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Separating Milliken-Taylor systems in $\mathbb{Q}$

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A finite sequence $\vec{a} = \langle a_i\rangle_{i=1}^k$ in $\mathbb{Q}\setminus\{0\}$ is *compressed* provided $a_i \neq a_{i+1}$ for $i < k$. Given a compressed sequence $\vec{a} = \langle a_i\rangle_{i=1}^k$ in $\mathbb{Z}\setminus\{0\}$ and given a sequence $\langle x_n\rangle_{n=1}^\infty$ in a commutative group $(G,+)$, the *Milliken-Taylor system* generated by $\vec{a}$ and $\langle x_n\rangle_{n=1}^\infty$ is $MT(\vec{a},\langle x_n\rangle_{n=1}^\infty) = \{\sum_{i=1}^k a_i \cdot \sum_{n\in F_i} x_n : F_1, F_2, \ldots, F_k$ are finite nonempty subsets of $\mathbb{N}$ with $\max F_i < \min F_{i+1}$ for $i < k\}$. It is an easy consequence of the Milliken-Taylor Theorem that Milliken-Taylor systems are partition regular in the strong sense that if $\langle y_n\rangle_{n=1}^\infty$ is any sequence in G, and $MT(\vec{a},\langle y_n\rangle_{n=1}^\infty)$ is partitioned into finitely many cells, there is a sequence $\langle x_n\rangle_{n=1}^\infty$ such that $MT(\vec{a},\langle x_n\rangle_{n=1}^\infty)$ is contained in one of those cells.

It is known that if $\vec{a}$ and $\vec{b}$ are compressed sequences in $\mathbb{Z}\setminus\{0\}$ which are not rational multiples of each other, then there is a partition of $\mathbb{Z}\setminus\{0\}$ into two cells, neither of which contains $MT(\vec{a},\langle x_n\rangle_{n=1}^\infty)\cup MT(\vec{b},\langle y_n\rangle_{n=1}^\infty)$ for any sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$. In this paper we establish the corresponding statement for Milliken-Taylor systems in $\mathbb{Q}$. (In fact, the entries of $\vec{a}$ and $\vec{b}$ are allowed to come from $\mathbb{Q}\setminus\{0\}$.)

1. Introduction

Given a set X, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X. If κ is a cardinal, then $[X]^\kappa$ is the set of subsets of X with κ elements. And given $F, H \in \mathcal{P}_f(\mathbb{N})$, where $\mathbb{N}$ is the set of positive integers, we write $F < H$ to indicate that $\max F < \min H$.

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Theorem 1.1 (Milliken-Taylor Theorem). *Let $k, r \in \mathbb{N}$ and let*

$$[\mathcal{P}_f(\mathbb{N})]^k = \bigcup_{i=1}^r \mathcal{A}_i\,.$$

There exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle F_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $F_n < F_{n+1}$ for each n and whenever $H_1, H_2, \ldots, H_k \in \mathcal{P}_f(\mathbb{N})$ with $H_t < H_{t+1}$ for $t < k$, one has $\{\bigcup_{n\in H_1} F_n, \bigcup_{n\in H_2} F_n, \ldots, \bigcup_{n\in H_k} F_n\} \in \mathcal{A}_i$.

Proof. [6, Theorem 2.2] or [7, Lemma 2.2]. □

Definition 1.2. Let $(S, +)$ be a commutative semigroup and let $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ be sequences in S. Then $\langle x_n\rangle_{n=1}^\infty$ is a *sum subsystem* of $\langle y_n\rangle_{n=1}^\infty$ if and only if there is a sequence $\langle F_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $F_n < F_{n+1}$ for each n and $x_n = \sum_{t\in F_n} y_t$ for each n.

As we remarked in the abstract, it is an immediate consequence of the Milliken-Taylor Theorem that Milliken-Taylor systems are partition regular.

Theorem 1.3. *Let $k, r \in \mathbb{N}$, let $\vec{a} = \langle a_1, \ldots, a_k\rangle$ be a compressed sequence in $\mathbb{Z}\setminus\{0\}$, let $(G, +)$ be a commutative group, and let $\langle y_n\rangle_{n=1}^\infty$ be a sequence in G. If $MT(\vec{a}, \langle y_n\rangle_{n=1}^\infty) = \bigcup_{i=1}^r A_i$, then there exist $i \in \{1, 2, \ldots, r\}$ and a sum subsystem $\langle x_n\rangle_{n=1}^\infty$ of $\langle y_n\rangle_{n=1}^\infty$ such that $MT(\vec{a}, \langle x_n\rangle_{n=1}^\infty) \subseteq A_i$.*

Proof. For $i \in \{1, 2, \ldots, r\}$, let

$$\mathcal{A}_i = \big\{\{K_1, K_2, \ldots, K_k\} \in [\mathcal{P}_f(\mathbb{N})]^k : \; K_1 < K_2 < \ldots < K_k \text{ and } \textstyle\sum_{j=1}^k a_j \cdot \sum_{n\in K_j} y_n \in A_i\big\}$$

and let $\mathcal{A}_0 = [\mathcal{P}_f(\mathbb{N})]^k \setminus \bigcup_{i=1}^r \mathcal{A}_i$. By Theorem 1.1, pick $i \in \{0, 1, \ldots, r\}$ and a sequence $\langle F_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $F_n < F_{n+1}$ for each n and whenever $H_1, H_2, \ldots, H_k \in \mathcal{P}_f(\mathbb{N})$ with $H_t < H_{t+1}$ for $t < k$, one has

$$\{\textstyle\bigcup_{n\in H_1} F_n, \bigcup_{n\in H_2} F_n, \ldots, \bigcup_{n\in H_k} F_n\} \in \mathcal{A}_i\,.$$

Notice that $i > 0$. For each $n \in \mathbb{N}$, let $x_n = \sum_{t\in F_n} y_t$. □

Given a compressed sequence $\vec{a}$ in $\mathbb{Z}\setminus\{0\}$, there is a matrix M such that $MT(\vec{a}, \langle x_n\rangle_{n=1}^\infty)$ is the set of entries of $M\vec{x}$, where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$. These matrices are examples of *image partition regular* matrices and were some of the first known examples of infinite image partition regular matrices. Finite image partition regular matrices with rational entries have the property that given any finite partition of $\mathbb{N}$, there is one cell which contains an image of all of these matrices. (See [5, Theorem 15.24].) By way of contrast, there is the following theorem.

Theorem 1.4. *Let $\vec{a}$ and $\vec{b}$ be compressed sequences in $\mathbb{Z}\setminus\{0\}$ such that $\vec{b}$ is not a rational multiple of $\vec{a}$. There exist a partition $\{A_1, A_2\}$ of $\mathbb{Z}\setminus\{0\}$ such that there do not exist $i \in \{1,2\}$ and sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ with $MT(\vec{a}, \langle x_n\rangle_{n=1}^\infty) \cup MT(\vec{b}, \langle y_n\rangle_{n=1}^\infty) \subseteq A_i$.*

Proof. [3, Corollary 3.9]. □

Theorem 1.4 extended a similar result in [2] in which the entries of $\vec{a}$ and $\vec{b}$ were assumed to be positive. The proof of Theorem1.4 was constructed by defining a partition of $\mathbb{Z}\setminus\{0\}$ into a very large number of cells, none of which contained $MT(\vec{a}, \langle x_n\rangle_{n=1}^\infty) \cup MT(\vec{b}, \langle y_n\rangle_{n=1}^\infty)$. The number of cells was then reduced to two by invoking Theorem 1.3.

Notice that, if $\vec{b}$ is a rational multiple of $\vec{a}$, then given any sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{Z}$ there is a sequence $\langle y_n\rangle_{n=1}^\infty$ in $\mathbb{Z}$ such that $MT(\vec{b}, \langle y_n\rangle_{n=1}^\infty) \subseteq MT(\vec{a}, \langle x_n\rangle_{n=1}^\infty)$. To see this, say $\vec{b} = \frac{m}{n}\cdot\vec{a}$. Choose a sum subsystem $\langle z_n\rangle_{n=1}^\infty$ of $\langle x_n\rangle_{n=1}^\infty$ such that every term of $\langle z_n\rangle_{n=1}^\infty$ is divisible by m, and let $\langle y_n\rangle_{n=1}^\infty = \frac{n}{m}\cdot\langle z_n\rangle_{n=1}^\infty$.

In Section 4 we establish the analogue of Theorem 1.4 for $\mathbb{Q}$. That is, we show that Milliken-Taylor systems can be separated in $\mathbb{Q}$.

We utilize the algebraic structure of βS, the Stone-Čech compactification of S, where $(S,+)$ is a discrete semigroup. (The reader should be cautioned that whenever we use βS, we are taking S to have the discrete topology. In particular, $\beta\mathbb{Q}$ refers to the Stone-Čech compactification of $\mathbb{Q}_d$, the set $\mathbb{Q}$ with the discrete topology.) We take the points of βS to be the ultrafilters on S, with the points of S being identified with the principal ultrafilters. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The operation $+$ on S extends to an operation on βS, also denoted by $+$, so that $(\beta S, +)$ is a right topological semigroup (meaning that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q + p$) with S contained in its topological center (meaning that for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\lambda_x(q) = x + q$). Given p and q in βS and $A \subseteq S$, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$. (Here $-x + A = \{y \in S : x + y \in A\}$. If S is a group, then this agrees with the usual definition as $\{-x + z : z \in A\}$.) As does any compact right topological semigroup, βS has idempotents. See [5, Part I] for an elementary introduction to the algebraic structure of βS.

The reader should be cautioned that, even though we denote the operation on βS by $+$, it is not commutative by [5, Theorem 6.54] if S is cancellative.

Given $a \in \mathbb{N}$, a commutative semigroup $(S,+)$, and $x \in S$, we let ax have its usual meaning – that is the sum of x with itself a times. If $a \in \mathbb{N}$ and $p \in S^* = \beta S \setminus S$, we define $ap = \widetilde{l_a}(p)$ where $l_a : S \to S$ is defined as $l_a(x) = ax$ and $\widetilde{l_a} : \beta S \to \beta S$ is its continuous extension. Thus, for example, $2p$ does not mean $p + p$. (In $\beta\mathbb{Z}$, by [5, Theorem 13.18], there is no $p \in \mathbb{Z}^*$ such that $2p = p + p$.)

Given a group G, $a \in \mathbb{N}$, and $x \in G$ we also let $-ax$ have its usual meaning and define $-ap$ for $p \in G^*$ as above. In either case, if $p \in \beta S$ and $a \in \mathbb{N}$ or $a \in \mathbb{Z} \setminus \{0\}$ as appropriate, then for each $A \subseteq S$, $A \in ap$ if and only if $a^{-1}A \in p$, where $a^{-1}A = \{x \in S : ax \in A\}$.

We will use coloring terminology throughout. By a *finite coloring* of a set X we mean a function from X to a finite set. It is said to be a k-coloring if its range has cardinality k. A set $A \subseteq X$ is said to be *monochromatic* with respect to a coloring ψ provided ψ is constant on A. In this terminology, Theorem 1.4 says that if $\vec{a}$ and $\vec{b}$ are compressed sequences in $\mathbb{Z} \setminus \{0\}$ such that $\vec{b}$ is not a rational multiple of $\vec{a}$, then there is a 2-coloring of $\mathbb{Z} \setminus \{0\}$ such that there do not exist sequences $\langle x_n \rangle_{n=1}^\infty$ and $\langle y_n \rangle_{n=1}^\infty$ with $MT(\vec{a}, \langle x_n \rangle_{n=1}^\infty) \cup MT(\vec{b}, \langle y_n \rangle_{n=1}^\infty)$ monochromatic. If ψ is a coloring of X, then D is a *color class of* ψ if and only if there is some i in the range of ψ such that $D = \psi^{-1}[\{i\}]$.

Our coloring of $\mathbb{Q} \setminus \{0\}$ is based on the negative factorial representation of rational numbers introduced by Budak, Işic, and Pym in [1]. Section 2 consists of several results about the arithmetic of this representation and construction of two colorings, one of initial segments of the negative factorial representation and the other of terminal segments. In Section 3 we use those two colorings to obtain a coloring of $\mathbb{Q}$ which separates expressions of the form $a_1p + a_2p + \ldots + a_mp$ and $b_1q + b_2q + \ldots b_kq$ where $\langle a_1, a_2, \ldots, a_m \rangle$ and $\langle b_1, b_2, \ldots, b_k \rangle$ are compressed sequences in $\mathbb{Z} \setminus \{0\}$ that are not rational multiples of each other and p and q are idempotents in $\bigcap_{\epsilon>0} c\ell_{\beta\mathbb{Q}}\big((-\epsilon,\epsilon) \cap \mathbb{Q}\big) \setminus \{0\}$. Section 4 then consists of the proof of our main results.

A connection between Milliken-Taylor systems and linear expressions in βG is provided by the following theorem. This result is well known by aficionados, but we cannot find an explicit statement in the literature.

Theorem 1.5. *Let G be a commutative group, let $\vec{a} = \langle a_1, a_2, \ldots, a_m \rangle$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$, and let $A \subseteq G$. There is a sequence $\langle x_n \rangle_{n=1}^\infty$ in G such that $MT(\vec{a}, \langle x_n \rangle_{n=1}^\infty) \subseteq A$ if and only if there is an idempotent $p \in \beta G$ such that $A \in a_1p + a_2p + \ldots + a_mp$.*

Proof. Necessity. Pick an idempotent $p \in \bigcap_{m=1}^\infty \overline{FS(\langle x_n \rangle_{n=m}^\infty)}$ by [5, Lemma 5.11]. The proof that $A \in a_1p + a_2p + \ldots + a_mp$ may then be taken verbatim from the proof of [5, Theorem 17.32] except that the one occurrence of "$x \in \mathbb{N}$" should be replaced by "$x \in G$".

Sufficiency. Let $S = T = G$, let $h : \bigcup_{k=1}^{\infty} G^k \to p$ be the function constantly equal to G, and define $f : G^m \to G$ by $f(x_1, x_2, \ldots, x_m) = a_1x_1 + a_2x_2 + \ldots + a_mx_m$. Then apply [4, Theorem 3.3]. □

2. Arithmetic in the negative factorial representation

The following is the negative factorial representation of rationals due to Budak, Işic, and Pym.

Theorem 2.1. *Each $x \in \mathbb{Q}$ has a unique representation of the form*

$$\sum_{t=2}^{\infty} \frac{a(x,t)}{t!}(-1)^t + \sum_{t=1}^{\infty} b(x,t) \cdot t! \cdot (-1)^{t+1}$$

where each $a(x,t) \in \{0, 1, \ldots, t-1\}$, each $b(x,t) \in \{0, 1, \ldots, t\}$, and all but finitely many of each are zero.

Proof. This is [1, Theorem 4.2] except the expression given there represents $-x$ as we have written it. □

The reason we take the negative of the Budak-Işic-Pym representation is that we will be concerned only with numbers close to zero, and these all have representations in the form $\sum_{t=2}^{\infty} \frac{a(x,t)}{t!}(-1)^t$. (We prefer to type an exponent of t rather than $t+1$.)

The proof of the following lemma is a rather tedious computation which we omit.

Lemma 2.2. *Let $k, m \in \mathbb{N}$ with $2 \leq k \leq m$ and let*

$$A = \{\sum_{t=k}^{m} \frac{a(t)}{t!}(-1)^t : (\forall t \in \{k, k+1, \ldots, m\})(a(t) \in \{0, 1, \ldots, t-1\})\}.$$

Then $A = \{\frac{t}{m!} : t \in \{a, a+1, \ldots, b\}\}$ where

(a) *if $m = 2s$ and either $k = 2r$ or $k = 2r+1$, then $a = -m! \cdot \sum_{t=r}^{s-1} \frac{2t}{(2t+1)!}$;*
(b) *if $m = 2s+1$ and either $k = 2r$ or $k = 2r+1$, then $a = -m! \cdot \sum_{t=r}^{s} \frac{2t}{(2t+1)!}$;*
(c) *if $k = 2r$ and either $m = 2s$ or $m = 2s+1$, then $b = m! \cdot \sum_{t=r}^{s} \frac{2t-1}{(2t)!}$; and*
(d) *if $k = 2r+1$ and either $m = 2s$ or $m = 2s+1$, then $b = m! \cdot \sum_{t=r+1}^{s} \frac{2t-1}{(2t)!}$.*

If $x = \sum_{t=2}^{\infty} \frac{a(x,t)}{t!}(-1)^t$ with each $a(x,t) \in \{0,1,\ldots,t-1\}$ and all but finitely many equal to 0, we write $\mathrm{supp}(x) = \{t : a(x,t) \neq 0\}$, and if $\mathrm{supp}(x) \neq \emptyset$, we let $\alpha(x) = \min\ \mathrm{supp}(x)$ and $\delta(x) = \max\ \mathrm{supp}(x)$. We then say simply "$x$ has a small negative factorial representation" and let $T = \{x \in \mathbb{Q} : x$ has a small negative factorial representation$\}$.

Lemma 2.3. *Let $x \in \mathbb{Q}$ and let $k = 2r \in \mathbb{N}$. Then $x \in T \setminus \{0\}$ with $\alpha(x) \geq k$ if and only if*

$$-\sum_{t=r}^{\infty} \frac{2t}{(2t+1)!} < x < \sum_{t=r}^{\infty} \frac{2t-1}{(2t)!}\,.$$

In particular $x \in T \setminus \{0\}$ if and only if $-\sum_{t=1}^{\infty} \frac{2t}{(2t+1)!} < x < \sum_{t=1}^{\infty} \frac{2t-1}{(2t)!}$.

Proof. We will use the fact that for each $m \in \mathbb{N}$, $\sum_{t=m+1}^{\infty} \frac{t-1}{t!} = \frac{1}{m!}$. (Prove by induction on s that $\sum_{t=m+1}^{s} \frac{t-1}{t!} = \frac{1}{m!} - \frac{1}{s!}$.)

The necessity is immediate. So assume that

$$-\sum_{t=r}^{\infty} \frac{2t}{(2t+1)!} < x < \sum_{t=r}^{\infty} \frac{2t-1}{(2t)!}$$

If $x = 0$, the conclusion is trivial, so assume first that $x > 0$. Pick the first $m = 2s$ in $\mathbb{N}$ such that $x = \frac{t}{m!}$ for some $t \in \mathbb{Z}$. By Lemma 2.2, it suffices to show that $t \leq m! \cdot \sum_{t=r}^{s} \frac{2t-1}{(2t)!}$. So suppose instead that $t \geq m! \cdot \sum_{t=r}^{s} \frac{2t-1}{(2t)!} + 1$. Then

$$\begin{array}{rcl} \frac{t}{m!} & \geq & \sum_{t=r}^{s} \frac{2t-1}{(2t)!} + \frac{1}{m!} \\ & = & \sum_{t=r}^{s} \frac{2t-1}{(2t)!} + \sum_{t=m+1}^{\infty} \frac{t-1}{t!} \\ & > & \sum_{t=r}^{s} \frac{2t-1}{(2t)!} + \sum_{t=s+1}^{\infty} \frac{2t-1}{(2t)!}\,, \end{array}$$

a contradiction.

The proof that if $x < 0$, then $x > -\sum_{t=r}^{\infty} \frac{2t}{(2t+1)!}$ is very similar. □

We think of our numbers being written using $\times_{t=2}^{\infty}\{0,1,\ldots,t-1\}$ in a fashion directly analogous to ordinary arithmetic. Thus, for example, if $y = \langle 1,2,0,3,0,0,\ldots\rangle$ and $x = \sum_{t=2}^{\infty} \frac{y_t}{t!}(-1)^t$, then $x = \frac{1}{2} - \frac{2}{6} - \frac{3}{120} = \frac{17}{120}$. We think of x as being written as .1203 and refer to 1 as the entry in column 2, 2 as the entry in column 3, 0 as the entry in column 4, and 3 as the entry in column 5. The next lemma describes how to do addition using the negative factorial representation.

Lemma 2.4. *Let $v, w \in \times_{t=2}^{\infty}\{0,1,\ldots,t-1\} \setminus \{0\}$ with $v_2 = v_3 = w_2 = w_3 = 0$, let $x = \sum_{t=2}^{\infty} \frac{v_t}{t!}(-1)^t$, and let $y = \sum_{t=2}^{\infty} \frac{w_t}{t!}(-1)^t$. Then $x + y = \sum_{t=2}^{\infty} \frac{z_t}{t!}(-1)^t$, where $z \in \times_{t=2}^{\infty}\{0,1,\ldots,t-1\}$ is obtained as follows. Let $m = \max\{t : v_t \neq 0$ or $w_t \neq 0\}$. For each $t \in \{2,3,\ldots,m-1\}$ we will have a "carry" $c_t \in \{-1,0,1\}$. Starting in column m add v_m and w_m.*

(1) If $v_m + w_m < m$*, let* $z_m = v_m + w_m$ *and set* $c_{m-1} = 0$*.*
(2) If $v_m + w_m \geq m$*, let* $z_m = v_m + w_m - m$ *and set* $c_{m-1} = -1$*.*

Assume we have added columns $t + 1$ *through* m *and have defined* c_t*.*

(1) If $v_t + w_t + c_t \in \{0, 1, \ldots, t-1\}$*, let* $z_t = v_t + w_t + c_t$ *and let* $c_{t-1} = 0$*.*
(2) If $v_t + w_t + c_t \geq t$*, let* $z_t = v_t + w_t + c_t - t$ *and let* $c_{t-1} = -1$*.*
(3) If $v_t + w_t + c_t = -1$*, let* $z_t = t - 1$ *and let* $c_{t-1} = 1$*.*

Proof. This analysis is carried out on pages 106 and 107 of [1]. □

Notice, as was remarked in [1], getting a carry of 1 is relatively rare. For this to happen one has to have $v_t = w_t = 0$ and $c_t = -1$.

Lemma 2.5. *Assume that* $x \in T \setminus \{0\}$ *and let* $k = \alpha(x)$*.*

(1) If k *is even, then* $\frac{k+1}{(k+2)!} < x < \frac{k^2}{(k+1)!}$*.*
(2) If k *is odd, then* $-\frac{k^2}{(k+1)!} < x < -\frac{k+1}{(k+2)!}$*.*

Proof. Assume $k = 2r$. We will show that $x < \frac{k^2}{(k+1)!}$. The other three conclusions are proved in a very similar fashion.

$$\begin{array}{rcl} x & = & \sum_{t=k}^{\infty} \frac{a(x,t)}{t!}(-1)^t \\ & \leq & \frac{2r-1}{(2r)!} + \sum_{t=r+1}^{\infty} \frac{a(x,2t)}{(2t)!} \\ & < & \frac{2r-1}{(2r)!} + \sum_{t=2r+2}^{\infty} \frac{t-1}{t!} \\ & = & \frac{2r-1}{(2r)!} + \frac{1}{(2r+1)!} \\ & = & \frac{k^2}{(k+1)!} \end{array}$$

□

Note that, in particular, if $\alpha(x)$ is even, then $x > 0$, while if $\alpha(x)$ is odd, then $x < 0$.

Lemma 2.6. *Assume that* $x \in T \setminus \{0\}$*, let* $k = \alpha(x)$*, and let* $b \in \mathbb{Z}$*.*

(1) If $0 < 2b < k$*, then* $\alpha(bx) = k$ *or* $\alpha(bx) = k - 2$*.*
(2) If $-k < 2b < 0$*, then* $\alpha(bx) = k + 1$ *or* $\alpha(bx) = k - 1$*.*

Proof. (1) Assume first that $x > 0$. Then $k = 2r$ and $\alpha(bx) = 2m$ for some $m, r \in \mathbb{N}$. If $m > r$, then $b > 1$ so, using Lemma 2.5,

$$\frac{4(r+1)^2}{(2r+3)!} \geq \frac{4m^2}{(2m+1)!} > bx \geq 2x > \frac{2k+2}{(k+2)!} = \frac{4r+2}{(2r+2)!}$$

so $4(r+1)^2 > (4r+2)(2r+3)$, a contradiction.

Now assume that $m < r-1$. Then

$$\tfrac{4br^2}{(2r+1)!} > bx > \tfrac{2m+1}{(2m+2)!} \geq \tfrac{2r-3}{(2r-2)!}$$

so $4r^3 > 4br^2 > (2r-3)(2r-1)(2r)(2r+1)$, a contradiction.

Now assume that $x < 0$. Then $k = 2r+1$ and $\alpha(bx) = 2m+1$ for some $m, r \in \mathbb{N}$. As above, we derive contradictions from the assumptions that $m > r$ or that $m < r-1$, the latter using the fact that $r \geq b$.

(2) Assume first that $x > 0$. Then $k = 2r$ and $\alpha(bx) = 2m+1$ for some $m, r \in \mathbb{N}$. We claim that $m = r$ or $m = r-1$. If $m > r$, then using Lemma 2.5,

$$-\tfrac{(2r+3)^2}{(2r+4)!} \leq -\tfrac{(2m+1)^2}{(2m+2)!} < bx \leq -x < -\tfrac{2r+1}{(2r+2)!}$$

so $(2r+3)^2 > (2r+1)(2r+3)(2r+4)$, a contradiction.

Now assume that $m < r-1$. Then $x < \frac{4r^2}{(2r+1)!}$ so

$$\tfrac{4br^2}{(2r+1)!} < bx < -\tfrac{2m+2}{(2m+3)!} \leq -\tfrac{2r-2}{(2r-1)!}$$

so, since $-b \leq r-1$, $4(r-1)r^2 \geq 4(-b)r^2 > (2r-2)(2r)(2r+1)$, a contradiction.

Now assume that $x < 0$. Then $k = 2r+1$ and $\alpha(bx) = 2m$ for some $m, r \in \mathbb{N}$. As above, we derive contradictions from the assumptions that $m > r+1$ or that $m < r$, the latter using the fact that $r \geq -b$. We thus conclude that $m = r$ or $m = r+1$. □

Lemma 2.7. *Assume that $x \in T \setminus \{0\}$, let $k = \alpha(x)$, and let $b \in \mathbb{Z}$ such that $|2b| < k$ and $|b| + 2 < k$.*

(1) If $b > 0$, then

$$\big(a(by, k-2), a(by, k-1)\big) \in \{(0,0), (1, k-2), (1, k-3), \ldots, (1, k-b)\}\,.$$

(2) If $b < 0$, then

$$\big(a(by, k-2), a(by, k-1)\big) \in \{(0,0), (0,1), (0,2), \ldots, (0, -b)\}\,.$$

Proof. We use the analysis of Lemma 2.4.

(1) We add x to itself $b-1$ times. Since $a(x,k) > 0$, each such addition results in a carry of 0 or -1 into column $k-1$. The first instance of a carry of -1 results in entries in columns $k-2$ and $k-1$ of $(1, k-2)$. The second results in $(1, k-3)$, and so on.

(2) We will add $-x$ to itself $-b-1$ times. By Lemma 2.6, $\alpha(-x)=k+1$ or $\alpha(-x)=k-1$. If $\alpha(-x)=k+1$, then the resulting entries in columns $k-1$ and k are in $\{(0,0),(1,k-2),(1,k-3),\ldots,(1,k+b)\}$ by (1) and so the entries in columns $k-2$ and $k-1$ are $(0,0)$ or $(0,1)$.

So assume that $\alpha(-x)=k-1$. In this case, $a(-x,k-1)=1$ and $a(-x,k)>0$ since all other possibilities lead to the conclusion that $|x|=|-x|>\frac{k^2}{(k-1)!}$, contradicting Lemma 2.5. Consequently, each addition of $-x$ results in a carry of 0 or -1 to column $k-1$. Assume that

$$d\in\{1,2,\ldots,-b-2\}\,,\ a(-dx,k-2)=0,\text{ and } a(-dx,k-1)\in\{1,2,\ldots,d\}\,.$$

Let $u=a(-dx,k-1)$. Then $a(-(d+1)x,k-2)=0$, and $a(-(d+1)x,k-1)\in\{u,u+1\}$ and thus $a(bx,k-2)=0$ and $a(bx,k-1)\in\{1,2,\ldots,-b\}$. □

The next lemma will be used to produce a relevant coloring of initial segments of negative factorial representations.

Lemma 2.8. *Assume that $x\in T\setminus\{0\}$. Let $b,c\in\mathbb{Z}$, let $m\in\mathbb{N}$, let $r=\alpha(x)$ such that $\frac{1}{r+1}\le\left(\frac{m}{m-1}\right)^{1/8}$ and $\frac{1}{r+2}\le\left(\frac{c}{b}\right)^{-2}$. Pick $i\in\mathbb{Z}$ such that*

$$\left(\tfrac{c}{b}\right)^{i/8}\cdot\tfrac{1}{r!}<\tfrac{a(x,r)}{r!}-\tfrac{a(x,r+1)}{(r+1)!}+\tfrac{a(x,r+2)}{(r+2)!}-\tfrac{a(x,r+3)}{(r+3)!}<\left(\tfrac{c}{b}\right)^{(i+1)/8}\cdot\tfrac{1}{r!}\,.$$

(1) If $x>0$, then $\left(\frac{c}{b}\right)^{(i-1)/8}\cdot\frac{1}{r!}<x<\left(\frac{c}{b}\right)^{(i+2)/8}\cdot\frac{1}{r!}$.
(2) If $x<0$, then $-\left(\frac{c}{b}\right)^{(i+2)/8}\cdot\frac{1}{r!}<x<-\left(\frac{c}{b}\right)^{(i-1)/8}\cdot\frac{1}{r!}$.

Proof. Note that

$$\begin{aligned}\tfrac{r+5}{(r+3)(r+4)(r+6)}\cdot\tfrac{1}{r!}&<\tfrac{r+4}{(r+2)(r+3)(r+5)}\cdot\tfrac{1}{r!}\\&<\tfrac{1}{r!}-\tfrac{r}{(r+1)!}-\tfrac{r+2}{(r+3)!}\\&\le\tfrac{a(x,r)}{r!}-\tfrac{a(x,r+1)}{(r+1)!}+\tfrac{a(x,r+2)}{(r+2)!}-\tfrac{a(x,r+3)}{(r+3)!}\\&<\left(\tfrac{c}{b}\right)^{(i+1)/8}\cdot\tfrac{1}{r!}\end{aligned}$$

so $\frac{r+5}{(r+3)(r+4)(r+6)}<\frac{r+4}{(r+2)(r+3)(r+5)}<\left(\frac{c}{b}\right)^{(i+1)/8}$. Consequently

$$\begin{aligned}\left(\tfrac{c}{b}\right)^{(i+1)/8}\cdot\tfrac{1}{r!}+\tfrac{(r+4)^2}{(r+5)!}&=\left(\tfrac{c}{b}\right)^{(i+1)/8}\cdot\tfrac{1}{r!}+\tfrac{r+4}{(r+2)(r+3)(r+5)}\cdot\tfrac{1}{r+1}\cdot\tfrac{1}{r!}\\&<\left(\tfrac{c}{b}\right)^{(i+1)/8}\cdot\tfrac{1}{r!}+\left(\tfrac{c}{b}\right)^{(i+1)/8}\cdot\Big(\left(\tfrac{c}{b}\right)^{1/8}-1\Big)\cdot\tfrac{1}{r!}\\&=\left(\tfrac{c}{b}\right)^{(i+2)/8}\cdot\tfrac{1}{r!}\text{ and}\\\left(\tfrac{c}{b}\right)^{i/8}\cdot\tfrac{1}{r!}-\tfrac{(r+5)^2}{(r+6)!}&=\left(\tfrac{c}{b}\right)^{i/8}\cdot\tfrac{1}{r!}-\tfrac{r+5}{(r+3)(r+4)(r+6)}\cdot\tfrac{1}{r+1}\cdot\tfrac{1}{r+2}\cdot\tfrac{1}{r!}\\&>\left(\tfrac{c}{b}\right)^{i/8}\cdot\tfrac{1}{r!}-\left(\tfrac{c}{b}\right)^{(i+1)/8}\cdot\Big(\left(\tfrac{c}{b}\right)^{1/8}-1\Big)\cdot\left(\tfrac{c}{b}\right)^{-2}\cdot\tfrac{1}{r!}\\&=\left(\tfrac{c}{b}\right)^{(i-1)/8}\cdot\tfrac{1}{r!}\,.\end{aligned}$$

If $x > 0$, then $x = \frac{a(x,r)}{r!} - \frac{a(x,r+1)}{(r+1)!} + \frac{a(x,r+2)}{(r+2)!} - \frac{a(x,r+3)}{(r+3)!} + z$ where $\alpha(z) \geq r+4$. By Lemma 2.5, $-\frac{(r+5)^2}{(r+6)!} < z < \frac{(r+4)^2}{(r+5)!}$ and so

$$\begin{array}{rcl} \left(\frac{c}{b}\right)^{(i-1)/8} \cdot \frac{1}{r!} & < & \left(\frac{c}{b}\right)^{i/8} \cdot \frac{1}{r!} - \frac{(r+5)^2}{(r+6)!} \\ & < & x \\ & < & \left(\frac{c}{b}\right)^{(i+1)/8} \cdot \frac{1}{r!} + \frac{(r+4)^2}{(r+5)!} \\ & < & \left(\frac{c}{b}\right)^{(i+2)/8} \cdot \frac{1}{r!} \end{array}$$

.

If $x < 0$, then $x = -\frac{a(x,r)}{r!} + \frac{a(x,r+1)}{(r+1)!} - \frac{a(x,r+2)}{(r+2)!} + \frac{a(x,r+3)}{(r+3)!} + z$ where $\alpha(z) \geq r+4$. By Lemma 2.5, $-\frac{(r+4)^2}{(r+5)!} < z < \frac{(r+5)^2}{(r+6)!}$ and so

$$\begin{array}{rcl} -\left(\frac{c}{b}\right)^{(i+2)/8} \cdot \frac{1}{r!} & < & -\left(\frac{c}{b}\right)^{(i+1)/8} \cdot \frac{1}{r!} - \frac{(r+4)^2}{(r+5)!} \\ & < & x \\ & < & -\left(\frac{c}{b}\right)^{i/8} \cdot \frac{1}{r!} + \frac{(r+5)^2}{(r+6)!} \\ & < & -\left(\frac{c}{b}\right)^{(i-1)/8} \cdot \frac{1}{r!} \end{array}$$

. □

Definition 2.9. Let $F \in \mathcal{P}_f(\mathbb{Z} \setminus \{0\})$ and let $\xi = \max(\{3\} \cup \{|b| : b \in F\})$.

$$\begin{array}{rl} \mathcal{T} = \{\langle r, t_0, t_1, t_2, t_3 \rangle : & r \in \mathbb{N}\,,\ \frac{1}{r} < \left(\frac{\xi}{\xi-1}\right)^{1/8} - 1\,,\ \frac{1}{r} < \left(\frac{1}{\xi}\right)^2, \\ & t_0 \in \{1, 2, \ldots, r-1\}, \text{ and for } j \in \{1,2,3\}\,, \\ & t_j \in \{0, 1, \ldots, r+j-1\}\} \end{array}$$

If $x \in T \setminus \{0\}$, then $\overline{\alpha}(x) = \langle \alpha(x), a(x, \alpha(x)), a(x, \alpha(x)+1), a(x, \alpha(x)+2), a(x, \alpha(x)+3) \rangle$.

Note that if $\frac{1}{\alpha(x)} < \left(\frac{\xi}{\xi-1}\right)^{1/8} - 1$ and $\frac{1}{\alpha(x)} < \left(\frac{1}{\xi}\right)^2$, then $\overline{\alpha}(x) \in \mathcal{T}$.

Theorem 2.10. *Let $F \in \mathcal{P}_f(\mathbb{Z} \setminus \{0\})$ and let $\xi = \max(\{3\} \cup \{|b| : b \in F\})$. There is a finite coloring ψ of $\mathcal{T}$ such that, if x and y have small negative factorial reprentations, $\frac{1}{\alpha(x)-2} < \left(\frac{\xi}{\xi-1}\right)^{1/8} - 1$, $\frac{1}{\alpha(x)-2} < \left(\frac{1}{\xi}\right)^2$, $2\xi < \alpha(x)$, $\frac{1}{\alpha(y)-2} < \left(\frac{\xi}{\xi-1}\right)^{1/8} - 1$, $\frac{1}{\alpha(y)-2} < \left(\frac{1}{\xi}\right)^2$, $2\xi < \alpha(y)$, $\psi(\overline{\alpha}(x)) = \psi(\overline{\alpha}(y))$, and b and c are distinct members of F, then $\psi(\overline{\alpha}(bx)) \neq \psi(\overline{\alpha}(cy))$.*

Proof. Given $\langle r, t_0, t_1, t_2, t_3 \rangle$ and $\langle n, s_0, s_1, s_2, s_3 \rangle$ in $\mathcal{T}$, agree that

$$\psi(\langle r, t_0, t_1, t_2, t_3 \rangle) = \psi(\langle n, s_0, s_1, s_2, s_3 \rangle)$$

if and only if

(1) $r \equiv n \pmod 4$ and
(2) if $b, c \in F$ and either $0 < b < c$ or $c < b < 0$, then
   (a) if $i, j \in \omega$, $\left(\frac{c}{b}\right)^{i/8} \le r < \left(\frac{c}{b}\right)^{(i+1)/8}$, and $\left(\frac{c}{b}\right)^{j/8} \le n < \left(\frac{c}{b}\right)^{(j+1)/8}$, then $i \equiv j \pmod{16}$;
   (b) if $i, j \in \omega$, $\left(\frac{c}{b}\right)^{i/8} \le r - 1 < \left(\frac{c}{b}\right)^{(i+1)/8}$, and $\left(\frac{c}{b}\right)^{j/8} \le n - 1 < \left(\frac{c}{b}\right)^{(j+1)/8}$, then $i \equiv j \pmod{16}$;
   (c) if $i, j \in \omega$, $\left(\frac{c}{b}\right)^{i/8} \le r + 1 < \left(\frac{c}{b}\right)^{(i+1)/8}$, and $\left(\frac{c}{b}\right)^{j/8} \le n + 1 < \left(\frac{c}{b}\right)^{(j+1)/8}$, then $i \equiv j \pmod{16}$; and
   (d) if $i, j \in \mathbb{Z}$, $\left(\frac{c}{b}\right)^{i/8} \cdot \frac{1}{r!} \le \frac{t_0}{r!} - \frac{t_1}{(r+1)!} + \frac{t_2}{(r+2)!} - \frac{t_3}{(r+3)!} < \left(\frac{c}{b}\right)^{(i+1)/8} \cdot \frac{1}{r!}$ and $\left(\frac{c}{b}\right)^{j/8} \cdot \frac{1}{n!} \le \frac{s_0}{n!} - \frac{s_1}{(n+1)!} + \frac{s_2}{(n+2)!} - \frac{s_3}{(n+3)!} < \left(\frac{c}{b}\right)^{(i+1)/8} \cdot \frac{1}{n!}$, then $i \equiv j \pmod{16}$.

Now suppose we have x, y, b, and c as in the statement of the theorem except that $\psi(\overline{\alpha}(bx)) = \psi(\overline{\alpha}(cy))$. Let $r = \alpha(x)$ and $n = \alpha(y)$. Since $r \equiv n \pmod 2$, we have that x and y have the same sign. Also, since $\alpha(bx) \equiv \alpha(cy) \pmod 2$ we have that b and c have the same sign. We assume without loss of generality that either $0 < b < c$ or $c < b < 0$. And since $\alpha(bx) \equiv \alpha(cy) \pmod 4$ we have by Lemma 2.6 that

(1) if $0 < b < c$, then
   (a) $\alpha(bx) = r$ and $\alpha(cy) = n$ or
   (b) $\alpha(bx) = r - 2$ and $\alpha(cy) = n - 2$; and
(2) if $c < b < 0$, then
   (a) $\alpha(bx) = r + 1$ and $\alpha(cy) = n + 1$ or
   (b) $\alpha(bx) = r - 1$ and $\alpha(cy) = n - 1$.

Pick i and j in $\mathbb{Z}$ such that

$$\left(\frac{c}{b}\right)^{i/8} \cdot \frac{1}{r!} < \frac{a(x,r)}{r!} - \frac{a(x,r+1)}{(r+1)!} + \frac{a(x,r+2)}{(r+2)!} - \frac{a(x,r+3)}{(r+3)!} < \left(\frac{c}{b}\right)^{(i+1)/8} \cdot \frac{1}{r!}$$

and

$$\left(\frac{c}{b}\right)^{j/8} \cdot \frac{1}{n!} < \frac{a(y,n)}{n!} - \frac{a(y,n+1)}{(n+1)!} + \frac{a(y,n+2)}{(n+2)!} - \frac{a(y,n+3)}{(n+3)!} < \left(\frac{c}{b}\right)^{(j+1)/8} \cdot \frac{1}{n!}\,.$$

By Lemma 2.8,

$$\left(\frac{c}{b}\right)^{(i-1)/8} \cdot \frac{1}{r!} < |x| < \left(\frac{c}{b}\right)^{(i+2)/8} \cdot \frac{1}{r!} \text{ and } \left(\frac{c}{b}\right)^{(j-1)/8} \cdot \frac{1}{n!} < |y| < \left(\frac{c}{b}\right)^{(j+2)/8} \cdot \frac{1}{n!}$$

so

$$(*)\qquad \left(\tfrac{c}{b}\right)^{(i-j-3)/8}\cdot\tfrac{n!}{r!}<\tfrac{x}{y}<\left(\tfrac{c}{b}\right)^{(i-j+3)/8}\cdot\tfrac{n!}{r!}\,.$$

We consider four cases.

<u>Case 1a</u>. $0<b<c$, $\alpha(bx)=r$, and $\alpha(cy)=n$.

Pick p and q in $\mathbb{Z}$ such that

$$\left(\tfrac{c}{b}\right)^{p/8}\cdot\tfrac{1}{r!}<\tfrac{a(bx,r)}{r!}-\tfrac{a(bx,r+1)}{(r+1)!}+\tfrac{a(bx,r+2)}{(r+2)!}-\tfrac{a(bx,r+3)}{(r+3)!}<\left(\tfrac{c}{b}\right)^{(p+1)/8}\cdot\tfrac{1}{r!}$$

and

$$\left(\tfrac{c}{b}\right)^{q/8}\cdot\tfrac{1}{n!}<\tfrac{a(cy,n)}{n!}-\tfrac{a(cy,n+1)}{(n+1)!}+\tfrac{a(cy,n+2)}{(n+2)!}-\tfrac{a(cy,n+3)}{(n+3)!}<\left(\tfrac{c}{b}\right)^{(q+1)/8}\cdot\tfrac{1}{n!}\,.$$

By Lemma 2.8,

$$\begin{array}{l}\left(\tfrac{c}{b}\right)^{(p-1)/8}\cdot\tfrac{1}{r!}<|bx|<\left(\tfrac{c}{b}\right)^{(p+2)/8}\cdot\tfrac{1}{r!}\text{ and}\\ \left(\tfrac{c}{b}\right)^{(q-1)/8}\cdot\tfrac{1}{n!}<|cy|<\left(\tfrac{c}{b}\right)^{(q+2)/8}\cdot\tfrac{1}{n!}\end{array}$$

so

$$\left(\tfrac{c}{b}\right)^{(p-q-3)/8}\cdot\tfrac{n!}{r!}<\tfrac{bx}{cy}<\left(\tfrac{c}{b}\right)^{(p-q+3)/8}\cdot\tfrac{n!}{r!}$$

and thus

$$\left(\tfrac{c}{b}\right)^{(p-q+5)/8}\cdot\tfrac{n!}{r!}<\tfrac{x}{y}<\left(\tfrac{c}{b}\right)^{(p-q+11)/8}\cdot\tfrac{n!}{r!}\,.$$

Combining the last inequalities with $(*)$ we get that $i-j-3<p-q+11$ and $p-q+5<i-j+3$ so $-14<(p-q)-(i-j)<-2$, which is a contradiction since $(p-q)-(i-j)\equiv 0 \pmod{16}$.

<u>Case 1b</u>. $0<b<c$, $\alpha(bx)=r-2$, and $\alpha(cy)=n-2$.

Pick p and q in $\mathbb{Z}$ such that

$$\left(\tfrac{c}{b}\right)^{p/8}\cdot\tfrac{1}{(r-2)!}<\tfrac{a(bx,r-2)}{(r-2)!}-\tfrac{a(bx,r-1)}{(r-1)!}+\tfrac{a(bx,r)}{r!}-\tfrac{a(bx,r+1)}{(r+1)!}<\left(\tfrac{c}{b}\right)^{(p+1)/8}\cdot\tfrac{1}{(r-2)!}$$

and

$$\left(\tfrac{c}{b}\right)^{q/8}\cdot\tfrac{1}{(n-2)!}<\tfrac{a(cy,n-2)}{(n-2)!}-\tfrac{a(cy,n-1)}{(n-1)!}+\tfrac{a(cy,n)}{n!}-\tfrac{a(cy,n+1)}{(n+1)!}<\left(\tfrac{c}{b}\right)^{(q+1)/8}\cdot\tfrac{1}{(n-2)!}$$

By Lemma 2.8,

$$\begin{array}{l}\left(\tfrac{c}{b}\right)^{(p-1)/8}\cdot\tfrac{1}{(r-2)!}<|bx|<\left(\tfrac{c}{b}\right)^{(p+2)/8}\cdot\tfrac{1}{(r-2)!}\text{ and}\\ \left(\tfrac{c}{b}\right)^{(q-1)/8}\cdot\tfrac{1}{(n-2)!}<|cy|<\left(\tfrac{c}{b}\right)^{(q+2)/8}\cdot\tfrac{1}{(n-2)!}\end{array}$$

so

$$\left(\tfrac{c}{b}\right)^{(p-q-3)/8} \cdot \tfrac{(n-2)!}{(r-2)!} < \tfrac{bx}{cy} < \left(\tfrac{c}{b}\right)^{(p-q+3)/8} \cdot \tfrac{(n-2)!}{(r-2)!}$$

and thus

$$(**) \qquad \left(\tfrac{c}{b}\right)^{(p-q+5)/8} \cdot \tfrac{(n-2)!}{(r-2)!} < \tfrac{x}{y} < \left(\tfrac{c}{b}\right)^{(p-q+11)/8} \cdot \tfrac{(n-2)!}{(r-2)!} \, .$$

Pick u, v, w, and z in ω such that $\left(\frac{c}{b}\right)^{u/8} \leq n \leq \left(\frac{c}{b}\right)^{(u+1)/8}$, $\left(\frac{c}{b}\right)^{v/8} \leq r \leq \left(\frac{c}{b}\right)^{(v+1)/8}$, $\left(\frac{c}{b}\right)^{w/8} \leq n-1 \leq \left(\frac{c}{b}\right)^{(w+1)/8}$, and $\left(\frac{c}{b}\right)^{z/8} \leq r-1 \leq \left(\frac{c}{b}\right)^{(z+1)/8}$. Then $\left(\frac{c}{b}\right)^{(u-v-1)/8} < \frac{n}{r} < \left(\frac{c}{b}\right)^{(u-v+1)/8}$ and $\left(\frac{c}{b}\right)^{(w-z-1)/8} < \frac{n-1}{r-1} < \left(\frac{c}{b}\right)^{(w-z+1)/8}$.

Combining $(*)$ and $(**)$ we have

$$\begin{aligned} \left(\tfrac{c}{b}\right)^{(p-q+11)/8} \cdot \tfrac{(n-2)!}{(r-2)!} &> \tfrac{x}{y} \\ &> \left(\tfrac{c}{b}\right)^{(i-j-3)/8} \cdot \tfrac{(n-2)!}{(r-2)!} \cdot \tfrac{n-1}{r-1} \cdot \tfrac{n}{r} \\ &> \left(\tfrac{c}{b}\right)^{(i-j-3)/8} \cdot \left(\tfrac{c}{b}\right)^{(w-z-1)/8} \cdot \left(\tfrac{c}{b}\right)^{(u-v-1)/8} \cdot \tfrac{(n-2)!}{(r-2)!} \end{aligned}$$

and

$$\begin{aligned} \left(\tfrac{c}{b}\right)^{(p-q+5)/8} \cdot \tfrac{(n-2)!}{(r-2)!} &< \tfrac{x}{y} \\ &< \left(\tfrac{c}{b}\right)^{(i-j+3)/8} \cdot \tfrac{(n-2)!}{(r-2)!} \cdot \tfrac{n-1}{r-1} \cdot \tfrac{n}{r} \\ &< \left(\tfrac{c}{b}\right)^{(i-j+3)/8} \cdot \left(\tfrac{c}{b}\right)^{(w-z+1)/8} \cdot \left(\tfrac{c}{b}\right)^{(u-v+1)/8} \cdot \tfrac{(n-2)!}{(r-2)!} \, . \end{aligned}$$

So we deduce that $0 < (i-j) + (w-z) + (u-v) - (p-q) < 16$, while $(i-j) + (w-z) + (u-v) - (p-q) \equiv 0 \pmod{16}$, a contradiction.

The proofs of Case 2a (namely $c < b < 0$, $\alpha(bx) = r+1$, and $\alpha(cy) = n+1$) and Case 2b (namely $c < b < 0$, $\alpha(bx) = r-1$, and $\alpha(cy) = n-1$) are very similar to the proof of Case 1b. We leave the details to the reader. □

We now turn our attention to coloring based on the least significant digits using the small negative factorial expansions.

Lemma 2.11. *Let $x \in T \setminus \{0\}$. Then $\delta(x)$ is the smallest positive integer m such that $m!x \in \mathbb{Z}$.*

Proof. Trivially $\delta(x)!x \in \mathbb{Z}$. Also, $(\delta(x)-1)!x = k + \frac{a(x,\delta(x))}{\delta(x)}$ for some integer k. □

Lemma 2.12. *Let $x \neq 0$ and let $b \in \mathbb{Z} \setminus \{0\}$. If x and bx both have small negative factorial representations, then $\delta(x) \geq \delta(bx) \geq \delta(x) - |b|$.*

Proof. Let $m = \delta(bx)$. By Lemma 2.11, $m!bx \in \mathbb{Z}$. Now

$$(m+1)(m+2)\dots(m+|b|) = br$$

for some $r \in \mathbb{Z}$ so $(m+|b|)!x = m!brx \in \mathbb{Z}$ and thus by Lemma 2.11, $\delta(x) \leq m + |b|$. □

Definition 2.13. Let $F \in \mathcal{P}_f(\mathbb{Z}\setminus\{0\})$ and let $\xi = \max(\{3\}\cup\{|b| : b \in F\})$.

$$\begin{aligned}\mathcal{S} = \big\{\langle l, s_0, s_1, \dots, s_\xi\rangle :\ & l \in \mathbb{N}, l > 2\xi, \text{ for } i \in \{0,1,\dots,\xi-1\}, \\ & s_i \in \{0,1,\dots,l-\xi+i-1\}, \\ & \text{and } s_\xi \in \{1,2,\dots,l-1\}\big\}.\end{aligned}$$

If $x \in T\setminus\{0\}$ and $\delta(x) > 2\xi$, then

$$\overline{\delta}(x) = \langle \delta(x), a(x,\delta(x)-\xi), a(x,\delta(x)-\xi+1), \dots, a(x,\delta(x))\rangle .$$

Note that $\overline{\delta}(x) \in \mathcal{S}$.

Lemma 2.14. *Let $F \in \mathcal{P}_f(\mathbb{Z}\setminus\{0\})$ and let $\xi = \max(\{3\}\cup\{|b| : b \in F\})$. Assume that b and c are in F with $b < c$ (not necessarily $|b| < |c|$). There is a coloring τ of $\mathcal{S}$ in $|b|+|c|+3$ colors such that, if x, bx, and cx have small negative factorial representations, $\delta(bx) > \xi$, and $\delta(cx) > \xi$, then $\tau(\overline{\delta}(bx)) \neq \tau(\overline{\delta}(cx))$.*

Proof. We define a directed graph on $\mathcal{S}$ such that, given $\vec{s} = \langle l, s_0, s_1, \dots, s_\xi\rangle$ and $\vec{r} = \langle n, r_0, r_1, \dots, r_\xi\rangle$ in $\mathcal{S}$, there is an edge from $\vec{s}$ to $\vec{r}$ if and only if $n = l$ and there is some x such that x, bx, and cx have small negative factorial representations with $\delta(bx) > 2\xi$ and $\delta(cx) > 2\xi$, $\overline{\delta}(bx) = \vec{s}$, and $\overline{\delta}(cx) = \vec{r}$.

We will show that this graph has no loops and given a vertex $\vec{s}$, the out degree is at most $|b|+1$ and the in degree is at most $|c|+1$. One can then color the vertices one at a time giving each vertex a color which no adjacent vertex has.

So suppose first we have an edge from $\vec{s} = \langle l, s_0, s_1, \dots, s_\xi\rangle$ to itself. Pick x such that $\delta(bx) = \delta(cx) = l$ and for $t \in \{l-\xi, l-\xi+1, \dots, l\}$, $s_{t-l+\xi} = a(bx,t) = a(cx,t)$. Let $w = \sum_{t=l-\xi}^{l} \frac{s_{t-l+\xi}}{t!}(-1)^t$. Then

$$bx = \textstyle\sum_{t=2}^{l-\xi-1} \frac{a(bx,t)}{t!}(-1)^t + w \text{ and } cx = \sum_{t=2}^{l-\xi-1} \frac{a(cx,t)}{t!}(-1)^t + w$$

so $bx = \frac{u}{(l-\xi-1)!} + w$ and $cx = \frac{v}{(l-\xi-1)!} + w$ for some u and v in $\mathbb{Z}$. Then $(c-b)x = \frac{v-u}{(l-\xi-1)!}$ and thus

$$x = \frac{(v-u)(l-\xi)(l-\xi+1)\cdots(l-\xi+c-b-1)}{(c-b)(l-\xi+c-b-1)!}.$$

So, since $c-b$ divides $l-\xi+i$ for some $i \in \{0,1,\ldots,c-b-1\}$, we have $x = \frac{z}{(l-\xi+c-b-1)!}$ for some $z \in \mathbb{Z}$. Therefore by Lemma 2.11, $\delta(x) \leq l-\xi+c-b-1$. Also, $l = \delta(bx) \leq \delta(x) \leq l-\xi+c-b-1$ so $\xi \leq c-b-1$, a contradiction.

The proofs that the out degree of $\vec{s}$ is at most $|b|+1$ and the in degree of $\vec{s}$ is at most $|c|+1$ are identical, so we will do only the first. Suppose instead we have $x_1, x_2, \ldots, x_{|b|+2}$, each with small negative factorial representations such that

(1) for each $j \in \{1,2,\ldots,|b|+2\}$, $\overline{\delta}(bx_j) = \vec{s}$ and
(2) if j and n are distinct members of $\{1,2,\ldots,|b|+2\}$, then $\overline{\delta}(cx_j) \neq \overline{\delta}(cx_n)$.

Let $s = \sum_{i=0}^{\xi} \frac{s_i}{(l-\xi+i)!}(-1)^{l-\xi+i}$. Then for each $j \in \{1,2,\ldots,|b|+2\}$, $bx_j = \sum_{t=2}^{l-\xi-1} \frac{a(bx_j,t)}{t!}(-1)^t + s$.

For $j \in \{1,2,\ldots,|b|+2\}$, let $y_j = x_j - \sum_{t=2}^{l-\xi-1} \frac{a(x_j,t)}{t!}(-1)^t$. Then $by_j = bx_j - b \cdot \sum_{t=2}^{l-\xi-1} \frac{a(x_j,t)}{t!}(-1)^t = \sum_{t=2}^{l-\xi-1} \frac{a(bx_j,t)-a(x_j,t)}{t!}(-1)^t + s$. Let $z = \sum_{t=2}^{l-\xi-1} \frac{a(bx_j,t)-a(x_j,t)}{t!}(-1)^t$. Since $(l-\xi-1)!z \in \mathbb{Z}$ we have by Lemma 2.11 that $\delta(z) \leq l-\xi-1$. Therefore $\overline{\delta}(by_j) = \vec{s} = \overline{\delta}(bx_j)$. Similarly $\overline{\delta}(cy_j) = \overline{\delta}(cx_j)$ for each $j \in \{1,2,\ldots,|b|+2\}$.

Let $j \in \{1,2,\ldots,|b|+2\}$. By Lemma 2.6, $\alpha(by_j) \geq \alpha(y_j) - 2 \geq l-\xi-2$. If $b > 0$ and $\alpha(y_j) = l-\xi$, then

$$\begin{aligned}&\big(a(by_j,l-\xi-2), a(by_j,l-\xi-1)\big) \in \\ &\{(0,0),(1,l-\xi-2),(1,l-\xi-3),\ldots,(1,l-\xi-b)\}\end{aligned}$$

by Lemma 2.7. If $\alpha(y_j) = l-\xi+1$, then by the same lemma,

$$\big(a(by_j,l-\xi-2), a(by_j,l-\xi-1)\big) \in \{(0,0),(0,1)\},$$

while if $\alpha(y_j) \geq l-\xi+2$, then $\big(a(by_j,l-\xi-2), a(by_j,l-\xi-1)\big) = (0,0)$.

There are thus a total of $|b|+1$ possibilities for $\big(a(by_j,l-\xi-2), a(by_j,l-\xi-1)\big)$.

If $b < 0$, then by Lemma 2.7,

$$\big(a(by_j,l-\xi-2), a(by_j,l-\xi-1)\big) \in \{(0,0),(0,1),\ldots,(0,-b)\}$$

so there are again a total of $|b|+1$ possibilities.

Therefore there exist j and n, distinct members of $\{1,2,\ldots,|b|+2\}$, such that $\big(a(by_j,l-\xi-2), a(by_j,l-\xi-1)\big) = \big(a(by_n,l-\xi-2), a(by_n,l-\xi-1)\big)$.

Now

$$by_j = \sum_{t=l-\xi-2}^{l-\xi-1} \frac{a(by_j,t)}{t!}(-1)^t + s = \sum_{t=l-\xi-2}^{l-\xi-1} \frac{a(by_n,t)}{t!}(-1)^t + s = by_n$$

so $y_j = y_n$ and thus

$$\overline{\delta}(cx_j) = \overline{\delta}(cy_j) = \overline{\delta}(cy_n) = \overline{\delta}(cx_n)\,,$$

a contradiction. □

Theorem 2.15. *Let $F \in \mathcal{P}_f(\mathbb{Z}\setminus\{0\})$ and let $\xi = \max(\{3\}\cup\{|b| : b \in F\})$. There is a finite coloring τ of $\mathcal{S}$ such that, for any distinct b and c in F and any x such that x, bx, and cx have small negative factorial representations, $\delta(bx) > \xi$, and $\delta(cx) > \xi$, one has $\tau(\overline{\delta}(bx)) \neq \tau(\overline{\delta}(cx))$.*

Proof. For each pair of distinct elements b and c of F with $b < c$, pick a finite coloring $\tau_{b,c}$ of $\mathcal{S}$ as guaranteed by Lemma 2.14. Define a coloring τ of $\mathcal{S}$ by for $\vec{s}$ and $\vec{t}$ in $\mathcal{S}$, $\tau(\vec{s}) = \tau(\vec{t})$ if and only if for each pair of distinct elements b and c of F with $b < c$, $\tau_{b,c}(\vec{s}) = \tau_{b,c}(\vec{t})$. □

3. Separating linear expressions in $\beta\mathbb{Q}$

This section is devoted to a proof of Theorem 3.3 below. We will assume throughout that we have m and k in $\mathbb{N}$ and compressed sequences $\vec{a} = \langle a_1, a_2, \ldots, a_m\rangle$ and $\vec{b} = \langle b_1, b_2, \ldots, b_k\rangle$ in $\mathbb{Z}\setminus\{0\}$. We let $A = \{a_1,\ldots,a_m\}$, $B = \{b_1,\ldots,b_k\}$, and $\xi = \max(\{3\}\cup\{|b| : b \in A\cup B\})$. Recall that

$$T = \{x \in \mathbb{Q} : x \text{ has a small negative factorial representation}\}\,.$$

For $x, y \in T\setminus\{0\}$, we write $x \ll y$ if and only if $\delta(x) + \xi < \alpha(y)$. (Note that if $x \ll y$, then $x > y$. We write $x \ll y$ because in the constructions y is chosen after x.)

Definition 3.1. $S_0 = \bigcap_{\epsilon>0} c\ell_{\beta\mathbb{Q}}((-\epsilon,\epsilon)\cap\mathbb{Q})\setminus\{0\}$. For $x \in T\setminus\{0\}$, $C_x = \{y \in T\setminus\{0\} : x \ll y\}$.

Lemma 3.2. *S_0 is a compact subsemigroup of $(\beta\mathbb{Q},+)$. If $c \in \mathbb{Z}\setminus\{0\}$ and $r \in S_0$, then $cr \in S_0$ and cr is an idempotent if r is an idempotent. If $x \in T\setminus\{0\}$ and $r \in S_0$, then $C_x \in r$.*

Proof. Given $p, q \in S_0$ and $0 < \epsilon < \frac{1}{4}$,

$$(-\tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}) \cap \mathbb{Q} \subseteq \{x \in T : -x + ((-\epsilon, \epsilon) \cap \mathbb{Q}) \in q\}$$

so $(-\epsilon, \epsilon) \cap \mathbb{Q} \in p + q$. Given $c \in \mathbb{Z} \setminus \{0\}$, $r \in S_0$, and $0 < \epsilon < \frac{1}{4}$, $(-\frac{\epsilon}{|c|}, \frac{\epsilon}{|c|}) \subseteq c^{-1}(-\epsilon, \epsilon)$ so $(-\epsilon, \epsilon) \cap \mathbb{Q} \in cr$. It is routine to verify that cr is an idempotent when r is an idempotent.

For the final conclusion let $x \in T \setminus \{0\}$, pick $n \in \mathbb{N}$ such that $2n > \delta(x) + \xi$, and let $\epsilon = \sum_{t=n}^{\infty} \frac{2t}{(2t+1)!}$. Then by Lemma 2.3, $(-\epsilon, \epsilon) \cap \mathbb{Q} \subseteq C_x$. $\square$

Recall that we have defined (with $F = A \cup B$) ξ,

$$\begin{aligned}\mathcal{T} = \{\langle r, t_0, t_1, t_2, t_3\rangle : \ & r \in \mathbb{N}\,,\ \tfrac{1}{r} < \left(\tfrac{\xi}{\xi-1}\right)^{1/8} - 1\,,\ \tfrac{1}{r} < \left(\tfrac{1}{\xi}\right)^2\,,\\ & t_0 \in \{1, 2, \ldots, r-1\},\text{ and for } j \in \{1,2,3\}\,,\\ & t_j \in \{0, 1, \ldots, r+j-1\}\}\,,\end{aligned}$$

and

$$\begin{aligned}\mathcal{S} = \{\langle l, s_0, s_1, \ldots, s_\xi\rangle : \ & l \in \mathbb{N}\,, l > 2\xi,\text{ for } i \in \{0, 1, \ldots, \xi-1\}\,,\\ & s_i \in \{0, 1, \ldots, l-\xi+i-1\}\,,\\ & \text{and } s_\xi \in \{1, 2, \ldots, l-1\}\}\,.\end{aligned}$$

The proof of the following theorem uses a gap counting technique based on that in [3].

Theorem 3.3. *Assume that $\vec{b}$ is not a rational multiple of $\vec{a}$. There is a finite coloring Γ of $T \setminus \{0\}$ such that there do not exist idempotents $p, q \in S_0$ and a color class D such that $D \in a_1p + a_2p + \ldots + a_mp$ and $D \in b_1q + b_2q + \ldots + b_kq$.*

Proof. Assume without loss of generality that $m \leq k$. Note also that if $m = k = 1$, then $\vec{b}$ is a rational multiple of $\vec{a}$, so we may assume that $k > 1$. Let ψ and τ be the colorings guaranteed by Theorems 2.10 and 2.15 respectively. We may assume that we have $w \in \mathbb{N}$ such that $\psi : \mathcal{T} \to \{1, 2, \ldots, w\}$ and $\tau : \mathcal{S} \to \{1, 2, \ldots, w\}$

Given $r \in S_0$ and $c \in A \cup B$, pick $i(r, c)$ and $j(r, c)$ in $\{1, 2, \ldots, w\}$ such that

$$\{x \in T \setminus \{0\} : \psi(\overline{\alpha}(cx)) = i(r, c) \text{ and } \tau(\overline{\delta}(cx)) = j(r, c)\} \in r$$

and note that

$$\{x \in T \setminus \{0\} : \psi(\overline{\alpha}(x)) = i(r, c) \text{ and } \tau(\overline{\delta}(x)) = j(r, c)\} \in cr\,.$$

Note also that if $c,d\in A\cup B$ and $c\neq d$, then $i(r,c)\neq i(r,d)$ and $j(r,c)\neq j(r,d)$.

Definition 3.4. For $x\in T\setminus\{0\}$, let $G(x)=$

$$\begin{aligned}\{(t,u,v):\ & t\in \operatorname{supp}(x)\setminus\{\delta(x)\} \text{ and if } s=\min\{\eta\in\operatorname{supp}(x):\eta>t\}\,,\\ & \text{then } \tau(\langle t,a(x,t-\xi),a(x,t-\xi+1),\ldots,a(x,t)\rangle)=u, \text{ and}\\ & \psi(\langle s,a(x,s),a(x,s+1),a(x,s+2),a(x,s+3)\rangle)=v\}\,.\end{aligned}$$

For $x\in T\setminus\{0\}$ and $(u,v)\in\{1,2,\ldots,w\}^2$, let $G_{u,v}(x)=\{t\in\mathbb{N}:(t,u,v)\in G(x)\}$ and let $\varphi_{u,v}\in\{-k,-k+1,\ldots,k-1,k\}$ such that $\varphi_{u,v}(x)\equiv|G_{u,v}(x)|\pmod{2k+1}$. Let $\widetilde{\varphi}_{u,v}:\beta(T\setminus\{0\})\to\{-k,-k+1,\ldots,k-1,k\}$ be the continuous extension of $\varphi_{u,v}$.

We embark on a sequence of lemmas that will allow us to compute $\widetilde{\varphi}_{u,v}$ at certain points.

Lemma 3.5. *Let $x,y\in T\setminus\{0\}$, let $u,v\in\{1,2,\ldots,w\}$, and assume that $x\ll y$. Then $\varphi_{u,v}(x+y)\equiv\varphi_{u,v}(x)+\varphi_{u,v}(y)+h\pmod{2k+1}$, where*

$$h=\begin{cases}1 & \text{if } u=\tau(\overline{\delta}(x)) \text{ and } v=\psi(\overline{\alpha}(y))\,;\\ 0 & \text{otherwise.}\end{cases}$$

Proof. $G_{u,v}(x+y)=G_{u,v}(x)\cup G_{u,v}(y)\cup H$ where

$$H=\begin{cases}\{\delta(x)\} & \text{if } u=\tau(\overline{\delta}(x)) \text{ and } v=\psi(\overline{\alpha}(y))\,;\\ \emptyset & \text{otherwise.}\end{cases}$$

□

Lemma 3.6. *Let r be an idempotent in S_0, let $u,v\in\{1,2,\ldots,w\}$, and let $c\in A\cup B$. If $u=j(r,c)$ and $v=i(r,c)$, then $\widetilde{\varphi}_{u,v}(cr)=-1$ and $\widetilde{\varphi}_{u,v}(cr)=0$ otherwise.*

Proof. Let $g=\widetilde{\varphi}_{u,v}(cr)$ and let $E=\{x\in T\setminus\{0\}:\varphi_{u,v}(x)=g\}$. Then $E\in cr$ so, since cr is an idempotent, $E^\star=\{z\in E:-z+E\in cr\}\in cr$. Pick $x\in E^\star$ such that $\tau(\overline{\delta}(x))=j(r,c)$. Pick $y\in E\cap(-x+E)\cap C_x$ such that $\psi(\overline{\alpha}(y))=i(r,c)$. By Lemma 3.5, $g=\varphi_{u,v}(x+y)\equiv\varphi_{u,v}(x)+\varphi_{u,v}(y)+h\pmod{2k+1}$, where

$$h=\begin{cases}1 & \text{if } u=\tau(\overline{\delta}(x)) \text{ and } v=\psi(\overline{\alpha}(y))\,;\\ 0 & \text{otherwise.}\end{cases}$$

Thus $g\equiv -h\pmod{2k+1}$ as required. □

Lemma 3.7. *Let $z \in T$, let $c \in A \cup B$, and let $r \in S_0$. Then* $\{x \in T \setminus \{0\} : \tau(\overline{\delta}(x)) = j(r,c)\} \in z + cr$.

Proof. Let $E = \{x \in T \setminus \{0\} : \tau(\overline{\delta}(x)) = j(r,c)\}$. Then $E \in cr$. We claim that for all $x \in T$, $-x + E \in cr$. So let $x \in T$. If $x = 0$, the conclusion is immediate so assume that $x \neq 0$. Then $C_x \cap E \subseteq -x + E$. □

Lemma 3.8. *Let $t \in \{1, 2, \ldots, k\}$, let $\vec{c} = \langle c_1, c_2, \ldots, c_t \rangle$ be a compressed sequence in $A \cup B$, and let r be an idempotent in S_0. For $d \in A \cup B$, let $\mu(\vec{c}, d) = |\{s \in \{1, 2, \ldots, t\} : c_s = d\}|$ and for $d, f \in A \cup B$, let $\nu(\vec{c}, d, f) = |\{s \in \{1, 2, \ldots, t-1\} : c_s = d$ and $c_{s+1} = f\}|$. Let $u, v \in \{1, 2, \ldots, w\}$.*

(1) If $d \in A \cup B$ and $(u,v) = (j(r,d), i(r,d))$, then $\widetilde{\varphi}_{u,v}(c_1 r + \ldots + c_t r) = -\mu(\vec{c}, d)$.
(2) If $d, f \in A \cup B$, $d \neq f$, and $(u,v) = (j(r,d), i(r,f))$, then $\widetilde{\varphi}_{u,v}(c_1 r + \ldots + c_t r) = \nu(\vec{c}, d, f)$.
(3) In all other cases $\widetilde{\varphi}_{u,v}(c_1 r + \ldots + c_t r) = 0$.

Proof. We proceed by induction on t. Assume first that $t = 1$. If $d, f \in A \cup B$ and $d \neq f$, then $\nu(\vec{c}, d, f) = 0$ so all conclusions follow from Lemma 3.6.

Now assume that $t > 1$ and the conclusions hold for $t - 1$. Let $\vec{c}\,' = \langle c_1, c_2, \ldots, c_{t-1} \rangle$ and let $E_1 = \{x \in T \setminus \{0\} : \tau(\overline{\delta}(x)) = j(r, c_{t-1})\}$. By Lemma 3.7, $E_1 \in c_1 r + \ldots + c_{t-1} r$. Let

$$E_2 = \{x \in T \setminus \{0\} : \varphi_{u,v}(x) = \widetilde{\varphi}(c_1 r + \ldots + c_{t-1} r)\}.$$

Then $E_2 \in c_1 r + \ldots + c_{t-1} r$.

If $(u,v) = (j(r,d), i(r,d))$ for some $d \in A \cup B$, let $g = -\mu(\vec{c}, d)$. If $(u,v) = (j(r,d), i(r,f))$, for some $d \neq f$ in $A \cup B$, let $g = \nu(\vec{c}, d, f)$. In all other cases let $g = 0$. Let $E_3 = \{x \in T \setminus \{0\} : \varphi_{u,v}(x) = g\}$. We shall show that $E_1 \cap E_2 \subseteq \{x \in T : -x + E_3 \in c_t r\}$ so that $E_3 \in c_1 r + \ldots + c_t r$ as required.

Let $x \in E_1 \cap E_2$. Let

$$E_4 = C_x \cap \{y \in T \setminus \{0\} : \psi(\overline{\alpha}(y)) = i(r, c_t) \text{ and } \varphi_{u,v}(y) = \widetilde{\varphi}_{u,v}(c_t r)\}.$$

Then $E_4 \in c_t r$ so it suffices to show that $E_4 \subseteq -x + E_3$. So let $y \in E_4$. Then $x \ll y$ so by Lemma 3.5, $\varphi_{u,v}(x+y) \equiv \varphi_{u,v}(x) + \varphi_{u,v}(y) + h \pmod{2k+1}$, where

$$h = \begin{cases} 1 & \text{if } u = \tau(\overline{\delta}(x)) \text{ and } v = \psi(\overline{\alpha}(y)); \\ 0 & \text{otherwise.} \end{cases}$$

First assume that $(u,v)=\big(j(r,d),i(r,d)\big)$ for some $d\in A\cup B$. Then $\varphi_{u,v}(x)=\widetilde{\varphi}_{u,v}(c_1r+\ldots+c_{t-1}r)=-\mu(\vec{c}\,',d)$. Also $\tau\big(\overline{\delta}(x)\big)=j(r,c_{t-1})$ and $\psi\big(\overline{\alpha}(y)\big)=i(r,c_t)$ so one cannot have $\tau\big(\overline{\delta}(x)\big)=u$ and $\psi\big(\overline{\alpha}(y)\big)=v$ since that would imply that $c_{t-1}=d=c_t$, while $\vec{c}$ is a compressed sequence. Thus $h=0$. If $(u,v)=\big(j(r,c_t),i(r,c_t)\big)$, then $\mu(\vec{c},c_t)=\mu(\vec{c}\,',c_t)+1$ and by Lemma 3.6, $\varphi_{u,v}(y)=\widetilde{\varphi}_{u,v}(c_tr)=-1$. If $(u,v)\neq\big(j(r,c_t),i(r,c_t)\big)$, then $\mu(\vec{c},c_t)=\mu(\vec{c}\,',c_t)$ and by Lemma 3.6, $\varphi_{u,v}(y)=\widetilde{\varphi}_{u,v}(c_tr)=0$. In either case $\varphi_{u,v}(x+y)=g$ as required.

Next assume that $(u,v)=\big(j(r,d),i(r,f)\big)$ for some $d\neq f$ in $A\cup B$. Then $\varphi_{u,v}(x)=\widetilde{\varphi}_{u,v}(c_1r+\ldots+c_{t-1}r)=\nu(\vec{c}\,',d,f)$. Also $\tau\big(\overline{\delta}(x)\big)=j(r,c_{t-1})$ and $\psi\big(\overline{\alpha}(y)\big)=i(r,c_t)$ so one cannot have $(u,v)=\big(j(r,c_t),i(r,c_t)\big)$ since that would imply that $d=f$. Consequently, $\varphi_{u,v}(y)=\widetilde{\varphi}_{u,v}(c_tr)=0$. If $d=c_{t-1}$ and $f=c_t$, then $\nu(\vec{c},d,f)=\nu(\vec{c}\,',d,f)+1$ and $h=1$. If $(d,f)\neq(c_{t-1},c_t)$, then $\nu(\vec{c},d,f)=\nu(\vec{c}\,',d,f)$ and $h=0$. In either case $\varphi_{u,v}(x+y)=g$ as required.

Finally assume that for all $d\in A\cup B$, $(u,v)\neq\big(j(r,d),i(r,d)\big)$ and for all distinct d and f in $A\cup B$, $(u,v)\neq\big(j(r,d),i(r,f)\big)$. Then $\varphi_{u,v}(x)=\varphi_{u,v}(y)=h=0$. □

Lemma 3.9. *Assume that p and q are idempotents in S_0 and for all $u,v\in\{1,2,\ldots,w\}$, $\widetilde{\varphi}_{u,v}(a_1p+\ldots+a_mp)=\widetilde{\varphi}_{u,v}(b_1q+\ldots+b_kq)$. Let μ be defined as in Lemma 3.8. There is a function $\gamma:A\xrightarrow[\text{onto}]{\text{1-1}}B$ such that*

(1) for all $c\in A$, $\mu(\vec{a},c)=\mu(\vec{b},\gamma(c))$ and
(2) for all $c\in A$ and all $d\in B$, the following statements are equivalent.
  (a) $\gamma(c)=d$;
  (b) $j(p,c)=j(q,d)$;
  (c) $i(p,c)=i(q,d)$;
  (d) $i(p,c)=i(q,d)$ and $j(p,c)=j(q,d)$.

Proof. Let $c\in A$. We claim there is exactly one $d\in B$ such that $i(p,c)=i(q,d)$ and $j(p,c)=j(q,d)$. There is at most one such d since if $j(q,d)=j(q,f)$, then $d=f$. Let $u=j(p,c)$ and $v=i(p,c)$. By Lemma 3.8,

$$\varphi_{u,v}(a_1p+\ldots+a_mp)=-\mu(\vec{a},c)\text{ so }\varphi_{u,v}(b_1q+\ldots+b_kq)=-\mu(\vec{a},c)\,.$$

Now $-\mu(\vec{a},c)\in\{-m,-m+1,\ldots,-1\}\subseteq\{-k,-k+1,\ldots,-1\}$ while for any $d\neq f$ in B, $\nu(\vec{b},d,f)\in\{0,1,\ldots,k\}$. Therefore, by Lemma 3.8, there is some $d\in A\cup B$ such that $(u,v)=\big(j(q,d),i(q,d)\big)$ and $\varphi_{u,v}(b_1q+\ldots+b_kq)=-\mu(\vec{b},d)$. Since $\mu(\vec{a},c)>0$, we must have $d\in B$.

We define $\gamma(c)$ to be the unique $d \in B$ such that $i(p,c) = i(q,d)$ and $j(p,c) = j(q,d)$. Notice that the argument above also establishes that $\mu(\vec{a},c) = \mu(\vec{b},\gamma(c))$.

Trivially γ is injective. The argument above starting with $d \in B$ shows that γ is surjective.

Now we establish (2). The only nontrivial implications are that (b) implies (a) and that (c) implies (a), and these proofs are essentially the same. If $i(p,c) = i(q,d)$, then, since we also know $i(p,c) = i(q,\gamma(c))$ we conclude that $d = \gamma(c)$. □

Lemma 3.10. *Assume that* p *and* q *are idempotents in* S_0 *and for all* $u, v \in \{1,2,\ldots,w\}$, $\widetilde{\varphi}_{u,v}(a_1p+\ldots+a_mp) = \widetilde{\varphi}_{u,v}(b_1q+\ldots+b_kq)$. *Then* $m = k$.

Proof. Let μ be defined as in Lemma 3.8. Then

$$m = \textstyle\sum_{c\in A} \mu(\vec{a},c) = \sum_{c\in A} \mu(\vec{b},\gamma(c)) = \sum_{d\in B} \mu(\vec{b},d) = k\,.$$

□

We now introduce some notation to assist us in our counting of gaps.

Definition 3.11. Let $x \in T \setminus \{0\}$.

(a) $P(x) = \{(u,v) \in \{1,2,\ldots,w\}^2 : \varphi_{u,v}(x) \in \{1,2,\ldots,k\}\}$.
(b) $G_P(x) = \{(t,u,v) \in G(x) : (u,v) \in P(x)\}$.
(c) For $t \in \mathbb{N}$, $R_t(x) = \{(t',u',v') \in G_P(x) : t' < t\}$.
(d) For $l \in \{0,1,\ldots,k-1\}$, $S_l(x) = \{(t,u,v) \in G_P(x) : |R_t(x)| \equiv l \pmod{2k+1}\}$.
(e) For $l \in \{0,1,\ldots,k-1\}$, $T_l(x) = \{(u,v) \in \{1,2,\ldots,w\}^2 : |\{t \in \mathbb{N} : (t,u,v) \in S_l(x)\}| \equiv 1 \pmod{2k+1}\}$.

We now define a finite coloring Γ of $T\setminus\{0\}$ by agreeing that $\Gamma(x) = \Gamma(y)$ if and only if

(1) $\tau(\overline{\delta}(x)) = \tau(\overline{\delta}(y))$,
(2) $\psi(\overline{\alpha}(x)) = \psi(\overline{\alpha}(y))$,
(3) $\varphi_{u,v}(x) = \varphi_{u,v}(y)$ for all $(u,v) \in \{1,2,\ldots,w\}^2$, and
(4) $T_l(x) = T_l(y)$ for all $l \in \{0,1,\ldots,k-1\}$.

We claim that Γ is as required for Theorem 3.3. So suppose instead that we have a color class D of Γ and idempotents $p, q \in S_0$ such that $D \in a_1p + a_2p + \ldots + a_mp$ and $D \in b_1q + b_2q + \ldots + b_kq$. Then $\widetilde{\varphi}_{u,v}(a_1p + a_2p + \ldots + a_mp) = \widetilde{\varphi}_{u,v}(b_1q + b_2q + \ldots + b_kq)$ for all $(u,v) \in \{1,2,\ldots,w\}^2$ so by Lemma 3.10, we have that $m = k$.

Lemma 3.12. *For all $x \in D$,*

$$\begin{aligned} P(x) &= \big\{\big(j(p,a_l),i(p,a_{l+1})\big) : l \in \{1,2,\ldots,k-1\}\big\} \\ &= \big\{\big(j(q,b_l),i(q,b_{l+1})\big) : l \in \{1,2,\ldots,k-1\}\big\}\,. \end{aligned}$$

Proof. Let $x \in D$. It suffices to establish the first equality. Let $(u,v) \in \{1,2,\ldots,w\}^2$ and let $g = \varphi_{u,v}(x)$. Since $\varphi_{u,v}$ is constant on D, $\widetilde{\varphi}_{u,v}(a_1p + a_2p + \ldots + a_kp) = g$. By Lemma 3.8, $\varphi_{u,v}(x) \in \{1,2,\ldots,k\}$ if and only if $(u,v) = \big(j(p,a_l),i(p,a_{l+1})\big)$ for some $l \in \{1,2,\ldots,k-1\}$. □

Definition 3.13. Let $z \in T$.

(a) $Q = \big\{\big(j(p,a_l),i(p,a_{l+1})\big) : l \in \{1,2,\ldots,k-1\}\big\}$.
(b) For $(u,v) \in \{1,2,\ldots,w\}^2$ and $l \in \{-k,-k+1,\ldots,k-1,k\}$, $\gamma_{l,u,v}(z) = \{t \in \mathbb{N} : (t,u,v) \in G(z) \text{ and } |R_t(z)| \equiv l \pmod{2k+1}\}$.

Note that $Q = \big\{\big(j(q,b_l),i(q,b_{l+1})\big) : l \in \{1,2,\ldots,k-1\}\big\}$ by Lemma 3.12.

Lemma 3.14. *Let $s \in \{1,2,\ldots,k\}$.*

(1) $\big(j(p,a_s),i(p,a_s)\big) \notin Q$.
(2) If $(u,v) \in Q$, *then* $\widetilde{\varphi}_{u,v}(a_sp) = 0$.

Proof. (1) Suppose $\big(j(p,a_s),i(p,a_s\big) \in Q$ so for some $l \in \{1,2,\ldots,k\}$, $\big(j(p,a_s),i(p,a_s)\big) = \big(j(p,a_l),i(p,a_{l+1})\big)$. But then $a_l = a_s = a_{l+1}$ while $\vec{a}$ is a compressed sequence.

(2) This follows immediately from (1) and Lemma 3.6. □

Lemma 3.15. *There exists $P \in p$ such that for all $(u,v) \in Q$, all $s \in \{1,2,\ldots,k\}$, all $l \in \{-k,-k+1,\ldots,k-1,k\}$, and all $\nu \in P$, $|\gamma_{l,u,v}(a_s\nu)| \equiv 0 \pmod{2k+1}$.*

Proof. Pick $P \in p$ such that for all $(u,v) \in Q$, all $s \in \{1,2,\ldots,k\}$, all $l \in \{-k,-k+1,\ldots,k-1,k\}$, and all $\nu,\nu' \in P$,

$$|\gamma_{l,u,v}(a_s\nu)| \equiv |\gamma_{l,u,v}(a_s\nu')| \pmod{2k+1}\,.$$

We may presume that for all $s \in \{1,2,\ldots,k\}$,

$$P \subseteq \{\nu \in T\setminus\{0\} : \psi\big(\overline{\alpha}(a_s\nu)\big) = i(p,a_s) \text{ and } \tau\big(\overline{\delta}(a_s\nu)\big) = j(p,a_s)\}\,.$$

And by Lemma 3.14 we may presume that for all $(u,v) \in Q$ and all $s \in \{1,2,\ldots,k\}$, $P \subseteq \{\nu \in T\setminus\{0\} : \varphi_{u,v}(a_s\nu) = 0\}$.

Let $(u,v) \in Q$, let $s \in \{1,2,\ldots,k\}$, and let $l \in \{-k,-k+1,\ldots,k-1,k\}$. Let $z \in \{-k,-k+1,\ldots,k-1,k\}$ such that for all $\nu \in P$, $|\gamma_{l,u,v}(a_s\nu)| \equiv z \pmod{2k+1}$. Since p is an idempotent in S_0 we may choose $\nu \ll \nu'$ in P such that $\alpha(\nu) > 2|a_s|$ and $\nu + \nu' \in P$.

We claim that $\gamma_{l,u,v}(a_s\nu) \cap \gamma_{l,u,v}(a_s\nu') = \emptyset$. Indeed, if $t \in \gamma_{l,u,v}(a_s\nu) \cap \gamma_{l,u,v}(a_s\nu')$, then $t \in \mathrm{supp}(a_s\nu) \cap \mathrm{supp}(a_s\nu')$, while by Lemmas 2.6 and 2.12, $\alpha(a_s\nu') > \delta(a_s\nu)$. Consequently, it suffices to show that

$$\gamma_{l,u,v}(a_s\nu + a_s\nu') = \gamma_{l,u,v}(a_s\nu) \cup \gamma_{l,u,v}(a_s\nu')\,,$$

since then $z \equiv z + z \pmod{2k+1}$. Now

$$G(a_s\nu + a_s\nu') = G(a_s\nu) \cup G(a_s\nu') \cup \big\{\big(\delta(a_s\nu), \tau(\overline{\delta}(a_s\nu)), \psi(\overline{\alpha}(a_s\nu'))\big)\big\}$$

and $\big(\delta(a_s\nu), \tau(\overline{\delta}(a_s\nu)), \psi(\overline{\alpha}(a_s\nu'))\big) = \big(\delta(a_s\nu), j(p,a_s), i(p,a_s)\big)$. By Lemma 3.14, $\big(j(p,a_s), i(p,a_s)\big) \notin Q$. Therefore there is no t such that $(t,u,v) = \big(\delta(a_s\nu), j(p,a_s), i(p,a_s)\big)$. Consequently we have that

$$\begin{aligned}&\gamma_{l,u,v}(a_s\nu + a_s\nu') = \\ &\{t \in \mathbb{N} : (t,u,v) \in G(a_s\nu) \text{ and } |R_t(a_s\nu + a_s\nu')| \equiv l \pmod{2k+1}\} \\ &\cup \{t \in \mathbb{N} : (t,u,v) \in G(a_s\nu') \text{ and } |R_t(a_s\nu + a_s\nu')| \equiv l \pmod{2k+1}\}\,.\end{aligned}$$

If $(t,u,v) \in G(a_s\nu)$, then $R_t(a_s\nu + a_s\nu') = R_t(a_s\nu)$ since $\nu \ll \nu'$. Therefore, it suffices to show that for $(t,u,v) \in G(a_s\nu')$, $|R_t(a_s\nu + a_s\nu')| \equiv |R_t(a_s\nu')| \pmod{2k+1}$.

Let $(t,u,v) \in G(a_s\nu')$ be given. By Lemma 3.14 $\big(j(p,a_s), i(p,a_s)\big) \notin Q$ so $R_t(a_s\nu + a_s\nu') = R_t(a_s\nu) \cup R_t(a_s\nu')$ so it suffices to show that $|R_t(a_s\nu)| \equiv 0 \pmod{2k+1}$. We have that

$$\begin{aligned}R_t(a_s\nu) &= \textstyle\bigcup_{(u',v')\in Q}\{(t',u',v') : (t',u',v') \in G(a_s\nu)\} \\ &= \textstyle\bigcup_{(u',v')\in Q}\{(t',u',v') : t' \in G_{u',v'}(a_s\nu)\}\,.\end{aligned}$$

Given $(u',v') \in Q$, $|\{(t',u',v') : t' \in G_{u',v'}(a_s\nu)\}| \equiv \varphi_{u',v'}(a_s\nu) \pmod{2k+1}$ and since $\nu \in P$, for $(u',v') \in Q$, $\varphi_{u',v'}(a_s\nu) = 0$. □

We shall complete the proof by showing that there is some $x \in D$ such that for all $l \in \{0,1,\ldots,k-2\}$,

$$(\dagger) \qquad T_l(x) = \big\{\big(j(p,a_{l+1}), i(p,a_{l+2})\big)\big\}$$

Assume that we have done this. Then similarly there exists $y \in D$ such that for all $l \in \{0,1,\ldots,k-2\}$, $T_l(y) = \big\{\big(j(q,b_{l+1}), i(q,b_{l+2})\big)\big\}$. Since x

and y are in D, we have for each $l \in \{0, 1, \ldots, k-2\}$, $j(p, a_{l+1}) = j(q, b_{l+1})$ and $i(p, a_{l+2}) = i(q, b_{l+2})$. Therefore by Lemma 3.9, for all $l \in \{1, 2, \ldots, k\}$, $i(p, a_l) = i(q, b_l)$.

Pick $g \in T \setminus \{0\}$ such that for all $l \in \{1, 2, \ldots, k\}$, $\psi(\overline{\alpha}(a_l g)) = i(p, a_l)$ and pick $z \in T \setminus \{0\}$ such that for all $l \in \{1, 2, \ldots, k\}$, $\psi(\overline{\alpha}(b_l z)) = i(q, b_l)$.

We are assuming that $\vec{b}$ is not a rational multiple of $\vec{a}$, so pick the first $s \in \{2, 3, \ldots, k\}$ such that $b_s/a_s \neq b_1/a_1$. Now $\psi(\overline{\alpha}(a_1 g)) = i(p, a_1) = i(q, b_1) = \psi(\overline{\alpha}(b_1 z))$ and $a_s/a_1 \neq b_s/b_1$ so

$$i(p, a_s) = \psi(\overline{\alpha}(a_s g)) = \psi(\overline{\alpha}(\tfrac{a_s}{a_1} a_1 g)) \neq \psi(\overline{\alpha}(\tfrac{b_s}{b_1} b_1 z)) = \psi(\overline{\alpha}(b_s z)) = i(q, b_s) .$$

This is a contradiction.

To establish (†), pick $P \in p$ as guaranteed by Lemma 3.15. We may presume that for all $s \in \{1, 2, \ldots, k\}$,

$$P \subseteq \{\nu \in T \setminus \{0\} : \psi(\overline{\alpha}(a_s \nu)) = i(p, a_s) \text{ and } \tau(\overline{\delta}(a_s \nu)) = j(p, a_s)\} .$$

And by Lemma 3.14 we may presume that for all $(u, v) \in Q$ and all $s \in \{1, 2, \ldots, k\}$, $P \subseteq \{\nu \in T \setminus \{0\} : \varphi_{u,v}(a_s \nu) = 0\}$.

Since $D \in a_1 p + a_2 p + \ldots + a_k p$ and p is an idempotent in S_0, pick $\nu_1 \ll \nu_2 \ll \ldots \ll \nu_k$ in P such that $a_1 \nu_1 + a_2 \nu_2 + \ldots + a_k \nu_k \in D$ and let $x = a_1 \nu_1 + a_2 \nu_2 + \ldots + a_k \nu_k$. It then suffices to show that for all $l \in \{0, 1, \ldots, k-2\}$, $T_l(x) = \{(\tau(\overline{\delta}(a_{l+1} \nu_{l+1})), \psi(\overline{\alpha}(a_{l+2} \nu_{l+2})))\}$.

Note that if $l \in \{0, 1, \ldots, k-2\}$ and $(u, v) \in T_l(x)$, then $\{t \in \mathbb{N} : (t, u, v) \in S_l(x)\} \neq \emptyset$. Picking t such that $(t, u, v) \in S_l(x)$, we have $(t, u, v) \in G_P(x)$ so $(u, v) \in P(x)$ so by Lemma 3.12, $(u, v) \in Q$. Therefore, for $l \in \{0, 1, \ldots, k-2\}$, $T_l(x) = \{(u, v) \in Q : |\{t \in \mathbb{N} : (t, u, v) \in S_l(x)\}| \equiv 1 \pmod{2k+1}\}$.

Lemma 3.16. *Let $l \in \{0, 1, \ldots, k-2\}$ and let*

$$(t, u, v) = (\delta(a_{l+1} \nu_{l+1}), \tau(\overline{\delta}(a_{l+1} \nu_{l+1})), \psi(\overline{\alpha}(a_{l+2} \nu_{l+2}))) .$$

Then $(t, u, v) \in S_l(x)$.

Proof. We have by Lemma 3.12 that $(u, v) \in P(x)$ so $(t, u, v) \in G_P(x)$. We need to show that $|R_t(x)| \equiv l \pmod{2k+1}$. Now

$$\begin{aligned} R_t(x) &= \{(t', u', v') \in G(x) : (u', v') \in Q \text{ and } t' < t\} \\ &= \{(\delta(a_s \nu_s), \tau(\overline{\delta}(a_s \nu_s)), \psi(\overline{\alpha}(a_{s+1} \nu_{s+1}))) : s \in \{1, 2, \ldots, l\}\} \cup \\ &\quad \textstyle\bigcup_{s=1}^{l+1} \{(t', u', v') \in G(a_s \nu_s) : (u', v') \in Q\} . \end{aligned}$$

Given $(u', v') \in Q$ and $s \in \{1, 2, \ldots, l+1\}$ we have

$$|\{(t', u', v') : (t', u', v') \in G(a_s\nu_s)\}| \equiv \varphi_{u',v'}(a_s\nu_s) \pmod{2k+1}$$

and $\varphi_{u',v'}(a_s\nu_s) = 0$, so $|R_t(x)| \equiv l \pmod{2k+1}$ as required. □

Lemma 3.17. *Let* $s \in \{1, 2, \ldots, k\}$*, let* $(t, u, v) \in G_P(a_s\nu_s)$*, and let* $l \in \{0, 1, \ldots, k-2\}$*. Then* $(t, u, v) \in S_l(x)$ *if and only if* $t \in \gamma_{c,u,v}(a_s\nu_s)$ *where* $c \in \{-k, k+1, \ldots, k-1, k\}$ *and* $c \equiv l - s + 1 \pmod{2k+1}$*.*

Proof. We use the fact from Lemma 3.12 that $P(x) = Q$. Since $(t, u, v) \in G_P(x)$, we have that $(t, u, v) \in S_l(x)$ if and only if $|R_t(x)| \equiv l \pmod{2k+1}$ and for any $c \in \{-k, k+1, \ldots, k-1, k\}$, $t \in \gamma_{c,u,v}(a_s\nu_s)$ if and only if $|R_t(a_s\nu_s)| \equiv c \pmod{2k+1}$.

Also $R_t(x) = \{(t', u', v') \in G(x) : t' < t \text{ and } (u', v') \in Q\}$. If $s = 1$, then $R_t(x) = R_t(a_1\nu_1)$ so $|R_t(x)| \equiv l \pmod{2k+1}$ if and only if $|R_t(a_1\nu_1)| \equiv l \pmod{2k+1}$. Since $l = l - s + 1$, we are done in this case, so assume that $s > 1$. Then

$$\begin{aligned} R_t(x) &= \{(t', u', v') \in G(x) : t' < \alpha(a_s\nu_s) \text{ and } (u', v') \in Q\} \\ &\quad \cup \{(t', u', v') \in G(a_s\nu_s) : t' < t \text{ and } (u', v') \in Q\} \\ &= R_{\delta(a_{s-1}\nu_{s-1})}(x) \cup R_t(a_s\nu_s) \\ &\quad \cup \left\{\left(\delta(a_{s-1}\nu_{s-1}), \tau(\overline{\delta}(a_{s-1}\nu_{s-1})), \psi(\alpha(a_s\nu_s)\right)\right\}. \end{aligned}$$

By Lemma 3.16, $|R_{\delta(a_{s-1}\nu_{s-1})}(x)| \equiv s - 2 \pmod{2k+1}$, so $|R_t(x)| \equiv l \pmod{2k+1}$ if and only if $(s-2) + 1 + |R_t(a_s\nu_s)| \equiv l \pmod{2k+1}$ as required. □

Let $l \in \{0, 1, \ldots, k-2\}$ and let $(u, v) = \left(\tau(\overline{\delta}(a_{l+1}\nu_{l+1})), \psi(\overline{\alpha}(a_{l+2}\nu_{l+2}))\right)$. To see that $(u, v) \in T_l(x)$, let $X = \{t \in \mathbb{N} : (t, u, v) \in S_l(x)\}$. We need to show that $|X| \equiv 1 \pmod{2k+1}$. By Lemma 3.16, $\delta(a_{l+1}\nu_{l+1}) \in X$ and for $s \in \{0, 1, \ldots, k-2\} \setminus \{l\}$, $\delta(a_{s+1}\nu_{s+1}) \notin X$. For $s \in \{1, 2, \ldots, k\}$, pick $c_s \in \{-k, -k+1, \ldots, k-1, k\}$ such that $c_s \equiv l - s + 1 \pmod{2k+1}$. Then by Lemma 3.17, $X = \{\delta(a_{l+1}\nu_{l+1})\} \cup \bigcup_{s=1}^{k} \gamma_{c_s,u,v}(a_s\nu_s)$ so by the choice of P, $|X| \equiv 1 \pmod{2k+1}$ as required. We have thus established that $\left\{\left(j(p, a_{l+1}), i(p, a_{l+2})\right)\right\} \subseteq T_l$.

To establish the reverse inclusion (thereby completing the proof of (†) and consequently completing the proof of the theorem), let

$$(u, v) \in Q \setminus \left\{\left(j(p, a_{l+1}), i(p, a_{l+2})\right)\right\}.$$

To see that $(u, v) \notin T_l(x)$, let $X = \{t \in \mathbb{N} : (t, u, v) \in S_l(x)\}$. As before, for $s \in \{1, 2, \ldots, k\}$, pick $c_s \in \{-k, -k+1, \ldots, k-1, k\}$ such that $c_s \equiv$

$l - s + 1 \pmod{2k+1}$. If $(t, u, v) \in \bigcup_{s=1}^{k} G(a_s \nu_s)$, then by Lemma 3.17, $(t, u, v) \in X$ if and only if $t \in \bigcup_{s=1}^{k} \gamma_{c_s,u,v}(a_s \nu_s)$. The only other possibility for $(t, u, v) \in S_l(x)$ is

$$(t, u, v) = \big(\delta(a_{s+1}\nu_{s+1}), \tau(\overline{\delta}(a_{s+1}\nu_{s+1})), \psi(\overline{\alpha}(a_{s+2}\nu_{s+2}))\big)$$

for some $s \in \{0, 1, \ldots, k-2\}$. But since $(u, v) \notin \{(j(p, a_{l+1}), i(p, a_{l+2}))\}$, $s \neq l$ so by Lemma 3.16, $(t, u, v) \in S_s \neq S_l$. Therefore, by Lemma 3.15, $|X| \equiv 0 \pmod{2k+1}$ so $(u, v) \notin T_l(x)$. $\square$

4. Separating Milliken-Taylor systems in $\mathbb{Q}$

We begin by showing that it is sufficient to separate Milliken-Taylor sytems in $\mathbb{Q}$ near zero. We will represent the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by the points in $[-\frac{1}{2}, \frac{1}{2})$.

Lemma 4.1. *Let $\psi : \mathbb{Q} \to \mathbb{Z}$ be the nearest integer function defined by $\psi(x) = \lfloor x + \frac{1}{2} \rfloor$ and let $\widetilde{\psi} : \beta\mathbb{Q} \to \beta\mathbb{Z}$ be its continuous extension. Let $\pi : \mathbb{Q} \to \mathbb{T}$ be the natural projection and let $\widetilde{\pi} : \beta\mathbb{Q} \to \mathbb{T}$ be its continuous extension. Let $S = \{p \in \beta\mathbb{Q} : \widetilde{\pi}(p) = 0\}$. Then S is a subsemigroup of $\beta\mathbb{Q}$ which contains the idempotents, $-p \in S$ whenever $p \in S$, and the restriction of $\widetilde{\psi}$ to S is a homomorphism.*

Proof. Note that, given our representation of $\mathbb{T}$, for $x \in \mathbb{Q}$, $\pi(x) = x - \psi(x)$. Since π is a homomorphism, we have by [5, Corollary 4.22] that $\widetilde{\pi}$ is a homomorphism, so S is a subsemigroup of $\beta\mathbb{Q}$ which contains the idempotents, and it is immediate that $-p \in S$ whenever $p \in S$.

To see that the restriction of $\widetilde{\psi}$ to S is a homomorphism, let $p, q \in S$. We need to show that $\widetilde{\psi}(p + q) = \widetilde{\psi}(p) + \widetilde{\psi}(q)$. Let $A = \pi^{-1}[(-\frac{1}{4}, \frac{1}{4})]$ and note that $A \in p \cap q$. Since $\widetilde{\psi}(p + q) = \widetilde{\psi} \circ \rho_q(p)$ and $\widetilde{\psi}(p) + \widetilde{\psi}(q) = \rho_{\tilde{\psi}(q)} \circ \widetilde{\psi}(p)$, it suffices to show that $\widetilde{\psi} \circ \rho_q$ and $\rho_{\tilde{\psi}(q)} \circ \widetilde{\psi}$ agree on A so let $x \in A$. Since $\widetilde{\psi}(x + q) = \widetilde{\psi} \circ \lambda_x(q)$ and $\widetilde{\psi}(x) + \widetilde{\psi}(q) = \lambda_{\tilde{\psi}(x)} \circ \widetilde{\psi}(q)$ it suffices to show that $\widetilde{\psi} \circ \lambda_x$ and $\lambda_{\tilde{\psi}(x)} \circ \widetilde{\psi}$ agree on A, so let $y \in A$. Then $\psi(x + y) = \psi(x) + \psi(y)$ as required. $\square$

Note that, since $-p \in S$ whenever $p \in S$, as a consequence of Lemma 4.1, if $a_1, a_2, \ldots, a_m \in \mathbb{Z}$, p is an idempotent in $\beta\mathbb{Q}$, and $p' = \widetilde{\psi}(p)$, then $\widetilde{\psi}(a_1 p + a_2 p + \ldots + a_m p) = a_1 p' + a_2 p' + \ldots + a_m p'$.

Theorem 4.2. *Let m, k in $\mathbb{N}$ and let*

$$\vec{a} = \langle a_1, a_2, \ldots, a_m \rangle \text{ and } \vec{b} = \langle b_1, b_2, \ldots, b_k \rangle$$

be compressed sequences in $\mathbb{Z}\setminus\{0\}$ *such that* $\vec{b}$ *is not a rational multiple of* $\vec{a}$. *Let* $\epsilon > 0$ *and assume that there is a finite coloring* μ *of* $(-\epsilon,\epsilon)\cap\mathbb{Q}\setminus\{0\}$ *such that there do not exist a color class* D *of* μ *and idempotents* p *and* q *in* $\beta\mathbb{Q}$ *with* $D \in a_1p+a_2p+\ldots+a_mp$ *and* $D\in b_1q+b_2q+\ldots+b_kq$. *Then there is a finite coloring* γ *of* $\mathbb{Q}\setminus\{0\}$ *such that there do not exist a color class* D *of* γ *and idempotents* p *and* q *in* $\beta\mathbb{Q}$ *with* $D\in a_1p+a_2p+\ldots+a_mp$ *and* $D\in b_1q+b_2q+\ldots+b_kq$.

Proof. Pick by Theorem 1.4 a finite coloring τ of $\mathbb{Z}\setminus\{0\}$ such that there do not exist a color class D of τ and sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ with $MT(\vec{a},\langle x_n\rangle_{n=1}^\infty)\cup MT(\vec{b},\langle y_n\rangle_{n=1}^\infty)\subseteq D$. By Theorem 1.5, there do not exist a color class D of τ and idempotents p and q in $\beta\mathbb{Z}$ with $D\in a_1p+a_2p+\ldots+a_mp$ and $D\in b_1q+b_2q+\ldots+b_kq$.

Let ψ be the nearest integer function as in Lemma 4.1. Define a finite coloring γ of $\mathbb{Q}\setminus\{0\}$ so that, for $x,y\in\mathbb{Q}\setminus\{0\}$, $\gamma(x)=\gamma(y)$ if and only if one of

(1) $\epsilon\le|x|\le\frac{1}{2}$ and $\epsilon\le|y|\le\frac{1}{2}$;
(2) $|x|<\epsilon$, $|y|<\epsilon$, and $\mu(x)=\mu(y)$; or
(3) $|x|>\frac{1}{2}$, $|y|>\frac{1}{2}$, and $\tau(\psi(x))=\tau(\psi(y))$.

Suppose we have a color class D of γ and idempotents p and q in $\beta\mathbb{Q}$ with $D\in a_1p+a_2p+\ldots+a_mp$ and $D\in b_1q+b_2q+\ldots+b_kq$. That color class cannot be $[\epsilon,\frac{1}{2}]\cup[-\frac{1}{2},-\epsilon]$ since this set does not contain any Milliken-Taylor systems, which it would have to by virtue of Theorem 1.5. And that color class cannot be a color class of μ by assumption.

Therefore we have some t in the range of τ such that $D=(\tau\circ\psi)^{-1}[\{t\}]$. Let $p'=\widetilde{\psi}(p)$ and $q'=\widetilde{\psi}(q)$. By Lemma 4.1, p' and q' are idempotents in $\beta\mathbb{Z}$ and $\tau^{-1}[\{t\}]\in a_1p'+a_2p'+\ldots+a_mp'$ and $\tau^{-1}[\{t\}]\in b_1q'+b_2q'+\ldots+b_kq'$. □

Theorem 4.3. *Let* $\vec{a}$ *and* $\vec{b}$ *be compressed sequences in* $\mathbb{Z}\setminus\{0\}$ *such that* $\vec{b}$ *is not a rational multiple of* $\vec{a}$. *There exists a finite coloring of* $\mathbb{Q}\setminus\{0\}$ *such that there do not exist a color class* D *and sequences* $\langle x_n\rangle_{n=1}^\infty$ *and* $\langle y_n\rangle_{n=1}^\infty$ *in* $\mathbb{Q}$ *with* $MT(\vec{a},\langle x_n\rangle_{n=1}^\infty)\cup MT(\vec{b},\langle y_n\rangle_{n=1}^\infty)\subseteq D$.

Proof. Pick by Theorem 3.3 a finite coloring Γ of $T\setminus\{0\}$ such that there do not exist idempotents $p,q\in S_0$ and a color class D such that $D\in a_1p+a_2p+\ldots+a_mp$ and $D\in b_1q+b_2q+\ldots+b_kq$. Let $\epsilon=\sum_{t=1}^\infty\frac{2t}{(2t+1)!}$. By Lemma 2.3, $(-\epsilon,\epsilon)\cap\mathbb{Q}\setminus\{0\}\subseteq T\setminus\{0\}$ so by Theorem 4.2, there is a finite coloring γ of $\mathbb{Q}\setminus\{0\}$ such that there do not exist a color class D of γ and idempotents p and q in $\beta\mathbb{Q}$ with $D\in a_1p+a_2p+\ldots+a_mp$ and $D\in b_1q+b_2q+\ldots+b_kq$. By Theorem 1.5, there do not exist a color class D and sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ in $\mathbb{Q}$ with $MT(\vec{a},\langle x_n\rangle_{n=1}^\infty)\cup MT(\vec{b},\langle y_n\rangle_{n=1}^\infty)\subseteq D$. □

Corollary 4.4. *Let $\vec{a}$ and $\vec{b}$ be compressed sequences in $\mathbb{Z}\setminus\{0\}$ such that $\vec{b}$ is not a rational multiple of $\vec{a}$. There exist a partition $\{A_1, A_2\}$ of $\mathbb{Q}\setminus\{0\}$ such that there do not exist $i \in \{1,2\}$ and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ with $MT(\vec{a}, \langle x_n\rangle_{n=1}^{\infty}) \cup MT(\vec{b}, \langle y_n\rangle_{n=1}^{\infty}) \subseteq A_i$.*

Proof. By Theorem 4.3 pick $r \in \mathbb{N}$ and a partition $\{D_1, D_2, \ldots, D_r\}$ of $\mathbb{Q}\setminus\{0\}$ such that there do not exist $i \in \{1, 2, \ldots, r\}$ and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ in $\mathbb{Q}$ with $MT(\vec{a}, \langle x_n\rangle_{n=1}^{\infty}) \cup MT(\vec{b}, \langle y_n\rangle_{n=1}^{\infty}) \subseteq D_i$. Let $J = \{i \in \{1, 2, \ldots, r\} : (\exists \langle x_n\rangle_{n=1}^{\infty})(MT(\vec{a}, \langle x_n\rangle_{n=1}^{\infty}) \subseteq D_i)\}$, let $A_1 = \bigcup_{i\in J} D_i$, and let $A_2 = \mathbb{Q}\setminus(A_1 \cup \{0\})$. Suppose there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $MT(\vec{a}, \langle x_n\rangle_{n=1}^{\infty}) \subseteq A_2$. Pick by Theorem 1.3 some $i \in \{1, 2, \ldots, r\}\setminus J$ and a sum subsystem $\langle z_n\rangle_{n=1}^{\infty}$ of $\langle x_n\rangle_{n=1}^{\infty}$ with $MT(\vec{a}, \langle z_n\rangle_{n=1}^{\infty}) \subseteq A_i$. But then, $i \in J$, a contradiction. Similarly, if there is some sequence $\langle y_n\rangle_{n=1}^{\infty}$ with $MT(\vec{b}, \langle y_n\rangle_{n=1}^{\infty}) \subseteq A_1$, there will be some $i \in J$ and a sum subsystem $\langle z_n\rangle_{n=1}^{\infty}$ of $\langle y_n\rangle_{n=1}^{\infty}$ with $MT(\vec{b}, \langle z_n\rangle_{n=1}^{\infty}) \subseteq A_i$. □

As promised in the abstract, we observe that we can allow the entries of our compressed sequences to be rational.

Corollary 4.5. *Let $\vec{a}$ and $\vec{b}$ be compressed sequences in $\mathbb{Q}\setminus\{0\}$ such that $\vec{b}$ is not a multiple of $\vec{a}$. There exist a partition $\{A_1, A_2\}$ of $\mathbb{Q}\setminus\{0\}$ such that there do not exist $i \in \{1,2\}$ and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ with $MT(\vec{a}, \langle x_n\rangle_{n=1}^{\infty}) \cup MT(\vec{b}, \langle y_n\rangle_{n=1}^{\infty}) \subseteq A_i$.*

Proof. Pick $M \in \mathbb{N}$ such that the entries of $M\vec{a}$ and $M\vec{b}$ are integers. Pick A_1 and A_2 as guaranteed by Corollary 4.4 for the compressed sequences $M\vec{a}$ and $M\vec{b}$. Suppose we have sequences $i \in \{1,2\}$ and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ with $MT(\vec{a}, \langle x_n\rangle_{n=1}^{\infty}) \cup MT(\vec{b}, \langle y_n\rangle_{n=1}^{\infty}) \subseteq A_i$. Then $MT(M\vec{a}, \frac{1}{M}\langle x_n\rangle_{n=1}^{\infty}) \cup MT(M\vec{b}, \frac{1}{M}\langle y_n\rangle_{n=1}^{\infty}) \subseteq A_i$. □

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