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Recent Progress in the Topological Theory of Semigroups and the Algebra of βS

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1. Introduction

Throughout this article, we shall assume that all hypothesized topological spaces are Hausdorff.

Let S be a semigroup which is also a topological space. S is said to be a *topological semigroup* if the operation $\cdot : S \times S \to S$ is continuous. Given $x \in S$, define $\lambda_x : S \to S$ and $\rho_x : S \to S$ by $\lambda_x(y) = x \cdot y$ and $\rho_x(y) = y \cdot x$. If one only assumes that each λ_x is continuous and each ρ_x is continuous, then S is a *semitopological* semigroup. If one only assumes that each ρ_x is continuous, then S is a *right topological* semigroup. (Some authors call this a *left topological* semigroup because multiplication is continuous in the left variable.)

From our point of view, probably the most fundamental theorem about right topological semigroups is the following.

1.1. THEOREM. *Let S be a compact right topological semigroup. Then S has a smallest two sided ideal $K(S)$. Further $K(S)$ is the union of all of the minimal left ideals of S and is also the union of all of the minimal right ideals of S. Given any minimal left ideal L of S and any minimal right ideal R of S, $L \cap R$ is a maximal subgroup of S. Also, any two minimal left ideals of S are isomorphic, any two minimal right ideals of S are isomorphic, and any two maximal subgroups of $K(S)$ are isomorphic.*

Theorem 1.1 was established for finite semigroups by SUSCHKEWITSCH [1928], for topological semigroups by WALLACE [1955], and for right topological semigroups by RUPPERT [1973]. A crucial contribution to the result for right topological semigroups was the proof by ELLIS [1969] that any compact right topological semigroup has an idempotent.

Classic (and neo-classic) references are the books by CLIFFORD and PRESTON [1961] on the algebraic theory of semigroups, by HOFMANN and MOSTERT [1996] on compact topological semigroups, by RUPPERT [1984] on semitopological semigroups, and by BERGLUND, JUNGHENN, and MILNES [1989] on right topological semigroups.

Suppose that S is both a semigroup and a topological space. A *semigroup compactification of S* is a pair (ϕ, T) such that T is a compact right topological semigroup, $\phi : S \to T$ is a continuous homomorphism, $\phi[S]$ is dense in T and $\lambda_{\phi(s)} : T \to T$ is continuous for every $s \in S$. (In this case, we may simply call T a semigroup compactification of S. Note that a semigroup compactification need not be a topological compactification, because ϕ is not required to be an embedding.)

Let $\mathcal{P}$ be a property of semigroups which are topological spaces. A semigroup compactification (ϕ, T) of S is said to be the *universal $\mathcal{P}$-semigroup compactification of S* if T has property $\mathcal{P}$ and if, for every semigroup compactification (ϕ', T') of S for which T' has property $\mathcal{P}$, there is a continuous homomorphism $\theta : T \to T'$ such that $\phi' = \theta \circ \phi$.

We shall discuss the *weakly almost periodic compactification wS* of S and the *$\mathcal{LMC}$ compactification $S^{\mathcal{LMC}}$* of S. We define (η, wS) to be the universal $\mathcal{P}$-semigroup compactification of S, where $\mathcal{P}$ denotes the property of being a semitopological semigroup. A bounded continuous function $f : S \to \mathbb{C}$ is *weakly almost periodic* if and only if there is a continuous $\gamma : wS \to \mathbb{C}$ such that $\gamma \circ \eta = f$.

We define $S^{\mathcal{LMC}}$ to be the universal $\mathcal{P}$-semigroup compactification of S, where $\mathcal{P}$ denotes the property of being a right topological semigroup.

It has been known for some time that if S is a discrete semigroup, the operation on S can be uniquely extended to the Stone-Čech compactification βS of S so that βS becomes a semigroup compactification of S, and in fact $\beta S = S^{\mathcal{LMC}}$. See the notes to Chapter 4 of Hindman and Strauss [1998b] for a discussion of the origins of this fact.

We shall also mention the *uniform compactification* uG of a topological group G. We define this in terms of the right uniform structure on G, which has the sets of the form $\{(x, y) \in G \times G : xy^{-1} \in U\}$, where U denotes a neighborhood of the identity in G, as a base for the vicinities. This compactification has the property that a continuous bounded real-valued function defined on G has a continuous extension to uG if and only if it is uniformly continuous. It is a semigroup compactification of G in which G is embedded. In the case in which G is locally compact, $uG = G^{\mathcal{LMC}}$.

The semigroup βS plays a significant role in topological dynamics. Whenever a discrete semigroup S acts on a compact topological space S, the enveloping semigroup (defined as the closure in ${}^{X}X$ of the functions corresponding to elements of S), is a semigroup compactification of S and therefore a quotient of βS. For this reason, some of the concepts related to the algebra of βS originated in topological dynamics. Several of these are described in Section 5 below.

Because the points of βS can be viewed as ultrafilters on S one obtains built in applications to the branch of combinatorics known as *Ramsey Theory*. That is, as soon as one knows that there is an ultrafilter on S which is contained in some set $\mathcal{G}$ of "good" subsets of S, one automatically has a corresponding Ramsey Theoretic result, namely that whenever S is divided into finitely many parts, one of these parts is a member of $\mathcal{G}$.

In this paper we propose to survey progress in the areas mentioned above in the last decade, i.e. since the publication of Hušek and van Mill [1992]. Section 2 of Comfort, Hofmann and Remus [1992] dealt primarily with topological semigroups, with brief mention of results in semitopological semigroups, right topological semigroups, and the algebra of βS. In Section 2 of this paper we shall only mention a few recent results of which we are aware from the theory of topological semigroups. This light treatment is dictated by two facts. Most importantly, neither of the authors is an expert in the theory of topological semigroups. Secondly, a thorough treatment of progress during the last decade of the theory of topological semigroups would consume much more space than is allocated for this paper.

In 1998 our book Hindman and Strauss [1998b] was published. Sections 3 through 5 of this paper will survey results in subjects covered in that book, and will concentrate on progress since the manuscript went to the publisher. Section 3 will deal with results in the theory of right topological semigroups. Section 4 will present recent progress in the algebra of βS. And Section 5 will survey recent progress in the applications of the algebra of βS to Ramsey Theory.

In Section 6 we deal with a subject, the Stone-Čech compactification of *partial semigroups*, that has only recently emerged as an area of productive research, both in terms of abstract algebra and in terms of combinatorial applications.

2. Topological and Semitopological Semigroups

For reasons mentioned in the introduction, we are unable to give a substantive review of recent progress in the theory of topological semigroups. The excellent article Hofmann [2000] discusses the theory beginning in antiquity (meaning in this case the 19^{th} century) and continues through results as recent as 1998, with an emphasis on the Lie theory of semigroups. See also the volume Hofmann and Mostert [1996], which has several relevant articles, and the survey Hofmann and Lawson [1996].

In a now classic result, Ellis [1957] showed that any semitopological semigroup which is locally compact and is algebraically a group is in fact a topological group. Bouziad [1993] describes a class $\mathcal{C}$ of Baire spaces and shows that if G is a left topological group which acts on a space X in a separately continuous fashion and if G and X both belong to the class $\mathcal{C}$, then the action is jointly continuous.

It is reasonably easy to see that if S is an infinite discrete cancellative semigroup, then βS contains at least $2^{\mathfrak{c}}$ idempotents. (See Hindman and Strauss [1998b, Section 6.3].) In the case of wS, even for S discrete, the situation is not so simple. Using techniques of harmonic analysis, Brown and Moran [1972] established in 1972 that $w\mathbb{Z}$ has $2^{\mathfrak{c}}$ idempotents. The proof of this result was simplified by an elementary (but still complicated) exhibition of specific weakly almost periodic functions on $\mathbb{Z}$ by Ruppert [1991]. Bordbar [1998], gave a much simpler construction of enough weakly almost periodic functions on $\mathbb{Z}$ to guarantee the existence of $2^{\mathfrak{c}}$ idempotents in $w\mathbb{Z}$.

Bordbar's construction used the base -2 expansion of an arbitrary integer. It is a simple, but not so well known, fact that for any $p \in \mathbb{N}$ with $p \geq 2$, any $x \in \mathbb{Z}$ has a unique expansion to the base $-p$ using only the digits $\{0, 1, 2, \ldots, p-1\}$. This expansion has the virtue that, so long as the supports of x and y are disjoint, there is no borrowing and no carrying when x and y are added. We shall have occasion to refer to another use of this representation in Section 4.

Berglund [1980] asked whether the set of idempotents in any compact monothetic semitopological semigroup must be closed. (A semigroup S with topology is *monothetic* provided there is some $x \in S$ for which $\{x^n : n \in \mathbb{N}\}$ is dense.) Bordbar and Pym [2000] used the base -2 expansion of integers to show that the set of idempotents in $w\mathbb{N}$ is not closed, thereby answering Berglund's question. They also showed that the set of idempotents in $w\mathbb{Z}$ is not closed. Independently, Bouziad, Memańczyk, and Mentzen [2001] also answered Berglund's question by constructing a class of compact semitopological semigroups, each containing a dense topological group which is monothetic (as a semigroup), in which the set of idempotents is not closed. Notice that because of the universal extension property of $w\mathbb{N}$ and $w\mathbb{Z}$, this latter result implies that the set of idempotents in $w\mathbb{N}$ is not closed and that the set of idempotents in $w\mathbb{Z}$ is not closed.

Bordbar and Pym [1998] investigated the structure of wG, where G is the direct sum of countably many finite groups. In any semigroup there is a natural ordering of the idempotents according to which one has $e \leq f$ if and only if $e = ef = fe$. Bordbar and Pym established that not only does wG have $2^{\mathfrak{c}}$ idempotents, it in fact has an antichain consisting of $2^{\mathfrak{c}}$ idempotents. They showed further that under the continuum hypothesis, there is also a chain of $2^{\mathfrak{c}}$ idempotents.

Notice that in the definition of the weak almost periodic compactification, the continuous homomorphism from S into wS is not required to be an embedding. Of course, if S is not a semitopological semigroup, then it could not possibly be an embedding. In the following remarkable result, M. Megrelishvili established that it can be very far from being an embedding, even when S is not only a topological semigroup, but in fact a topological group.

2.1. Theorem (Megrelishvili). *Let G be the set of all orientation preserving self homeomorphisms of the interval $[0, 1]$ with the compact-open topology. Then G is a topological group and all weakly almost periodic functions on G are constant. Consequently $|wG| = 1$.*

□ Megrelishvili [2001, Theorem 3.1] □

If G is a locally compact group, the homomorphism mapping G into wG is an embedding. However, wG need not be much larger than G. Ruppert [1984, Theorem 6.3] has shown that, if G is a simple non-compact Lie group, then wG is the one-point compactification of G. It was recently shown by Ferri [2001] that wG is large if G is an IN group (i.e. a group in which the identity has a compact neighborhood invariant under conjugation). More precisely, let κ be the cardinal denoting the smallest number of compact subsets of G required to cover G. Assuming that G is non-compact, wG has at least 2^{2^κ} points. If G is a non-compact SIN group (i.e. a group in which the identity has a basis of compact neighborhoods invariant under conjugation), S. Ferri showed that $uG \setminus G$ has a dense open subset W of cardinality 2^{2^κ} with the following property: for every $w \in W$, $\{w\} = \phi^{-1}[\{\phi(w)\}]$, where $\phi : uG \to wG$ denotes the natural homomorphism. This extends a result due to Ruppert [1973], who had previously proved this fact for a discrete group G.

3. Right (or Left) Topological Semigroups

As we mentioned in the introduction, if S is a discrete semigroup, then βS is in a natural way a right topological semigroup.

If S is a right topological semigroup, its topological center $\Lambda(S)$ is the set of points $s \in S$ for which $\lambda_s : S \to S$ is continuous. In the case of discrete commutative S, it is easy to see that the topological center and the algebraic center of βS coincide by a simple consideration of the functions λ_x and ρ_x. (See Hindman and Strauss [1998b, Theorem 4.24].) If S is weakly left cancellative (meaning that for all $u, v \in S$, $\{x \in S : ux = v\}$ is finite), then the algebraic center of βS is equal to the algebraic center of S, and the algebraic center of $S^* = \beta S \setminus S$ is empty (See Hindman and Strauss [1998b, Theorem 6.54].)

If $q \in S^*$, the question of the continuity of λ_q restricted to S^* is not straightforward. The following is an old result of E. van Douwen. (The date on the paper is 1991, but the result was established in 1979.)

3.1. Theorem (van Douwen). *Let S be a countable cancellative semigroup.*

(a) *There is a dense subset D of S^* such that for all $p \in D$ and all $q \in S^*$, the restriction of λ_q to S^* is discontinuous at p.*

(b) *There is a P-point in $\mathbb{N}^*$ if and only if there is a dense subset E of S^* such that for all $p \in E$ and all $q \in S^*$, the restriction of the operation $\cdot$ to $S^* \times S^*$ is continuous at (q, p).*

□ These conclusions follow from Theorems 9.7 and 9.8 of van Douwen [1991] respectively. □

The following theorem about joint continuity was proved by Protasov.

3.2. Theorem (Protasov). [1996] *If G is a countable discrete abelian group, with only a finite number of elements of order 2, then there is no point in $G^* \times G^*$ at which the operation $\cdot$ from $\beta G \times \beta G$ to βG is continuous.*

□ Protasov[1996, Theorem 4.1] □

In the same paper, Protasov [1996, Example 4.4] showed that, if G denotes a discrete abelian group for which $|G|$ is Ulam measurable, then $G^* \times G^*$ does contain a point at which the operation $\cdot$ from $\beta G \times \beta G$ to βG is continuous. Zelenyuk [1996b] showed that Martin's Axiom implies that the same statement holds if G is a countable Boolean group. We do not know whether examples of this kind of joint continuity can be constructed in ZFC.

We shall continue with the discussion of continuity in G^* momentarily. However, in this discussion we shall use the notion of strongly summable ultrafilters, which we pause now to introduce. An ultrafilter on a semigroup $(S, +)$ is said to be *strongly summable* if it has a base of sets of the form $FS(\langle x_n \rangle_{n=1}^{\infty})$, where $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$. Blass and Hindman [1987] showed that Martin's Axiom implies the existence of strongly summable ultrafilters on $\mathbb{N}$, but that their existence cannot be established in ZFC. This result was extended from $\mathbb{N}$ to arbitrary countable abelian groups by Hindman, Protasov, and Strauss [1998a]. If p is a strongly summable ultrafilter of a certain kind on a countable abelian group G, it has a remarkable algebraic property. The equation $x + y = p$ can only hold with $x, y \in G^*$ if $x = a + p$ and $y = -a + p$ for some $a \in G$. The existence of ultrafilters p with this property follows from Martin's Axiom. This extends to many non-commutative groups. If G is any countable group which can be embedded algebraically in a compact topological group, MA guarantees the existence of ultrafilters p on G with the property that whenever $xy = p$, with $x, y \in G^*$, one must have that $x = pa^{-1}$ and $y = ap$ for some $a \in G$.

Strongly summable ultrafilters on a countable Boolean group are particularly interesting, because they can be used to define topologies for which the group is an extremally disconnected non-discrete topological group. This construction is due to Malykhin [1975]. It is not known whether extremally disconnected non-discrete topological groups can be defined in ZFC.

Suppose that G is a countable discrete group. For each $p \in \beta G$, let λ_p^* denote the restriction of λ_p to G^*. It is easy to see that, if q is a P-point in G^*, then λ_p^* is continuous at q for every $p \in G^*$. Conversely, Protasov [20∞] has announced that if G can be algebraically embedded in a compact topological group and if $q \in G^*$ has the property that λ_p^* is continuous at q for every $p \in G^*$, then q is a P-point in G^*. Protasov [20∞] has also announced that for any countable discrete group, if $p \in G^*$ is idempotent, the continuity of λ_p^* at p implies the existence of a P-point in ω^*. So the existence of an idempotent p with the property that λ_p^* is continuous at p cannot be established in ZFC. However, as we have just observed, if G is a countable abelian group, then Martin's Axiom implies that there is an idempotent $p \in G^*$ which is strongly summable. If G is Boolean and countable and $p \in G^*$ is strongly summable, it is easy to show that λ_p^* is continuous at p. We do not know whether it is consistent with ZFC that there exists an infinite discrete group G with the property that λ_p^* is discontinuous at q for all $p, q \in G^*$.

Recall that, if G is a locally compact group, then $G^{\mathcal{LMC}} = uG$, and if S is discrete, then $S^{\mathcal{LMC}} = \beta S$. Since in general G is embedded in uG we may pretend that $G \subseteq uG$ (just as we pretend that $S \subseteq \beta S$) and one may then let $G^* = uG \backslash G$. I. Protasov and J. Pym proved that the topological center of G^* is empty for any locally compact topological group G. They also obtained the following generalization of Theorem 3.1(a).

3.3. Theorem (Protasov and Pym). *Let G be a locally compact, noncompact, σ-compact topological group. There is a dense subset D of G^* such that for all $p \in D$ and all $q \in G^*$, the restriction of λ_q to G^* is discontinuous at p.*

□ Protasov and Pym [2001, Theorem 1]. □

Recall from Theorem 1.1 that any compact right topological semigroup S has a smallest two sided ideal which is the union of all minimal left ideals, and each minimal left ideal is the union of pairwise isomorphic groups. Further, given a minimal left ideal L of S and a point $x \in L$, $L = Sx = \rho_x[S]$ so L is compact, and thus closed.

3.4. Theorem (Lau, Milnes, and Pym). *Let G be a locally compact noncompact topological group and let L be a minimal left ideal of uG. Then L is not a group.*

□ Lau, Milnes, and Pym [1999]. □

In the process of proving Theorem 3.4, Lau, Milnes, and Pym establish for "nearly all groups" the stronger result that no maximal subgroup of the smallest ideal can be closed.

The following result is a *local structure theorem* for uG, when G is a locally compact topological group.

3.5. Theorem (Lau, Medghalchi, and Pym). *Let G be a locally compact topological group and let U be an open symmetric neighborhood of the identity with $c\ell_G(U)$ compact. Let $X \subseteq G$ be maximal with respect to the property that $\{Ux : x \in X\}$ is a disjoint family. Then $\overline{X} = c\ell_{uG}(X)$ is homeomorphic with βX. Also, for each open neighborhood V of the identity with $c\ell_G(V) \subseteq U$, the subspace $V\overline{X}$ is open in uG and homeomorphic with $V \times \beta X$.*

Moreover, given any $\mu \in uG$ one may choose an open symmetric neighborhood of the identity with $c\ell_G(U)$ compact and $X \subseteq G$ maximal with respect to the property that $\{Ux : x \in X\}$ is a disjoint family such that $\mu \in \overline{X}$.

□ Lau, Medghalchi, and Pym [1993, Theorem 2.10] and Pym [1999]. □

Pym [1999] used Theorem 3.5 to provide a short proof of a theorem of W. Veech, namely that if G is a locally compact group, $s \in G$, and s is not the identity of G, then for all $\mu \in uG$, $s\mu \neq \mu$ (Veech [1977, Theorem 2.2.1]).

M. Filali and J. Pym have recently extended some results known to hold for βS (for a discrete semigroup S) to $uG = G^{\mathcal{LMC}}$ for a locally compact group G.

3.6. THEOREM (FILALI). *Let G be a locally compact noncompact abelian topological group. Then the set of points in G^* which are right cancelable in uG has dense interior in G^*. If, in addition, G is countable, then for each $x \in G^*$, $\{y \in G^* : (G^* + y) \cap (G^* + x) \neq \emptyset\}$ is nowhere dense in G^*.*

□ FILALI [1997, Corollary 1] □

In FILALI and PYM [20∞] this result was extended and the commutativity assumption was eliminated.

3.7. THEOREM (FILALI and PYM). *Let G be a locally compact noncompact topological group. Then the set of points in G^* which are right cancellable in uG has dense interior in G^*. If κ is the cardinal denoting the smallest number of compact subsets of G required to cover G, then $G^{\mathcal{LMC}}$ has 2^{2^κ} minimal left ideals.*

□ FILALI and PYM [20∞, Theorem 1 and Corollary 3] □

S. Ferri and one of the authors have obtained results of this kind for a class of topological groups larger than the class of locally compact groups, in which case one need not have $uG = G^{\mathcal{LMC}}$.

3.8. THEOREM (FERRI and STRAUSS). *Let G be a topological group. For each neighborhood U of the identity in G, let κ_U be the cardinal denoting the smallest number of sets of the form Uy, where $y \in G$, required to cover G, and let $\kappa = \sup\{\kappa_U : U$ is a neighborhood of the identity in $G\}$. If κ is infinite and there is a neighborhood U of the identity in G for which G cannot be covered by fewer than κ sets of the form xUy with $x, y \in G$, then there are at least 2^{2^κ} points in G^* which are right cancelable in uG and at least 2^{2^κ} minimal left ideals in uG.*

□ FERRI and STRAUSS [2001, Theorem 1.3] □

Observe that the hypotheses of Theorem 3.8 are satisfied if G is a topological group which is not totally bounded and is either locally compact or separable. It is an open problem whether there exists a topological group G, which is not totally bounded, for which uG has precisely one minimal left ideal.

In collaboration with I. Protasov, we described a method for obtaining topologies on a semigroup S that are completely determined by the algebra of S and make S into a *left* topological semigroup by using idempotents in the *right* topological compactification βS. (Of course, if one takes βS to be left topological, the resulting topologies are right topological.)

3.9. THEOREM (HINDMAN, PROTASOV, and STRAUSS). *Let S be a cancellative semigroup. For any idempotent $p \in \beta S$, let $\mathcal{T}_p = \{V \subseteq S :$ for all $x \in V$, $V \in xp\}$ and let $\mathcal{V}_p = \{\rho_p^{-1}[U] \cap S : U$ is open in $\beta S\}$. Then for each idempotent $p \in \beta S$, $\mathcal{V}_p$ and $\mathcal{T}_p$ are Hausdorff topologies on S making S into a left topological semigroup. If $|S| = \kappa$, then there are 2^{2^κ} noncomparable topologies of the form $\mathcal{V}_p$. One always has that $\mathcal{V}_p \subseteq \mathcal{T}_p$ and the inclusion is proper unless p has the property that $\{q \in \beta S : q \cdot p = p\} = \{p\}$. If S is a group, the property that $\{q \in \beta S : q \cdot p = p\} = \{p\}$ guarantees that $\mathcal{V}_p = \mathcal{T}_p$.*

□ HINDMAN, PROTASOV, and STRAUSS [1998b, Theorems 3.4, 3.6, 4.1, 4.2, and 5.1 and Corollary 3.13]. □

An idempotent $p \in \beta S$ such that $\{q \in \beta S : q \cdot p = p\} = \{p\}$ is said to be *strongly right maximal*. They are certainly rare birds, but it is a result of I. Protasov that their existence can be established in ZFC. (See Hindman and Strauss [1998b, Theorem 9.10].) If S is an infinite group and p a strongly right maximal idempotent in βS, then $\mathcal{V}_p = \mathcal{T}_p$ and this topology on S is extremally disconnected and maximal subject to having no isolated points. (See Hindman and Strauss [1998b, Corollary 9.17].) This fact answers an old question posed by E. van Douwen: is it possible in ZFC to define a regular homogeneous topology on $\mathbb{Z}$ which is maximal subject to having no isolated points?

Protasov has obtained results about ω-resolvability by using the algebra of the Stone-Čech compactification. He showed that any non-discrete left topological group G, which is not of first category, is ω-resolvable; i.e. it can be partitioned into infinitely many disjoint dense subsets Protasov [2001a].

In Hindman and Strauss [1995d] we investigated topological properties of certain algebraically defined subsets of βS, where S denoted a countable commutative discrete semigroup. In any compact right topological semigroup, all minimal left ideals are homeomorphic as well as isomorphic. However, we showed that, if the minimal left ideals of βS are infinite, then the minimal right ideals of βS belong to $2^{\mathfrak{c}}$ different homeomorphism classes. The same statement is true for the maximal groups contained in any minimal left ideal of βS. If, in addition, S is cancellative, then the sets of the form $S + e$, where e denotes an idempotent in S^*, also belong to $2^{\mathfrak{c}}$ homeomorphism classes. We also showed that, if e and e' are idempotents in $\beta\mathbb{N}$, with e' being non-minimal, then there is no continuous surjective homomorphism from $\beta\mathbb{N} + e$ onto $\beta\mathbb{N} + e'$, apart from the identity.

4. Algebra of βS

Let us begin with a little history about a difficult and annoying open problem which has attracted some significant attention. In 1979, E. van Douwen asked (in van Douwen [1991], published much later) whether there are topological and algebraic copies of the right topological semigroup $(\beta\mathbb{N}, +)$ contained in $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$. This question was answered in Strauss [1992a], where it was in fact established that if φ is a continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$, then $\varphi[\beta\mathbb{N}]$ is finite. The problem to which we refer is whether one can have such a continuous homomorphism with $|\varphi[\beta\mathbb{N}]| > 1$. We conjecture that one cannot.

Another old and difficult problem in the algebra of $\beta\mathbb{N}$ was solved by Zelenyuk [1996a] who showed that there are no nontrivial finite groups contained in $\mathbb{N}^*$. (See Hindman and Strauss [1998b, Section 7.1] for a presentation of this proof.)

Protasov has generalized Zelenyuk's Theorem by characterising the subgroups of βG, where G denotes a countable discrete group.

4.1. Theorem (Protasov). *If G is a countable discrete group, every finite subgroup of G^* has the form Hp, where H is a finite subgroup of G and p an idempotent in G^* which commutes with all the elements of H.*

□ Protasov [1998]. □

Using Zelenyuk's Theorem, it is not hard to show that there is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$ if and only if there exist distinct p and

q in $\mathbb{N}^*$ such that $p+p=q=q+q=q+p=p+q$. (See HINDMAN and STRAUSS [1998b, Corollary 10.20].)

The question of which finite semigroups can exist in $\mathbb{N}^*$ has implications for a large class of semigroups of the form βS. It is not hard to prove that any finite semigroup in $\mathbb{N}^*$ is contained in $\mathbb{H} = \bigcap_{n\in\mathbb{N}} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$. Now if S is any infinite discrete semigroup which is right cancellative and weakly left cancellative, S^* contains copies of $\mathbb{H}$. (See HINDMAN and STRAUSS [1998b, Theorem 6.32].) Thus a finite semigroup which occurs in $\mathbb{N}^*$ also occurs in S^*, if S is any semigroup of this kind.

In collaboration with I. Protasov and J. Pym, one of us established a technical lemma that has several corollaries relating to continuous homomorphisms. We combine a few of these in the following.

4.2. THEOREM (PROTASOV, PYM, and STRAUSS). *Let G be a countable discrete group.*

(a) *If S is a cancellative discrete semigroup, then any continuous injective homomorphism from βS to βG is the extension of an injective homomorphism from S to G.*

(b) *If S is a countable discrete semigroup and $\varphi : \beta S \to G^*$ is a continuous homomorphism, then every element of $\varphi[S]$ has finite order.*

(c) *If $\varphi : \beta\mathbb{N} \to G^*$ is a continuous homomorphism, then $\varphi[\beta\mathbb{N}]$ is finite and $\varphi[\mathbb{N}^*]$ is a finite group.*

(d) *If C is a compact subsemigroup of G^*, then every element of the topological center of C has finite order.*

□ PROTASOV, PYM, and STRAUSS [2000, Theorems 6.5 and 6.6 and Corollaries 6.7 and 6.8]. □

The conjecture above can be stated equivalently by saying that $\mathbb{N}^*$ contains no elements of finite order, other than idempotents. This conjecture has implications about the nature of possible continuous homomorphisms from βS into $\mathbb{N}^*$, where S is any countable semigroup at all. It follows from Theorem 4.2(b) that, if this conjecture is true, then any continuous homomorphism from βS into $\mathbb{N}^*$ must map all the elements of S to idempotents.

DAVENPORT, HINDMAN, LEADER, and STRAUSS [2000] showed that the existence of the two element subsemigroup of $\mathbb{N}^*$ mentioned above implies the existence of a three element semigroup $\{p,q,r\}$ where $p+p=q=q+q=q+p=p+q$, $r+r=r$, $p=p+r=r+p$, and $q=q+r=r+q$. We also showed that if there is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ into $\mathbb{N}^*$, then there is a subset A of $\mathbb{N}$ with the property that, whenever A is finitely colored, there must exist a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}\backslash A$ such that $\{\sum_{t\in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\text{and } |F| \geq 2\}$ is a monochrome subset of A. (When we refer to a "k-coloring" of a set X we mean a function $\phi : X \to \{1,2,\ldots,k\}$. The assertion that a set B is "monochrome" is the assertion that ϕ is constant on B.)

Finite subsemigroups of $\mathbb{N}^*$ of any size do exist, for trivial reasons. Any minimal right or left ideal of $\beta\mathbb{N}$ contains $2^{\mathfrak{c}}$ idempotents and if e and f are idempotents in the same minimal left (respectively right) ideal then $e + f = e$ (respectively $e + f = f$). It was shown some time ago in Berglund and Hindman [1992] that there are idempotents in the smallest ideal of $\beta\mathbb{N}$ whose sum is not idempotent. (Idempotents in the smallest ideal are *minimal* idempotents.) This raised the question of whether there are any minimal idempotents whose sum is again idempotent but not equal to either of them. This question has recently been answered affirmatively by Zelenyuk [2001] in a grand fashion.

4.3. Definition. A semigroup S is an *absolute coretract* if and only if for any continuous homomorphism f from a compact Hausdorff right topological semigroup T onto a compact Hausdorff right topological semigroup containing S algebraically there exists a homomorphism $g : S \to T$ such that $f \circ g$ is the identity on S.

There is a copy of any absolute coretract in $\beta\mathbb{N}$. Zelenyuk [2001] produced a class of countable semigroups of idempotents, showed that each of them is an absolute coretract, and showed that any finite semigroup of idempotents which is an absolute coretract is a member of this class. The self contained proof of the following special case of Zelenyuk's result can be found in Hindman [2001].

4.4. Theorem (Zelenyuk [2001]). *There exist $p \in \mathbb{H}$ and $\{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\} \subseteq K(\mathbb{H}) = K(\beta\mathbb{N}) \cap \mathbb{H}$ such that the listed elements are all distinct and the operation $+$ satifies*

$+$	p	α_{11}	α_{12}	α_{21}	α_{22}
p	p	α_{11}	α_{12}	α_{21}	α_{22}
α_{11}	α_{12}	α_{11}	α_{12}	α_{11}	α_{12}
α_{12}	α_{12}	α_{11}	α_{12}	α_{11}	α_{12}
α_{21}	α_{22}	α_{21}	α_{22}	α_{21}	α_{22}
α_{22}	α_{22}	α_{21}	α_{22}	α_{21}	α_{22}

In particular, α_{11}, α_{22}, and α_{12} are idempotents in $K(\beta\mathbb{N})$ and $\alpha_{11} + \alpha_{22} = \alpha_{12}$.

Some recent results deal with the ability to solve certain equations in βS. An element e of βS satisfying the equation $xe = x$ for all $x \in \beta S$ is a *right identity* for βS. Recall that for any ultrafilter p, the *norm* of p, $||p|| = \min\{|A| : A \in p\}$. J. Baker, A. Lau, and J. Pym recently obtained the following result, which implies that if βS has a two sided identity e, then $e \in S$.

4.5. Theorem (Baker, Lau, and Pym). *Let S be a discrete semigroup, let $e \in \beta S \backslash S$ be a right identity for βS, and let $\kappa = ||e||$. Then βS has 2^{2^κ} right identities.*

□ Baker, Lau, and Pym [1999, Theorem 6]. □

Hindman, Maleki, and Strauss [2000] showed that for any distinct positive integers a and b, if $(S, +)$ is any commutative cancellative semigroup, and the equation $n \cdot s = n \cdot t$ has at most finitely many solutions with $s, t \in S$ and $n = ab|a - b|$, then the equation $u + a \cdot p = v + b \cdot p$ has no solutions with $u, v \in \beta S$ and $p \in \beta S \backslash S$. (Note for example that $2 \cdot p$ is the continuous extension of the function $s \mapsto 2 \cdot s$ to βS applied at p and it is usually not true that $2 \cdot p = p + p$.) We also showed that if S can be embedded in the circle group $\mathbb{T}$, then the equation $a \cdot p + u = b \cdot p + v$ has no solutions with $u, v \in \beta S$ and $p \in \beta S \backslash S$.

Adams [2001] has shown that the above statements hold if S is a countable commutative group and a and b are distinct elements of $\mathbb{Z} \setminus \{0\}$.

We mentioned above a Ramsey Theoretic consequence of the (unknown) existence of a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$. In Section 5 we shall present several Ramsey Theoretic results that have been obtained recently using the algebraic structure of βS. The relationship between combinatorics and topological algebra goes both ways. Recently, in collaboration with I. Leader, we established a Ramsey Theoretic result which had the following as a corollary. We shall not attempt to explain the Ramsey Theoretic result of which it is a corollary, but remark that the proof used the idea of expansion of numbers to negative bases which we mentioned in Section 2.

4.6. Theorem (Hindman, Leader, and Strauss). *Let $n, m \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in \mathbb{Z}\setminus\{0\}$ such that $a_i \neq a_{i+1}$ and $b_j \neq b_{j+1}$ for all $i \in \{1, 2, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, m-1\}$. If $p + p = p \in \beta\mathbb{N}$ and $a_1 \cdot p + a_2 \cdot p + \ldots + a_n \cdot p = b_1 \cdot p + b_2 \cdot p + \ldots + b_m \cdot p$, then $\langle a_1, a_2, \ldots, a_n\rangle = \langle b_1, b_2, \ldots, b_m\rangle$.*

□ Hindman, Leader, and Strauss [20∞c, Corollary 4.2]. □

It is an open question whether the assumption that $p = p + p$ in Theorem 4.6 can be replaced by the weaker assumption that $p \in \mathbb{N}^*$.

The choice to make βS a right topological semigroup rather than a left topological semigroup is an arbitrary one. Let us denote by $\odot$ the operation on βS making βS a left topological semigroup with S contained in its topological center (in this case, $\{p \in \beta S : \rho_p \text{ is continuous}\}$). One might suspect that results for $(\beta S, \odot)$ and $(\beta S, \cdot)$ would be simply left-right switches of each other. If S is commutative, this is correct because for any $p, q \in \beta S$, $p \odot q = q \cdot p$. In particular a subset of βS is a subsemigroup under one operation if and only if it is a subsemigroup under the other and the smallest ideals $K(\beta S, \cdot)$ and $K(\beta S, \odot)$ are identical. It has been known since 1994 that both conclusions can fail given sufficient noncommutativity of S. El-Mabhouh, Pym, and Strauss [1994a] showed that if S is the free semigroup on countably many generators, then there is a subsemigroup H of $(\beta S, \cdot)$ with the property that given any $p, q \in H$, $p \odot q \notin H$. And it was shown by Anthony [1994a] that if S is the free semigroup or free group on two generators, then $K(\beta S, \cdot)\setminus c\ell K(\beta S, \odot) \neq \emptyset$. On the other hand, it was also shown in Anthony [1994a] that for any semigroup S whatever, $K(\beta S, \cdot) \cap c\ell K(\beta S, \odot) \neq \emptyset$. It was recently shown by Burns [2001] that if S is the free semigroup or free group on two generators, then $K(\beta S, \cdot) \cap K(\beta S, \odot) = \emptyset$. In fact the following much stronger result was established in the same paper.

4.7. Theorem. *Let S be the free semigroup on two generators. If $p \in c\ell K(\beta S, \cdot) \cap c\ell K(\beta S, \odot)$, then p is right cancelable in $(\beta S, \cdot)$ or left cancelable in $(\beta S, \odot)$.*

□ Burns [2001, Theorem 3.13]. □

Adams [2001] has proved the corresponding theorem for the free group on two generators.

5. Applications to Ramsey Theory

We were first led to study the algebra of βS because of the very simple proof given in 1975 by F. Galvin and S. Glazer of the Finite Sums Theorem (whose proofs had previously been very complicated). See the notes to Chapter 5 of Hindman and Strauss [1998b] for details of the discovery of this proof. Over a quarter of a century later, new applications of the algebra of βS to Ramsey Theory continue to be discovered.

One of the classic results of Ramsey Theory is the Hales-Jewett Theorem Hales and Jewett [1963]. Given an alphabet A, a *variable word* over A is a word over the alphabet $A \cup \{v\}$ in which v actually occurs (where v is a "variable" not in A). Given a variable word w and $a \in A$, $w(a)$ has its obvious meaning, namely the replacement of all occurrences of v by a. The Hales-Jewett Theorem says that whenever A is a finite alphabet, $r \in \mathbb{N}$, and the set of finite words over A are r-colored, there is a variable word w over A such that $\{w(a) : a \in A\}$ is monochrome. For a simple algebraic proof of the Hales-Jewett Theorem see Hindman and Strauss [1998b, Section 14.2].

Notice that one can color words based on what their leftmost and rightmost letters are. Consequently, the variable word guaranteed by the Hales-Jewett Theorem cannot be a *left variable* word (i.e., one whose leftmost letter is v) or a *right variable word*. However, in collaboration with R. McCutcheon, one of us obtained the the following theorem which extends previous generalizations of the Hales-Jewett Theorem due to Carlson [1988] and to Carlson and Simpson [1984]. The proof of Theorem 5.1 uses in an intricate fashion the structure of the smallest ideal $K(\beta S)$. The products that are "obviously forbidden" are those beginning with a left variable word, ending with a right variable word, or having a right variable word immediately followed by a left variable word. The latter is forbidden because one may count the number of occurrences of a 1 followed immediately by a 2 and divide $\mathcal{W}_{k+1}$ according to whether this count is even or odd. (See Hindman and McCutcheon [20∞b, Theorem 2.10].) In an expression of the form $\prod_{n\in F} x_n$, the terms occur in the order of increasing indices.

5.1. Theorem (Hindman and McCutcheon). *Let $\mathcal{W}_k$ be the free semigroup on the alphabet $\{1,2,\ldots,k\}$. Let $\mathcal{W}_k$ and $\mathcal{W}_{k+1}\setminus\mathcal{W}_k$ be finitely colored. There exists a sequence $\langle w_n\rangle_{n=1}^\infty$ of variable words over $\mathcal{W}_k$ such that*

(1) for each $n \in \mathbb{N}$, if $n \equiv 1 \pmod 3$, then w_n is a right variable word;

(2) for each $n \in \mathbb{N}$, if $n \equiv 0 \pmod 3$, then w_n is a left variable word; and

(3) all products of the form $\prod_{n\in F} w_n\big(f(n)\big)$ that lie in $\mathcal{W}_k$ are monochrome and all of those that lie in $\mathcal{W}_{k+1}$ are monochrome, except for those that are obviously forbidden.

□ Hindman and McCutcheon [20∞a, Theorem 2.9]. □

Deuber, Hindman, Gunderson, and Strauss [1997] obtained results in graph theory which depended on properties of idempotents in βS. A. Hajnal had asked whether, for every triangle-free graph on $\mathbb{N}$, there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ for which $FS\langle x_n\rangle_{n=1}^\infty$ is an independent set. We showed that the answer is "no". However, we showed that for every K_m-free graph G on a semigroup S, there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that $\{\prod_{n\in F} x_n, \prod_{n\in H} x_n\} \notin E(G)$ whenever F and H are disjoint nonempty finite subsets of $\mathbb{N}$. We also showed that, for every $K_{m,m}$-free graph on a cancellative semigroup S, there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ for which $FP(\langle x_n\rangle_{n=1}^\infty)$ is independent, where $FP(\langle x_n\rangle_{n=1}^\infty) = \{\prod_{n\in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$.

In another purely combinatorial result whose proof relies heavily on facts about idempotents in βS, Hindman and Strauss [20∞b] have shown, extending (and using) a result of Gunderson, Leader, Prömel, and Rödl [2001], that given any $m \in \mathbb{N}$ and any graph on $\mathbb{N}$ which does not include a complete graph on m vertices, there is a sequence of arithmetic progressions of all lengths such that there are not edges within or between the progressions nor between certain specified sums of the terms of those progressions.

As mentioned in the introduction, the relationship between the algebra of βS and topological dynamics has always been strong. Several notions from topological dynamics are important in describing the algebraic structure of βS. For example given an ultrafilter p on S, $p \in c\ell K(\beta S)$ if and only if every member of p is *piecewise syndetic*. Another notion, originally defined in terms of topological dynamics, is *central*. A central set is quite simply characterized as one which is a member of a minimal idempotent in βS. Central sets are guaranteed to have substantial combinatorial structure. For example, the chosen monochrome sets in Theorem 5.1 above can both be chosen to be central (in $\mathcal{W}_k$ and in $\mathcal{W}_{k+1}$ respectively). Two other notions of largeness that originated in topological dynamics, namely *syndetic* and *thick* have simple characterizations in terms of βS. A set A is thick if and only if $\overline{A}$ contains a left ideal of βS, while A is syndetic if and only if $\overline{A}$ meets every left ideal of βS.

Let $u, v \in \mathbb{N} \cup \{\omega\}$. A $u \times v$ matrix with rational entries and only finitely many nonzero entries in each row is *image partition regular* provided that whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are monochrome. Such a matrix is *kernel partition regular* provided that whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = \vec{0}$ and the entries of $\vec{x}$ are monochrome. A computable characterization of finite kernel partition regular matrices was found by Rado [1933] and several characterizations of finite image partition regular matrices were found by Hindman and Leader [1993].

For finite matrices which are either image partition regular or kernel partition regular, one may always choose the color class in which solutions are found to be a central set. It was shown by Deuber, Hindman, Leader, and Lefmann [1995] that this need not hold for infinite image partition regular matrices. Hindman, Leader, and Strauss investigated infinite matrices with entries from $\mathbb{Z}$ which satisfied the requirement that images could be found in any central set, which we call *centrally image partition regular*. We defined the compressed form of a finite vector with entries in $\mathbb{Z} \setminus \{0\}$ to be the vector obtained from the given one by deleting every entry equal to its predecessor. Let A be any matrix with entries from $\mathbb{Z}$ with finitely many nonzero entries in each row and no row equal to $\vec{0}$. Assume that the rows of A have the same compressed form with positive last entry and for some $s \in \mathbb{Z} \setminus \{0\}$, each row of A has a sum of terms equal to s. By using extensively the algebraic properties of $\beta\mathbb{N}$, we showed that, for every central subset C of $\mathbb{N}$, there is an infinite increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ with the property that $\sum_{i=1}^\infty a_i \cdot x_i \in C$ for every row $\vec{a}$ of A. This implies the following new result in Ramsey Theory.

5.2. Theorem (Hindman, Leader, and Strauss). *Let $\mathcal{E}$ denote the set of all finite vectors of the form $\langle a_1, a_2, \ldots, a_m\rangle$ where each $a_i \in \mathbb{Z}\backslash\{0\}$, $a_m > 0$ and $a_1 + a_2 + \ldots + a_m \neq 0$. Let a finite coloring of $\mathbb{N}$ be given. For each $\epsilon = \langle a_1, a_2, \ldots, a_m\rangle \in \mathcal{E}$, there is an infinite increasing sequence $\langle x_n(\epsilon)\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that, if $Y_\epsilon = \{a_1 x_{n_1}(\epsilon) + a_2 x_{n_2}(\epsilon) + \ldots + a_m x_{n_m}(\epsilon) : n_1 < n_2 < \ldots < n_m\}$, then $\bigcup_{\epsilon\in\mathcal{E}} Y_\epsilon$ is monochrome. Furthermore, the sequences $\langle x_n(\epsilon)\rangle_{n=1}^\infty$ can be chosen so that the sets Y_ϵ are pairwise disjoint.*

□ Hindman, Leader, and Strauss [20∞b, Corollary 3.8]. □

Hindman and Strauss [20∞a] provide, again using the algebra of $\beta\mathbb{N}$ as well as some elementary combinatorics, ways of producing new centrally image partition regular matrices from old ones.

Furstenberg and Glasner [1998] showed, in an extension of van der Waerden's Theorem, that whenever B is a piecewise syndetic subset of $\mathbb{Z}$ and $l \in \mathbb{N}$, then the set of length l arithmetic progressions in B is not only nonempty, but is in fact piecewise syndetic in the set of all arithmetic progressions. Using some simple facts about the algebra of βS, Bergelson and Hindman [2001, Theorem 3.7] generalized this result by showing that for a large number of notions of largeness (including "piecewise syndetic", "central", and "thick"), if S is a semigroup, $l \in \mathbb{N}$, E is a subsemigroup of S^l with $\{(a, a, \ldots, a) : a \in S\} \subseteq E$, I is an ideal of E, and B is a large subset of S, then $B^l \cap I$ is a large subset of I.

In a similar vein, Hindman, Leader, and Strauss [20∞a, Theorem 4.5] showed for the same notions of largeness mentioned above, that if $u, v \in \mathbb{N}$, A is a $u \times v$ matrix with entries from $\mathbb{Q}$, $I = \{A\vec{x} : \vec{x} \in \mathbb{N}^v\} \cap \mathbb{N}^u$, $\vec{1} \in I$, and C is a large subset of $\mathbb{N}$, then $I \cap C^u$ is a large subset of I.

It is a simple fact that if $A \subseteq \mathbb{N}$ has positive upper density, then $A - A = \{x - y : x, y \in A \text{ and } y < x\}$ meets $FS(\langle x_n\rangle_{n=1}^\infty)$ for every sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$. Bergelson, Hindman, and McCutcheon [1998] investigated the relationship between "left" and "right" versions of *syndetic*, *thick*, and *piecewise syndetic*, in an arbitrary semigroup S. (The "right" versions are the usual notions. The "left" versions correspond to the left topological structure on βS.) They then investigated the conditions under which AA^{-1} or $A^{-1}A$ can be guaranteed to meet $FP(\langle x_n\rangle_{n=1}^\infty)$ for every sequence $\langle x_n\rangle_{n=1}^\infty$ in S, where $AA^{-1} = \{x \in S : (\exists y \in A)(xy \in A)\}$ and $A^{-1}A = \{x \in S : (\exists y \in A)(yx \in A)\}$.

6. Partial Semigroups

The study of algebraic operations defined for only some members of $S \times S$ has a long history. (See the book Evseef and Ljapin [1997].) Its relationship to algebra in the Stone-Čech compactification is of much more recent origin. In 1987 Pym [1987] introduced the concept of an "oid". He showed that the oid structure of $\mathbb{N}$, in which the sum of two numbers is defined as usual but only when they have disjoint binary supports, already induces all of the semigroup structure of the set

$\mathbb{H} = \bigcap_{n\in\mathbb{N}} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$. This approach was extended in Bergelson, Blass, and Hindman [1994].

6.1. Definition. A *partial semigroup* is a pair $(S,\cdot)$ where S is a set and there is some set $D \subseteq S \times S$ such that $\cdot : D \to S$ and the operation is associative where it is defined (in the sense that for any $x, y, z \in S$, if either of $(x \cdot y)\cdot z$ or $x\cdot(y\cdot z)$ is defined, then so is the other and they are equal). Given $x \in S$, $\varphi(x) = \{y \in S : (x,y) \in D\}$. The partial semigroup $(S,\cdot)$ is *adequate* if and only if for every finite nonempty set $F \subseteq S$, $\bigcap_{x\in F}\varphi(x) \neq \emptyset$. If S is adequate, then $\delta S = \bigcap_{x\in S}\overline{\varphi(x)}$.

Notice that the requirement that S be adequate is exactly what is needed to have $\delta S \neq \emptyset$. From our point of view, the most important thing about adequate partial semigroups is that δS is a (compact right topological) *semigroup*, with all of the structure known for such objects. In Bergelson, Blass and Hindman [1994] several Ramsey Theoretic results related to the Hales-Jewett Theorem were obtained.

In 1992, W. Gowers established a Ramsey Theoretic result as a tool to solve a problem about Banach spaces. While he did not state it this way, his result is naturally stated in terms of partial semigroups. Let $k \in \mathbb{N}$ and let $Y = \{f : f : \mathbb{N} \to \{0,1,\ldots,k\}$ and $\{x \in \mathbb{N} : f(x) \neq 0\}$ is finite$\}$. Given $f \in Y$, let $\mathrm{supp}(f) = \{x \in \mathbb{N} : f(x) \neq 0\}$ and for $f, g \in Y$, define $f+g$ pointwise, but only when $\mathrm{supp}(f)\cap\mathrm{supp}(g) = \emptyset$. Then $(Y,+)$ is an adequate partial semigroup. Let $Y_k = \{f \in Y : \max(f[\mathbb{N}]) = k\}$. Define $\sigma : Y \to Y$ by

$$\sigma(f)(x) = \begin{cases} f(x)-1 & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) = 0\,. \end{cases}$$

Notice that σ is a partial semigroup homomorphism in the sense that $\sigma(f+g) = \sigma(f)+\sigma(g)$ whenever $f+g$ is defined.

6.2. Theorem (Gowers). *Let k, Y, Y_k and σ be as defined above, let $r \in \mathbb{N}$, and let $Y = \bigcup_{i=1}^r C_i$. Then there exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle f_n\rangle_{n=1}^\infty$ in Y_k such that $\mathrm{supp}(f_n)\cap\mathrm{supp}(f_m) = \emptyset$ for all $m, n \in \mathbb{N}$ and $\{\sum_{n\in F}\sigma^{t(n)}(f_n) : F \in \mathcal{P}_f(\mathbb{N}),\, t : F \to \{0,1,\ldots,k-1\},$ and $t^{-1}[\{0\}] \neq \emptyset\} \subseteq C_i$.*

□ Gowers [1992, Theorem 1]. □

Farah, Hindman, and McLeod derived a simultaneous generalization of Theorem 6.2 and one of the results of Bergelson, Blass, and Hindman [1994]. This generalization is quite complicated to state in its entirety, but we shall describe a reasonably simple corollary.

6.3. Theorem (Farah, Hindman, and McLeod). *Let S, T, and R be the free semigroups with identity e on the alphabets $\{a,b,c\}$, $\{a,b\}$, and $\{a\}$ respectively. Given $x, y, z \in \{a,b,c,e\}$, let f_{xyz} be the endomorphism of S determined by $f(a) = x$, $f(b) = y$, and $f(c) = z$. For every $r \in \mathbb{N}$ and every partition $S = \bigcup_{j=1}^r C_j$ there exist an infinite $\langle x_n\rangle_{n=1}^\infty$ in $S \setminus T$ and $\gamma\colon\{a,b,c\} \to \{1,2,\ldots,r\}$ such that if $\sigma \in \{f_{eab}, f_{aeb}, f_{aab}\}$ and $\mathcal{F} = \{f_{abc}, f_{abb}, f_{aba}, f_{abe}, \sigma\} \cup \big\{f_{xyz} | x, y, z \in \{a,e\}\big\}$,* *then we have*

$$\{\prod_{n\in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N})\,,\ \textit{and for each } n \in F\,,\ g_n \in \mathcal{F}\} \cap (S \setminus T) \subseteq C_{\gamma(a)}$$

$$\{\prod_{n\in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N})\,,\ \textit{and for each } n \in F\,,\ g_n \in \mathcal{F}\} \cap (T \setminus R) \subseteq C_{\gamma(b)}$$

$$\{\prod_{n\in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N})\,,\ \textit{and for each } n \in F\,,\ g_n \in \mathcal{F}\} \cap R \setminus \{e\}) \subseteq C_{\gamma(c)}\,.$$

□ Farah, Hindman, and McLeod [20∞, Corollary 3.14]. □

As we have previously mentioned, several dynamical notions of largeness in a semigroup S, including "syndetic", "thick", and "piecewise syndetic" have simple characterizations in terms of the algebra of βS. These notions (for a discrete semigroup) also have simple combinatorial characterizations. For example, a subset A of S is syndetic if and only if there is a finite nonempty subset H of S such that $S = \bigcup_{t\in H} t^{-1}A$, where $t^{-1}A = \{s \in S : ts \in A\}$. Each of these notions has a completely obvious analogue for partial semigroups in terms of the algebra of δS. (So that, for example, a subset A of the partial semigroup S is syndetic if and only if for every left ideal of δS, $\overline{A} \cap S \neq \emptyset$.) There are also natural, though somewhat less obvious, analogues of the combinatorial characterizations. For example A is $\check{c}$-syndetic if and only if there exists finite nonempty $H \subseteq S$ such that $\bigcap_{t\in H} \varphi(t) \subseteq \bigcup_{t\in H} t^{-1}A$.

McLeod [2001] and [20∞], showed that for each of these (and other) notions of largeness, the natural algebraic and the natural combinatorial versions (the ones preceeded by $\check{c}$) need not be equivalent. She also showed that in each case one of the notions implies the others. (For example "syndetic" implies "$\check{c}$-syndetic", while "$\check{c}$-thick" implies "thick".)

A *VIP system* is a polynomial type generalization of the notion of an IP system, i.e., a set of finite sums. Hindman and McCutcheon [2001], extended the notion of VIP system to commutative partial semigroups and obtained an analogue of the Central Sets Theorem for these systems which extends the polynomial Hales-Jewett Theorem of Bergelson and Leibman [1996]. Several Ramsey Theoretic consequences, including the Central Sets Theorem itself, were then derived from these results.

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