_{pp}(y) = T_p(T_p(y)) = T_p(x)$.

To see that (c) implies (d), pick $p \in S^*$ such that $T_p(x) = x$. Let V be a neighborhood of x . Then $\{s \in S : T_s(x) \in V\} \in p$ so $\{s \in S : T_s(x) \in V\}$ is infinite.

To see that (d) implies (c), assume that x is recurrent and for each neighborhood V of x , let $D_V = \{s \in S : T_s(x) \in V\}$. Then any finite subfamily of $\{D_V : V \text{ is a neighborhood of } x\}$ has infinite intersection so pick by [4, Corollary 3.14] some $p \in S^*$ such that $\{D_V : V \text{ is a neighborhood of } x\} \subseteq p$. Then $T_p(x) = x$.

Now assume that S^* is a semigroup. To see that (c) implies (a), pick $p \in S^*$ such that $T_p(x) = x$ and let $E = \{q \in S^* : T_q(x) = x\}$. Since S^* is a subsemigroup of βS , we have that E is a subsemigroup of βS . We claim that E is closed. To see this, let $q \in \beta S \setminus E$. If $q \in S$, then $\{q\}$ is a neighborhood of q missing E , so assume that $q \in S^*$. Pick an open neighborhood U of $T_q(x)$ such that $x \notin \text{cl}U$ and let $A = \{s \in S : T_s(x) \in U\}$. Then \bar{A} is a neighborhood of q which misses E . Since E is a compact right topological semigroup, there is an idempotent in E . \square

Recall that in any dynamical system, $(X, \langle T_s \rangle_{s \in S})$, $K(\beta S) \subseteq \bigcap_{x \in X} U_X(x)$ and we have obtained sufficient conditions for equality.

Theorem 5.3. *Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system, let $p \in \beta S$, and assume that $T_p : X \rightarrow X$ is surjective and $K(\beta S) = \bigcap_{x \in X} U_X(x)$. Then for any $q \in \beta S$, $qp \in K(\beta S)$ if and only if $q \in K(\beta S)$.*

Proof. Let $q \in \beta S$. The sufficiency is trivial, so assume that $qp \in K(\beta S)$. It suffices to show that $q \in \bigcap_{x \in X} U(x)$, so let $x \in X$ be given. Pick $y \in X$ such that $T_p(y) = x$. Then $T_q(x) = T_q(T_p(y)) = T_{qp}(y)$. Since $qp \in U(y)$ we have $T_{qp}(y)$ is uniformly recurrent, and so $T_q(x) \in U(x)$ as required. \square

Definition 5.4. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. Then $NS = NS_X = \{p \in \beta S : T_p \text{ is not surjective}\}$.

We have seen that $U(x)$ is always a left ideal of βS .

Lemma 5.5. *Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. If $NS \neq \emptyset$, then NS is a right ideal of βS .*

Proof. Given $p \in NS$ and $q \in \beta S$, the range of T_{pq} is contained in the range of T_p . \square

Lemma 5.6. *Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. If there is some $x \in X$ such that x is not recurrent, then $\{p \in S^* : pp = p\} \subseteq NS$.*

Proof. Pick $x \in X$ such that x is not recurrent and let p be an idempotent in S^* . We claim that x is not in the range of T_p , so suppose instead we have $y \in X$ such that $T_p(y) = x$. Then by Theorem 5.2, x is recurrent. \square

We shall establish a strong connection between the surjectivity of T_p and p being right cancelable in βS . The purely algebraic result in Theorem 5.8 will be useful.

Lemma 5.7. *Let S be a countable right cancellative and weakly left cancellative semigroup and let B be an infinite subset of S . There is an infinite subset D of B with the property that whenever s and t are distinct members of S , there is a finite subset F of D such that $sa \neq tb$ whenever $a, b \in D \setminus F$.*

Proof. Let $\Delta = \{(s, s) : s \in S\}$ and enumerate $(S \times S) \setminus \Delta$ as $\langle (s_n, t_n) \rangle_{n=1}^\infty$. Pick $a_1 \in B$. Assume $n \in \mathbb{N}$ and we have chosen $\langle a_i \rangle_{i=1}^n$. Let $W_n = \{b \in S : \text{there exist } i, j \in \{1, 2, \dots, n\} \text{ such that } s_i a_j = t_i b \text{ or } s_i b = t_i a_j\}$. Then W_n is the union of finitely many left solution sets, so is finite. Pick $a_{n+1} \in B \setminus (W_n \cup \{a_1, a_2, \dots, a_n\})$.

Let $D = \{a_n : n \in \mathbb{N}\}$. Let s and t be distinct members of S and pick n such that $(s, t) = (s_n, t_n)$. Let $F = \{a_i : i \in \{1, 2, \dots, n\}\}$. To see that F is as required, let $a, b \in D \setminus F$ and suppose $sa = tb$. Then by right cancellation, $a \neq b$. Pick $m > n$ and $r > n$ such that $a = a_m$ and $b = a_r$. If $m < r$, then $a_r \in W_{r-1}$. If $r < m$, then $a_m \in W_{m-1}$. \square

Theorem 5.8. *Let S be a countable cancellative semigroup. If $p \in \beta S \setminus K(\beta S)$, then there exists an infinite $D \subseteq S$ such that for every $r \in D^*$, rp is right cancelable in βS .*

Proof. Choose $q \in K(\beta S)$. We first claim that for each $s \in S$, $sp \notin K(\beta S)$ and in particular, $sp \notin \beta Sqp$. So suppose we have $sp \in K(\beta S)$. Then sp is in a minimal left ideal L of βS . Pick an idempotent $r \in L$. By [4, Lemma 1.30], $sp = spr$. By [4, Lemma 8.1] s is left cancelable in βS so $p = pr$, and thus $p \in K(\beta S)$. This contradiction establishes the claim. For each $s \in S$, pick $U_s \in sp$ such that $\overline{U_s} \cap \beta Sqp = \emptyset$. For each $s, t \in S$, there exists $V_{s,t} \in q$ such that $\overline{U_s} \cap t\overline{V_{s,t}p} = \emptyset$ because $\lambda_t \circ \rho_p(q) \in \beta S \setminus \overline{U_s}$.

By [4, Theorem 3.36], there exists an infinite subset B of S such that $B^* \subseteq \bigcap_{s,t \in S} \overline{V_{s,t}}$. Then for every $r \in B^*$ and every $s, t \in S$, $trp \notin \overline{U_s}$.

By Lemma 5.7 pick an infinite subset D of B such that, whenever s and t are distinct elements of S , there is a finite subset F of D such that $sa \neq tb$

whenever $a, b \in D \setminus F$. Enumerate D as $\langle d_n \rangle_{n=1}^\infty$ and for each distinct s and t in S , pick $n_{s,t} \in \mathbb{N}$ such that $sd_m \neq td_n$ whenever $m, n > n_{s,t}$.

We claim that, for every $r \in D^*$, rp is right cancelable in βS . We shall apply [4, Theorem 3.40] three times.

Assume that $q_1rp = q_2rp$, where q_1 and q_2 are distinct elements of βS . Let A_1 and A_2 be disjoint subsets of S which are members of q_1 and q_2 respectively. Since $q_1rp \in cl(A_1rp)$ and $q_2rp \in cl(A_2rp)$, an application of [4, Theorem 3.40] shows that either $A_1rp \cap cl(A_2rp) \neq \emptyset$ or $A_2rp \cap cl(A_1rp) \neq \emptyset$, and without loss of generality, we may assume that the former holds. Thus we have some $s \in A_1$ and $q' \in \overline{A_2}$ such that $srp = q'rp$. Now $srp \in cl(sDp)$ and $q'rp \in cl((S \setminus \{s\})rp)$, so either $sDp \cap cl((S \setminus \{s\})rp) \neq \emptyset$ or $(S \setminus \{s\})rp \cap cl(sDp) \neq \emptyset$. We thus have either

- (i) $sDp \cap cl((S \setminus \{s\})rp) \neq \emptyset$, in which case we choose $d \in D$ and $y \in \beta S$ such that $sdp = yrp$; or
- (ii) $sDp \cap cl((S \setminus \{s\})rp) = \emptyset$, in which case we pick $t \in S \setminus \{s\}$ and $r' \in \overline{D}$ such that $sr'p = trp$. Since $sDp \cap cl((S \setminus \{s\})rp) = \emptyset$, we have $r' \in D^*$.

Suppose that (i) holds. Then $U_{sd} \in sdp$ so $\{v \in S : v^{-1}U_{sd} \in rp\} \in y$, so pick $v \in S$ such that $U_{sd} \in vrp$. But $r \in V_{sd,v}$, so this is a contradiction. Thus (ii) holds.

Now $sr'p \in cl\{sd_m p : m > n_{s,t}\}$ and $trp \in cl\{td_m p : m > n_{s,t}\}$ so, essentially without loss of generality, we have $\{sd_m p : m > n_{s,t}\} \cap cl\{td_m p : m > n_{s,t}\} \neq \emptyset$. (We have distinguished between s and t at this stage, but the arguments below with s and t interchanged remain valid.) Thus either

- (iii) there exist $m, n > n_{s,t}$ such that $sd_m p = td_n p$; or
- (iv) there exist $m > n_{s,t}$ and $r'' \in D^*$ such that $sd_m p = tr''p$.

If (iii) holds, then by [4, Lemma 6.28], $sd_m = td_n$, contradicting the choice of $n_{s,t}$. So (iv) holds. But $r'' \in V_{sd_m,t}$ so $tr''p \notin U_{sd_m}$, a contradiction. \square

We now present several results about the dynamical systems considered in Section 3.

Lemma 5.9. *Let S be a semigroup and let p be a right cancelable element of βS . Then for any clopen subset E of βSp , there is some $A \subseteq S$ such that $E = \overline{A} \cap \beta Sp$.*

Proof. Let E be a clopen subset of βSp . Let $\mathcal{D} = \{\overline{D} \cap \beta Sp : D \subseteq S \text{ and } \overline{D} \cap \beta Sp \subseteq E\}$. Since $\{\overline{D} \cap \beta Sp : D \subseteq S\}$ is a basis for the topology of βSp and E is open in βSp , we have that $E = \bigcup \mathcal{D}$. Since E is compact, pick finite $\mathcal{F} \subseteq \mathcal{P}(S)$ such that $E = \bigcup_{D \in \mathcal{F}} (\overline{D} \cap \beta Sp)$ and let $A = \bigcup \mathcal{F}$. \square

Theorem 5.10. *Let S be a semigroup. Let $(Y, \langle T_s \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. Let $p \in \beta S$. If p is right cancelable in βS , then $T_p : Y \rightarrow Y$ is surjective.*

Proof. Note that since $\rho_p : \beta S \rightarrow \beta Sp$ is injective and takes closed sets to closed sets, it is a homeomorphism.

To see that T_p is surjective, let $z \in Y$, let $B = \{s \in S : z(s) = 1\}$, and let $E = \rho_p[\overline{B}]$. Then E is clopen in βSp so by Lemma 5.9 pick $A \subseteq S$ such that $E = \overline{A} \cap \beta Sp$. Let x be the characteristic function of A in Y . We claim that $T_p(x) = z$. For this, it suffices that for each $s \in S$, $\{t \in S : T_t(x)(s) = z(s)\} \in p$. So let $s \in S$. Note that $\{t \in S : T_t(x)(s) = 1\} = \{t \in S : x(st) = 1\} = s^{-1}A$. Also $s^{-1}A \in p$ if and only if $s \in \rho_p^{-1}[\overline{A} \cap \beta Sp]$ so $s \in B$ if and only if $s^{-1}A \in p$.

If $z(s) = 1$, then $s \in B$ so $s^{-1}A \in p$ so $\{t \in S : T_t(x)(s) = z(s)\} \in p$. If $z(s) = 0$, then $s \notin B$ so $s^{-1}A \notin p$ so $\{t \in S : T_t(x)(s) = z(s)\} \in p$. \square

Notice that the hypotheses of the following corollary hold if S has any right cancelable element.

Corollary 5.11. *Let S be a semigroup. Let $(Y, \langle T_s \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. Let $p \in \beta S$. Assume that for whenever q and r are distinct elements of βS , there exists $s \in S$ such that $sq \neq sr$. Then $T_p : Y \rightarrow Y$ is surjective if and only if p is right cancelable in βS .*

Proof. The necessity is Theorem 5.10.

So assume that T_p is surjective and suppose that we have distinct q and r in βS such that $qp = rp$. We claim that $T_q = T_r$. To see this, let $x \in Y$ be given. Pick $z \in Y$ such that $T_p(z) = x$. Then $T_q(x) = T_q(T_p(z)) = T_{qp}(z) = T_{rp}(z) = T_r(T_p(z)) = T_r(x)$.

Pick $s \in S$ such that $sq \neq sr$, pick $A \in sq \setminus sr$, and let x be the characteristic function of A in Y . Then $A \subseteq \{t \in S : T_t(x)(s) = 1\}$ so $T_q(x)(s) = 1$ and $S \setminus A \subseteq \{t \in S : T_t(x)(s) = 0\}$ so $T_r(x)(s) = 1$. \square

Theorem 5.12. *Let S be a semigroup and let $Q = S \cup \{e\}$ where e is an identity adjoined to S . Let $(X, \langle T_s \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = Q$ and let $p \in \beta S$. Then $T_p : X \rightarrow X$ is surjective if and only if p is right cancelable in βQ .*

Proof. Sufficiency. Note that $\rho_p : \beta S \rightarrow \beta Sp$ is a homeomorphism. Note also that $p \notin \beta Sp$. (If we had $p = qp$ for some $q \in \beta S$, then we would have $ep = qp$.) Let $x \in X$ and let $B = \{s \in S : x(s) = 1\}$. By Lemma 5.9, pick $A \subseteq S$ such that $\rho_p[\overline{B}] = \overline{A} \cap \beta Sp$. Pick $P \in p$ such that $\overline{P} \cap \beta Sp = \emptyset$. If $x(e) = 1$, let $D = A \setminus P$. If $x(e) = 0$, let $D = A \cup B$. Let z be the characteristic function of D in X .

We claim that $T_p(z) = x$. As in the proof of Theorem 5.10, we see that for $s \in S$, $T_p(z)(s) = x(s)$. Regardless of the value of $x(e)$, we have that $P \subseteq \{s \in S : T_s(z)(e) = x(e)\}$, so $T_p(z)(e) = x(e)$.

Necessity. Suppose that T_p is surjective and we have $q \neq r$ in βQ such that $qp = rp$. Assume first that $e \in \{q, r\}$, so without loss of generality, $q = e$. Let x be the characteristic function of S in X and pick $z \in X$ such that $T_p(z) = x$. Then $0 = x(e) = T_p(z)(e) = T_{rp}(z)(e) = T_r(T_p(z))(e) = T_r(x)(e) = 1$, a contradiction.

So we can assume that q and r are in βS . Pick $A \in q \setminus r$ and let A be the characteristic function of A in X . Pick $z \in X$ such that $T_p(z) = x$. Then $0 = T_r(x)(e) = T_{rp}(z)(e) = T_{qp}(z)(e) = T_q(T_p(z))(e) = T_q(x)(e) = 1$, a contradiction. \square

Theorem 5.13. *Let S be a countable semigroup which can be embedded in a group and assume that S can be enumerated as $\langle s_t \rangle_{t=0}^\infty$ so that if $u, v \in S$, $i, j \in \omega$ with $i < j$, and $s_i u = s_j v$, then $s_0 s_i^{-1} s_j \in S$. Let $(Y, \langle T_s \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$ and let $p \in \beta S$. The T_p is surjective if and only if there exists $x \in Y$ such that $T_p(x)$ is the characteristic function of $\{s_0\}$ in Y .*

Proof. The necessity is trivial. Assume that we have $x \in Y$ such that $T_p(x)$ is the characteristic function of $\{s_0\}$ in Y . For $m \in \mathbb{N}$, let $D_m = \{s_0 s_i^{-1} s_j : i, j \in \{0, 1, \dots, m\}, i < j, \text{ and } s_0 s_i^{-1} s_j \in S\}$ and note that $s_0 \notin D_m$. For each $m \in \mathbb{N}$, let

$$B_m = \{s \in S : T_s(x) \in \pi_{s_0}^{-1}[\{1\}] \cap \bigcap_{i=1}^m \pi_{s_i}^{-1}[\{0\}] \cap \bigcap_{r \in D_m} \pi_r^{-1}[\{0\}]\},$$

and note that $B_m \in p$. We claim that if $m, k \in \mathbb{N}$, $u \in B_m$, $v \in B_k$, $i \in \{0, 1, \dots, m\}$, $j \in \{0, 1, \dots, k\}$, and $s_i u = s_j v$, then $i = j$. Suppose instead we have such m, k, u, v, i, j with $i \neq j$ and assume without loss of generality that $i < j$. Then $u = s_i^{-1} s_j v$. By assumption $s_0 s_i^{-1} s_j \in S$ so $s_0 s_i^{-1} s_j \in D_k$. Since $u \in B_m$, $1 = T_u(x)(s_0) = x(s_0 u)$. Since $v \in B_k$ and $s_0 s_i^{-1} s_j \in D_k$, $0 = T_v(x)(s_0 s_i^{-1} s_j) = x(s_0 s_i^{-1} s_j v)$, a contradiction.

Now to see that T_p is surjective, let $y \in Y$ be given. Define $w \in Y$ as follows. If $m \in \mathbb{N}$, $u \in B_m$, and $i \in \{0, 1, \dots, m\}$, then $w(s_i u) = y(s_i)$. For $s \in S$ which is not of the form $s_i u$ for some $m \in \mathbb{N}$, $u \in B_m$, and $i \in \{0, 1, \dots, m\}$, define $w(s)$ at will. To see that $T_p(w) = y$, let U be a neighborhood of y . Pick $m \in \mathbb{N}$ such that $\bigcap_{i=0}^m \pi_i^{-1}[\{y(s_i)\}] \subseteq U$. Then $B_m \subseteq U$. \square

The following is an immediate corollary of Theorem 5.13.

Corollary 5.14. *Let S be a countable group with identity e , let $(Y, \langle T_s \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$, and let $p \in \beta S$. The following statements are equivalent.*

- (a) T_p is surjective.

(b) For each $s \in S$, there exists $x \in Y$ such that $T_p(x)$ is the characteristic function of $\{s\}$.

(c) There exists $x \in Y$ such that $T_p(x)$ is the characteristic function of $\{e\}$.

Notice that the hypotheses of the following theorem hold if S is very weakly left cancellative and right cancellative. If κ is regular, the assumption that for any subset D of S with fewer than κ members, $|\{e \in S : (\exists s \in D)(\exists t \in D \setminus \{s\})(se = te)\}| < \kappa$ can be replaced by the assumption that for all distinct s and t in S , $|\{e \in S : se = te\}| < \kappa$.

Theorem 5.15. *Let S be a semigroup with $|S| = \kappa \geq \omega$ which is very weakly left cancellative and has the property that for any subset D of S with fewer than κ members, $|\{e \in S : (\exists s \in D)(\exists t \in D \setminus \{s\})(se = te)\}| < \kappa$. Let $(Y, \langle T_s \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. There is a dense open subset W of $U_\kappa(S)$ such that for every $p \in W$, p is right cancelable in βS and $T_p : Y \rightarrow Y$ is surjective.*

Proof. We show that for any $C \in [S]^\kappa$, there exists $B \in [C]^\kappa$ such that for every $p \in \overline{B} \cap U_\kappa(S)$, p is right cancelable in βS and $T_p : Y \rightarrow Y$ is surjective.

Enumerate S as $\langle s_\gamma \rangle_{\gamma < \kappa}$. Choose $t_0 \in C$. Let $0 < \alpha < \kappa$ and assume that we have chosen $\langle t_\delta \rangle_{\delta < \alpha}$ in C satisfying the following inductive hypotheses:

- (1) If $\gamma < \delta$, then $t_\gamma \neq t_\delta$.
- (2) If $\gamma < \delta$, $\mu < \beta \leq \delta$, and $\mu \neq \gamma$, then $s_\gamma t_\delta \neq s_\mu t_\beta$.

The hypotheses are satisfied for $\delta = 0$. Let $E = \{e \in S : (\exists \mu < \beta \leq \alpha)(s_\mu e = s_\beta e)\}$. For $\mu < \beta < \alpha$ and $\gamma < \alpha$ let $A_{\gamma, \mu, \beta} = \{t \in S : s_\gamma t = s_\mu t_\beta\}$. Then each $A_{\gamma, \mu, \beta}$ is a left solution set. Pick

$$t_\alpha \in C \setminus (\{t_\gamma : \gamma < \alpha\} \cup E \cup \bigcup_{\gamma < \alpha} \bigcup_{\beta < \alpha} \bigcup_{\mu < \beta} A_{\gamma, \mu, \beta}).$$

Hypothesis (1) is trivially satisfied and if $\mu < \beta < \alpha$ and $\gamma < \alpha$, then $t_\alpha \notin A_{\gamma, \mu, \beta}$ so $s_\gamma t_\alpha \neq s_\mu t_\beta$. If $\mu < \beta = \alpha$ and $\gamma < \alpha$, then $t_\alpha \notin E$ so $s_\gamma t_\alpha \neq s_\mu t_\beta$.

Let $B = \{t_\alpha : \alpha < \kappa\}$ and let $p \in \overline{B} \cap U_\kappa(S)$. To see that p is right cancelable in βS , let $q \neq r \in \beta S$ and suppose that $qp = rp$. Pick subsets C and D of S such that $C \cap D = \emptyset$ and $C \in q$ and $D \in r$. Then $H = \{s_\gamma t_\alpha : \gamma < \alpha \text{ and } s_\gamma \in C\} \in qp$. (To see this, let $s_\gamma \in C$. Then $\{t_\alpha : \gamma < \alpha < \kappa\} \subseteq s_\gamma^{-1}H$.) Similarly, $\{s_\mu t_\beta : \mu < \beta \text{ and } s_\mu \in D\} \in rp$. Since these sets are disjoint by hypothesis (2), we have a contradiction.

The fact that T_p is surjective follows from Theorem 5.10. \square

Lemma 5.16. *Let S be a cancellative semigroup, let $a \in S$, and let $(Y, \langle T_s \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. If x is the characteristic function of $\{a\}$ in Y , then x is not a recurrent point.*

Proof. We claim that $|\{s \in S : T_s(x)(a) = 1\}| \leq 1$. Indeed, if $x(as) = 1$, then $as = a$ so by left cancellation, s is a left identity for S and then by right cancellation, s is a two sided identity for S . \square

We have seen that $U(x)$ is always a left ideal of βS and that NS is a right ideal of βS provided it is nonempty.

Theorem 5.17. *Let S be a countable cancellative semigroup. Let $(Y, \langle T_s \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. Then NS_Y is not a left ideal of βS .*

Proof. By [4, Corollary 6.33] pick an idempotent $p \in \beta S \setminus K(\beta S)$. By Theorem 5.8 pick $r \in \beta S$ such that rp is right cancelable in βS . By Lemma 5.16 and Theorem 5.6, $p \in NS$ and by Theorem 5.10, $rp \notin NS$. \square

If S is commutative, then by [4, Exercise 4.4.9] and Theorem 5.5, if $NS \neq \emptyset$, then $clNS$ is a two sided ideal of βS . The following theorem shows that this may fail if S is not commutative.

Theorem 5.18. *Let S be the free semigroup on the alphabet $\{a, b\}$ (where $a \neq b$). Let $(Y, \langle T_s \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. Then $NS \neq \emptyset$ and $clNS$ is not a left ideal of βS .*

Proof. Let p be an idempotent in βS with $\{a^n : n \in \mathbb{N}\} \in p$. By Lemma 5.16 and Theorem 5.6, $p \in NS$. We will show that $bp \notin clNS$. Let $B = \{ba^n : n \in \mathbb{N}\}$. Then $B \in bp$. We shall show that $\overline{B} \cap NS = \emptyset$. So let $q \in \overline{B}$. Let $s_0 = a$ and let $\langle s_n \rangle_{n=1}^\infty$ enumerate $S \setminus \{a\}$ so that if the length of s_i is less than the length of s_j , then $i < j$. By Theorem 5.13, to see that T_q is surjective, it suffices to show that there is some $x \in Y$ such that $T_q(x)$ is the characteristic function of $\{a\}$.

Let x be the characteristic function of $\{aba^n : n \in \mathbb{N}\}$ in Y . Let U be a neighborhood of $\chi_{\{a\}}$ and pick $F \in \mathcal{P}_f(S \setminus \{a\})$ such that $\pi_a^{-1}[\{1\}] \cap \bigcap_{y \in F} \pi_y^{-1}[\{0\}] \subseteq U$. It suffices to show that $B \subseteq \{w \in S : T_w(x) \in \pi_a^{-1}[\{1\}] \cap \bigcap_{y \in F} \pi_y^{-1}[\{0\}]\}$. So let $ba^n \in B$. Then $T_{ba^n}(x)(a) = x(aba^n) = 1$ and for $y \in F$, $T_{ba^n}(x)(y) = x(yba^n) = 0$. \square

We remark that Theorem 5.18 remains valid if S is the free semigroup on a countably infinite alphabet.

References

- [1] D. Davenport and N. Hindman, *Subprincipal closed ideals in $\beta\mathbb{N}$* , Semigroup Forum **36** (1987), 223-245.
- [2] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, 1981.

- [3] N. Hindman and D. Strauss, *Prime properties of the smallest ideal of $\beta\mathbb{N}$* , Semigroup Forum **52** (1996), 357-364.
- [4] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, second edition*, de Gruyter, Berlin, 2012.