LONG INCREASING CHAINS OF IDEMPOTENTS IN βG

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Abstract. We show that there is a sequence $\langle p_\alpha \rangle_{\alpha<\omega_1}$ of idempotents in $(\beta\mathbb{Z}, +)$ with the property that whenever $\alpha < \delta < \omega_1$, $p_\alpha <_R p_\delta$, where $p <_R q$ means that $p = q + p$ and $q \neq p + q$. More generally, if G is any countably infinite discrete group, p is an element of $\beta G \setminus G$ which is right cancelable in βG, and q is any minimal idempotent in the smallest compact subsemigroup of βG with p as a member, then there is a a sequence $\langle q_\alpha \rangle_{\alpha<\omega_1}$ of idempotents in βG which is $<_R$-increasing with $q_0 = q$.

1. Introduction

Given a discrete space X we take the Stone-Čech compactification βX of X to be the set of ultrafilters on X, identifying the principal ultrafilters with the points of X. A basis for the open sets of βX (as well as a basis for the closed sets) is $\{\overline{A} : A \subseteq X\}$, where $\overline{A} = \{p \in \beta X : A \in p\}$. We write $X^\star = \beta X \setminus X$.

Sometime in the 1970's or 1980's Mary Ellen Rudin was asked by some, now anonymous, analysts whether every point of $\mathbb{Z}^*$ is a member of a maximal orbit closure under the continuous extension $\widetilde{\sigma}$ of the shift map $\sigma : \mathbb{Z} \to \mathbb{Z}$ defined by $\sigma(n) = 1 + n$. This question is still open. Although it was not known at the time, this question turned out to be a question about the algebraic structure of the compact right topological semigroup $(\beta\mathbb{Z}, +)$. We pause to give a brief introduction to this structure.

Given a discrete semigroup $(S, \cdot)$, there is a unique extension of the operation to βS, also denoted by $\cdot$, with the property that $(\beta S, \cdot)$ is right topological (meaning that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). We systematically use the same symbol for the extended operation on βS as for the operation on S. The reader should be cautioned that the operation on βS is very unlikely to be commutative, even if the operation on S is commutative and is denoted by $+$. Given p and q in βS and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$,

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where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. If the operation is denoted by $+$ we have $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$.

As with any compact Hausdorff right topological semigroup, βS has a smallest 2-sided ideal, $K(\beta S)$, which is the union of all minimal right ideals of βS as well as the union of all minimal left ideals of βS. The intersection of any minimal right ideal with any minimal left ideal is a group. In particular, βS has idempotents. For detailed information about the algebraic structure of βS see [5].

Returning to the analysts' question above, one sees that the continuous functions $\widetilde{\sigma}$ and λ_1 from $\beta\mathbb{Z}$ to $\beta\mathbb{Z}$ agree on the dense subspace $\mathbb{Z}$, and so for $p \in \beta\mathbb{Z}$, $\widetilde{\sigma}(p) = 1 + p$. Consequently, the orbit of $\widetilde{\sigma}$ at p, $\{\widetilde{\sigma}^n(p) : n \in \mathbb{Z}\} = \mathbb{Z} + p$, and so the orbit closure is $c\ell(\mathbb{Z} + p) = \beta\mathbb{Z} + p$. Thus their question in algebraic terms was whether every point of $\mathbb{Z}^*$ is a member of some maximal left ideal of $\beta\mathbb{Z}$ of the form $\beta\mathbb{Z} + p$. Since for $p \in \mathbb{Z}^*$, $p = 0 + p \in \beta\mathbb{Z} + p$, one could answer this question in the affirmative by showing that there is no strictly increasing sequence of principal left ideals, that is left ideals of the form $\beta\mathbb{Z} + p$. This question also remains open. However, in [3, Corollary 1.8], it was shown that there is a strictly increasing sequence of principal right ideals $\langle p_n + \beta\mathbb{Z}\rangle_{n=0}^{\infty}$ in $\beta\mathbb{Z}$. (They were called "left ideals" in [3] because there $\beta\mathbb{Z}$ was taken to be left topological rather than right topological.) In [6, Theorem 5.4], this result was extended by showing that one could take each p_n to be an idempotent. The extension is intimately related to one of the orderings of idempotents.

Given idempotents p and q in $(\beta S, \cdot)$,

(1) $p \leq_L q$ if and only if $p \cdot q = p$;
(2) $p \leq_R q$ if and only if $q \cdot p = p$; and
(2) $p \leq q$ if and only if $p \cdot q = q \cdot p = p$.

The partial orders $\leq_L$ and $\leq_R$ are not antisymmetric. We write $p <_R q$ when $p \leq_R q$ and it is not the case that $q \leq_R p$, and define $p <_L q$ similarly. Given idempotents p and q in βS, one has that $p \leq_R q$ if and only if $p \cdot \beta S \subseteq q \cdot \beta S$. (The necessity is trivial. For the sufficiency, observe that $p = p \cdot p \in p \cdot \beta S$ and if $p \in q \cdot \beta S$, then $p = q \cdot r$ for some $r \in \beta S$ so $q \cdot p = q \cdot q \cdot r = q \cdot r = p$.) Therefore $p <_R q$ if and only if $p \cdot \beta S \subsetneq q \cdot \beta S$.

In these terms, the result of [6, Theorem 5.4] mentioned above says that there is a sequence $\langle p_n\rangle_{n=0}^{\infty}$ of idempotents in $\beta\mathbb{N}$ such that $p_n <_R p_{n+1}$ for each $n \in \omega$. (We take $\omega = \{0, 1, 2, \ldots\}$ to be the first infinite ordinal and let $\mathbb{N} = \{1, 2, 3, \ldots\} = \omega \setminus \{0\}$.) Similarly, if one could show that there is a

sequence $\langle p_n \rangle_{n=0}^{\infty}$ of idempotents in $\beta\mathbb{N}$ such that $p_n <_L p_{n+1}$ for each $n \in \omega$, one would obtain a strictly increasing sequence of principal left ideals of $\beta\mathbb{Z}$.

The main result of this paper is that there is a sequence $\langle q_\sigma \rangle_{\sigma<\omega_1}$ of idempotents in $\beta\mathbb{N}$ such that $q_\sigma <_R q_\delta$ whenever $\sigma < \delta < \omega_1$, where ω_1 is the first uncountable ordinal. It is also true that if $(G, \cdot)$ is any countably infinite discrete group, $p \in G^*$ is right cancelable in βG, and q is any idempotent which is minimal (see below for the definition) in the smallest compact subsemigroup C_p of βG which has p as a member, then there is a sequence $\langle q_\sigma \rangle_{\sigma<\omega_1}$ of idempotents in βG such that $q_\sigma <_R q_\delta$ whenever $\sigma < \delta < \omega_1$ and $q_0 = q$.

If e is an idempotent in a compact right topological semigroup $(T, \cdot)$, the following statements are equivalent.

(1) e is minimal with respect to $\leq_L$;
(2) e is minimal with respect to $\leq_R$;
(3) e is minimal with respect to $\leq$;
(4) e is a member of the smallest ideal $K(T)$ of T.

For the proof of the above equivalence see [5, Theorems 1.36 and 2.9]. An idempotent satisfying these equivalent statements is said to be *minimal in* T.

The semigroups C_p have a very rich algebraic structure, as shown in [5, Section 8.5], and have been useful in studying the algebra of $\beta\mathbb{N}$. For example, the authors [4, Corollary 2.4] were able to answer the open question of whether there are any idempotents in $c\ell K(\beta\mathbb{N}) \setminus K(\beta\mathbb{N})$, by showing that this set contains a semigroup of the form C_p and therefore contains $2^{\mathfrak{c}}$ idempotents, where $\mathfrak{c} = |\mathbb{R}|$. The results in this paper show, in particular, that that there are uncountable increasing $<_R$-chains of idempotents contained in $c\ell K(\beta\mathbb{N}) \setminus K(\beta\mathbb{N})$.

We remark that much more is known about decreasing chains of idempotents. By [4, Theorem 3.1], given any nonminimal idempotent p in $\beta\mathbb{N}$, there are $2^{\mathfrak{c}}$ nonminimal idempotents q such that $q < p$ and there does not exist an idempotent r such that $q < r < p$, from which it follows immediately that there exists a sequence $\langle p_n \rangle_{n<\omega}$ of idempotents such that $p_{n+1} < p_n$ for each n.

For the relation $<_L$, even more is known. Let λ be any ordinal with the property that $|\lambda| \leq \mathfrak{c}$ (such as λ equal to the ordinal sum $\mathfrak{c} + \mathfrak{c} + \omega_1$) and let q be any nonminimal idempotent in $\beta\mathbb{N}$. There is a sequence $\langle q_\sigma \rangle_{\sigma<\lambda}$ of idempotents such that $q_0 = q$, $q_{\sigma+1} < q_\sigma$ for each σ with $\sigma + 1 < \lambda$, and $q_\tau <_L q_\sigma$ whenever $\sigma < \tau < \lambda$.

It is an old result [7, Lemma I.2.6] that given any idempotent $p \in \beta\mathbb{N}$, there is a $\leq_R$-maximal idempotent q, with $p \leq_R q$. But it was not known until recently whether it could be shown in ZFC that $\leq_L$-maximal idempotents exist. It was shown by Y. Zelenyuk in [8, Corollary 1.2] that if G is any countably infinite discrete group, then there are idempotents in G^* that are both minimal and $\leq_L$-maximal. As a consequence, the same result holds for $\mathbb{N}^*$.

We remark that properties of idempotents in βS have had numerous applications to Ramsey Theory. In particular, order relations between idempotents have provided algebraic proofs of important Ramsey theoretic results. For example, A. Blass in [1, Theorem 8] gave a short and elegant proof of the powerful Hales-Jewett Theorem, based on the order relation between two idempotents in the Stone-Čech compactification of a semigroup of variable words. The authors, in collaboration with T. Carlson, were able to prove the even more powerful Graham Rothschild Parameter Sets Theorem by proving the existence of an infinite decreasing chain of idempotents in the Stone-Čech compactification of a larger semigroup of variable words [2].

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2. Preliminary results

Most of this section consists of a presentation of some of the details of [5, Exercise 8.5.1]. Throughout this section we will assume that we have an element p of $\mathbb{N}^*$ which is right cancelable in $\beta\mathbb{N}$. We will also assume we have a strictly increasing sequence $\langle b_n \rangle_{n=1}^{\infty}$ such that for each $k \in \mathbb{N}$, if $P_k = \{b_n : b_n + k < b_{n+1}\}$, then $P_k \in p$. (The existence of such a sequence is guaranteed by [5, Theorem 8.27].) We let $P = \{b_n : n \in \mathbb{N}\}$.

Definition 2.1. (a) $T = \{b_{n_1} + b_{n_2} + \ldots + b_{n_k}$: if $k > 1$, then $n_1 < n_2$ and for each $i \in \{2, 3, \ldots, k\}$, $b_{n_i+1} > (1 + 2 + \ldots + b_{n_{i-1}}) + b_{n_i}\}$.

(b) For $n \in \mathbb{N}, T_n = \{b_{n_1} + b_{n_2} + \ldots + b_{n_k} : n_1 > n$, $b_{n_1+1} > 1 + 2 + \ldots + b_n + b_{n_1}$ and if $k > 1$, then $n_1 < n_2$ and for each $i \in \{2, 3, \ldots, k\}$, $b_{n_i+1} > 1 + 2 + \ldots + b_{n_{i-1}} + b_{n_i}\}$.

(c) $T_\infty = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}} T_n$.

An expression of the form $b_{n_1} + b_{n_2} + \ldots + b_{n_k}$ as in the definition of T will be called a *P-sum*. As an example, the requirements for $b_2 + b_5 + b_9$ to be a P sum are that $b_6 > 1 + 2 + \ldots + b_2 + b_5$ and $b_{10} > 1 + 2 + \ldots + b_5 + b_9$.

Lemma 2.2. *Let $a, k, l \in \mathbb{N}$ and assume that $b_{m_1} + \ldots + b_{m_k}$ and $b_{n_1} + \ldots + b_{n_l}$ are P-sums, $b_{m_1+1} > (1 + 2 + \ldots + a) + b_{m_1}$, $b_{m_1} > a$, and $a + b_{m_1} + \ldots + b_{m_k} =$*

$b_{n_1} + \ldots + b_{n_l}$. *Then* $l > k$ *and, if* $i = l - k$, *then* $a = b_{n_1} + \ldots + b_{n_i}$ *and for* $j \in \{1, 2, \ldots, k\}$, $b_{m_j} = b_{n_{i+j}}$.

Proof. Suppose the conclusion fails and pick a counterexample with $k + l$ a minimum among all counterexamples. Assume first that $k > 1$ and $l > 1$. We cannot have $m_k = n_l$, for then the equation $a + b_{m_1} + \ldots + b_{m_{k-1}} = b_{n_1} + \ldots + b_{n_{l-1}}$ would provide a smaller counterexample.

If $m_k < n_l$, then $m_k + 1 \leq n_l$, so

$$b_{n_l} \geq b_{m_k+1} > 1+2+\ldots+b_{m_{k-1}}+b_{m_k} \geq a+b_{m_1}+\ldots+b_{m_k} = b_{n_1}+\ldots+b_{n_l},$$

a contradiction. If $n_l < m_k$, then $n_l + 1 \leq m_k$ so

$$b_{m_k} \geq b_{n_l+1} > 1+2+\ldots+b_{n_{l-1}}+b_{n_l} \geq b_{n_1}+\ldots+b_{n_l} = a+b_{m_1}+\ldots+b_{m_k},$$

again a contradiction.

Thus we must have $k = 1$ or $l = 1$.

Case 1. $k = 1$ and $l = 1$. Then $a + b_{m_1} = b_{n_1}$ so $b_{n_1} > b_{m_1}$ and thus $m_1 + 1 \leq n_1$. Therefore $b_{n_1} \geq b_{m_1+1} > 1 + 2 + \ldots + a + b_{m_1} \geq b_{n_1}$, a contradiction.

Case 2. $l = 1$ and $k > 1$. Then $a + b_{m_1} + \ldots + b_{m_k} = b_{n_1}$ so $n_1 \geq m_k + 1$. Therefore $b_{n_1} \geq b_{m_k+1} > 1+2+\ldots+b_{m_{k-1}}+b_{m_k} \geq a+b_{m_1}+\ldots+b_{m_k} = b_{n_1}$, a contradiction.

Case 3. $l > 1$ and $k = 1$. Then $a+b_{m_1} = b_{n_1}+\ldots+b_{n_l}$. If $m_1 < n_l$ or $m_1 > n_l$ we derive a contradiction as in cases 1 and 2. So $m_1 = n_l$ and thus the conclusion of the lemma holds, and we did not have a counterexample. □

Lemma 2.3. *The expression of an element of* T *as a* P*-sum is unnique.*

Proof. Suppose that we have P-sums $b_{m_1} + \ldots + b_{m_k}$ and $b_{n_1} + \ldots + b_{n_l}$ such that $b_{m_1} + \ldots + b_{m_k} = b_{n_1} + \ldots + b_{n_l}$ but $(m_1, m_2, \ldots, m_k) \neq (n_1, n_2, \ldots, n_l)$ and pick such an example with $k + l$ a minimum among all examples.

Case 1. $k > 1$ and $l > 1$. Then $m_k \neq n_l$ or else the equation $b_{m_1} + \ldots + b_{m_{k-1}} = b_{n_1} + \ldots + b_{n_{l-1}}$ provides a smaller example. So assume without loss of generality that $m_k+1 \leq n_l$. Then $b_{n_l} \geq b_{m_k+1} > 1+2+\ldots+b_{m_{k-1}}+b_{m_k} \geq b_{m_1} + \ldots + b_{m_k} = b_{n_1} + \ldots + b_{n_l}$, a contradiction.

Case 2. $k = 1$ or $l = 1$. Assume without loss of generality that $k = 1$. If $l = 1$ also, then there was not a counterexample, so $l > 1$. Then $m_1 \geq n_l+1$ so $b_{m_1} \geq b_{n_l+1} > 1 + 2 + \ldots + b_{n_{l-1}} + b_{n_l} \geq b_{m_1}$, a contradiction. □

Definition 2.4. Define $\psi : T \to \mathbb{N}$ by $\psi(b_{n_1} + b_{n_2} + \ldots + b_{n_k}) = k$ and let $\widetilde{\psi} : c\ell_{\beta\mathbb{N}} T \to \beta\mathbb{N}$ be its continuous extension.

Recall that by we are denoting the smallest compact subsemigroup of $\beta\mathbb{N}$ with p as a member by C_p.

Theorem 2.5. *T_∞ is a compact subsemigroup of $\mathbb{N}^*$, $C_p \subseteq T_\infty$, the restriction of $\widetilde{\psi}$ to T_∞ is a homomorphism, $\widetilde{\psi}(p) = 1$, and $\widetilde{\psi}[C_p] = \beta\mathbb{N}$.*

Proof. We first claim that for each $n \in \mathbb{N}$, if $k = 1 + 2 + \ldots + b_n$, then $\{b_m \in P_k : m > n\} \subseteq T_n$. To see this let $b_m \in P_k$ such that $m > n$. Then $b_{m+1} > 1 + 2 + \ldots + b_n + b_m$ so $b_m \in T_n$. Thus, given n, since $\{b_m \in P_k : m > n\} \in p$, we have that $p \in c\ell_{\beta\mathbb{N}} T_n$. Consequently, $p \in T_\infty$ and $\widetilde{\psi}(p) = 1$.

To see that T_∞ is a subsemigroup of $\beta\mathbb{N}$, let $m \in \mathbb{N}$ and let $x = b_{m_1} + \ldots + b_{m_k} \in T_m$, where $b_{m_1} + \ldots + b_{m_k}$ is a P-sum as in the definition of T_m. By [5, Theorem 4.20], it suffices to show that $x + T_{m_k} \subseteq T_m$. So let $b_{n_1} + \ldots + b_{n_l} \in T_{m_k}$. To see that $x + b_{n_1} + \ldots + b_{n_l} \in T_m$ we need that $b_{m_1} + \ldots + b_{m_k} + b_{n_1} + \ldots + b_{n_l}$ is as in the definition of T_m. If $k > 1$, we only need to note that $b_{n_1+1} > 1 + 2 + \ldots + b_{m_k} + b_{n_1}$. If $k = 1$, we also need to note that $n_1 > m_k$.

Further, with $x = b_{m_1} + \ldots + b_{m_k}$ and $y = b_{n_1} + \ldots + b_{n_l}$ as in the above paragraph, we have that $\psi(x + y) = k + l = \psi(x) + \psi(y)$, so by [5, Theorem 4.21], the restriction of $\widetilde{\psi}$ to T_∞ is a homomorphism.

Since $p \in T_\infty$, we have $C_p \subseteq T_\infty$. Since $D = \{p, p+p, p+p+p, \ldots\} \subseteq C_p$ and $\psi[D] = \mathbb{N}$, we have $\widetilde{\psi}[C_p] = \beta\mathbb{N}$. □

Lemma 2.6. *Let $x \in \beta\mathbb{N}$, let $y \in T_\infty$, and assume that $x + y \in T_\infty$. Then $x \in T_\infty$.*

Proof. Suppose that $x \notin T_\infty$ and pick $r \in \mathbb{N}$ such that $x \notin c\ell_{\beta\mathbb{N}} T_r$. Let $X = \mathbb{N} \setminus T_r$ and let $Z = \{a + b_{m_1} + \ldots + b_{m_k} : a \in X\,,\ b_{m_1} + \ldots + b_{m_k}$ is a P-sum, $b_{m_1+1} > 1 + 2 + \ldots + a + b_{m_1}$, and $m_1 > a\}$. We claim that $Z \in x + y$ for which it suffices that $X \subseteq \{a \in \mathbb{N} : -a + Z \in y\}$. So let $a \in X$. We claim that $T_a \subseteq -a + Z$. To see this, let $b_{m_1} + \ldots + b_{m_k}$ be a P-sum in T_a. Then $m_1 > a$ and $b_{m_1+1} > 1 + 2 + \ldots + b_a + b_{m_1} \geq 1 + 2 + \ldots + a + b_{m_1}$. so $a + b_{m_1} + \ldots + b_{m_k} \in Z$ as claimed.

Now $x + y \in T_\infty \subseteq c\ell_{\beta\mathbb{N}} T_r$ so pick $w \in Z \cap T_r$. Since $w \in Z$, pick $a \in X$ and a P-sum $b_{m_1} + \ldots + b_{m_k}$ such that $b_{m_1+1} > 1 + 2 + \ldots + a + b_{m_1}$, $m_1 > a$, and $w = a + b_{m_1} + \ldots + b_{m_k}$. Since $w \in T_r$, pick a P-sum $b_{n_1} + b_{n_2} + \ldots + b_{n_l}$ such that $w = b_{n_1} + b_{n_2} + \ldots + b_{n_l}$, $n_1 > r$, $b_{n_1+1} > 1 + 2 + \ldots + b_r + b_{n_1}$ and if $k > 1$, then $n_1 < n_2$. By Lemma 2.2, there is some $i < l$ such that $a = b_{n_1} + \ldots + b_{n_i}$, so that $a \in T_r$, a contradiction. □

Definition 2.7. (a) For $n \in \mathbb{N}$, $\mathrm{supp}(n)$ is the finite set $F \subseteq \omega$ such that $n = \sum_{t \in F} 2^t$.

(b) Define $\phi : \mathbb{N} \to \omega$ by $\phi(n) = \max \mathrm{supp}(n)$.

(c) Define $M : \mathbb{Z} \to \omega$ by $M(n) = \begin{cases} \phi(|n|) & \text{if } n \neq 0 \\ 0 & \text{if } n = 0\,. \end{cases}$

Lemma 2.8. (1) *$M[\mathbb{N}] = \omega$;*

(2) *for all $n < \omega$, $\{m \in \mathbb{Z} : M(m) < n\}$ is finite; and*

(3) *for all r and s in $\mathbb{Z}$, if $M(s) + 1 < M(r)$, then $M(r + s) \in \{M(r) - 1, M(r), M(r) + 1\}$.*

Proof. Conclusions (1) and (2) are immediate. To verify conclusion (3), let $r, s \in \mathbb{Z}$ and assume that $M(s) + 1 < M(r)$. Note that $r \neq 0$. If $s = 0$, then $M(r + s) = M(r)$. So we assume $s \neq 0$.

If $s > 0$ and $r > 0$, then $s < 2^{M(s)+1}$ and $2^{M(r)} \leq r < 2^{M(r)+1}$ so $2^{M(r)} < s + r < 2^{M(s)+1} + 2^{M(r)+1} < 2^{M(r)+2}$ and thus $M(r + s) \in \{M(r), M(r) + 1\}$.

If $s < 0$ and $r < 0$, then $M(r + s) = M(-r + -s) \in \{M(-r), M(-r) + 1\} = \{M(r), M(r) + 1\}$.

If $s < 0$ and $r > 0$, then $2^{M(s)} \leq -s < 2^{M(s)+1} \leq 2^{M(r)-1}$ and $2^{M(r)} \leq r < 2^{M(r)+1}$ so $2^{M(r)-1} = 2^{M(r)} - 2^{M(r)-1} < s + r < 2^{M(r)+1}$ and consequently $M(r + s) \in \{M(r) - 1, M(r)\}$.

If $s > 0$ and $r < 0$, then $M(r + s) = M(-r + -s) \in \{M(-r) - 1, M(-r)\} = \{M(r) - 1, M(r)\}$. □

3. Increasing right chains

We write $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}} 2^n\mathbb{N}$. Given any $p \in \beta\mathbb{N}$, C_p is a compact right topological semigroup, so it has a smallest ideal and idempotents minimal in C_p.

Lemma 3.1. *Assume that $p \in \mathbb{N}^*$, p is right cancelable in $\beta\mathbb{N}$, and q is an idempotent which is minimal in C_p. There exist $p' \in C_p \cap \mathbb{H}$ and an idempotent q' which is minimal in $C_{p'}$ such that p' is right cancelable in $\beta\mathbb{N}$, $q <_R q'$, and $p' + q = q$.*

Proof. Let T_∞ and ψ be as defined in Section 2 for p. Then by Theorem 2.5, $\widetilde{\psi}$ is a homomorphism on T_∞, $C_p \subseteq T_\infty$, and $\widetilde{\psi}[C_p] = \beta\mathbb{N}$. Let ϕ and M be as in Definition 2.7. By [5, Lemma 6.8] if $r \in \beta\mathbb{N}$ and $s \in \mathbb{H}$, then $\widetilde{\phi}(r + s) = \widetilde{\phi}(s)$. And, of course, for $r \in \beta\mathbb{N}$, $\widetilde{M}(r) = \widetilde{\phi}(r)$. By Lemma 2.8, M satisfies the conclusion (as f) of [5, Lemma 6.47] with $S = \mathbb{Z}$, $T = \mathbb{N}$, and $\kappa = \omega$.

Pick an injective sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\{2^t : t \in \mathbb{N}\}^* = c\ell_{\beta\mathbb{N}}\{2^t : t \in \mathbb{N}\} \setminus \mathbb{N}$. We claim $\langle \widetilde{\phi}(x_n) \rangle_{n=1}^{\infty}$ is also injective. So let $n \neq m$ be given. Pick $A \subseteq \{2^t : t \in \mathbb{N}\}$ such that $A \in x_n$ and $\{2^t : t \in \mathbb{N}\} \setminus A \in x_m$. Let $B = \{t \in \mathbb{N} : 2^t \in A\}$. Then $\phi[A] = B$ and $\phi[\{2^t : t \in \mathbb{N}\} \setminus A] = \mathbb{N} \setminus B$ so $B \in \widetilde{\phi}(x_n)$ and $\mathbb{N} \setminus B \in \widetilde{\phi}(x_m)$.

By thinning the sequence we may assume that $\{\widetilde{\phi}(x_n) : n \in \mathbb{N}\}$ is discrete. For each $n \in \mathbb{N}$ pick $y_n \in C_p$ such that $\widetilde{\psi}(y_n) = x_n$. Then $C_p + y_n$ is a

left ideal of C_p which therefore contains a minimal left ideal of C_p and $q+C_p$ is a minimal right ideal of C_p. Recalling that in any compact Hausdorff right topological semigroup, the intersection of a minimal left ideal and a minimal right ideal is a group, we may pick an idempotent $q_n \in (C_p + y_n) \cap (q + C_p)$ and pick $s_n \in C_p$ such that $q_n = s_n + y_n$. Let p' be a cluster point of the sequence $\langle q_n \rangle_{n=1}^{\infty}$. Since by [5, Lemma 6.6] all idempotents of $\beta\mathbb{N}$ are in $\mathbb{H}$, we have that $p' \in C_p \cap \mathbb{H}$.

Let $r = \widetilde{\psi}(p')$ and note that r is a cluster point of $\langle \widetilde{\psi}(q_n) \rangle_{n=1}^{\infty}$. Now, given $n \in \mathbb{N}$, $\widetilde{\psi}(q_n) = \widetilde{\psi}(s_n + y_n) = \widetilde{\psi}(s_n) + \widetilde{\psi}(y_n) = \widetilde{\psi}(s_n) + x_n$ and since $x_n \in \mathbb{H}$, $\widetilde{\phi}(\widetilde{\psi}(s_n) + x_n) = \widetilde{\phi}(x_n)$. That is, $\widetilde{M}\big(\widetilde{\psi}(q_n)\big) = \widetilde{\phi}\big(\widetilde{\psi}(q_n)\big) = \widetilde{\phi}(x_n)$. Since $\{\widetilde{\phi}(x_n) : n \in \mathbb{N}\}$ is discrete and r is a cluster point of $\langle \widetilde{\psi}(q_n) \rangle_{n=1}^{\infty}$, we have by [5, Theorem 6.54.4] with $S = \mathbb{Z}$ and $T = \mathbb{N}$ that $(\mathbb{Z} + r) \cap (\mathbb{Z}^* + \mathbb{Z}^*) = \emptyset$.

We claim that r is right cancelable in $\beta\mathbb{Z}$. By (9) $\Rightarrow$ (3) of [5, Theorem 8.11] with $S = T = \mathbb{Z}$, it suffices to show that for $a \in \mathbb{Z}$ and $s \in \beta\mathbb{Z} \setminus \{a\}$, $a + r \neq s + r$. If $s \in \mathbb{Z}$, this holds by [5, Corollary 8.2]. If $s \in \mathbb{Z}^*$, this holds because $(\mathbb{Z} + r) \cap (\mathbb{Z}^* + \mathbb{Z}^*) = \emptyset$.

Next we claim that p' is right cancelable in $\beta\mathbb{N}$. Suppose not and by [5, Theorem 8.18] pick an idempotent $e \in \mathbb{N}^*$ such that $p' = e + p'$. Now $p' \in C_p \subseteq T_\infty$ so by Lemma 2.6, $e \in T_\infty$ and thus by Theorem 2.5, $r = \widetilde{\psi}(p') = \widetilde{\psi}(e) + \widetilde{\psi}(p') = \widetilde{\psi}(e) + r$ so by [5, Theorem 8.18], r is not right cancelable in $\beta\mathbb{N}$, hence not right cancelable in $\beta\mathbb{Z}$, a contradiction.

For each $n \in \mathbb{N}$, $q_n \in q + C_p$ so $q_n + C_p \subseteq q + C_p$ and, since q is minimal in C_p, $q + C_p$ is a minimal right ideal of C_p, so $q_n + C_p = q + C_p$ and therefore $q_n + q = q$. That is ρ_q is constantly equal to q on $\{q_n : n \in \mathbb{N}\}$, so $p' + q = q$.

Since $p' \in \{y \in \beta\mathbb{N} : y + q = q\} = \rho_q^{-1}[\{q\}]$ we have $\{y \in \beta\mathbb{N} : y + q = q\}$ is a compact subsemigroup of $\beta\mathbb{N}$ with p' as a member and thus $C_{p'} \subseteq \{y \in \beta\mathbb{N} : y + q = q\}$. Let q' be a minimal idempotent in $C_{p'}$. Then $q' + q = q$ so $q \leq_R q'$. It remains only to show that the inequality is strict.

We have that $\widetilde{\psi}^{-1}[C_r] \cap C_p$ is a compact subsemigroup of $\beta\mathbb{N}$ with p' as a member, so $C_{p'} \subseteq \widetilde{\psi}^{-1}[C_r]$. We have that r is right cancelable in $\beta\mathbb{Z}$ (not just in $\beta\mathbb{N}$), so by [5, Theorem 8.57], $C_r \cap K(\beta\mathbb{Z}) = \emptyset$ and therefore $C_r \cap K(\beta\mathbb{N}) = \emptyset$. Further, $\widetilde{\psi}^{-1}[K(\beta\mathbb{N})] \cap C_p$ is an ideal of C_p so $K(C_p) \subseteq \widetilde{\psi}^{-1}[K(\beta\mathbb{N})]$. If we had some $s \in C_{p'} \cap K(C_p)$, we would have $\widetilde{\psi}(s) \in C_r \cap K(\beta\mathbb{N})$. Therefore $C_{p'} \cap K(C_p) = \emptyset$. If $q' \leq_R q$, then $q' = q + q' \in K(C_p) \cap C_{p'}$. Therefore $q <_R q'$. $\square$

In the above proof note the distinction between right cancelability in $\beta\mathbb{N}$ versus $\beta\mathbb{Z}$. (One has p and p' right cancelable in $\beta\mathbb{N}$ while r is right

cancelable in $\beta\mathbb{Z}$.) By [5, Example 8.29], right cancelability in $\beta\mathbb{N}$ does not imply right cancelability in $\beta\mathbb{Z}$.

The above proof cites [5, Theorem 8.57] and the proof of Corollary 3.3 below cites [5, Corollary 8.62]. These results use [5, Lemma 8.48] which is incorrect as stated. We include a corrected statement and proof as Section 4.

In the proof of the following theorem we shall inductively construct two ω_1 sequences, $\langle p_\sigma\rangle_{\sigma<\omega_1}$ and $\langle q_\sigma\rangle_{\sigma<\omega_1}$ where each p_σ is right cancelable in $\beta\mathbb{N}$ and $\langle q_\sigma\rangle_{\sigma<\omega_1}$ is a $<_R$-increasing chain of idempotents, with each q_σ being a minimal idempotent in C_{p_σ}.

Theorem 3.2. *Let p be a right cancelable element of $\beta\mathbb{N}$ and let q be a minimal idempotent in C_p. There exists a sequence $\langle q_\sigma\rangle_{\sigma<\omega_1}$ of idempotents in $\beta\mathbb{N}$ such that $q_0 = q$ and $q_\sigma <_R q_\delta$ whenever $\sigma < \delta < \omega_1$.*

Proof. Let $p_0 = p$ and $q_0 = q$. Let $0 < \alpha < \omega_1$ and assume we have chosen $\langle p_\sigma\rangle_{\sigma<\alpha}$ and $\langle q_\sigma\rangle_{\sigma<\alpha}$ such that

(1) if $0 < \delta < \alpha$, then $p_\delta \in \mathbb{H}$;
(2) if $\delta < \alpha$, then p_δ is right cancelable in $\beta\mathbb{N}$;
(3) if $\delta < \alpha$, then q_δ is a minimal idempotent in C_{p_δ};
(4) if $\delta < \sigma < \alpha$, then $q_\delta <_R q_\sigma$;
(5) if $\delta < \sigma < \alpha$, then $p_\sigma \in C_{p_\delta}$; and
(6) if $\delta < \sigma < \alpha$, then $p_\sigma + q_\delta = q_\delta$.

The hypotheses hold for $\alpha = 1$, all but (2) amd (3), vacuously.

Case 1. $\alpha = \gamma + 1$ for some γ. By hypotheses (2) and (3) and Lemma 3.1 we may pick $p_\alpha \in C_{p_\gamma} \cap \mathbb{H}$ which is right cancelable in $\beta\mathbb{N}$ and an idempotent q_α which is minimal in C_{p_α} such that $q_\gamma <_R q_\alpha$ and $p_\alpha + q_\gamma = q_\gamma$. One sees immediately that hypotheses (1) through (4) hold at $\alpha + 1$. To verify hypothesis (5), let $\delta < \alpha$. If $\delta = \gamma$, we have $p_\alpha \in C_{p_\delta}$ directly. Otherwise, by assumption $p_\gamma \in C_{p_\delta}$ by assumption so $p_\alpha \in C_{p_\gamma} \subseteq C_{p_\delta}$.

To verify hypothesis (6), again if $\delta = \gamma$ we have $p_\alpha + q_\delta = q_\delta$ directly, so assume $\delta < \gamma$. Then $p_\alpha + q_\gamma = q_\gamma$ and, since $q_\delta <_R q_\gamma$, $q_\gamma + q_\delta = q_\delta$ so $p_\alpha + q_\delta = p_\alpha + q_\gamma + q_\delta = q_\gamma + q_\delta = q_\delta$.

Case 2. α is a limit ordinal. Choose a cofinal sequence $\langle\delta(n)\rangle_{n<\omega}$ in α such that $\delta(0) > 0$ and $\delta(n) < \delta(n+1)$ for each $n < \omega$. Let p_α be a cluster point of the sequence $\langle p_{\delta(n)}\rangle_{n<\omega}$. Let q_α be a minimal idempotent in C_{p_α}. Since $p_{\delta(n)} \in \mathbb{H}$ for each $n < \omega$, we have $p_\alpha \in \mathbb{H}$.

We claim that p_α is right cancelable in $\beta\mathbb{N}$. Suppose not and by [5, Theorem 8.18] pick an idempotent $e \in \mathbb{N}^*$ such that $p_\alpha = e + p_\alpha$. Then

$p_\alpha \in \beta\mathbb{N}+p_\alpha = c\ell_{\beta\mathbb{N}}(\mathbb{N}+p_\alpha)$ and $p_\alpha \in c\ell_{\beta\mathbb{N}}\{p_{\delta(n)} : n < \omega\}$ so by [5, Theorem 3.40], either there is some $n \in \mathbb{N}$ such that $n + p_\alpha \in c\ell_{\beta\mathbb{N}}\{p_{\delta(n)} : n < \omega\}$ or there is some $n < \omega$ such that $p_{\delta(n)} \in \beta\mathbb{N} + p_\alpha$. The first alternative is impossible because $p_\alpha \in \mathbb{H}$ and $\{p_{\delta(n)} : n < \omega\} \subseteq \mathbb{H}$. So pick $n < \omega$ and $x \in \beta\mathbb{N}$ such that $p_{\delta(n)} = x + p_\alpha$. Let T_∞ and ψ be as defined in Section 2 for $p_{\delta(n)}$. Since $p_{\delta(m)} \in C_{p_{\delta(n)}}$ for all $m > n$ by hypothesis (5), we have $p_\alpha \in C_{p_{\delta(n)}} \subseteq T_\infty$. Since also $p_{\delta(n)} \in T_\infty$, we have by Lemma 2.6 that $x \in T_\infty$. But now, by Theorem 2.5, $1 = \widetilde{\psi}(p_{\delta(n)}) = \widetilde{\psi}(x) + \widetilde{\psi}(p_\alpha)$ which is impossible. Thus hypothesis (2) holds.

Hypothesis (3) holds directly. To verify hypotheses (4), (5), and (6), let $\sigma < \alpha$ and pick $n < \omega$ such tht $\sigma < \delta(n) < \alpha$. For each m with $n < m < \omega$, we have by hypothesis (6) that $p_{\delta(m)} + q_{\delta(n)} = q_{\delta(n)}$ so $p_\alpha + q_{\delta(n)} = q_{\delta(n)}$. Therefore $\{y \in \beta\mathbb{N} : y + q_{\delta(n)} = q_{\delta(n)}\}$ is a compact subsemigroup of $\beta\mathbb{N}$ with p_α as a member so $C_{p_{\delta(n)}} \subseteq \{y \in \beta\mathbb{N} : y + q_{\delta(n)} = q_{\delta(n)}\}$. Therefore $q_\alpha + q_{\delta(n)} = q_{\delta(n)}$ so $q_\sigma <_R q_{\delta(n)} \leq_R q_\alpha$ and we have verified hypothesis (4). Also, for each $m \geq n$ we have $p_{\delta(n)} \in C_{p_\sigma}$ so $p_\alpha \in C_{p_\sigma}$ as required by hypothesis (5). Since for all $m \geq n$, $p_{\delta(m)} + q_\sigma = q_\sigma$, we have $p_\alpha + q_\sigma = q_\sigma$ as required by hypothesis (6). □

Corollary 3.3. *Let G be a countably infinite discrete group, let $p \in G^*$ be a right cancelable element of βG and let q be a minimal idempotent in C_p. There exists a sequence $\langle q_\sigma \rangle_{\sigma<\omega_1}$ of idempotents in βG such that $q_0 = q$ and $q_\sigma <_R q_\delta$ whenever $\sigma < \delta < \omega_1$.*

Proof. By [5, Corollary 8.62] there exist an element $r \in \mathbb{N}^*$ such that $\{2^n : n \in \mathbb{N}\} \in r$ and a function f taking C_p (which is a subset of βG) onto C_r such that f is an isomorphism and a homeomorphism. By [5, Theorem 8.27] r is right cancelable in $\beta\mathbb{N}$. Since $f(q)$ is a minimal idempotent in C_r, Theorem 3.2 applies. □

Corollary 3.4. *Let G be a countably infinite discrete group and let q be a minimal idempotent in βG. Then there is a sequence $\langle q_\sigma \rangle_{\sigma<\omega_1}$ of idempotents in βG such that $q_0 = q$ and $q_\sigma <_R q_\delta$ whenever $\sigma < \delta < \omega_1$.*

Proof. By [5, Lemma 6.47] pick a function $f : G \to \omega$ such that

(1) $f[G] = \omega$;
(2) for each $n < \omega$, $f^{-1}[\{n\}]$ is finite; and
(3) for $r, s \in G$, if $f(s)+1 < f(r)$, then $f(sr) \in \{f(r)-1, f(r), f(r)+1\}$.

Let $\widetilde{f} : \beta G \to \beta\omega$ be the continuous extension of f and note that by (1), $\widetilde{f}[\beta G] = \beta\omega$.

Pick an injective sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\overline{3\mathbb{N}}\setminus\mathbb{N}$ such that $\{x_n : n\in\mathbb{N}\}$ is discrete and note that $\big\{x_n+i : n\in\mathbb{N} \text{ and } i\in\{-1,0,1\}\big\}$ is discrete. Given $n\in\mathbb{N}$, pick $v_n\in\beta G$ such that $\widetilde{f}(v_n)=x_n$ and note that $v_n\in G^*$. Then by [5, Lemma 6.54.2], $\beta G v_n\subseteq \widetilde{f}^{-1}[\{x_n-1,x_n,x_n+1\}$. Pick an idempotent $p_n\in\beta G v_n\cap q\beta G$. Note that $p_n\beta G\subseteq q\beta G$ so $p_n\beta G=q\beta G$ and consequently $p_nq=q$.

Pick an accumulation point p of the sequence $\langle p_n\rangle_{n=1}^{\infty}$. By [5, Theorem 6.54.4], $Gp\cap G^*G^*=\emptyset$ so by [5, Theorem 8.11(9) and Corollary 8.2], p is right cancelable in βG. Since $p_nq=q$ for each $n\in\mathbb{N}$, $pq=q$ and thus $C_p\subseteq\{x\in\beta G : xq=q\}$ because $\{x\in\beta G : xq=q\}$ is a compact subsemigroup of βG. Pick a minimal idempotent $r\in C_p$ and pick by Corollary 3.3 a sequence $\langle q_\sigma\rangle_{\sigma<\omega_1}$ of idempotents in βG such that $q_0=r$ and $q_\sigma<_R q_\delta$ whenever $\sigma<\delta<\omega_1$. Since $r\in C_p$, we have $rq=q$ so $q\leq_R r$ and thus we may replace q_0 by q. □

A sequence $\langle x_n\rangle_{n=1}^{\infty}$ in a semigroup $(S,\cdot)$ has *distinct finite products* if and only if, whenever F and H are finite nonempty subsets of $\mathbb{N}$ and $\prod_{t\in F}x_t=\prod_{t\in H}x_t$ (where both products are written in increasing order of indices), one has $F=H$.

Corollary 3.5. *Let S be a semigroup which has some sequence which has distinct finite products. Then there is a sequence $\langle q_\sigma\rangle_{\sigma<\omega_1}$ of idempotents in βS such that $q_\sigma<_R q_\delta$ whenever $\sigma<\delta<\omega_1$.*

Proof. By [5, Theorem 6.27], βS contains an algebraic and topological copy of $\mathbb{H}$ and all idempotents of $\beta\mathbb{N}$ are in $\mathbb{H}$ so Theorem 3.2 applies. □

Question 3.6. *Are there increasing $<_R$ chains of idempotents in $\beta\mathbb{N}$ indexed by $\mathfrak{c}$? Can there be such chains indexed by a cardinal greater than $\mathfrak{c}$?*

Idempotents that are minimal in C_p for some right cancelable $p\in\mathbb{N}^*$ played a crucial role in the proof of Theorem 3.2. By induction hypothesis (5) in the proof, the sequence $\langle q_\sigma\rangle_{\sigma<\omega_1}$ lies in C_p, so no idempotent which is maximal with respect to $<_R$ can be minimal in C_p. Further, by [5, Theorem 8.57], q can not be minimal in $\beta\mathbb{N}$, since $C_p\cap K(\beta\mathbb{N})=\emptyset$. We have not been able to find any other restrictions.

Question 3.7. *Can one characterize those idempotents q in $\beta\mathbb{N}$ such that there is some right cancelable $p\in\mathbb{N}^*$ for which q is minimal in C_p?*

4. Corrections to Section 8.5 of [5]

[5, Lemma 8.48] is incorrect as stated. Following is a corrected version with its proof.

Lemma 8.48. *Let $a \in G \setminus \{e\}$ and pick $r \in \mathbb{N}$ such that $a < b_r$. Let $k, l \in \mathbb{N}$ and assume that $b_{m_1} b_{m_2} \cdots b_{m_k}$ and $b_{n_1} b_{n_2} \cdots b_{n_l}$ are P-products such that $a b_{m_1} b_{m_2} \cdots b_{m_k} = b_{n_1} b_{n_2} \cdots b_{n_l}$ and $b_{m_1} \in P_r$. Then $k < l$ and, if $i = l - k$, then $a = b_{n_1} b_{n_2} \cdots b_{n_i}$ and $m_j = n_{i+j}$ for each $j \in \{1, 2, \ldots, k\}$.*

Proof. Suppose the conclusion fails and pick a counterexample with $k + l$ a minimum among all counterexamples. Note that $b_{m_1} \in P_r$ and

$$\{b_1, b_2, \ldots, b_r\} \subseteq F_r$$

so $m_1 > r$.

Assume first that $k > 1$ and $l > 1$. If $n_{l-1} \geq m_{k-1}$, we have

$$b_{n_l} = (b_{n_1} \cdots b_{n_{l-1}})^{-1} (a b_{m_1} \cdots b_{m_{k-1}}) b_{m_k}$$

and, since $b_{n_l} \notin F_{n_{l-1}} P$, we must have that $a b_{m_1} \cdots b_{m_{k-1}} = b_{n_1} \cdots b_{n_{l-1}}$ and $b_{n_l} = b_{m_k}$ so there is a smaller counterexample. Similarly, if $n_{l-1} < m_{k-1}$, we get a smaller counterexample because of the equation

$$b_{m_k} = (a b_{m_1} \cdots b_{m_{k-1}})^{-1} (b_{n_1} \cdots b_{n_{l-1}}) b_{n_l} \,.$$

Therefore we must have $k = 1$ or $l = 1$. Suppose that $l = 1$. Then $b_{m_1} \cdots b_{m_k} = a^{-1} b_{n_1}$. If $k = 1$, this says that $b_{m_1} = a^{-1} b_{n_1} \in F_r P$, a contradiction. If $k > 1$, this says $b_{m_k} = (a b_{m_1} \cdots b_{m_{k-1}})^{-1} b_{n_1}$ so $b_{m_k} \in F_{m_{k-1}} P$ unless $a b_{m_1} \cdots b_{m_{k-1}} = e$. But if $k - 1 = 1$, the equation $a b_{m_1} = e$ says that $b_{m_1} \in F_r$ while if $k - 1 > 1$, the equation $a b_{m_1} \cdots b_{m_{k-1}} = e$ says that $b_{m_{k-1}} \in F_{m_{k-2}}$.

Thus we must have $k = 1$ and $l > 1$. If $n_{l-1} \leq r$ we get

$$b_{m_1} = a^{-1} (b_{n_1} \cdots b_{n_{l-1}}) b_{n_l}$$

so $a^{-1}(b_{n_1} \cdots b_{n_{l-1}}) = e$; that is $a = b_{n_1} \cdots b_{n_{l-1}}$ and $m_1 = n_l$, so this is not a counterexample. If $n_{l-1} > r$, we get $b_{n_l} = (b_{n_1} \cdots b_{n_{l-1}})^{-1} a b_{m_1}$ so $(b_{n_1} \cdots b_{n_{l-1}})^{-1} a = e$ and we again conclude that we don't have a counterexample. □

The lemmas and the theorem in [5] that cited Lemma 8.48 are all correct as stated (except for a typo in the statement of Lemma 8.64), but all need adjustments to their proofs – in the case of Lemma 8.49, the proof needs replacement.

Lemma 8.49. *The expression for an element of T as a P-product is unique.*

Proof. Assume that there are P-products $b_{m_1}b_{m_2}\cdots b_{m_k}$ and $b_{n_1}b_{n_2}\cdots b_{n_l}$ such that $b_{m_1}b_{m_2}\cdots b_{m_k} = b_{n_1}b_{n_2}\cdots b_{n_l}$ but

$$(m_1, m_2, \ldots, m_k) \neq (n_1, n_2, \ldots, n_l)$$

and pick such products with $k + l$ a minimum. As in the proof of Lemma 8.48 above, if $k > 1$ and $l > 1$, then $b_{n_k} = b_{n_l}$ and so the equation $b_{m_1}b_{m_2}\cdots b_{m_{k-1}} = b_{n_1}b_{n_2}\cdots b_{n_{l-1}}$ provides a smaller example.

Thus we can assume without loss of generality that $k = 1$. If also $l = 1$, then $b_{m_1} = b_{n_1}$, so we must have $l > 1$. But then $(b_{n_1}\cdots b_{n_{l-1}})^{-1}b_{m_1} = b_{n_l}$ and so $b_{n_l} \in F_{n_{l-1}}P$, a contradiction. □

For the proof of Lemma 8.57, the sentence "For each $a \in G$, the set $X_a = \{b_n : n \in Q\,,\, b_n > a, \text{ and } b_n > a^{-1}\} \in x$." should be replaced by "For each $a \in G$, pick $r_a \in \mathbb{N}$ such that $a < b_{r_a}$ and let $X_a = \{b_n : n \in Q \text{ and } b_n \in P_{r_a}\}$. Note that $X_a \in x$."

For the proof of Lemma 8.59, the sentence "For each $a \in G$, let Q_a denote the set of P-products $b_{n_1}b_{n_2}\cdots b_{n_k}$ with $b_{n_1} > a$ and $b_{n_1} > a^{-1}$." should be replaced by "For each $a \in G$, pick $r_a \in \mathbb{N}$ such that $a < b_{r_a}$ and let Q_a denote the set of P-products $b_{n_1}b_{n_2}\cdots b_{n_k}$ such that $b_{n_1} \in P_{r_a}$."

For the proof of Theorem 8.63, the sentence "If we choose n such that $a < b_n$ and $a < b_n^{-1}$, it follows from Lemma 8.48 that $aT_n \cap T_m = \emptyset$." should be replaced by "If we choose $r \in \mathbb{N}$ such that $a < b_r$ and choose n such that $b_n \in P_r$, it follows from Lemma 8.48 that $aT_n \cap T_m = \emptyset$."

Finally, the statement of Lemma 8.64 needs to specify that $x \neq e$ and the proof needs revision.

Lemma 8.64. *Let G be a countably infinite discrete group and let p be a right cancelable element of G^*. Suppose that $x \in \beta G \setminus \{e\}$, $y \in T_\infty$, and $xy \in \overline{T}$. Then $x \in \overline{T}$.*

Proof. Suppose that $x \notin \overline{T}$ and let $X = G \setminus (T \cup \{e\})$. For each $a \in X$, pick $r_a \in \mathbb{N}$ such that $a < r_a$. Let Z be the set of all products of the form $ab_{n_1}b_{n_2}\cdots b_{n_k}$ where $b_{n_1}b_{n_2}\cdots b_{n_k}$ is a P-product, $a \in X$, and $b_{n_1} \in P_{r_a}$. By Theorem 4.15, $Z \in xy$. Since $T \in xy$, pick $a \in X$ and a P-product $b_{n_1}b_{n_2}\cdots b_{n_k}$ such that $b_{n_1} \in P_{r_a}$ and $ab_{n_1}b_{n_2}\cdots b_{n_k} \in T$. Then by Lemma 8.48, $a \in T$, a contradiction. □

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