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The Mathematics of Bruce Rothschild

By

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Abstract. A review is given of some of the mathematical research of Bruce Rothschild, emphasizing his results in combinatorial theory, especially that part known as Ramsey Theory. Special emphasis is given to the *Graham-Rothschild Parameter Sets Theorem*, its consequences, and some extensions.

1. Introduction

This paper is being written to honor the accomplishments of Bruce Rothschild who, along with Basil Gordon, brought the *Journal of Combinatorial Theory, Series A* to its current prestigious standing. The bibliography at the end of this paper lists 56 papers and a book authored by Bruce Rothschild. Three of these papers ([R42], [R54], and [R57]) are applications of mathematics to biology. Two others ([R14] and [R15]) deal with Lie algebras. All of the rest are combinatorial in nature, and a majority of these deal with subjects which I would classify as Ramsey Theory. I have neither the time nor the ability to describe all of these results, and shall instead present a modest sample.

I acknowledge a personal honor at being asked to write this paper. When I first met Bruce in 1972 he was very kind to a young and not too confident mathematician. Ever since he has been a good friend.

At the time of our first meeting Bruce showed me a desk in his office which was piled high with papers. He told me that these were papers in Ramsey Theory and that he (along with Ronald Graham and Joel Spencer) was in the process of writing a book on the subject. That book [R40] is the defining source for the field. Since Bruce is so strongly identified with Ramsey Theory, that will be the main emphasis of this paper. In Section 2 I will give a brief historical introduction to Ramsey Theory in general. In Section 3 I will discuss several of Bruce's contributions to the field, reserving the Parameter Sets Theorem for its own section. In Section 5 I will discuss some of Bruce's other combinatorial results. A final section provides some brief remarks about Bruce's mathematical genealogy.

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2. Ramsey Theory

Of course all of the material in this section is covered in [R40] so the reader who wants to see more detail is referred there.

Ramsey Theory began in 1892 with the following result of D. Hilbert [8]. (Here $FS(\langle x_t\rangle_{t=1}^n) = \big\{\sum_{t\in F}\ x_t : \emptyset \neq F \subseteq \{1,2,\ldots,n\}\big\}$. We let $\mathbb{N}$ be the set of positive integers.)

2.1 Theorem (Hilbert). *Given any $n \in \mathbb{N}$, whenever $\mathbb{N}$ is partitioned into finitely many cells (or "finitely colored") there exist $a \in \mathbb{N}$ and $\langle x_t\rangle_{t=1}^n$ in $\mathbb{N}$ such that $a + FS(\langle x_t\rangle_{t=1}^n)$ is contained in one cell (or "is monochromatic").*

The next major result in the field was the 1916 result of Schur [18].

2.2 Theorem (Schur). *Whenever $\mathbb{N}$ is finitely colored there exist x and y with $\{x, y, x+y\}$ monochromatic.*

(Thus Schur's Theorem is Hilbert's Theorem with $n = 2$ and without the translate a.) This was followed in 1927 by van der Waerden's Theorem [21].

2.3 Theorem (van der Waerden). *Whenever $\mathbb{N}$ is finitely colored, there exist arbitrarily long (but finite) monochromatic arithmetic progressions.*

Next appeared Ramsey's Theorem itself [17], proved in 1930. Given a set A and $k \in \mathbb{N}$, $[A]^k = \{B : B \subseteq A \text{ and } |B| = k\}$. Ramsey's Theorem says:

2.4 Theorem (Ramsey). *Given any infinite set A, any $k \in \mathbb{N}$ and any finite coloring of $[A]^k$, there is some infinite $C \subseteq A$ with $[C]^k$ monochromatic.*

One could be excused for asking why the field is called *Ramsey Theory* given that Ramsey's Theorem is the fourth major result in the area. Notice however, that Ramsey's Theorem is of a more general structural variety than the earlier results which applied only to the semigroup $(\mathbb{N}, +)$.

Notice that by a standard "compactness" argument, a finite version of Ramsey's Theorem follows. (The finite version can also be derived directly, and indeed was in the original paper.) There are several ways of phrasing a compactness argument. I shall present one which uses topological compactness.

2.5 Corollary (Ramsey). *Let $k, m, r \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $\varphi : [\{1,2,\ldots,n\}]^k \to \{1,2,\ldots,r\}$, there must exist $B \in [\{1,2,\ldots,n\}]^m$ such that φ is constant on $[B]^k$.*

Proof. Suppose that for each $n \in \mathbb{N}$ one has $\psi_n : [\{1,2,\ldots,n\}]^k \to \{1,2,\ldots,r\}$ so that for every $B \in [\{1,2,\ldots,n\}]^m$, ψ_n is not constant on $[B]^k$. For each $n \in \mathbb{N}$, define $\psi'_n : [\mathbb{N}]^k \to \{1,2,\ldots,r\}$ so that for $C \in [\mathbb{N}]^k$,

$$\psi'_n(C) = \begin{cases} \psi_n(C) & \text{if } C \subseteq \{1,2,\ldots,n\} \\ 1 & \text{if } C \setminus \{1,2,\ldots,n\} \neq \emptyset \,. \end{cases}$$

Let μ be a cluster point of the sequence $\langle \psi'_n \rangle_{n=m}^{\infty}$ in the compact product space $Y = \times_{C \in [\mathbb{N}]^k} \{1,2,\ldots,r\}$, where $\{1,2,\ldots,r\}$ has the discrete topology.

Pick by Theorem 2.4 an infinite subset A of $\mathbb{N}$ and $i \in \{1,2,\ldots,r\}$ such that $\mu(C) = i$ for all $C \in [A]^k$. Pick $B \in [A]^m$ and let $U = \{\tau \in Y : \text{for all } C \in [B]^k \,,\, \tau(C) = \mu(C)\}$. Then U is a neighborhood of τ in Y so pick $n > \max B$ such that $\psi'_n \in U$. Then ψ_n is constantly equal to i on $[B]^k$, a contradiction. □

Of special interest in Corollary 2.5 is the case $k = 2$. Then one may rephrase the result in graph theoretic terminology as follows.

2.6 Corollary. *Let $m, r \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever the edges of the complete graph K_n on n vertices are r-colored, there must exist a monochromatic copy of K_m.*

A great deal of effort has gone in to finding bounds (or in rare cases exact values) for the number n in Corollary 2.6. For up to date information on these efforts see the dynamic survey [16]. Because of the interest in computing Ramsey numbers, the following special case of Corollary 2.6 has also received substantial interest.

2.7 Corollary. *Let G and H be finite graphs. There exists $n \in \mathbb{N}$ such that whenever the edges of the complete graph K_n on n vertices are 2-colored (red and blue), there must exist a red copy of G or a blue copy of H.*

The next major result in Ramsey Theory was Rado's 1933 solution [15] to the problem of partition regularity of systems of homogeneous linear equations. (The result generalizes Hilbert's, Schur's and van der Waerden's Theorems). A routinely checkable (though NP-complete) condition called the "columns condition" is defined for a finite matrix.

2.8 Definition. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with rational entries and let $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ be the columns of A. Then A satisfies the *columns condition* if and only if there exist $m \in \{1, 2, \ldots, v\}$ and a partition $\{I_1, I_2, \ldots, I_m\}$ of $\{1, 2, \ldots, v\}$ such that $\sum_{t\in I_1} \vec{c_t} = \vec{0}$ and for each $j \in \{2, 3, \ldots, m\}$, if any, $\sum_{t\in I_j} \vec{c_t}$ is a linear combination of the columns from $\bigcup_{i=1}^{j-1} I_i$.

Given a matrix A with rational entries, one says that A is *partition regular* over $\mathbb{N}$ if and only if for any finite coloring of $\mathbb{N}$, there is some monochromatic vector $\vec{x}$ with $A\vec{x} = \vec{0}$.

2.9 Theorem (Rado). *A matrix A is partition regular over $\mathbb{N}$ if and only if A satisfies the columns condition.*

Another fundamental result which rivals Ramsey's Theorem itself in generality is the Hales-Jewett Theorem [7]. Let A be a nonempty finite set and let S be the free semigroup over the alphabet A. The members of S are "words" and concatenation is the operation. When discussing the Graham-Rothschild Parameter Sets Theorem in Section 4 we will need to be a little more formal, but for now an intuitive description is quite adequate. If for example, the alphabet $A = \{a, b, c\}$ then $u = acbaccb$ and $w = abba$ are typical members of S and $u \cdot w = acbaccbabba$. Let v be a "variable" which is not a member of A. A *variable word* is a word over the alphabet $A \cup \{v\}$ in which v occurs. Given a variable word w and $a \in A$, $w\langle a\rangle$ is the result of replacing each occurrence of v with a. For example, if $A = \{a, b, c\}$ and $w = avbvva$, then $w\langle a\rangle = aabaaa$ and $w\langle c\rangle = acbcca$.

2.10 Theorem (Hales and Jewett). *Let A be a finite nonempty alphabet, let S be the free semigroup over A and let S be finitely colored. Then there exists a variable word w such that $\{w\langle a\rangle : a \in A\}$ is monochromatic.*

Using a compactness argument, one also can derive a finite version of Theorem 2.10.

2.11 Corollary (Hales and Jewett). *Let $k, r \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever the length n words over an alphabet A with k members are r-colored, there is a length n variable word w such that $\{w\langle a\rangle : a \in A\}$ is monochromatic.*

In a remarkable result, S. Shelah [19] proved that n in Corollary 2.11 is bounded by a primitive recursive function of k and r. This result is presented in the second edition of [R40].

I will conclude this introductory section with a discussion of finite sums and finite unions. This subject is important to me in the context of this paper, because it was this subject which introduced me to Bruce in the first place. We have already introduced the notation $FS(\langle x_t\rangle_{t=1}^n)$. Similarly, if $\langle D_t\rangle_{t=1}^n$ is a sequence of sets, then $FU(\langle D_t\rangle_{t=1}^n) = \big\{\bigcup_{t\in F} D_t : \emptyset \neq F \subseteq \{1,2,\ldots,n\}\big\}$. Given any set X, let $\mathcal{P}_f(X)$ be the set of finite non-empty subsets of X. Then given infinite sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle D_n\rangle_{n=1}^\infty$ one defines analogously $FS(\langle x_n\rangle_{n=1}^\infty) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ and $FU(\langle D_n\rangle_{n=1}^\infty) = \{\bigcup_{n\in F} D_n : F \in \mathcal{P}_f(\mathbb{N})\}$.

The following theorem, which I will refer to as the finite version of the Finite Sums Theorem, is a consequence of Rado's Theorem (Theorem 2.9). Notice that the theorem eliminates the need for the translate a from Hilbert's Theorem (Theorem 2.1).

2.12 Theorem. *Let $k \in \mathbb{N}$. Whenever $\mathbb{N}$ is finitely colored, there exists a sequence $\langle x_t\rangle_{t=1}^k$ such that $FS(\langle x_t\rangle_{t=1}^k)$ is monochromatic.*

To see, for example, that the $k = 3$ instance of Theorem 2.12 follows from Rado's Theorem, simply note that the following matrix satisfies the columns condition with $I_1 = \{4,5,6,7\}$, $I_2 = \{2,3\}$, and $I_3 = \{1\}$.

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

The following theorem does not seem to follow from Rado's Theorem nor from Theorem 2.12. It was first proved in [R13] as a consequence of the Parameter Sets Theorem. (I shall present its derivation in Section 4.)

2.13 Theorem. *Let $k \in \mathbb{N}$. Whenever $\mathcal{P}_f(\mathbb{N})$ is finitely colored, there exists a sequence $\langle D_t\rangle_{t=1}^k$ of pairwise disjoint members of $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle D_t\rangle_{t=1}^k)$ is monochromatic.*

In [R13] the authors asked whether the infinite version of Theorem 2.12 was valid. I eventually succeeded in proving that it is [9]. Subsequently a simpler proof was found by J. Baumgartner [1] and a much simpler algebraic proof was found by F. Galvin and S. Glazer. (See [10, Section 5.2] and the notes to Chapter 5 of [10] for this simple proof and a historical account.)

2.14 Theorem (Finite Sums Theorem). *Let $\mathbb{N}$ be finitely colored. There is a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^\infty)$ is monochromatic.*

It was at the time of the proof of Theorem 2.14 that I first met Bruce. He informed me that the infinite Finite Unions Theorem is a corollary of Theorem 2.14. Notice the contrast with the fact that Theorem 2.13 does not seem to be a corollary to Theorem 2.12 (though the other implication is trivial).

2.15 Corollary (Finite Unions Theorem). *Let $\mathcal{P}_f(\mathbb{N})$ be finitely colored. There is a sequence $\langle D_n\rangle_{n=1}^{\infty}$ of pairwise disjoint members of $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle D_n\rangle_{n=1}^{\infty})$ is monochromatic.*

The key to the derivation of Corollary 2.15 is that given any sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$, one can find a sequence $\langle y_n\rangle_{n=1}^{\infty}$ such that $FS(\langle y_n\rangle_{n=1}^{\infty}) \subseteq FS(\langle x_n\rangle_{n=1}^{\infty})$ and for each $n \in \mathbb{N}$, $\max \operatorname{supp}(y_n) < \min \operatorname{supp}(y_{n+1})$, where $x = \sum_{t\in \operatorname{supp}(x)} 2^t$.

3. Bruce Rothschild and Ramsey Theory

In this section I will discuss some of Bruce's results that I classify as Ramsey Theoretic.

Sometime before 1967 G. Rota made a conjecture that was to play an important role in the mathematics of Bruce Rothschild.

3.1 Conjecture (Rota). *Let $m, k, r \in \mathbb{N}$ with $m < k$ and let F be a finite field. There exists a vector space V over F with the property that whenever the m-dimensional subspaces of V are r-colored, there must exist a k-dimensional subspace W of V whose m-dimensional subspaces are monochromatic.*

Related to this is the corresponding statement about affine subspaces. (An affine subspace of a vector space is a translate of a vector subspace.)

3.2 Conjecture. *Let $m, k, r \in \mathbb{N}$ with $m < k$ and let F be a finite field. There exists a vector space V over F with the property that whenever the m-dimensional affine subspaces of V are r-colored, there must exist a k-dimensional affine subspace W of V whose m-dimensional affine subspaces are monochromatic.*

In his dissertation [R0] Bruce established that if Conjecture 3.2 is true for a fixed m and all k, r, and F, then Conjecture 3.1 is valid for $m + 1$ and all k, r, and F. In [R11] (joint with R. Graham) the equivalence of Conjectures 3.1 and 3.2 was established.

We shall see in the next section that the $m = 1$ case of Conjecture 3.2 is a consequence of the Graham-Rothschild Parameter Sets Theorem. Rota's conjecture was proved in its entirety in [R18]. A simplified proof can be found in [R40].

The paper [R12] is a rarity – one of only three of Bruce's papers written without a coauthor. This is not surprising because Bruce is a friendly person and for many mathematicians (including myself) mathematics is a social affair. It is much more fun when you have someone with whom you can talk about your results.

In this paper Bruce considered generalizations of a theorem of T. Motzkin [12]. A set $S \subseteq \mathbb{N}$ is *blocked* by a family $\mathcal{C} \subseteq \mathcal{P}(S)$ if and only if whenever π is a permutation of S, there is some $k \in \mathbb{N}$ such that $\big\{\pi(i) : i \in S \cap \{1, 2, \ldots, k\}\big\} \in \mathcal{C}$. The simplest examples of blocking families for S are $\mathcal{C} = [S]^k$ for some $k \leq |S|$. Other examples are easy to come by. For instance $\big\{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\big\}$ blocks $\{1, 2, 3\}$.

3.3 Theorem (Motzkin). *Let $k, m, r \in \mathbb{N}$ with $m \geq k$. There exists $n \in \mathbb{N}$ such that whenever $\mathcal{C} \subseteq \mathcal{P}(\{1, 2, \ldots, n\})$, if $\mathcal{C}$ blocks $\{1, 2, \ldots, n\}$, for each $F \in \mathcal{C}$, $|F| \leq k$, and $\mathcal{C} = \bigcup_{i=1}^{r} \mathcal{D}_i$, then there exist $T \in [\{1, 2, \ldots, n\}]^m$ and $i \in \{1, 2, \ldots, r\}$ such that $\mathcal{D}_i \cap \mathcal{P}(T)$ blocks T.*

Notice that if the requirement $|F| \leq k$ is changed to $|F| = k$, the substance of Theorem 3.3 is exactly the same as the finite version of Ramsey's Theorem, Corollary 2.5. (A set of k-element subsets blocks T if and only if it consists of all of the k-element subsets of T.)

Erdős conjectured that a particular infinite version of Theorem 3.3 was valid.

3.4 Conjecture (Erdős). *Let $\mathcal{C} \subseteq \mathcal{P}(\mathbb{N})$ and assume that $\mathcal{C}$ blocks $\mathbb{N}$. Whenever $r \in \mathbb{N}$ and $\mathcal{C} = \bigcup_{i=1}^{r} \mathcal{D}_i$, there exist $i \in \{1, 2, \ldots, r\}$ and an infinite subset T of $\mathbb{N}$ such that $\mathcal{D}_i \cap \mathcal{P}(T)$ blocks T.*

In [R12] Bruce proved some other infinite extensions of Theorem 3.3. These extensions involve more terminology than I care to introduce at this point. In that paper he also presented a counterexample to Conjecture 3.4, obtained with the assistance of M. Perles and E. Straus.

In collaboration with P. Erdős, R. Graham, P. Montgomery, J. Spencer, and E. Straus, Bruce wrote a series of three papers ([R21], [R24], and [R25]) dealing with "Euclidean Ramsey Theory".

3.5 Definition. Let K be a finite set of points in $\mathbb{R}^m$ for some $m \in \mathbb{N}$ and let $n, r \in \mathbb{N}$.

(a) $R(K, n, r)$ denotes the statement "whenever $\mathbb{R}^n$ is r-colored, there is a monochromatic set L which is congruent to K."

(b) The set K is *Ramsey* if and only if for each $r \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that $R(K, n, r)$ holds.

In [R21] the truth of $R(K, n, r)$ was investigated for several specific sets K and various values of n and r. And the following two basic theorems were proved.

3.6 Theorem (Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus). *Let K be a finite set of points in $\mathbb{R}^m$ for some $m \in \mathbb{N}$. If K is Ramsey, then K can be embedded in the surface of some sphere.*

3.7 Theorem (Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus). *Let K be a finite Cartesian product of 2-point sets. Then K is Ramsey (and so every subset of K is Ramsey).*

It has since been shown that all triangles are Ramsey by Frankl and Rödl [6]. In a later paper [11], Igor Kříž showed that all regular polygons are Ramsey and all regular polyhedra in $\mathbb{R}^3$ are Ramsey.

In [R24] some asymmetric versions of $R(K, n, r)$ along the line of Corollary 2.7 were investigated. The authors obtained several theorems of which the following is typical.

3.8 Theorem (Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus). *Let $\mathbb{R}^2$ be 2-colored (red and blue). Then either there exist two red points at distance one or there exist four blue points in a line at intervals of length one.*

Also in [R24] infinite questions were investigated, dealing both with infinite dimensional spaces and with infinite sets in finite dimensional spaces. As an example of the latter consider the following.

3.9 Theorem (Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus). *There is a set $S \subseteq \mathbb{R}$ with $|S| = \mathfrak{c}$ such that $\mathbb{R}$ can be 2-colored with no monochromatic points x and y with $x - y \in S$. Such a set cannot have positive measure.*

The third paper on Euclidean Ramsey Theory [R25] concentrated on $R(K, 2, 2)$ where K is a 3-element subset of $\mathbb{R}^2$. The starting point is the following theorem from [R21].

3.10 Theorem (Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus). *Let K be an equilateral triangle. Then $R(K, 2, 2)$ is false.*

Proof. Let d be the length of a side of K. Color a point $(x, y) \in \mathbb{R}^2$ red if $\left\lfloor \frac{2y}{d\sqrt{3}} \right\rfloor$ is even and blue otherwise. (Thus $\mathbb{R}^2$ is divided into half open strips of height $d\frac{\sqrt{3}}{2}$.) If one had a monochromatic copy of K, two of the vertices would have to lie in the same strip, which is then not tall enough to accomodate the third vertex. $\square$

In [R25] it was conjectured that for any non equilateral triangle K, $R(K, 2, 2)$ is true. The conjecture was verified for a substantial collection of triangles K. As far as I know, the conjecture remains unsettled.

Some years later, half of the authors of [R21], [R24], and [R25], namely Erdős, Rothschild, and Straus, returned to the subject in [R47]. There they concentrated, given $k \in \mathbb{N}$, on producing sets K with the property that for each $r \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that whenever $\mathbb{R}^n$ is r-colored, there is a congruent copy of K which uses at most k colors, but for each $r \in \mathbb{N}$ and for each $n \in \mathbb{N}$ there is an r-coloring of $\mathbb{R}^n$ such that any congruent copy of K uses at least k colors.

In [R26], in collaboration with R. Graham, a simplification of a standard combinatorial proof of van der Waerden's Theorem was given.

In [R33], in collaboration with M. Cates, P. Erdős, and me the following question was addressed. (This is the unique joint paper which I have with Bruce.)

3.11 Question. *Let α, β, γ, and δ be cardinals with $\delta \leq \omega$. If V is an α-dimensional vector space over $\{0, 1\}$ and V is γ-colored, must there exist $U \in [V]^\beta$ such that $\{\sum W : W \subseteq U$ and $|W| < \delta\}$ is monochromatic.*

The question is answered for most cases under the assumption of GCH and the nonexistence of regular limit cardinals greater than ω. It was left open whether this statement is true for $\alpha = \beta = \aleph_\omega$, $\gamma = 2$, and $\delta = 4$. As far as I know that question is still open.

Given $m \in \mathbb{N}$ and a graph G, if G contains some K_{m+1}, then whenever the vertices are colored with m colors, some two adjacent vertices must get the same color. (That is "G is not m-colorable".) On the other hand, it has been known for some time that for any $m \in \mathbb{N}$ there exist graphs which are contain no triangle but are not m-colorable. (See [2] for details about the history of this fact.) In particular, not every K_{m+1}-free graph is m-colorable. However in [R34] and [R52] Bruce had a hand in showing that almost all K_{m+1}-free graphs are m-colorable. Specifically, for $n, m \in \mathbb{N}$ let $L_n(m)$ be the number of labeled K_{m+1}-free graphs on $\{1, 2, \ldots, n\}$ and let and let $C_n(m)$ be the number of labeled m-colorable graphs on $\{1, 2, \ldots, m\}$. In [R34] the following theorem was proved.

3.12 Theorem (Erdős, Kleitman, and Rothschild). *For all $m \in \mathbb{N}$,*

$$\lim_{n \to \infty} \frac{\log\big(L_n(m)\big)}{\log\big(C_n(m)\big)} = 1\,.$$

In [R52] this result was improved.

3.13 Theorem (Kolaitis, Prömel, and Rothschild). *For all* $m \in \mathbb{N}$*,*

$$\lim_{n\to\infty} \frac{L_n(m)}{C_n(m)} = 1\,.$$

The final result in this section is not a result about Ramsey Theory, but rather a result which uses Ramsey Theory. Ramsey Theory has been widely applied from its beginning. In fact a perusal of the titles of several of the original papers reveals that many of these results were obtained with applications in mind. (The Hales-Jewett Theorem – Game Theory; Hilbert's Theorem – Algebra; Schur's Theorem – Number Theory; Ramsey's Theorem – Logic). In this case the result is an application of Ramsey's Theorem to solve instances of the Generalized Banach Contraction Conjecture.

Banach's famous Fixed Point Theorem says that if (X, d) is a complete metric space, $T : X \to X$, and there is some $M \in (0, 1)$ such that for all $x, y \in X$, $d\big(T(x), T(y)\big) \leq M \cdot d(x, y)$, then there is some $x \in X$ such that $T(x) = x$.

3.14 Definition. The *Generalized Banach Contraction Conjecture* is the assertion that for all $J \in \mathbb{N}$, if (X, d) is a complete metric space, $T : X \to X$, and there is some $M \in (0, 1)$ such that for all $x, y \in X$, $\min\big\{d\big(T^k(x), T^k(y)\big) : k \in \{1, 2, \ldots, J\}\big\} \leq M \cdot d(x, y)$, then there is some $x \in X$ such that $T(x) = x$.

Thus, Banach's Fixed Point Theorem is the case $J = 1$ of the Generalized Banach Contraction Conjecture. In [R56] the Generalized Banach Contraction Conjecture was proved under the additional assumption that T is continuous.

3.15 Theorem (Merryfield, Rothschild, and Stein). *Let* (X, d) *be a complete metric space, let* T *be a continuous function from* X *to* X*, and let* $J \in \mathbb{N}$*. If there is some* $M \in (0, 1)$ *such that for all* $x, y \in X$*,* $\min\big\{d\big(T^k(x), T^k(y)\big) : k \in \{1, 2, \ldots, J\}\big\} \leq M \cdot d(x, y)$*, then there is some* $x \in X$ *such that* $T(x) = x$*.*

The proof used Ramsey's Theorem. And the authors were able to use Theorem 3.15 to prove the case $J = 3$ without the assumption of the continuity of T.

4. The Parameter Sets Theorem

A separate section is devoted to the Graham-Rothschild Parameter Sets Theorem because, in my view, it is a monumental result. This is not my view alone. Consider the following statement by H. Prömel and B. Voigt in [14].

> This is a complete analogue to Ramsey's theorem carried over to the structures of parameter sets and, as it turns out, Ramsey's theorem itself

is an immediate consequence of the Graham-Rothschild theorem. But the concept of parameter sets does not only glue arithmetic progressions and finite sets together. Also, it provides a natural framework for seemingly different structures like Boolean lattices, partition lattices, hypergraphs and Deuber's (m, p, c)-sets, just to mention a few. So, the Graham-Rothschild theorem can be viewed as a starting point of *Ramsey Theory*.

We shall state a simplified version of the Parameter Sets Theorem. For the full version, one may see of course [R13] itself. Also it can be found in [14] and in [4]. (Although it will not be clear at first glance that these three statements are all saying the same thing.) It is shown in [4, Theorem 5.1] that the full version is in fact derivable from the version which will be stated here.

The Parameter Sets Theorem, like the Hales-Jewett Theorem, uses variable words over finite alphabets. However, one has infinitely many variables, and in order to state certain things, one must be more careful about formalities. Throughout this section, let A denote a nonempty alphabet. Let $\omega = \mathbb{N} \cup \{0\}$. Choose a set $V = \{v_n : n \in \omega\}$ (of *variables*) such that $A \cap V = \emptyset$ and define W to be the semigroup of words over the alphabet $A \cup V$, including the empty word. (Formally a *word* w is a function from an initial segment $\{0, 1, \ldots, k-1\}$ of ω to the alphabet and the length $\ell(w)$ of w is k. We shall occasionally need to resort to this formal meaning, so that if $i \in \{0, 1, \ldots, \ell(w)-1\}$, then $w(i)$ denotes the $(i+1)^{\mathrm{st}}$ letter of w.)

4.1 Definition. Let $n \in \mathbb{N}$, let $k \in \omega$ with $k \leq n$, and let $\emptyset \neq B \subseteq A$. Then $[B]\binom{n}{k}$ is the set of all words w over the alphabet $B \cup \{v_0, v_1, \ldots, v_{k-1}\}$ of length n such that

(1) for each $i \in \{0, 1, \ldots, k-1\}$, if any, v_i occurs in w and

(2) for each $i \in \{0, 1, \ldots, k-2\}$, if any, the first occurrence of v_i in w precedes the first occurrence of v_{i+1}.

4.2 Definition. Let $k \in \omega$. Then the *set of k-variable words* is $S_k = \bigcup_{n=k}^{\infty} [A]\binom{n}{k}$.

Given $w \in S_n$ and $u \in W$ with $\ell(u) = n$, we define $w\langle u \rangle$ to be the word with length $\ell(w)$ such that for $i \in \{0, 1, \ldots, \ell(w) - 1\}$

$$w\langle u \rangle(i) = \begin{cases} w(i) & \text{if } w(i) \in A \\ u(j) & \text{if } w(i) = v_j\,. \end{cases}$$

That is, $w\langle u \rangle$ is the result of substituting $u(j)$ for each occurrence of v_j in w. For example, if $A = \{a, b, c\}$, $w = av_0bv_0v_1cbv_2v_0ba$, and $u = cv_0a$, then $w\langle u \rangle = acbcv_0cbacba$.

4.3 Theorem (Graham and Rothschild). *Assume that the alphabet A is finite, let $m, k \in \omega$ with $m < k$, and let S_m be finitely colored. There exists $w \in S_k$ such that $\left\{w\langle u \rangle : u \in [A]\binom{k}{m}\right\}$ is monochromatic.*

Note that the Hales-Jewett Theorem is the case $m = 0$ and $k = 1$ of the Parameter Sets Theorem.

One might wonder whether the restriction on the order of first occurrence of the variables in Definition 4.1(2) is needed. To see that it is, we note that without it the instance $m = 2$ and $k = 3$ of Theorem 4.3 is false. One can simply define $\varphi : S_2 \to \{1, 2\}$ by $\varphi(w) = 1$ if and only if the first occurrence of v_0 in w preceeds the first occurrence of v_1. Then given any $w \in S_3$, $\varphi(w\langle v_0 v_1 v_1\rangle) \neq \varphi(w\langle v_1 v_0 v_0\rangle)$.

Using a standard compactness argument one obtains the following finite version of the Parameter Sets Theorem. (In [R13] it was a finite version that was established and the proof yielded (very large) bounds.)

4.4 Theorem (Graham and Rothschild). *Assume that the alphabet A is finite, let $m, k \in \omega$ with $m < k$ and let $r \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $[A]\binom{n}{m}$ is r-colored, there exists $w \in [A]\binom{n}{k}$ such that $\big\{w\langle u\rangle : u \in [A]\binom{k}{m}\big\}$ is monochromatic.*

I interrupt the discussion of the Parameter Sets Theorem to present another result which requires the notation which has just been introduced.

In [20] Joel Spencer proved a conjecture of Erdős that for each $k, r \in \mathbb{N}$ there is a set $T \subseteq \mathbb{N}$ which contains no $(k + 1)$-term arithmetic progression but whenever T is r-colored, T contains a monochromatic k-term arithmetic progression. In [R46] Deuber, Prömel, Rothschild, and Voigt proved the following analogous restricted version of the Hales-Jewett Theorem.

4.5 Theorem (Deuber, Prömel, Rothschild, and Voigt). *Assume that the alphabet A is finite, let $m, r \in \omega$ with $m < k$. Then there exist $n \in \mathbb{N}$ and $T \subseteq [A]\binom{n}{0}$ such that*

(1) there is no $w \in [A]\binom{n}{m+1}$ with $\{w\langle u\rangle : u \in [A]\binom{m+1}{0}\} \subseteq T$ but

(2) whenever T is r-colored, there is some $w \in [A]\binom{n}{m}$ such that $\{w\langle u\rangle : u \in [A]\binom{m}{0}\}$ is monochromatic.

In [R53] Prömel and Rothschild produced the following extension of Spencer's result which allows one to use infinitely many colors.

4.6 Theorem (Prömel and Rothschild) Let $k \in \mathbb{N}$. *There exists $T \subseteq \mathbb{N}$ such that T contains no $(k+1)$-term arithmetic progression but for any coloring of T by any number of colors, there is a k-term arithmetic progression which is either monochromatic or else has no two terms with the same color.*

Section 9 of [R13] contains 13 corollaries. Included among these are four results that were known at the time (namely the Hales-Jewett Theorem, van der Waerden's Theorem, Ramsey's Theorem, and the finite version of the Finite Sums Theorem). I believe that the other nine were new. We have remarked in Section 2 that one of these results is the finite version of the Finite Unions Theorem. We demonstrate here how easily this result follows. (For notational convenience we shift the set being colored from $\mathcal{P}_f(\mathbb{N})$ to $\mathcal{P}_f(\omega)$.)

2.13 Theorem. *Let $k \in \mathbb{N}$. Whenever $\mathcal{P}_f(\omega)$ is finitely colored, there exists a sequence $\langle D_t\rangle_{t=1}^k$ of pairwise disjoint members of $\mathcal{P}_f(\omega)$ such that $FU(\langle D_t\rangle_{t=1}^k)$ is monochromatic.*

Proof. Let $A = \{0\}$. Let $r \in \mathbb{N}$ and let $\varphi : \mathcal{P}_f(\omega) \to \{1, 2, \dots, r\}$. Define $\tau : S_1 \to \mathcal{P}_f(\omega)$ by $\tau(w) = \{i \in \{0, 1, \dots, \ell(w) - 1\} : w(i) = v_0\}$. Pick by Theorem 4.3 some $w \in S_k$ and $i \in \{1, 2, \dots, r\}$ such that for all $u \in [A]\binom{k}{1}$, $\varphi \circ \tau(w\langle u\rangle) = i$. For $t \in \{1, 2, \dots, k\}$, let $D_t = \{i \in \{0, 1, \dots, \ell(w) - 1\} : w(i) = v_{t-1}\}$. □

For example, if $k = 3$ and the word $w \in S_3$ produced in the proof above is $0v_000v_0v_100v_2v_10v_0$, then $D_1 = \{1, 4, 11\}$, $D_2 = \{5, 9\}$, and $D_3 = \{8\}$. Then, $D_1 \cup D_3 = \tau(w\langle v_00v_0\rangle)$ and $D_2 \cup D_3 = \tau(w\langle 0v_0v_0\rangle)$.

A major motivation for [R13] was Rota's conjecture (Conjecture 3.1). The $m = 1$ instance of Rota's conjecture is a consequence of the Parameter Sets Theorem (and is one of the corollaries stated in [R13]).

4.7 Corollary. *Let $k, r \in \mathbb{N}$ with $1 < k$ and let F be a finite field. There exists a vector space V over F with the property that whenever the 1-dimensional subspaces of V are r-colored, there must exist a k-dimensional subspace W of V whose 1-dimensional subspaces are monochromatic.*

Proof. Let $q = |F \setminus \{0\}|$ and enumerate $F \setminus \{0\}$ as $\{a_0, a_1, \dots, a_{q-1}\}$. Define $\lambda : F \setminus \{0\} \to \{0, 1, \dots, q-1\}$ by $\lambda(a_t) = t$. Pick by Theorem 4.4 some $n \in \mathbb{N}$ such that whenever $[F]\binom{n}{q}$ is r-colored there exists $w \in [F]\binom{n}{qk}$ such that $\{w\langle u\rangle : u \in [F]\binom{qk}{q}\}$ is monochromatic. Let $V = F^n$. (Formally V and $[F]\binom{n}{0}$ are identical.)

Let $\mathcal{O}$ be the set of 1-dimensional subspaces of V and let $\varphi : \mathcal{O} \to \{1, 2, \dots, r\}$. Define $\sigma : [F]\binom{n}{q} \to V$ by, for $w \in [F]\binom{n}{q}$ and $i \in \{0, 1, \dots, n-1\}$,

$$\sigma(w)(i) = \begin{cases} 0 & \text{if } w(i) \in F \\ a_t & \text{if } w(i) = v_t\,. \end{cases}$$

Define $\tau : [F]\binom{n}{q} \to \mathcal{O}$ by $\tau(w) = \{b \cdot \sigma(w) : b \in F\}$. Then $\varphi \circ \tau$ r-colors $[F]\binom{n}{q}$ so pick $w \in [F]\binom{n}{qk}$ such that $\{w\langle u\rangle : u \in [F]\binom{qk}{q}\}$ is monochromatic. For $j \in \{0, 1, \dots, k-1\}$

define $z_j \in V$ by, for $i \in \{0, 1, \ldots, n-1\}$

$$z_j(i) = \begin{cases} a_t & \text{if } w(i) = v_{qj+t} \text{ for some } t \in \{0, 1, \ldots, q-1\} \\ 0 & \text{otherwise.} \end{cases}$$

The z_j's are linearly independent, having pairwise disjoint supports. Let W be the subspace of V generated by the z_j's.

To complete the proof it suffices to show that if $s \in W$, then there is some $u \in [F]\binom{qk}{q}$ such that $\{b \cdot s : b \in F\} = \tau(w\langle u\rangle)$. To this end let $s \in W$ and pick $b_0, b_1, \ldots, b_{k-1} \in F$ such that $s = \sum_{j=0}^{k-1} b_j \cdot z_j$. We may assume that if l is the first such that $b_l \neq 0$, then $b_l = 1$ since the subspaces generated by s and ${b_l}^{-1} \cdot s$ are the same. Define $u \in [F]\binom{qk}{q}$ as follows. If $j \in \{0, 1, \ldots, k-1\}$ and $b_j = 0$, then for all $t \in \{0, 1, \ldots, q-1\}$, $u(qj + t) = 0$. If $b_j \neq 0$, then for all $t \in \{0, 1, \ldots, q-1\}$, $u(qj + t) = v_{\lambda(b_j \cdot a_t)}$.

Since the first l such that $b_l \neq 0$ has $b_l = 1$, one has for that l and all $t \in \{0, 1, \ldots, k-1\}$, $u(ql + t) = v_t$ and one has $u(i) = 0$ for all $i < ql$. Thus each v_t occurs and the first occurrence of v_t preceeds the first occurrence of v_{t+1} if $t + 1 < q$. That is $u \in [F]\binom{qk}{q}$.

Now we show that $s = \sigma(w\langle u\rangle)$. So let $i \in \{0, 1, \ldots, n-1\}$ be given. If $w(i) \in F$, then $z_j(i) = 0$ for all $j \in \{0, 1, \ldots, k-1\}$ so $s(i) = 0$. Also in this case $w\langle u\rangle(i) = w(i)$ so $\sigma(w\langle u\rangle)(i) = 0$. So assume that $w(i) = v_{qj+t}$ for some $j \in \{0, 1, \ldots, k-1\}$ and some $t \in \{0, 1, \ldots, q-1\}$. Then $s(i) = b_j \cdot z_j(i) = b_j \cdot a_t$. If $b_j = 0$, then $u(qj + t) = 0$ so $w\langle u\rangle(i) = 0$ so $\sigma(w\langle u\rangle)(i) = 0 = s(i)$. If $b_j \neq 0$, then $u(qj + t) = v_{\lambda(b_j \cdot a_t)}$ so $w\langle u\rangle(i) = v_{\lambda(b_j \cdot a_t)}$ and thus $\sigma(w\langle u\rangle)(i) = a_{\lambda(b_j \cdot a_t)} = b_j \cdot a_t$. □

In [3] T. Carlson obtained some very strong Ramsey Theoretic results. One of these, a direct consequence of [3, Theorem 10], is the following extension of Theorem 4.3. For an alternate derivation see [4].

4.8 Theorem (Carlson). *Assume that A is finite and for each $n \in \omega$, S_n has been finitely colored. Then there exists a sequence $\langle w_n\rangle_{n=0}^{\infty}$ with each $w_n \in S_n$ such that for every $m \in \omega$,*

$$S_m \cap \left\{ \prod_{n\in F} w_n\langle u_n\rangle : F \in \mathcal{P}_f(\omega) \text{ and for all } n \in F\,,\ u_n \in \bigcup_{i=0}^{\min F} [A]\binom{n}{i} \right\}$$

is monochromatic. (That is, the color of $\prod_{n\in F} w_n\langle u_n\rangle$ is determined solely by the number of variables in $\prod_{n\in F} w_n\langle u_n\rangle$.)

Applying Theorem 4.3 directly one sees for example that if S_0, S_1, and S_2 are finitely colored there exist $w, w', w'' \in S_3$ such that $\left\{w\langle u\rangle : u \in [A]\binom{3}{0}\right\}$ is monochromatic in S_0, $\{w'\langle u\rangle : u \in [A]\binom{3}{1}\}$ is monochromatic in S_1, and $\{w''\langle u\rangle : u \in [A]\binom{3}{2}\}$ is monochromatic in S_2. Applying Theorem 4.8 one sees that one can in fact choose $w = w' = w''$.

5. Other Combinatorial Results of Bruce Rothschild

In this section I will discuss some (but not all) of Bruce's combinatorial results which I would not classify as Ramsey Theoretical.

In an undirected graph (with loops allowed) an infinite path is *one way infinite* if it has an end point, and is otherwise *two way infinite*. In [13] Nash-Williams solved the problem of when a graph can be composed into k two way infinite paths, but no fewer. In [R1] Bruce solved the corresponding problem where the paths are allowed to be two way infinite, one way infinite, or finite.

In two papers with A. Whinston ([R3] and [R4]) Bruce investigated multiple flows in certain kinds of networks. I won't try to state the results of these papers as they require a substantial amount of terminology. However, in [R8] these authors together with D. Kleitman and A. Martin-Löf established a more abstract version which is easier to state.

5.1 Theorem (Kleitman, Martin-Löf, and Rothschild). *Let the vertices of an undirected graph be given labels $1, 2, \ldots, n, 1', 2', \ldots, n'$ in such a way that each vertex has at least $n-1$ different labels but no vertex has labels i and i' for any i. Then among all paths between a vertex labeled i and a vertex labeled i' for any i, the maximum number which are mutually edge disjoint equals the minimum size of an edge cut-set separating all vertices labeled j from all vertices labeled j' for any j.*

In the above theorem, a set C of edges of the graph G is an edge cut set provided when the edges in C are removed from G, none of the components of the resulting graph has a vertex labeled j and a vertex labeled j' for any j.

In two papers with D. Kleitman ([R9] and [R30]) Bruce investigated the number of partial orders on a given finite set. (Definitions of *partial order* vary. The only thing one can consistently count on is that it is a transitive relation. Here it is taken to be reflexive, antisymmetric, and transitive.) As is well known, the number of partial orders on $\{1, 2, \ldots, n\}$ is the same as the number of T_0 topologies on $\{1, 2, \ldots, n\}$. To see this let $\mathcal{PO}(n) = \{R : R$ is a reflexive, antisymmetric and transitive relation on $\{1, 2, \ldots, n\}\}$ and let $\mathcal{TO}(n) = \{\mathcal{V} : \mathcal{V}$ is a T_0 topology on $\{1, 2, \ldots, n\}\}$. Then the function $\varphi : \mathcal{TO}(n) \to \mathcal{PO}(n)$ defined by $\varphi(\mathcal{V}) = \{(x, y) : y \in c\ell_{\mathcal{V}}(\{x\})\}$ is a bijection.

5.2 Theorem (Kleitman and Rothschild). *For $n \in \mathbb{N}$, let $p(n) = |\mathcal{PO}(n)|$. There is a positive constant c such that for all n, $\frac{n^2}{4} \le \log_2 p(n) \le \frac{n^2}{4} + cn^{3/2}\log_2(n)$.*

In [R30] this result was improved.

5.3 Theorem (Kleitman and Rothschild). *For $n \in \mathbb{N}$, let $p(n) = |\mathcal{PO}(n)|$. There is a positive constant c such that for all n, $\frac{n^2}{4} + \frac{3n}{2} \le \log_2 p(n) \le \frac{n^2}{4} + \frac{3n}{2} + c\log_2(n)$.*

In [R20] the same two authors establish a result in game theory. They define a game called the (n_1, n_2)*-game* as follows. Players alternate choosing points in $\mathbb{R}^2$. Player i wins if she is the first to choose n_i points on a line on which the other player has made no choices.

5.4 Theorem (Kleitman and Rothschild). *For every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that the second player has a winning strategy in the (k, n)-game.*

The Erdős-Ko-Rado Theorem says that for all $n, k \in \mathbb{N}$ with $n \ge 2k$, if $\mathcal{A} \subseteq [\{1, 2, \ldots, n\}]^k$ and for all $B, C \in \mathcal{A}$, if $B \ne C$ then $B \cap C \ne \emptyset$, then $|\mathcal{A}| \le \binom{n-1}{k-1}$. (Equality is easily seen to hold if there is some $t \in \{1, 2, \ldots, k\}$ such that $\mathcal{A} = \{B \in [\{1, 2, \ldots, n\}]^k : t \in B\}$.) With A. Hajnal, Bruce established the following generalization [R22].

5.5 Theorem (Hajnal and Rothschild). *Let $k, r, s \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that for all $n \ge N$, if $\mathcal{A} \subseteq [\{1, 2, \ldots, n\}]^k$ and*

$$\max\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{A} \text{ and } (\forall A, B \in \mathcal{B})(A \ne B \Rightarrow |A \cap B| < s)\} \le r\,,$$

then $|\mathcal{A}| \le \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \binom{n-js}{k-js}$. Equality holds only if there is some $\mathcal{C} \subseteq [\{1, 2, \ldots, n\}]^s$ such that $|\mathcal{C}| = r$ and the members of $\mathcal{C}$ are pairwise disjoint and $\mathcal{A} = \{A \in [\{1, 2, \ldots, n\}]^k : (\exists B \in \mathcal{C})(B \subseteq A)\}$.

Notice that the Erdős-Ko-Rado Theorem is the case $s = r = 1$ of Theorem 5.5.

In [R27] the following three results were established.

5.6 Theorem (Graham, Rothschild, and Straus). *Let $n \in \mathbb{N}$.*

(a) There exist $n + 2$ points in $\mathbb{R}^n$ such that the distance between any two of them is an odd integer if and only if $n + 2 \equiv 0 \pmod{16}$.

(b) There exist $n + 2$ points in $\mathbb{R}^n$ such that the distance between any two of them is an integer relatively prime to 3 if and only if $n \equiv 1 \pmod{3}$.

(c) *There exist $n+2$ points in $\mathbb{R}^n$ such that the distance between any two of them is an integer relatively prime to* 6 *if and only if* $n \equiv -2 \pmod{48}$.

In [R31] the following theorem was proved.

5.7 Theorem (Kleitman, Rothschild, and Spencer). *For $n \in \mathbb{N}$, the number of semigroups on $\{1, 2, \ldots, n\}$ is asymptotically equal to $\sum_{t=1}^{n} \binom{n}{t} \cdot t^{1+(n-t)^2}$.*

In that paper, they also show that almost all semigroups S have the property that there exist disjoint sets A and B and an element $e \in B$ such that $S = A \cup B$, whenever $x, y \in A$, $xy \in B$, and whenever $x, y \in B$, $xy = e$.

Formally, of course an undirected graph without loops or multiple edges is a pair (V, E) where $E \subseteq [V^2]$. A *hypergraph* is a pair (V, E) where $E \subseteq \mathcal{P}(V)$. (There may or may not be restrictions on the size of elements of E.) A hypergraph (V, E) is called a *k-clique of rank r* if there is some $Y \in [V]^k$ such that $E = [Y]^r$. (So a K_m in an ordinary graph is an m-clique of rank 2.) In [R36], in collaboration with Erdős and Singhi, Bruce studied when k-cliques are characterized by their intersections with $[V]^t$ for various values of t. A sample result from this paper is the following.

5.8 Theorem (Erdős, Rothschild, and Singhi). *Let $n, l, r, j \in \mathbb{N}$ and let (V, E) be a hypergraph such that $|V| = n$, $|E| = l$, $E \subseteq [V]^r$, and for every $S \in [V]^{n-j}$ there is some $h \in \{0, 1, \ldots, n\}$ such that $|E \cap [S]^r| = \binom{h}{r}$. If $n \geq \max\{l + r, j + 2r\}$ and $r \leq j$, then there is some k such that (V, E) is a k-clique.*

In a series of papers ([R28], [R32], [R39], and [R41]) with different combinations of authors from among A. Bruen, J. van Lint, and N. Singhi, Bruce studied characterization of subspaces of different kinds of spaces (such as vector spaces and projective spaces). A typical result is the following from [R32]. Notice the similarity of the flavor of this result with Theorem 5.8. For this theorem, given a vector space V over a field F, let $\begin{bmatrix} V \\ r \end{bmatrix} = \{W : W$ is an r-dimensional subspace of V.

5.9 Theorem (Rothschild and Singhi). *Let F be a finite field and let $k, r, j \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that if $n > N$, V is an n-dimensional vector space over F, $\mathcal{S} \subseteq \begin{bmatrix} V \\ r \end{bmatrix}$, $U \in \begin{bmatrix} V \\ k \end{bmatrix}$, $|\mathcal{S}| = \left|\begin{bmatrix} U \\ r \end{bmatrix}\right|$, and for each $W \in \begin{bmatrix} V \\ n-j \end{bmatrix}$ there exists $W' \in \begin{bmatrix} V \\ n-j \end{bmatrix}$ such that $\left|\mathcal{S} \cap \begin{bmatrix} W \\ r \end{bmatrix}\right| = \left|\begin{bmatrix} U \cap W' \\ r \end{bmatrix}\right|$, then there is some $U' \in \begin{bmatrix} V \\ k \end{bmatrix}$ such that $\mathcal{S} = \begin{bmatrix} U' \\ r \end{bmatrix}$.*

Notice that a graph G is 2-colorable if and only if its vertices can be divided into two classes with no edges between vertices in the same class. That is G is a *bipartite* graph. Thus it is a consequence of Theorem 3.13 that the number of triangle free graphs on $\{1, 2, \ldots, n\}$ is asymptotically equal to the number of bipartite graphs on $\{1, 2, \ldots, n\}$. In [R45] it was shown that the same statement applies to any odd cycle.

5.10 Theorem (Lamken and Rothschild). *Let $k \in \mathbb{N}$. Then the number of graphs on $\{1, 2, \ldots, n\}$ that have no cycles of length $2k+1$ is asymptotically equal to the number of bipartite graphs on $\{1, 2, \ldots, n\}$.*

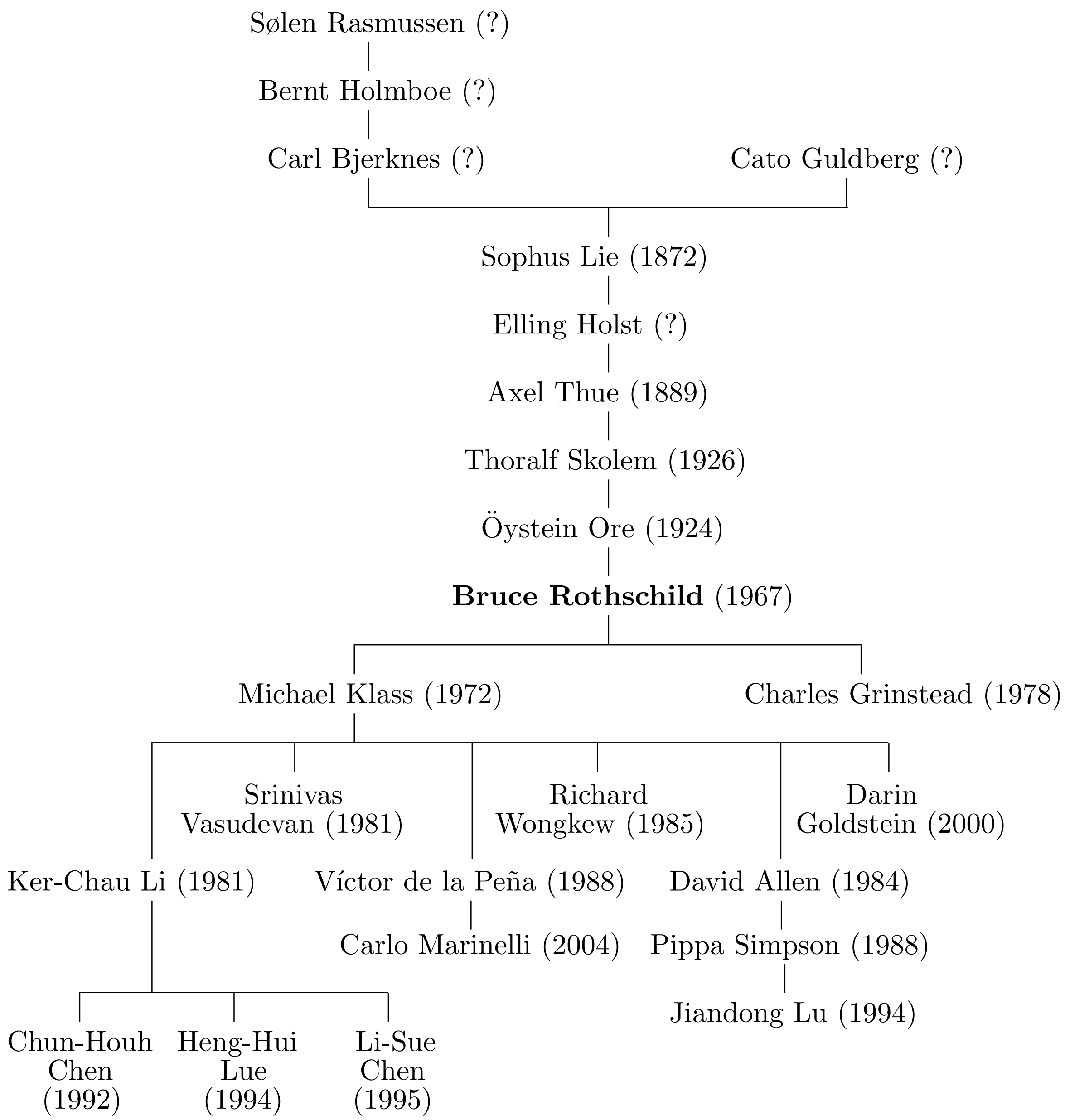
A graph is *sparse* if it has relatively few edges. If G is a subgraph of F, then the maximum sparseness of subgraphs of F is certainly as large as that of G. A graph is said to be *balanced* provided none of its subgraphs has a larger ration of edges to vertices than it does itself. In [R51] it was established that every graph G can be embedded in a balanced graph which was as sparse as possible given that G was a subgraph.

5.11 Theorem (Győri, Rothschild, and Ruciński). *Let $G = (V, E)$ be a graph. There exists a graph F such that*

$$\begin{aligned}\frac{|E(F)|}{|V(F)|} &= \max\left\{\frac{|E(H)|}{|V(H)|} : H \textit{ is a subgraph of } F\right\}\\ &= \max\left\{\frac{|E(H)|}{|V(H)|} : H \textit{ is a subgraph of } G\right\}.\end{aligned}$$

6. Some Genealogical Remarks

The family tree presented here is based on information from the *Mathematics Genealogy Project* (`http://www.genealogy.ams.org/`).



The Mathematical Family Tree of Bruce Rotschild

It is a curious fact that, if the information from the Genealogy Project is correct, Öystein Ore received his Ph.D. with Thoralf Skolem as his advisor two years before Skolem received his own Ph.D.

Based on information in *Mathematical Reviews* the mathematical descendents of Bruce Rothschild have published at least 154 research papers.

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