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The Relationships Among Many Notions of Largeness for Subsets of a Semigroup

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Abstract

We deal with 26 notions of largeness in a semigroup. These notions have their origins in topological dynamics and the algebraic theory of Stone-Čech compactifications, mostly as applied to Ramsey Theory. We establish exactly the patterns of implications that must hold among 24 of these. We also note which of them are partition regular in the sense that whenever the union of two sets is large, one of them must be large.

1 Introduction

There are several natural notions of size of subsets in a semigroup that are defined and motivated by topics in topological dynamics. Many of them have strong combinatorial properties and applications in Ramsey Theory. A few such notions are thick, syndetic, and AP. We mention these particularly because they can be defined more simply and possess straightforward examples and consequences in $\mathbb{N}$, the set of positive integers. A set A in $(\mathbb{N}, +)$ is *thick* if and only if it has arbitrarily long integer intervals. Further, $A \subseteq \mathbb{N}$ is *syndetic* if and only if it has bounded gaps, and an *AP set* if and only if it contains arbitrarily long arithmetic progressions.

Of great significance is the notion of *central*, first introduced by Furstenberg in [7]. His original version defined central sets in $\mathbb{N}$ using the notions of *proximal* and *uniformly recurrent* points from topological dynamics. For a set X, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.

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Theorem 1.1 (Central Sets Theorem)**.** *Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Let A be a central subset of $\mathbb{N}$. Then there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

1. *for all $n, \max H_n < \min H_{n+1}$ and*
2. *for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n\in F}(a_n + \sum_{t\in H_n} y_{i,t}) \in A$.*

Proof. [7, Proposition 8.21] □

For the statement of the Central Sets Theorem for an arbitrary semigroup, the reader is referred to [11, §14.4].

Central sets have a nice characterization that can be given in terms of the algebraic structure of βS, where βS is the Stone-Čech compactification of the discrete semigroup S. Before providing this characterization, we present necessary details and background information. One can find much greater detail in [11].

Let $(S, \cdot)$ be any discrete semigroup and denote its Stone-Čech compactification as βS. βS is the set of all ultrafilters on S, where the points of S are identified with the principal ultrafilters. The basis for the topology is $\{\overline{A} : A \subseteq S\}$, where $\overline{A} = \{p \in \beta S : A \in p\}$. The operation of S can be extended to βS making $(\beta S, \cdot)$ a compact, right topological semigroup with S contained in its topological center. That is, for all $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and for all $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\lambda_x(q) = x \cdot q$. For $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

Since βS is a compact Hausdorff right topological semigroup, it has a smallest two sided ideal denoted $K(\beta S)$, which is the union of all of the minimal right ideals of S, as well as the union of all of the minimal left ideals of S. Every left ideal of βS contains a minimal left ideal and every right ideal of βS contains a minimal right ideal. The intersection of any minimal left ideal and any minimal right ideal is a group and any two such groups are isomorphic. Any idempotent p in βS is said to be *minimal* if and only if $p \in K(\beta S)$. A subset A of S is then *central* if and only if there is some minimal idempotent p such that $A \in p$.

We defined the operation in $(\beta S, \cdot)$ so that βS is right topological with S contained in its topological center. We could equally well have defined the operation so that βS is left topological with S contained in its topological center. (In fact, that used to be the customary choice of the first named author of this paper.) We shall denote this operation by $\odot$. Then for $x, y \in S$, $x \odot y = x \cdot y$, for all $p \in \beta S$ the function $\lambda_p : \beta S \to \beta S$ is continuous, where $\lambda_p(q) = p \odot q$, and for all $x \in S$, the function $\rho_x : \beta S \to \beta S$ is continuous, where $\rho_x(q) = q \odot x$. For $p, q \in \beta S$ and $A \subseteq S$, $A \in p \odot q$ if and only if $\{x \in S : Ax^{-1} \in p\} \in q$, where $Ax^{-1} = \{y \in S : y \odot x \in A\}$.

If S is commutative, then for any $p, q \in \beta S$, $p \odot q = q \cdot p$, so a minimal left ideal in $(\beta S, \cdot)$ is a minimal right ideal in $(\beta S, \odot)$. Consequently, if S is commutative, then $K(\beta S, \cdot) = K(\beta S, \odot)$ and the minimal idempotents of $(\beta S, \cdot)$ are the same as the minimal idempotents of $(\beta S, \odot)$.

If S is not commutative, the situation can be vastly different. El-Mabhouh, Pym, and Strauss [6] showed that if S is the free semigroup on countably many generators, then there is a subsemigroup H of $(\beta S, \cdot)$ such that for all p and q in H, $p \odot q \notin H$. Anthony [1] showed that for any semigroup S, $K(\beta S, \cdot) \cap c\ell K(\beta S, \odot) \neq \emptyset$. On the other hand, Burns [5] showed that if S is the free semigroup on two generators, then $K(\beta S, \cdot) \cap K(\beta S, \odot) = \emptyset$. In fact, Burns showed that any element in $c\ell(K(\beta S, \cdot)) \cup c\ell(K(\beta S, \odot))$ is right cancelable in $(\beta S, \cdot)$ or left cancelable in $(\beta S, \odot)$. This raised the possibility that members of minimal idempotents in $(\beta S, \cdot)$ might not be members of minimal idempotents in $(\beta S, \odot)$. Thus we are, at least potentially, dealing with two notions of *central*. We shall refer to a member of a minimal idempotent in $(\beta S, \cdot)$ as "right central" and a member of a minimal idempotent in $(\beta S, \odot)$ as "left central"

We shall deal in this paper with a total of 14 notions of size in a semigroup, all but one of which have been introduced in other papers and studied because of their Ramsey Theoretic properties. Of these 14, all but two have distinct right and left versions. (The exceptions are *Prog* sets and *weak Prog* sets, notions generalizing AP sets in $\mathbb{N}$.)

In Section 2 we introduce the notions and provide a combinatorial and an algebraic characterization for all but one of them. In Section 3 we derive some basic facts and establish implications that must hold among the notions. In Section 4 we show that most of the missing implications are not valid in general. In Section 5 we establish which of the notions are partition regular.

2 The notions

We start this section by introducing some notation.

Definition 2.1. Let $(S, \cdot)$ be a semigroup.

(1) ${}^{\mathbb{N}}S$ is the set of sequences in S.

(2) For $m \in \mathbb{N}$, $\mathcal{J}_m = \big\{\big(t(1), t(2), \ldots, t(m)\big) \in \mathbb{N}^m : t(1) < t(2) < \cdots < t(m)\big\}$.

(3) For $\langle x_n \rangle_{n=1}^\infty \in {}^{\mathbb{N}}S$, $FP(\langle x_n \rangle_{n=1}^\infty) = \{\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ where $\prod_{n \in F} x_n$ is computed in increasing order of indices.

(4) Given $l \in \mathbb{N}$, a set $B \subseteq S$ is a *length l progression* if and only if there exist $a \in S^2$ and $d \in S$ such that $B = \big\{a(1)d^t a(2) : t \in \{1, 2, \ldots, l\}\big\}$.

(5) Given $l \in \mathbb{N}$, a set $B \subseteq S$ is a *length l weak progression* if and only if there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $d \in S$ such that

$$B = \big\{a(1)d^t a(2)d^t \cdots a(m)d^t a(m+1) : t \in \{1, 2, \ldots, l\}\big\}.$$

In $(\mathbb{N}, +)$ a length l progression is a length l arithmetic progression. The converse is almost true. The set $\{1 + d, 1 + 2d, \ldots, 1 + ld\}$ is not a progression, but if $a > 1$, then $\{a + d, \ldots, a + ld\}$ is.

The first notions that we introduce are those that have simple combinatorial definitions. We will only write out the "right" versions.

Definition 2.2. Let $(S,\cdot)$ be a semigroup and let $A\subseteq S$.

(1) A is *right thick* if and only if $\big(\forall F\in\mathcal{P}_f(S)\big)(\exists x\in S)(Fx\subseteq A)$.

(2) A is *right syndetic* if and only if $\big(\exists H\in\mathcal{P}_f(S)\big)(S=\bigcup_{t\in H}t^{-1}A)$.

(3) A is *right piecewise syndetic* if and only if $\big(\exists H\in\mathcal{P}_f(S)\big)\big(\forall F\in\mathcal{P}_f(S)\big)(\exists x\in S)(Fx\subseteq\bigcup_{t\in H}t^{-1}A)$.

(4) A is *right strongly piecewise syndetic* if and only if $\big(\exists H\in\mathcal{P}_f(S)\big)\big(\forall F\in\mathcal{P}_f(S)\big)(\exists x\in S)(Fx\subseteq\bigcup_{t\in H}At^{-1})$.

(5) A is a *right IP* set if and only if there exists $\langle x_n\rangle_{n=1}^\infty\in{}^{\mathbb{N}}S$ such that $FP(\langle x_n\rangle_{n=1}^\infty)\subseteq A$.

(6) A is a *right Q* set if and only if there exists $\langle x_n\rangle_{n=1}^\infty\in{}^{\mathbb{N}}S$ such that whenever $n<m$ in $\mathbb{N}$, $x_m\in x_nA$.

(7) A is a *right weak Q* set if and only if there exists $\langle x_n\rangle_{n=1}^\infty\in{}^{\mathbb{N}}A$ such that for all $n\in\mathbb{N}$, $x_nx_{n+1}\in A$.

(8) A is a *right J* set if and only if $\big(\forall F\in\mathcal{P}_f({}^{\mathbb{N}}S)\big)(\exists m\in\mathbb{N})(\exists a\in S^{m+1})$ $(\exists t\in\mathcal{J}_m)(\forall f\in F)\big(a(1)f\big(t(1)\big)a(2)f\big(t(2)\big)\cdots a(m)f\big(t(m)\big)a(m+1)\in A\big)$.

(9) A is a *Prog* set if and only if for each $l\in\mathbb{N}$, A contains a length l progression.

(10) A is a *weak Prog* (wProg) set if and only if for each $l\in\mathbb{N}$, A contains a length l weak progression.

Notice that "Prog" and "weak Prog" are two sided notions. That is, if for $m\in\mathbb{N}$, $a\in S^{m+1}$, and $d\in S$, we let $b(t)=a(m+2-t)$ for each $t\in\{1,2,\ldots,m+1\}$, then

$$a(1)d^ta(2)d^t\cdots a(m)d^ta(m+1)=b(m+1)d^tb(m)d^t\cdots d^tb(1)\,.$$

The "Q" in "Q set" is intended to represent "quotient"; if S is a group, then to say that $x_m\in x_nA$, says that $x_n^{-1}x_m\in A$. (The "J" in "J set" and the "C" in "C set" which will be defined below don't stand for anything in particular. They are names that have been used for several years.)

In most cases, the change necessary to define the "left" versions are obvious. For example, A is *left syndetic* if and only if $\big(\exists H\in\mathcal{P}_f(S)\big)(S=\bigcup_{t\in H}At^{-1})$. For a left IP set, one demands that the products be in decreasing order of indices.

The other four notions with which we will be concerned all have simple characterizations in terms of βS, which we take as the definitions.

Definition 2.3. Let $(S,\cdot)$ be a semigroup. $J(S,\cdot)=\{p\in\beta S:(\forall A\in p)(A$ is a right J set$)\}$.

By [11, Theorem 14.14.4], $J(S,\cdot)$ is a compact two sided ideal of $(\beta S,\cdot)$. (Theorem 14.14.4 should have been preceded by Lemma 14.14.6 to show that $J(S,\cdot)\neq\emptyset$.)

Definition 2.4. Let $(S,\cdot)$ be a semigroup and let $A\subseteq S$.

(1) A is *right central* if and only if there is an idempotent $p\in K(\beta S,\cdot)\cap\overline{A}$.

(2) A is a *right C* set if and only if there is an idempotent $p\in J(S,\cdot)\cap\overline{A}$.

(3) A is *right strongly central* if and only if, for every minimal left ideal L of $(\beta S,\cdot)$, there is an idempotent $p\in L\cap\overline{A}$.

(4) A is *right thickly central* if and only if there is some minimal left ideal L of $(\beta S,\cdot)$ such that $\{p\in L:p=p\cdot p\}\subseteq\overline{A}$.

We chose to define C sets as we did because that characterization is the easiest for us to use. We should point out that the reason C sets are interesting from the point of view of Ramsey Theory is that they are precisely the sets that satisfy the conclusion of the Central Sets Theorem.

Most of the notions with which we are concerned have both algebraic and combinatorial characterizations. Three of the algebraic characterizations refer to the following subsets of βS.

Definition 2.5. Let $(S,\cdot)$ be a semigroup.

(1) $\mathrm{Prog}(S)=\{p\in\beta S:(\forall A\in p)(A$ is a Prog set$)\}$.

(2) $\mathrm{wProg}(S)=\{p\in\beta S:(\forall A\in p)(A$ is a weak Prog set$)\}$.

(3) Given $p\in\beta S$, $D(p,\cdot)=\{q\in\beta S:(\forall A\in q)(\{x\in S:xA\in p\}\in p)\}$.

We shall show in the next section that $\mathrm{Prog}(S)$ and $\mathrm{wProg}(S)$ are compact two sided ideals of both $(\beta S,\cdot)$ and $(\beta S,\odot)$

We remark that, if S is commutative, then the right and left versions of all of our notions are equivalent. We invite the reader to work through the verification of this fact for the notions of left strongly central and right strongly central.

We present now those notions for which we have both algebraic and combinatorial characterizations. We do not have a purely algebraic characterization for right strongly piecewise syndetic, and present there a hybrid description. We do not know of any algebraic characterization of right weak Q sets.

The reader is referred to [11, Definition 14.19] for the definition of *right collectionwise piecewise syndetic* mentioned in Theorem 2.6(10) below.

Theorem 2.6. *Let $(S,\cdot)$ be a semigroup and let $A\subseteq S$. For each of the following notions, statements (a), (b), and (c) are equivalent.*

(1) (a) *A is right thick.*

(b) $\big(\forall F \in \mathcal{P}_f(S)\big)(\exists x \in S)(Fx \subseteq A)$.

(c) *There is a minimal left ideal L of $(\beta S, \cdot)$ such that $L \subseteq \overline{A}$.*

(2) (a) *A is right syndetic.*

(b) $\big(\exists H \in \mathcal{P}_f(S)\big)(S = \bigcup_{t \in H} t^{-1}A)$.

(c) *For every minimal left ideal L of $(\beta S, \cdot)$, $L \cap \overline{A} \neq \emptyset$.*

(3) (a) *A is right piecewise syndetic.*

(b) $\big(\exists H \in \mathcal{P}_f(S)\big)\big(\forall F \in \mathcal{P}_f(S)\big)(\exists x \in S)(Fx \subseteq \bigcup_{t \in H} t^{-1}A)$.

(c) $K(\beta S, \cdot) \cap \overline{A} \neq \emptyset$.

(4) (a) *A is right strongly piecewise syndetic.*

(b) $\big(\exists H \in \mathcal{P}_f(S)\big)\big(\forall F \in \mathcal{P}_f(S)\big)(\exists x \in S)(Fx \subseteq \bigcup_{t \in H} At^{-1})$.

(c) *There exist a minimal left ideal L of $(\beta S, \cdot)$ and $H \in \mathcal{P}_f(S)$ such that $L \subseteq \bigcup_{t \in H} \overline{At^{-1}}$.*

(5) (a) *A is a right IP set.*

(b) *There exists $\langle x_n \rangle_{n=1}^{\infty} \in {}^{\mathbb{N}}S$ such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.*

(c) *There exists an idempotent p in $(\beta S, \cdot)$ such that $A \in p$.*

(6) (a) *A is a right Q set.*

(b) *There exists $\langle x_n \rangle_{n=1}^{\infty} \in {}^{\mathbb{N}}S$ such that whenever $n < m$ in $\mathbb{N}$, $x_m \in x_nA$.*

(c) *There exists $p \in \beta S$ such that $\overline{A} \cap D(p, \cdot) \neq \emptyset$.*

(7) (a) *A is a right J set.*

(b) $\big(\forall F \in \mathcal{P}_f({}^{\mathbb{N}}S)\big)(\exists m \in \mathbb{N})(\exists a \in S^{m+1})(\exists t \in \mathcal{J}_m)(\forall f \in F)$
$\big(a(1)f\big(t(1)\big)a(2)f\big(t(2)\big) \cdots a(m)f\big(t(m)\big)a(m+1) \in A\big)$.

(c) $J(S, \cdot) \cap \overline{A} \neq \emptyset$.

(8) (a) *A is a Prog set.*

(b) *For each $l \in \mathbb{N}$, A contains a length l progression.*

(c) $\overline{A} \cap$ *Prog*$(S) \neq \emptyset$.

(9) (a) *A is a wProg set.*

(b) *For each $l \in \mathbb{N}$, A contains a length l weak progression.*

(c) $\overline{A} \cap$ *wProg*$(S) \neq \emptyset$.

(10) (a) *A is a right central set.*

(b) *There is a downward directed family $\langle C_F \rangle_{F \in I}$ of subsets of A such that*

(i) *for each* $F \in I$ *and each* $x \in C_F$ *there exists* $G \in I$ *with* $C_G \subseteq x^{-1}C_F$ *and*

(ii) $\{C_F : F \in I\}$ *is right collectionwise piecewise syndetic.*

(c) *There is an idempotent* $p \in K(\beta S, \cdot) \cap \overline{A}$.

(11) (a) *A is a right C set.*

(b) *There is a downward directed family* $\langle C_F \rangle_{F \in I}$ *of subsets of A such that*

(i) *for each* $F \in I$ *and each* $x \in C_F$ *there exists* $G \in I$ *with* $C_G \subseteq x^{-1}C_F$ *and*

(ii) *for each* $\mathcal{F} \in \mathcal{P}_f(I)$, $\bigcap_{F \in \mathcal{F}} C_F$ *is a J set.*

(c) *There is an idempotent* $p \in J(S, \cdot) \cap \overline{A}$.

(12) (a) *A is right strongly central.*

(b) *Whenever* $\mathcal{A}$ *is a family of subsets of S such that every finite intersection of members of* $\mathcal{A}$ *is right thick, there is a downward directed family* $\langle C_F \rangle_{F \in I}$ *of subsets of A such that*

(i) *for each* $F \in I$ *and each* $x \in C_F$ *there exists* $G \in I$ *with* $C_G \subseteq x^{-1}C_F$ *and*

(ii) $\mathcal{A} \cup \{C_F : F \in I\}$ *has the finite intersection property.*

(c) *For every minimal left ideal L of* $(\beta S, \cdot)$, *there is an idempotent* $p \in L \cap \overline{A}$.

(13) (a) *A is right thickly central.*

(b) *There is a family* $\mathcal{A}$ *of subsets of S such that every finite intersection of members of* $\mathcal{A}$ *is right thick and whenever* $\langle C_F \rangle_{F \in I}$ *is a downward directed family of subsets of* $S \setminus A$ *such that for each* $F \in I$ *and each* $x \in C_F$ *there exists* $G \in I$ *with* $C_G \subseteq x^{-1}C_F$, *one has* $\mathcal{A} \cup \{C_F : F \in I\}$ *does not have the finite intersection property.*

(c) *There is some minimal left ideal L of* $(\beta S, \cdot)$ *such that* $\{p \in L : p = p \cdot p\} \subseteq \overline{A}$.

Proof. In each case, either (b) or (c) is the definition of (a). We need to show that (b) and (c) are equivalent. When the proof is available in [11] we will cite that, referring the reader to the notes at the end of the chapters for the origins of the proof.

(1) The equivalence follows from [11, Theorem 4.48(a)] and the fact that every left ideal contains a minimal left ideal.

(2) The equivalence follows from [11, Theorem 4.48(b)] and the fact that every left ideal contains a minimal left ideal.

(3) [11, Theorem 4.40]

(4) This is an immediate consequence of (1).

(5) [11, Theorem 5.12]

(6) [2, Lemma 1.9(a)]

(7) [11, Theorem 14.14.7]

(8) In the proof of (9) below, replace m by 1.

(9) That (c) implies (b) is trivial. To see that (b) implies (c), it suffices by [11, Theorem 5.7] to show that if $B_1 \cup B_2$ is a wProg set in S, then either B_1 is a wProg set or B_2 is a wProg set. Suppose instead there is some $l \in \mathbb{N}$ such that neither B_1 nor B_2 contains a length l weak progression. Pick by van der Waerden's Theorem some $n \in \mathbb{N}$ such that whenever $\{1, 2, \ldots, n\} = C_1 \cup C_2$, there exist $i \in \{1, 2\}$ and $b, c \in \mathbb{N}$ such that $\{b, b+c, b+2c, \ldots, b+lc\} \subseteq C_i$. (See [11, Exercise 14.1.1] for this version of van der Waerden's Theorem.) Now $B_1 \cup B_2$ contains a length n weak progression, so pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $d \in S$ such that $\big\{a(1)d^t a(2)d^t \cdots a(m)d^t a(m+1) : t \in \{1, 2, \ldots, n\}\big\} \subseteq B_1 \cup B_2$. For $i \in \{1, 2\}$ let $C_i = \{t \in \{1, 2, \ldots, n\} : a(1)d^t a(2)d^t \cdots a(m)d^t a(m+1) \in B_i$. Pick $i \in \{1, 2\}$ and $b, c \in \mathbb{N}$ such that $\{b, b+c, b+2c, \ldots, b+lc\} \subseteq C_i$. For $j \in \{1, 2, \ldots, m\}$, let $f(j) = a(j)d^b$ and let $f(m+1) = a(m+1)$. Let $e = d^c$. Then for each $t \in \{1, 2, \ldots, l\}$, $f(1)e^t f(2)e^t \cdots f(m)e^t f(m+1) \in B_i$, so B_I contains a length l progression.

(10) [11, Theorem 14.25]

(11) [11, Theorems 14.15.1 and 14.27]

(12) To see that (b) implies (c), let L be a minimal left ideal of $(\beta S, \cdot)$ and let $\mathcal{A} = \{B \subseteq S : L \subseteq \overline{B}\}$, noting that $L = \bigcap_{B \in \mathcal{A}} \overline{B}$. Given $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, $L \subseteq \overline{\bigcap \mathcal{F}}$ so $\bigcap \mathcal{F}$ is right thick. Pick $\langle C_F \rangle_{F \in I}$ as guaranteed by (b). Let $M = \bigcap_{F \in I} \overline{C_F}$. By [11, Theorem 4.20] M is a subsemigroup of $(\beta S, \cdot)$. By (ii), $L \cap M \neq \emptyset$, so $L \cap M$ is a compact right topological semigroup, and therefore has an idempotent which is in $\overline{A}$ since $M \subseteq \overline{A}$.

To see that (c) implies (b), let $\mathcal{A}$ be a family of subsets of S such that every finite intersection of members of $\mathcal{A}$ is right thick. By [4, Lemma 2.7] pick a left ideal L of $(\beta S, \cdot)$ with $L \subseteq \bigcap_{B \in \mathcal{A}} \overline{B}$. We may assume that L is minimal. Pick an idempotent $p \in L \cap \overline{A}$. By [11, Lemmas 14.24 and 14.23.1], pick a tree T in A such that for each $f \in T$, $B_f = \{x \in A : f^\frown x \in T\} \in p$ and for each $f \in T$ and each $x \in B_f$, $B_{f^\frown x} \subseteq x^{-1} B_f$. (For the definition of tree that we are using, see [11, Definition 14.22].) Let $I = \mathcal{P}_f(T)$ and for $F \in I$, let $C_F = \bigcap_{f \in F} B_f$. Direct I by inclusion. Given $F, G \in I$, $C_{F \cup G} \subseteq C_F \cap C_G$ so $\langle C_F \rangle_{F \in I}$ is downward directed. To see that (i) holds, let $F \in I$ and let $x \in C_F$. Let $G = \{f^\frown x : f \in F\}$. Then $C_G \subseteq x^{-1} C_F$. To verify (ii) note that $\mathcal{A} \cup \{C_F : F \in I\} \subseteq p$.

(13) This equivalence follows from (12) and the fact that A is thickly central if and only if $S \setminus A$ is not strongly central. $\square$

3 The implications

We show in this section that all of the implications displayed in Figure 1 hold.

We shall need some lemmas for some of the implications. The first of these, while very easy, allows us to answer [9, Question 3.10]. It had not occurred

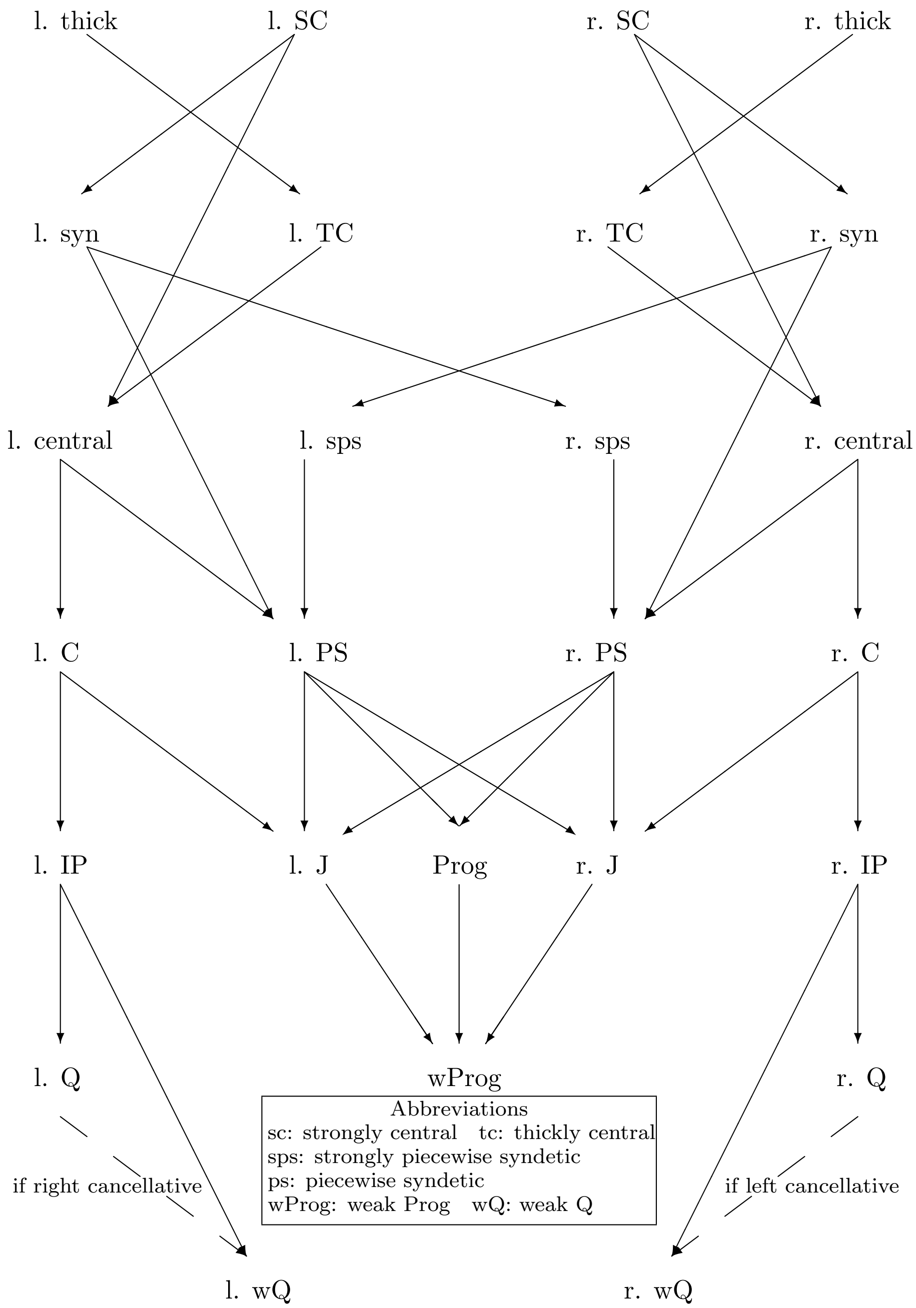

Figure 1: Implications among all of the notions.

to the authors of [9] to answer the question algebraically, since the question involved both operations $\cdot$ and $\odot$.

Lemma 3.1. *$J(S,\cdot)$ is a two-sided ideal of $(\beta S,\odot)$.*

Proof. $J(S,\cdot)\neq\emptyset$ by [11, Lemma 14.14.7 and Theorem 3.11]. Let $p\in J(S,\cdot)$ and $q\in\beta S$. To see that $p\odot q\in J(S,\cdot)$, let $A\in p\odot q$ and let $K\in\mathcal{P}_f({}^{\mathbb{N}}S)$. Then $\{x\in S: Ax^{-1}\in p\}\in q$. So pick $x\in S$ such that $Ax^{-1}\in p$. Pick $m\in\mathbb{N}, b\in S^{m+1}$ and $t\in\mathcal{J}_m$ such that for each $f\in K$,

$$b(1)f\big(t(1)\big)b(2)\cdots b(m)f\big(t(m)\big)b(m+1)\in Ax^{-1}.$$

Let $c(i)=b(i)$ for each $i\in\{1,2,\ldots,m\}$ and $c(m+1)=b(m+1)x$. Then for each $f\in K$, $c(1)f\big(t(1)\big)c(2)\cdots c(m)f\big(t(m)\big)c(m+1)\in A$.

Now to see that $q\odot p\in J(S,\cdot)$, let $A\in q\odot p$ and let $K\in\mathcal{P}_f({}^{\mathbb{N}}S)$ be given. Let $B=\{x\in S: Ax^{-1}\in q\}$. Then $B\in p$. So pick $m\in\mathbb{N}, b\in S^{m+1}$, and $t\in\mathcal{J}_m$ such that for each $f\in K$,

$$x_f=b(1)f\big(t(1)\big)b(2)\cdots b(m)f\big(t(m)\big)b(m+1)\in B.$$

Pick $y\in\bigcap_{f\in K}Ax_f^{-1}$ Let $c(1)=yb(1)$ and for $j\in\{2,3,\ldots,m+1\}$, let $c(j)=b(j)$. Then for each $f\in K$, $c(1)f\big(t(1)\big)c(2)\cdots c(m)f\big(t(m)\big)c(m+1)\in A$. □

The next lemma provides motivation for our definition of progressions and weak progressions. That is, if we had defined a length l progression as $\big\{ad^t: t\in\{1,2,\ldots,l\}\big\}$ we could not even conclude that $\mathrm{Prog}(S)$ is even a semigroup.

Lemma 3.2. *$Prog(S)$ and $wProg(S)$ are both two sided ideals of $(\beta S,\cdot)$ and of $(\beta S,\odot)$.*

Proof. All four conclusions are easy exercises. □

We shall establish the implications whose hypotheses involve the right notions. We take care of the trivial implications first.

Theorem 3.3. *Let $(S,\cdot)$ be a semigroup and let $A\subseteq S$.*

(1) If A is right strongly central, then A is right syndetic.

(2) If A is right strongly central, then A is right central.

(3) If A is right thick, then A is right thickly central.

(4) If A is right thickly central, then A is right central.

(5) If A is right syndetic, then A is right piecewise syndetic.

(6) If A is right central, then A is right piecewise syndetic.

(7) If A is a right C set, then A is a right J set.

Proof. All of these statements are immediate consequences of the algebraic charaterizations, that is the statements labelled (c), in Theorem 2.6. $\square$

Theorem 3.4. *Let $(S,\cdot)$ be a semigroup and let $A \subseteq S$.*

(1) If A is right syndetic, then A is left strongly piecewise syndetic.

(2) If A is a right C set, then A is a right IP set.

(3) If A is a right IP set, then A is a right Q set.

(4) If A is a right IP set, then A is a right weak Q set.

(5) If A is a Prog set, then A is a wProg set.

Proof. Statements (1), (2), and (5) are immediate consequences of the combinatorial characterizations, that is the statements labelled (b), in Theorem 2.6 and statement (4) is an immediate consequence of the combinatorial characterization of IP sets and the definition of weak Q set. To verify statement (3), let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence with $FP(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$. For $n \in \mathbb{N}$, let $y_n = \prod_{t=1}^{n} x_t$. If $n < m$, then $y_m = y_n(\prod_{t=n+1}^{m} y_t)$. $\square$

Theorem 3.5. *Let $(S,\cdot)$ be a semigroup and let $A \subseteq S$.*

(1) If A is right central, then A is a right C set.

(2) If A is right piecewise syndetic, then A is a right J set.

(3) If A is right piecewise syndetic, then A is a left J set.

(4) If A is right piecewise syndetic, then A is a Prog set.

(5) If A is right strongly piecewise syndetic, then A is right piecewise syndetic.

(6) If A is a right J set, then A is a wProg set.

Proof. As we have noted, by [11, Theorem 14.14.4], $J(S,\cdot)$ is a two sided ideal of $(\beta S,\cdot)$ so $K(\beta S,\cdot) \subseteq J(S,\cdot)$ and thus (1) and (2) follow from the algebraic characterizations. By the left-right switch of Lemma 3.1, $J(S,\odot)$ is a two sided ideal of $(\beta S,\cdot)$, so $K(\beta S,\cdot) \subseteq J(S,\odot)$ and thus (3) follows. By Lemma 3.2, $\text{Prog}(S)$ is a two sided ideal of $(\beta S,\cdot)$ so (4) follows.

To verify (5), pick a minimal left ideal L of $(\beta S,\cdot)$ and $H \in \mathcal{P}_f(S)$ such that $L \subseteq \bigcup_{t\in H} \overline{At^{-1}}$. Pick $p \in L$ and pick $t \in H$ such that $At^{-1} \in p$. Since $p \in K(\beta S,\cdot)$, $pt \in K(\beta S,\cdot) \cap \overline{A}$.

To verify (6), let $l \in \mathbb{N}$ be given. Pick any $d \in S$ and for $k \in \{1,2,\ldots,l\}$ and $n \in \mathbb{N}$, let $f_k(n) = d^k$. Let $F = \{f_1, f_2, \ldots, f_l\}$. Pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F$, $a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \in A$. $\square$

Theorem 3.6. *Let $(S,\cdot)$ be a left cancellative semigroup and let $A \subseteq S$. If A is a right Q set, then A is a right weak Q set.*

Proof. Pick a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S such that whenever $n < m$, $x_m \in x_n A$. For each pair $(n,m) \in \mathbb{N}\times\mathbb{N}$ with $n < m$, pick $y_{n,m} \in A$ such that $x_m = x_n y_{n,m}$. It suffices to show that for each $n \in \mathbb{N}$, $y_{n,n+1}y_{n+1,n+2} \in A$, so let $n \in \mathbb{N}$. Then $x_n y_{n,n+2} = x_{n+2} = x_{n+1}y_{n+1,n+2} = x_n y_{n,n+1}y_{n+1,n+2}$ so by left cancellation, $y_{n,n+1}y_{n+1,n+2} = y_{n,n+2}$. $\square$

We shall see in the next section that the left cancellativity assumption cannot be replaced by weakly left cancellative, even if right cancellativity is assumed.

We now show that, if S is a countably infinite cancellative semigroup, then any central subset of S is a member of $2^{\mathfrak{c}}$ idempotents which belong to the same minimal right ideal, but to distinct minimal left ideals of βS. The proof of the following theorem is essentially contained in that of [10, Theorem 2.12], in which a stronger theorem is proved for the case in which $S = \mathbb{N}$.

Theorem 3.7. *Let S be a countably infinite cancellative semigroup and let p be a minimal idempotent in $(\beta S,\cdot)$. Let R denote the minimal right ideal of $(\beta S,\cdot)$ which contains p, and let $C \in p$. Then there are $2^{\mathfrak{c}}$ minimal idempotents of $(\beta S,\cdot)$ in $R\cap\overline{C}$.*

Proof. Let $C^\star = \{x \in S : x^{-1}C \in p\}$. By [11, Lemma 4.14], $C^\star \in p$ and, for each $x \in C^\star$, $x^{-1}C^\star \in p$. For each $F \in \mathcal{P}_f(C^\star)$, we put $V_F = C^\star \cap \bigcap_{x\in F} x^{-1}C^\star$ and $V = \bigcap\{\overline{V_F} : F \in \mathcal{P}_f(C^\star)\}$. Since $V_F \in p$ for every $F \in \mathcal{P}_f(C^\star)$, $p \in V$. If $F \in \mathcal{P}_f(C^\star)$ and $y \in V_F$, then $G = \{y\}\cup Fy \subseteq C^\star$ and, if $z \in V_G$, $z \in y^{-1}C^\star \cap \bigcap_{x\in F} y^{-1}x^{-1}C^\star$. So $yz \in V_F$. It follows from [11, Theorem 4.20] that V is a subsemigroup of $(\beta S,\cdot)$.

Now V contains a copy of $\mathbb{H} = \bigcap_{n=1}^{\infty}\overline{2^n\mathbb{N}}$, by [11, Theorem 6.32]. By [11, Theorem 6.9], $(\beta\mathbb{N},+)$ has $2^{\mathfrak{c}}$ minimal left ideals. Each of these contains an idempotent, and by [11, Lemma 6.6] every idempotent of $(\beta\mathbb{N},+)$ is in $\mathbb{H}$. So V contains a set W of idempotents with the property that $uv \neq u$ whenever u and v are distinct elements of W, because $\mathbb{H}$ contains a set of this kind, since uv is in the same minimal left ideal as v. We claim that $\beta S u \cap \beta S v = \emptyset$ for every distinct u and v in W. To see this, assume the contrary. We may then assume that $su = yv$ for some $s \in S$ and some $y \in \beta S$, by [11, Theorem 6.19]. This implies that $suv = yvv = yv = su$ and hence, by [11, Lemma 8.1], that $uv = u$, a contradiction.

For every $u \in W$, the left ideal Vu of V contains a minimal left ideal of V, by [11, Corollary 2.6]. By [11, Theorem 2.7], the intersection of this minimal left ideal with the right ideal pV of V, contains an idempotent which is minimal in V and is therefore minimal in βS, by [11, Theorem 1.65]. We have thus shown that there are $2^{\mathfrak{c}}$ minimal idempotents of βS in $R\cap\overline{C}$. $\square$

4 Examples

We show in this section that none of the missing implications in Figure 2 are valid in general, and show which implications involving the notion of strongly central that we know do not hold. Most of the counterexamples involve known results. We begin with the new results that we will need.

Definition 4.1. For $n \in \mathbb{N}$, let $\theta(n) = \min \mathrm{supp}(n)$, where $\mathrm{supp}(n) \subseteq \omega = \mathbb{N} \cup \{0\}$ and $n = \sum_{t \in \, \mathrm{supp}(n)} 2^t$.

Lemma 4.2. *Let B be an infinite subset of $\mathbb{N}$ and let $A = \{n \in \mathbb{N} : \theta(n) \in B\}$. Then A is a (right) strongly central subset of $(\mathbb{N}, +)$.*

Proof. Let $\widetilde{\theta} : \beta\mathbb{N} \to \beta\omega$ be the continuous extension of θ. Let $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$. By [11, Lemma 6.8], $\mathbb{H}$ is a compact subsemigroup of $(\beta\mathbb{N}, +)$ which contains all the idempotents of $(\beta\mathbb{N}, +)$ and whenever $p \in \beta\mathbb{N}$ and $q \in \mathbb{H}$, $\widetilde{\theta}(p + q) = \widetilde{\theta}(p)$. Pick $p \in \mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ such that $\{2^n : n \in B\} \in p$. Let $C = \{r \in \mathbb{H} : \widetilde{\theta}(r) = \widetilde{\theta}(p)\}$. Then C is a right ideal of $\mathbb{H}$. Pick a minimal right ideal R of $\mathbb{H}$ such that $R \subseteq C$. Since $\mathbb{H}$ contains all of the idempotents of $\beta\mathbb{N}$, by [11, Theorem 1.65], there is a minimal right ideal T of $\beta\mathbb{N}$ such that $T \cap \mathbb{H} = R$.

To see that A is strongly central, let L be a minimal left ideal of $\beta\mathbb{N}$. Let s be the identity of the group $L \cap T$. We claim that $A \in s$. Suppose instead that $\mathbb{N} \setminus A \in s$. Now $s \in \mathbb{H}$ so $s \in R$ and thus $\widetilde{\theta}(s) = \widetilde{\theta}(p)$. By [11, Lemma 3.30], $\theta[\{2^n : n \in B\}] \in \widetilde{\theta}(p) = \widetilde{\theta}(s)$ so $\theta^{-1}\big[\theta[\{2^n : n \in B\}]\big] \in s$. Pick $x \in \mathbb{N} \setminus A$ such that $\theta(x) \in \theta[\{2^n : n \in B\}]$. Pick $m \in B$ such that $\theta(x) = \theta(2^m) = m$. But then $x \in A$, a contradiction. □

We now proceed to construct a left C set in the free semigroup on countably many generators which is not a right J set, not a right weak Q set, and not a Prog set. The construction is based on the construction in [9, Section 3] of a left J set which is not a right J set. The notation used here is similar but not identical to that of [9].

Definition 4.3. Let S be the free semigroup on the alphabet $\{a_n : n \in \mathbb{N}\}$.

(1) $\mathcal{M} = \bigcup_{r=2}^{\infty} {}^{\{1,2,\ldots,r\}}S = \{f : (\exists r \in \mathbb{N} \setminus \{1\})(f : \{1, 2, \ldots, r\} \to S)\}$.

(2) Define $\psi : \mathcal{P}_f(\mathcal{M}) \to \mathbb{N}$ by, for $H \in \mathcal{P}_f(\mathcal{M})$,
$\psi(H) = \max\{n \in \mathbb{N} : (\exists f \in H)\big(a_n \text{ occurs in } f(1)\big)\}$.

(3) $\mathcal{F} = \big\{H \in \mathcal{P}_f(\mathcal{M}) : (\forall f \in H)(\mathrm{dom}(f) = \{1, 2, \ldots, \psi(H)\})$ and
$(\forall f, g \in H)\big(f \neq g \Rightarrow (\forall t \in \{1, 2, \ldots, \psi(H)\})\big(f(t) \neq g(t)\big)\big)\big\}$.

Notice that $\mathcal{M}$ is countable and so $\mathcal{F}$ is countable. Consequently, we may choose δ as in the following definition.

Definition 4.4. Choose $\delta : \mathbb{N} \times \mathcal{F} \xrightarrow{1\text{-}1} 4\mathbb{N}$ such that for $n \in \mathbb{N}$ and $H \in \mathcal{F}$, if there exist $t \in \{1, 2, \ldots, \psi(H)\}$, $f \in H$, and $k \in \mathbb{N}$ such that a_k occurs in $f(t)$, then $k < \delta(n, H)$.

Definition 4.5. Given $n \in \mathbb{N}$,

$$\begin{aligned} B_n = \ & \{a_{\delta(n,H)} h\big(\psi(H)\big) a_{\delta(n,H)} h(\psi(H) - 1) \cdots a_{\delta(n,H)} h(1) a_{\delta(n,H)} : \\ & H \in \mathcal{F} \text{ and } h \in H\}\,. \end{aligned}$$

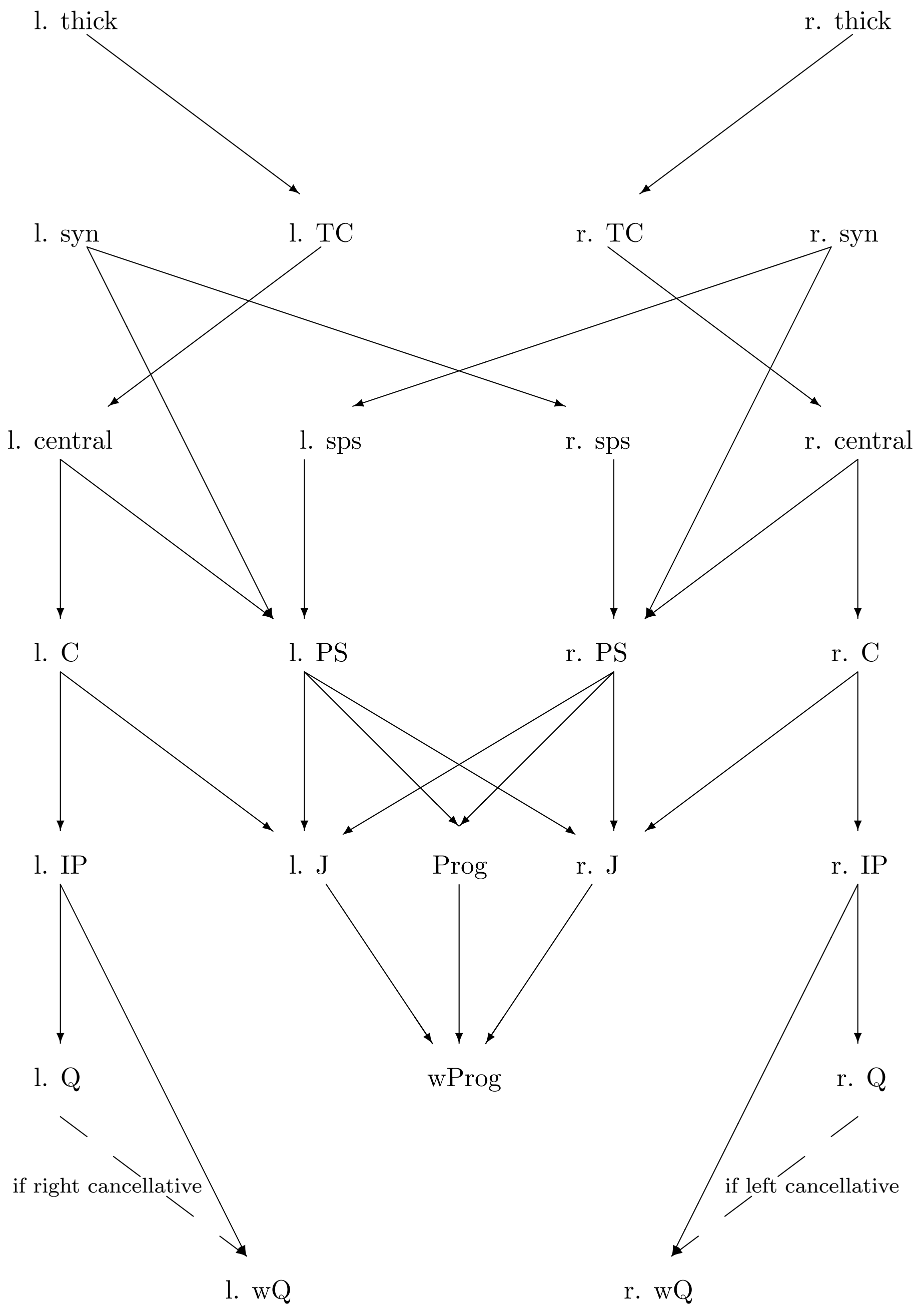

Figure 2: None of the missing implications are valid.

We write $\overleftarrow{\prod}_{i=1}^{l} y_i$ for the product in decreasing order of indices.

Definition 4.6. Given $n \in \mathbb{N}$,

$$\begin{aligned} A_n = \;& \big\{x \in S : (\exists l \in \mathbb{N})(\exists \text{ increasing } \langle m_i \rangle_{i=1}^{l} \in \mathbb{N}^l) \\ & (\exists \langle H_i \rangle_{i=1}^{l} \in \mathcal{F}^l)(\exists \langle h_i \rangle_{i=1}^{l})\big(\text{each } h_i \in H_i\,, \\ & x = \overleftarrow{\prod}_{i=1}^{l}(a_{\delta(m_i,H_i)} h_i\big(\psi(H_i)\big) a_{\delta(m_i,H_i)} \cdots a_{\delta(m_i,H_i)} h_i(1) a_{\delta(m_i,H_i)})\,, \\ & m_1 \geq n, \text{ and for } i \in \{1,2,\ldots,l-1\}\,,\ \delta(m_{i+1},H_{i+1}) > \delta(m_i,H_i)\big)\big\}\,. \end{aligned}$$

For example, assume that $l = 2$, $m_1 = 2$, $m_2 = 4$, $H_1 = \{f,g\}$, $H_2 = \{h\}$, $\psi(H_1) = 2$, $\psi(H_2) = 3$, and $\delta(2,H_1) < \delta(4,H_2)$. Then

$$a_{\delta(4,H_2)} h(3) a_{\delta(4,H_2)} h(2) a_{\delta(4,H_2)} h(1) a_{\delta(4,H_2)} a_{\delta(2,H_1)} f(2) a_{\delta(2,H_1)} f(1) a_{\delta(2,H_1)}$$
and
$$a_{\delta(4,H_2)} h(3) a_{\delta(4,H_2)} h(2) a_{\delta(4,H_2)} h(1) a_{\delta(4,H_2)} a_{\delta(2,H_1)} g(2) a_{\delta(2,H_1)} g(1) a_{\delta(2,H_1)}$$

are elements of A_1 and A_2, but not of A_n for any $n > 2 = m_1$.

Notice that for an element x of A_n as written in the definition, $\delta(m_l,H_l)$ is the largest index of any letter occurring in x, and its only occurrences are the listed ones. Similarly, while $a_{\delta(m_{l-1},H_{l-1})}$ may have many occurrences before the last occurrence of $a_{\delta(m_l,H_l)}$, beyond that point it only has the listed occurrences.

Lemma 4.7. *Let S be the free semigroup on the alphabet $\{a_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, B_m is a left J set in S.*

Proof. Let $n \in \mathbb{N}$. We need to show that for each $G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ there exist $m \in \mathbb{N}$, $\alpha \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for all $f \in G$,

$$\alpha(m+1) f\big(t(m)\big) \alpha(m) f\big(t(m-1)\big) \cdots \alpha(2) f\big(t(1)\big) \alpha(1) \in B_n\,.$$

So let $G \in \mathcal{P}_f({}^{\mathbb{N}}S)$. Let $G' = G \cup \{\overline{a_2}\}$, where $\overline{a_2} \in {}^{\mathbb{N}}S$ is the function constantly equal to a_2. By [9, Lemma 3.1], pick an infinite subset C of $\mathbb{N}$ such that for all $f, g \in G'$, either $(\forall n \in C)\big(f(n) = g(n)\big)$ or $(\forall n \in C)\big(f(n) \neq g(n)\big)$.

Enumerate C in increasing order as $\langle c_i \rangle_{i=1}^{\infty}$. Let

$$m = \max\{n \in \mathbb{N} : (\exists f \in G')(a_n \text{ occurs in } f(c_1)\}\,,$$

and note that $m \geq 2$. For $f \in G'$, define $h_f : \{1,2,\ldots,m\} \to S$ by, for $i \in \{1,2,\ldots,m\}$, $h_f(i) = f(c_i)$.

Let $H = \{h_f : f \in G'\}$ and observe that $\psi(H) = m$ so that $H \in \mathcal{F}$. Define $\alpha \in S^{m+1}$ by $\alpha(1) = \alpha(2) = \ldots = \alpha(m+1) = a_{\delta(n,H)}$ and define $t \in \mathcal{J}_m$ by $t(i) = c_i$ for $i \in \{1,2,\ldots,m\}$. Let $f \in G$. Then $h_f \in H$ and

$$\begin{aligned} & \alpha(m+1) f\big(t(m)\big) \alpha(m) f\big(t(m-1)\big) \cdots \alpha(2) f\big(t(1)\big) \alpha(1) = \\ & a_{\delta(n,H)} h_f(m) a_{\delta(n,H)} h_f(m-1) \cdots a_{\delta(n,H)} h_f(1) a_{\delta(n,H)} \in B_n\,. \end{aligned}$$

$\square$

Theorem 4.8. *Let S be the free semigroup on the alphabet $\{a_n : n \in \mathbb{N}\}$. A_1 is a left C set in S.*

Proof. By Lemma 4.7, each B_m is a left J set and $B_m \subseteq A_m$, so by the left-right switch of Theorem 2.6(11) it suffices to show that for each $n \in \mathbb{N}$ and each $x \in A_n$, there exists $m \in \mathbb{N}$ such that $A_m \subseteq A_n x^{-1}$. So let $n \in \mathbb{N}$ and $x \in A_n$ be given. Then

$$x = \coprod_{i=1}^{l} (a_{\delta(m_i,H_i)} h_i\big(\psi(H_i)\big) a_{\delta(m_i,H_i)} \cdots a_{\delta(m_i,H_i)} h_i(1) a_{\delta(m_i,H_i)})$$

for some $l \in \mathbb{N}$, some increasing $\langle m_i \rangle_{i=1}^{l} \in \mathbb{N}^l$, some $\langle H_i \rangle_{i=1}^{l} \in \mathcal{F}^l$, and some $\langle h_i \rangle_{i=1}^{l}$ with each $h_i \in H_i$, such that $m_1 \geq n$ and for $i \in \{1, 2, \ldots, l-1\}$, $\delta(m_{i+1}, H_{i+1}) > \delta(m_i, H_i)$.

There are only finitely many $k \in \mathbb{N}$ such that there is some $H \in \mathcal{F}$ with $\delta(k, H) \leq \delta(m_l, H_l)$. So pick $r \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq r$ and all $H \in \mathcal{F}$, $\delta(k, H) > \delta(m_l, H_l)$. Then $A_r \subseteq A_n x^{-1}$. □

It is a fact that A_1 is not a right weak Q set in the free semigroup on the alphabet $\{a_n : n \in \mathbb{N}\}$, but we will not need that fact. (The curious reader can construct the proof along the lines of the proof of [9, Theorem 3.7(a)].)

Theorem 4.9. *Let S be the free semigroup on the alphabet $\{a_n : n \in \mathbb{N}\}$. A_1 is not a right J set in S.*

Proof. Define h, k in ${}^{\mathbb{N}}S$ by, for $n \in \mathbb{N}$, $h(n) = a_{4n-1}$ and $k(n) = a_{4n+2} a_{4n+3}$. Let $K = \{h, k\}$. Suppose that A_1 is a right J set. Pick $m \in \mathbb{N}$, $\alpha \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that x and y are in A_1 where

$$\begin{aligned} x &= \alpha(1) h\big(t(1)\big) \alpha(2) \cdots \alpha(m) h\big(t(m)\big) \alpha(m+1) \text{ and} \\ y &= \alpha(1) k\big(t(1)\big) \alpha(2) \cdots \alpha(m) k\big(t(m)\big) \alpha(m+1)\,. \end{aligned}$$

Since x and y are in A_1, pick l and u in $\mathbb{N}$, increasing $\langle r_i \rangle_{i=1}^{l} \in \mathbb{N}^l$, increasing $\langle s_i \rangle_{i=1}^{u} \in \mathbb{N}^u$, $\langle C_i \rangle_{i=1}^{l} \in \mathcal{F}^l$, $\langle D_i \rangle_{i=1}^{u} \in \mathcal{F}^u$, and $\langle f_i \rangle_{i=1}^{l}$ and $\langle g_i \rangle_{i=1}^{u}$ such that each $f_i \in C_i$, each $g_i \in D_i$, and

$$\begin{aligned} x = \ & a_{\delta(r_l,C_l)} f_l\big(\psi(C_l)\big) a_{\delta(r_l,C_l)} \cdots a_{\delta(r_l,C_l)} f_l(1) a_{\delta(r_l,C_l)} \\ & \cdot a_{\delta(r_{l-1},C_{l-1})} f_{l-1}\big(\psi(C_{l-1})\big) \cdots a_{\delta(r_{l-1},C_{l-1})} f_{l-1}(1) a_{\delta(r_{l-1},C_{l-1})} \\ & \vdots \\ & \cdot a_{\delta(r_1,C_1)} f_1\big(\psi(C_1)\big) a_{\delta(r_1,C_1)} \cdots a_{\delta(r_1,C_1)} f_1(1) a_{\delta(r_1,C_1)}, \text{ and} \end{aligned}$$

$$\begin{aligned} y = \ & a_{\delta(s_u,D_u)} g_u\big(\psi(D_u)\big) a_{\delta(s_u,D_u)} \cdots a_{\delta(s_u,D_u)} g_u(1) a_{\delta(s_u,D_u)} \\ & \cdot a_{\delta(s_{u-1},D_{u-1})} g_{u-1}\big(\psi(D_{u-1})\big) \cdots a_{\delta(s_{u-1},D_{u-1})} g_{u-1}(1) a_{\delta(s_{u-1},D_{u-1})} \\ & \vdots \\ & \cdot a_{\delta(s_1,D_1)} g_1\big(\psi(D_1)\big) a_{\delta(s_1,D_1)} \cdots a_{\delta(s_1,D_1)} g_1(1) a_{\delta(s_1,D_1)}\,, \end{aligned}$$

where for each $i \in \{1, 2, \ldots, l-1\}$ and each $j \in \{1, 2, \ldots, u-1\}$ (if any) $\delta(r_i, C_i) < \delta(r_{i+1}, C_{i+1})$ and $\delta(s_j, D_j) < \delta(s_{j+1}, D_{j+1})$.

Notice that, given $n \in \mathbb{N}$, $H \in \mathcal{F}$, and $j \in \{1, 2, \ldots, m\}$, there are no occurrences of $a_{\delta(n,H)}$ in $h\big(t(j)\big)$ or in $k\big(t(j)\big)$, because $\delta(n, H)$ is divisible by 4.

From the expansions of x we see that the first letter of $\alpha(1)$ is $a_{\delta(r_l,C_l)}$ and from the expansions of y we see that the first letter of $\alpha(1)$ is $a_{\delta(s_u,D_u)}$ so $(r_l, C_l) = (s_u, D_u)$. The last occurrence of $a_{\delta(r_l,C_l)}$ in the expansion of either x or y is in some $\alpha(j)$. In the expansion of x we then deduce that the last occurrence of $a_{\delta(r_l,C_l)}$ is followed immediately in $\alpha(j)$ by $a_{\delta(r_{l-1},C_{l-1})}$ (since $a_{\delta(r_{l-1},C_{l-1})}$ cannot occur in $h\big(t(j)\big)$. We are using the fact that there are no occurrences of $a_{\delta(r_l,C_l)}$ after the first line of the displayed characterization of x as a member of A_1. Similarly, there may be occurrences of $a_{\delta(r_{l-1},C_{l-1})}$ in the first line of that display, but there are none beyond the second line.

Looking at the expansion of y we deduce that the last occurrence of $a_{\delta(r_l,C_l)}$ is followed immediately in $\alpha(j)$ by $a_{\delta(s_{u-1},D_{u-1})}$. Consequently, we have $(s_{u-1}, D_{u-1}) = (r_{l-1}, C_{l-1})$. Continuing in this fashion we see that $u = l$ and for each $i \in \{1, 2, \ldots, l\}$, $(r_i, C_i) = (s_i, D_i)$.

We have that $x \neq y$ so there is a smallest $p \in \{1, 2, \ldots, l\}$ such that $f_p \neq g_p$. Since $C_p \in \mathcal{F}$ and f_p and g_p are in C_p, we have that for all $i \in \{1, 2, \ldots, \psi(C_p)\}$, $f_p(i) \neq g_p(i)$. Now we claim that the rightmost occurrence of $a_{\delta(r_p,C_p)}$ is in $\alpha(m+1)$. If $p = 1$, this is trivial so assume that $p > 1$. Then

$$\begin{aligned} & a_{\delta(r_{p-1},C_{p-1})} f_{p-1}\big(\psi(C_{p-1})\big) \cdots a_{\delta(r_{p-1},C_{p-1})} f_{p-1}(1) a_{\delta(r_{p-1},C_{p-1})} \\ & \quad \vdots \\ & \cdot a_{\delta(r_1,C_1)} f_1\big(\psi(C_1)\big) a_{\delta(r_1,C_1)} \cdots a_{\delta(r_1,C_1)} f_1(1) a_{\delta(r_1,C_1)} \\ = \; & a_{\delta(r_{p-1},C_{p-1})} g_{p-1}\big(\psi(C_{p-1})\big) \cdots a_{\delta(r_{p-1},C_{p-1})} g_{p-1}(1) a_{\delta(r_{p-1},C_{p-1})} \\ & \quad \vdots \\ & \cdot a_{\delta(r_1,C_1)} g_1\big(\psi(C_1)\big) a_{\delta(r_1,C_1)} \cdots a_{\delta(r_1,C_1)} g_1(1) a_{\delta(r_1,C_1)} \end{aligned}$$

and the letter immediately to the left of $\alpha(m+1)$ in $h\big(t(m)\big)\alpha(m+1)$ is $a_{4t(m)-1}$ while the letter immediately to the left of $\alpha(m+1)$ in $k\big(t(m)\big)\alpha(m+1)$ is $a_{4t(m)+3}$. Therefore there is some $u \in S$ such that

$$\begin{aligned} & \alpha(m+1) = \\ & u \cdot a_{\delta(r_{p-1},C_{p-1})} f_{p-1}\big(\psi(C_{p-1})\big) a_{\delta(r_{p-1},C_{p-1})} \cdots a_{\delta(r_{p-1},C_{p-1})} f_{p-1}(1) a_{\delta(r_{p-1},C_{p-1})} \\ & \quad \vdots \\ & \cdot a_{\delta(r_1,C_1)} f_1\big(\psi(C_1)\big) a_{\delta(r_1,C_1)} \cdots a_{\delta(r_1,C_1)} f_1(1) a_{\delta(r_1,C_1)} \,. \end{aligned}$$

In this case, the rightmost letter of u is $a_{\delta(r_p,C_p)}$.

Let $q = \psi(C_p)$. We note that there exist $j_1 < j_2 < \ldots < j_{q+1} = m + 1$ such that the rightmost $q + 1$ occurrences of $a_{\delta(r_p,C_p)}$ in x are in $\alpha(j_1), \alpha(j_2), \ldots, \alpha(j_{q+1})$ respectively. This is true because if $a_{\delta(r_p,C_p)} f_p(w) a_{\delta(r_p,C_p)}$ occurs as part of $\alpha(j)$ for some $w \in \{1, 2, \ldots, q\}$, then also $a_{\delta(r_p,C_p)} g_p(w) a_{\delta(r_p,C_p)}$ occurs as part of $\alpha(j)$ and $a_{\delta(r_p,C_p)} f_p(w) a_{\delta(r_p,C_p)} \neq a_{\delta(r_p,C_p)} g_p(w) a_{\delta(r_p,C_p)}$.

Since $1 \leq j_1 < j_2 < \ldots < j_{q+1} = m + 1$, we have that $q \leq m$. But the letter of x immediately to the left of $\alpha(m+1)$ is $a_{4t(m)-1}$ and the letter of y

immediately to the left of $\alpha(m+1)$ is $a_{4t(m)+3}$. Since $f_p(1) \neq g_p(1)$, this says that $a_{4t(m)-1}$ occurs in $f_p(1)$ (and, though we won't use that fact, $a_{4t(m)+3}$ occurs in $g_p(1)$). Since $a_{4t(m)-1}$ occurs in $f_p(1)$ and $f_p \in C_p$, we have that $q = \psi(C_p) \geq 4t(m) - 1 \geq 4m - 1$, contradicting the fact that $q \leq m$. $\square$

Theorem 4.10. *Let S be the free semigroup on the alphabet $\{a_n : n \in \mathbb{N}\}$. A_1 is not a Prog set in S.*

Proof. We shall show that there do not exist $d \in S$ and $b \in S^2$ such that $b(1)db(2) \in A_1$ and $b(1)d^2b(2) \in A_1$. Suppose such do exist and pick d and b such that $l\big(b(1)\big) + l\big(b(1)\big)$ is a minimum among all examples.

We thus have some $l, u \in \mathbb{N}$, some increasing $\langle m_i \rangle_{i=1}^{l} \in \mathbb{N}^l$, some increasing $\langle r_i \rangle_{i=1}^{u} \in \mathbb{N}^u$, some $\langle C_i \rangle_{i=1}^{l} \in \mathcal{F}^l$, some $\langle D_i \rangle_{i=1}^{u} \in \mathcal{F}^u$, some $\langle f_i \rangle_{i=1}^{l}$ with each $f_i \in C_i$, some $\langle g_i \rangle_{i=1}^{u}$ with each $g_i \in D_i$, such that for $i \in \{1, 2, \ldots l-1\}$, $\delta(m_{i+1}, C_{i+1}) > \delta(m_i, C_i)$, for $i \in \{1, 2, \ldots, u-1\}$, $\delta(r_{i+1}, D_{i+1}) > \delta(r_i, D_i)$, and

$$b(1)db(2) = \prod_{i=1}^{l}\big(a_{\delta(m_i,C_i)} f_i\big(\psi(C_i)\big) a_{\delta(m_i,C_i)} \cdots a_{\delta(m_i,C_i)} f_i(1) a_{\delta(m_i,C_i)}\big)\,, \text{and}$$
$$b(1)d^2b(2) = \prod_{i=1}^{u}\big(a_{\delta(r_i,D_i)} g_i\big(\psi(D_i)\big) a_{\delta(r_i,D_i)} \cdots a_{\delta(r_i,D_i)} g_i(1) a_{\delta(r_i,D_i)}\big)\,.$$

Since $a_{\delta(m_l,C_l)} = a_{\delta(r_u,D_u)}$ and $a_{\delta(m_1,C_1)} = a_{\delta(r_1,D_1)}$ we have that $(m_l, C_l) = (r_u, D_u)$ and $(m_1, C_1) = (r_1, D_1)$.

Suppose first that $f_l \neq g_u$, so that for each $t \in \{1, 2, \ldots, \psi_(C_l)\}$, $f_l(t) \neq g_u(t)$ because $C_l = D_u \in \mathcal{F}$. Then $b(1)$ ends before $f_l\big(\psi(C_l)\big)$ ends and $b(2)$ starts after $f_l(1)$ starts – possibly after $f_l(1)a_{\delta(m_l,C_l)}$. If $b(2)$ starts after $f_l(1)a_{\delta(m_l,C_l)}$, then there are $\psi(C_l)$ occurrences of $a_{\delta(m_l,C_l)}$ in d and in d^2. Otherwise there are $\psi(C_l)-1$ occurrences of $a_{\delta(m_l,C_l)}$ in d and in d^2. In the first case $\psi(C_l) = 2\psi(C_l)$ while in the second, $\psi(C_l) - 1 = 2\psi(C_l) - 2$. So $\psi(C_l) \leq 1$. But $C_l \in \mathcal{F}$ so $\psi(C_l) \geq 2$, a contradiction.

Since $(m_l, C_l) = (r_u, D_u)$ and $(m_1, C_1) = (r_1, D_1)$, if either $l = 1$ or $u = 1$, then $l = u = 1$ and thus $f_l \neq g_u$ which we have seen is impossible.

Thus we may assume that $l > 1$, $u > 1$, and $f_l = g_u$.

If $b(1) = a_{\delta(m_l,C_l)} f_l\big(\psi(C_l)\big) \cdots a_{\delta(m_l,C_l)} f_l(1) a_{\delta(m_l,C_l)} \gamma$ for some $\gamma \in S$, then $\gamma db(2) \in A_1$ and $\gamma d^2 b(2) \in A_1$, contradicting the minimality of $l\big(b(1)\big) + l\big(b(2)\big)$. Similarly, if $b(2) = \gamma a_{\delta(m_1,C_1)} f_1\big(\psi(C_1)\big) \cdots a_{\delta(m_1,C_1)} f_1(1) a_{\delta(m_1,C_1)}$ for some $\gamma \in S$, then $b(1)d\gamma \in A_1$ and $b(1)d^2\gamma \in A_1$, again a contradiction.

Consequently, we have that $b(1)$ ends at or before the end of

$$a_{\delta(m_l,C_l)} f_l\big(\psi(C_l)\big) \cdots a_{\delta(m_l,C_l)} f_l(1) a_{\delta(m_l,C_l)}$$

and $b(2)$ starts at or after the start of

$$a_{\delta(m_1,C_1)} f_1\big(\psi(C_1)\big) \cdots a_{\delta(m_1,C_1)} f_1(1) a_{\delta(m_1,C_1)}\,.$$

Thus we have some γ and τ in $S \cup \{\emptyset\}$ such that

$$a_{\delta(m_l,C_l)} f_l\big(\psi(C_l)\big) \cdots a_{\delta(m_l,C_l)} f_l(1) a_{\delta(m_l,C_l)} = b(1)\gamma \text{ and}$$
$$a_{\delta(m_1,C_1)} f_1\big(\psi(C_1)\big) \cdots a_{\delta(m_1,C_1)} f_1(1) a_{\delta(m_1,C_1)} = \tau b(2)\,.$$

Therefore

$$d=\gamma\big(\prod_{i=2}^{l-1}(a_{\delta(m_i,C_i)}f_i\big(\psi(C_i)\big)a_{\delta(m_i,C_i)}\cdots a_{\delta(m_i,C_i)}f_i(1)a_{\delta(m_i,C_i)})\big)\tau\,,\text{ and}$$
$$d^2=\gamma\big(\prod_{i=2}^{u-1}(a_{\delta(r_i,D_i)}g_i\big(\psi(D_i)\big)a_{\delta(r_i,D_i)}\cdots a_{\delta(r_i,D_i)}g_i(1)a_{\delta(r_i,D_i)})\big)\tau\,.$$

Assume that $l>2$, and consequently also $u>2$. Then d starts out $\gamma a_{\delta(m_{l-1},C_{l-1})}$ and d^2 starts out $\gamma a_{\delta(r_{u-1},D_{u-1})}$ so $(m_{l-1},C_{l-1})=(r_{u-1},D_{u-1})$. Let k be the number of occurrences of $a_{\delta(m_{l-1},C_{l-1})}$ in γ. Then there are $k+\psi(C_{l-1})+1$ occurrences of $a_{\delta(m_{l-1},C_{l-1})}$ in d and in d^2 so $k+\psi(C_{l-1})+1=2k+2\psi(C_{l-1})+2$, which is impossible.

We must then have that $l=2$. Since d^2 is longer than d, $u>2$. Thus $d=\gamma\tau$ and

$$d^2=\gamma\big(\prod_{i=2}^{u-1}(a_{\delta(r_i,D_i)}g_i\big(\psi(D_i)\big)a_{\delta(r_i,D_i)}\cdots a_{\delta(r_i,D_i)}g_i(1)a_{\delta(r_i,D_i)})\big)\tau\,.$$

Since $d^2=\gamma\tau\gamma\tau$ we then have that

$$\tau\gamma=\prod_{i=2}^{u-1}(a_{\delta(r_i,D_i)}g_i\big(\psi(D_i)\big)a_{\delta(r_i,D_i)}\cdots a_{\delta(r_i,D_i)}g_i(1)a_{\delta(r_i,D_i)})\,.$$

If $\tau\neq\emptyset$, then from the choice of τ we have the leftmost letter of τ is $a_{\delta(m_1,C_1)}$, while from the above equation, the leftmost letter of τ is $a_{\delta(r_{u-1},D_{u-1})}$ while $\delta(r_{u-1},D_{u-1})>\delta(r_1,D_1)=\delta(m_1,C_1)$, a contradiction. So $\tau=\emptyset$ and thus $\gamma\neq\emptyset$. Then from the choice of γ, the rightmost letter of γ is $a_{\delta(m_l,C_l)}$ while from the above equation, the rightmost letter of γ is $a_{\delta(r_2,D_2)}$ while $\delta(r_2,D_2)<\delta(r_u,D_u)=\delta(m_l,C_l)$, a contradiction. □

Recall that a semigroup $(S,\cdot)$ is *weakly left cancellative* if and only if, whenever $u,v\in S$, $\{x\in S:u=vx\}$ is finite.

Theorem 4.11. *There exist a right cancellative and weakly left cancellative semigroup $(S,\cdot)$ and a subset A of S which is a right Q set but not a right weak Q set.*

Proof. Let $D=\{x_k:k\in\mathbb{N}\}\cup\{y_{n,m}:n,m\in\mathbb{N}\text{ and }n<m\}$ and let S be the set of words over D that have no occurrences of $x_ny_{n,m}$ for any $n<m$. Given $u,v\in S$, we let $u\cdot v=uv$, the ordinary concatenation of words, unless there exist $n<m$ in $\mathbb{N}$ such that u ends in x_n and v begins with $y_{n,m}$. In the latter case, pick $z,w\in S\cup\{\emptyset\}$, $l\in\mathbb{N}$, and $m_l>m_{l-1}>\ldots>m_1=m$ such that $u=zx_n$, $v=y_{n,m_1}y_{m_1,m_2}\cdots y_{m_{l-1},m_l}w$, and w does not begin with $y_{m_l,r}$ for any $r>m_l$. Then define $u\cdot v=zx_{m_l}w$.

It is routine, though mildly tedious, to verify that the operation on S is associative. To see that S is right cancellative, assume we have s, v, and u in S and that $u\cdot v=s\cdot v$. If $u\cdot v=uv$ and $s\cdot v=sv$, then we are done. So we assume without loss of generality that we have $z,w\in S\cup\{\emptyset\}$, $n,l\in\mathbb{N}$, and $m_l>m_{l-1}>\ldots>m_1$ such that $u=zx_n$, $v=y_{n,m_1}y_{m_1,m_2}\cdots y_{m_{l-1},m_l}w$, and w does not begin with $y_{m_l,r}$ for any $r>m_l$. Then $u\cdot v=zx_{m_l}w$. If s does not end in x_n, then $s\cdot v=sv$. The letter immediately preceding w in sv is y_{m_{l-1},m_l} while the letter immediately preceding w in $zx_{m_l}w$, is x_{m_l} so

$u \cdot v \neq s \cdot v$. So we have some $t \in S \cup \{\emptyset\}$ such that $s = tx_n$ and $s \cdot v = tx_{m_l}w$. Since $zx_{m_l}w = tx_{m_l}w$, we have $z = t$ so $u = s$ as required.

To see that S is weakly left cancellative, note that if $u, w \in S$ and $\{v \in S : u \cdot v = w\}$ has more than one member, then we must have $n < m$ in $\mathbb{N}$ and $z, s \in S \cup \{\emptyset\}$ such that s does not begin with $y_{m,r}$ for any $r > m$, $u = zx_n$, and $w = zx_m s$. In that event, if $u \cdot v = w$, then there exist $l \in \mathbb{N}$ $n = k_1 < k_2 < \ldots < k_l = m$ such that $v = y_{k_1,k_2} y_{k_2,k_3} \cdots y_{k_{l-1},k_l} s$. So $\{v \in S : u \cdot v = w\}$ is finite.

Finally, let $A = \{y_{n,m} : n, m \in \mathbb{N} \text{ and } n < m\}$. Then whenever $n < m$ in $\mathbb{N}$, we have $x_m = x_n y_{n,m}$ so A is a Q set in S. Since no two members of A have a product in A, A is not a weak Q set. $\square$

Theorem 4.12. *(1) There is a subset A of $(\mathbb{N}, +)$ which is right strongly central and is neither right nor left thickly central.*

(2) There is a subset A of $(\mathbb{N}, +)$ which is right thickly central and is not right thick.

(3) There is a subset A of $(\mathbb{N}, +)$ which is right syndetic and is neither a right weak Q set nor a left weak Q set.

(4) There is a subset A of $(\mathbb{N}, +)$ which is a right IP set and is not a weak Prog set.

(5) There is a subset A of $(\mathbb{N}, +)$ which is a Prog set and is not a right J set.

Proof. (1) Let B be an infinite subset of $\mathbb{N}$ such that $\mathbb{N} \setminus B$ is infinite and let $A = \{n \in \mathbb{N} : \theta(n) \in B\}$. By Lemma 4.2, A and $\mathbb{N} \setminus A$ are strongly central, so A is not thickly central. Since $(\mathbb{N}, +)$ is commutative, left and right thickly central are equivalent.

(2) Let $A = \{2^n + 2m : n, m \in \mathbb{N} \text{ and } m < n\}$. Since $\{2^n + m : n, m \in \mathbb{N} \text{ and } m < \frac{n}{2}\}$ is thick and $2\mathbb{N}$ is a member of any idempotent, A is thickly central. It is trivially not thick.

(3) Let $A = 2\mathbb{N} - 1$. One cannot get x_1 and x_2 with $\{x_1, x_2, x_1 + x_2\} \subseteq A$.

(4) Let $A = \{\sum_{n \in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$. We leave it as an exercise to show that there do not exist $a, d \in \mathbb{N}$ such that $\{a + d, a + 2d, a + 3d\} \subseteq A$.

(5) Let $A = \{2^{2n} + m2^n + 1 : n, m \in \mathbb{N} \text{ and } m < n\}$. A is trivially a Prog set. By [8, Lemma 4.3], A is not a J set. $\square$

Theorem 4.13. *Let S be the free semigroup on the alphabet $\{a_n : n \in \mathbb{N}\}$.*

(1) There is a subset A of S which is right thick and is not right strongly piecewise syndetic, not left piecewise syndetic, and not a left weak Q set.

(2) There is a subset A of S which is a right C set and is not a left J set and not a Prog set.

(3) There is a subset A of S which is a right weak Q set and is not a right Q set and not a right IP set.

Proof. (1) Enumerate $\mathcal{P}_f(S)$ as $\langle F_n\rangle_{n=1}^{\infty}$. For $n \in \mathbb{N}$, let $\tau(n) = 1 + \max\{m : (\exists w \in F_n)(a_m \text{ occurs in } w)\}$. Let $A = \bigcup_{n=1}^{\infty} F_n a_{\tau(n)}$. Trivially A is right thick. By the left-right switch of [9, Theorem 2.4] A is not left piecewise syndetic and is not a left weak Q set. To see that A is not right strongly piecewise syndetic, suppose we have $H \in \mathcal{P}_f(S)$ such that for all $F \in \mathcal{P}_f(S)$ there is some $x \in S$ such that $Fx \subseteq \bigcup_{t\in H} At^{-1}$. Now $H = F_k$ for some $k \in \mathbb{N}$. Let $r = \tau(k)$ and let $F = \{a_r\}$. Pick $x \in S$ such that $Fx \subseteq \bigcup_{t\in H} At^{-1}$ and pick $t \in H$ such that $a_r xt \in A$. Pick $n \in \mathbb{N}$ and $w \in F_n$ such that $a_r xt = wa_{\tau(n)}$. Then $a_{\tau(n)}$ occurs in t and since $t \in H$, $\tau(n) < \tau(k) = r$. This is a contradiction because $a_{\tau(n)}$ is the largest letter occurring in $wa_{\tau(n)}$.

(2) This is a consequence of the left-right switches of Theorems 4.8, 4.9, and 4.10.

(3) Let $A = \{a_n : n \in \mathbb{N}\} \cup \{a_n a_{n+1} : n \in \mathbb{N}\}$. □

The proof of (2) in the next theorem is adapted from [3, Theorem 2.18].

Theorem 4.14. *Let S be the free semigroup on the alphabet $\{a, b\}$.*

(1) There is a subset A of S which is right strongly central and is not right strongly piecewise syndetic.

(2) There is a subset A of S which is right strongly piecewise syndetic and is not left piecewise syndetic.

Proof. (1) Let $A = aS$. Then A is a right ideal of S so by [11, Corollary 4.18] $\overline{A}$ is a right ideal of $(\beta S, \cdot)$. Pick a minimal right ideal R of $(\beta S, \cdot)$ such that $R \subseteq \overline{A}$. Given any minimal left ideal L of $(\beta S, \cdot)$, there is an idempotent $p \in L \cap R$.

Now suppose that A is right strongly piecewise syndetic and pick $H \in \mathcal{P}_f(S)$ such that for each $F \in \mathcal{P}_f(S)$, there is some $x \in S$ such that $Fx \subseteq \bigcup_{t\in H} At^{-1}$. Letting $F = \{b\}$, we obtain a contradiction.

(2) Let $A = \{wa^n : w \in S \text{ and } n > l(w)\}$, where $l(w)$ denotes the length of w. To see that A is right strongly piecewise syndetic, let $H = \{a\}$. Let $F \in \mathcal{P}_f(S)$ be given and let $n = \max\{l(w) : w \in F\}$. Let $x = a^n$. Then $Fxa \subseteq A$.

Suppose now we have $H \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ such that $xF \subseteq \bigcup_{t\in H} At^{-1}$. We may presume $a \in H$. Let $m = \max\{n : a^n \in H\}$ and let $F = \{b^m\}$. For any $x \in S$ and $t \in H$, $xb^m t \notin A$. □

Theorems 4.11, 4.12, 4.13, and 4.14 establish that none of the missing implications in Figure 2 are valid in general. We do not know whether every right strongly central set must be left central, left C, left IP, left Q, or left weak Q. If, for example, we knew that there is a right cancellative semigroup with a right strongly central set which is not a left weak Q set, then we would know that none of the missing implications in Figure 1 are valid in general.

In some cases some mental gymnastics are needed to see that the listed examples are sufficient. For example, in addition to the example of Theorem

4.14(2), to see that right strongly piecewise syndetic does not imply any of the properties that are not shown, one needs to know that that there is a right strongly piecewise syndetic set which is not a right weak Q set (and similarly for left weak Q). We know that there is a right syndetic set which is neither right weak Q nor left weak Q. So we know that there is a left syndetic set which is neither left weak Q nor right weak Q, and such a set must be right strongly piecewise syndetic.

Similarly, we need to know that there is a right Q set which is not a right IP set. It is trivial to construct such an example in $(\mathbb{N},+)$, but the way that follows from our listed examples is that we have a right Q set which is not a right weak Q set, so it cannot be a right IP set.

We conclude this section by showing that to answer the questions about strongly central sets, it suffices to consider free semigroups.

As usual, in a free semigroup S on an alphabet A we identify the elements of A with the length 1 words.

Theorem 4.15. *Let $(T,\cdot)$ be a semigroup and let S be the free semigroup on the alphabet T, Let $h : S \to T$ be the homomorphism which extends the identity function and let $A \subseteq T$.*

(1) If $h^{-1}[A]$ is a left J set in S, then A is a left J set in T.

(2) If A is a right strongly central set in T, then $h^{-1}[A]$ is a right strongly central set in S.

(3) If $h^{-1}[A]$ is a left central set, a left C set, a left IP set, a left Q set, or a left weak Q set in S, then A is respectively a left central set, a left C set, a left IP set, a left Q set, or a left weak Q set in T.

Proof. Let $\widetilde{h} : \beta S \to \beta T$ be the continuous extension of h. Note that by [11, Corollary 4.22], $\widetilde{h}$ is a homomorphism from $(\beta S,\cdot)$ to $(\beta T,\cdot)$ and from $(\beta S,\odot)$ to $(\beta T,\odot)$. Since h is surjective, so is $\widetilde{h}$.

(1) Assume that $h^{-1}[A]$ is a left J set in S. To see that A is a left J set in T, let $F \in \mathcal{P}_f({}^{\mathbb{N}}T)$ be given. Since $T \subseteq S$, $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, so pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F$,

$$a(m+1)f\big(t(m)\big)a(m)\cdots a(2)f\big(t(1)\big)a(1) \in h^{-1}[A]\,.$$

Note that since each $f\big(t(i)\big) \in T$, we have that $h\big(f\big(t(i)\big)\big) = f\big(t(i)\big)$. So we have $h \circ a \in T^{m+1}$ and for each $f \in F$,

$$h\big(a(m+1)\big)f\big(t(m)\big)h\big(a(m)\big)\cdots h\big(a(2)\big)f\big(t(1)\big)h\big(a(1)\big) \in A\,.$$

(2) Assume that A is right strongly central in T. To see that $h^{-1}[A]$ is right strongly central in S, let L be a minimal left ideal in $(\beta S,\cdot)$. By [11, Exercise 1.7.3], $\widetilde{h}[L]$ is a minimal left ideal of $(\beta T,\cdot)$ so pick an idempotent p of $(\beta S,\cdot)$ with $p \in \widetilde{h}[L] \cap \overline{A}$. Then $\widetilde{h}^{-1}[\{p\}] \cap L$ is a compact subsemigroup of $(\beta S,\cdot)$ so pick an idempotent $q \in \widetilde{h}^{-1}[\{p\}] \cap L$. Then $h^{-1}[A] \in q$.

(3) Assume that $h^{-1}[A]$ is a left central set in S. Pick an idempotent p in $K(\beta S, \odot)$ such that $h^{-1}[A] \in p$ and pick a minimal right ideal R of $(\beta S, \odot)$ such that $p \in R$. By [11, Exercise 1.7.3] $\widetilde{h}[R]$ is a minimal right ideal of $(\beta T, \odot)$ so $\widetilde{h}(p)$ is an idempotent in $K(\beta T, \odot)$ and $A \in \widetilde{h}(p)$.

Assume that $h^{-1}[A]$ is a left C set in S. Pick an idempotent p in $J(\beta S, \odot)$ such that $h^{-1}[A] \in p$. By (1) $\widetilde{h}(p)$ is an idempotent in $J(\beta T, \odot)$ and $A \in \widetilde{h}(p)$.

Assume that $h^{-1}[A]$ is a left IP set in S. Pick an idempotent p in $(\beta S, \odot)$ such that $h^{-1}[A] \in p$. Then $\widetilde{h}(p)$ is an idempotent in $(\beta T, \odot)$ and $A \in \widetilde{h}(p)$.

Assume that $h^{-1}[A]$ is a left Q set in S. Pick a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that whenever $n, m \in \mathbb{N}$ with $n < m$, one has $x_m \in h^{-1}[A]x_n$. Then $\langle h(x_n) \rangle_{n=1}^{\infty}$ is a sequence in T and whenever $n, m \in \mathbb{N}$ with $n < m$, one has $h(x_m) \in Ah(x_n)$.

Assume that $h^{-1}[A]$ is a left weak Q set in S. Pick a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $h^{-1}[A]$ such that whenever $n \in \mathbb{N}$ one has $x_{n+1}x_n \in h^{-1}[A]$. Then $\langle h(x_n) \rangle_{n=1}^{\infty}$ is a sequence in A and whenever $n \in \mathbb{N}$ one has $x_{n+1}x_n \in A$. $\square$

5 Partition regularity

A notion that is closed under passage to supersets, as all of the properties we have been considering are, is *partition regular* if and only if, whenever the union of two sets has the property, one of them does. This notion is an important concept in Ramsey Theory. In this section we establish which of our properties are partition regular.

It is well known and trivial that right syndetic is not partition regular and it is easy to see that right strongly central is not partition regular. (If $A_1 = \bigcup_{n=0}^{\infty} \{2^{2n}, 2^{2n}+1, 2^{2n}+2, \ldots, 2^{2n+1}-1\}$ and $A_2 = \mathbb{N} \setminus A_1$, then both A_1 and A_2 are right thick, so neither is right syndetic nor right strongly central.) Similarly the even and odd positive integers show that right thick is not partition regular.

All of our notions that are implied by right central except right weak Q have an algebraic characterization that is witnessed by a single ultrafilter, and so these notions are all partition regular. That leaves three properties (and of course, their corresponding left versions). That is, right thickly central, right strongly piecewise syndetic, and right weak Q. We show now that none of these notions is partition regular.

Theorem 5.1. *In $(\mathbb{N}, +)$ the notion of right thickly central is not partition regular.*

Proof. Let B be an infinite subset of $\mathbb{N}$ such that $\mathbb{N} \setminus B$ is infinite and let $A = \{n \in \mathbb{N} : \theta(n) \in B\}$. By Lemma 4.2, A and $\mathbb{N} \setminus A$ are strongly central, so A and $\mathbb{N} \setminus A$ are not thickly central. $\square$

Theorem 5.2. *In the free semigroup S on the alphabet $\{a_n : n \in \mathbb{N}\}$ the notion of strongly right piecewise syndetic is not partition regular.*

Proof. Enumerate $\mathcal{P}_f(S)$ as $\langle F_n \rangle_{n=1}^{\infty}$. Define $\phi : S \to \{a_1, a_2\}$ by $\phi(w) = a_1$ if w begins with a_1 and $\phi(w) = a_2$ otherwise.

Pick $\sigma(1) \in \mathbb{N}$ larger than the index of any letter of any word in F_1, and let $K_1 = \{za_{\sigma(1)}\phi(z) : z \in F_1\}$. Inductively assume that $K_1, K_2, \ldots, K_{n-1}$ and $\sigma(1), \sigma(2), \ldots, \sigma(n-1)$ have been chosen. Pick $\sigma(n)$ larger than the index of any letter that occurs in any word in $F_n \cup \bigcup_{l=1}^{n-1} K_l$. Let $K_n = \{za_{\sigma(n)}\phi(z) : z \in F_n\}$. Let $A = \bigcup_{n=1}^{\infty} K_n$. Then given F_n, $F_n a_{\sigma(n)} \subseteq \bigcup_{t \in \{a_1, a_2\}} At^{-1}$. So A is strongly right piecewise syndetic. (Note that if $z \in F_n$, $u \in F_m$, and $za_{\sigma(n)}\phi(z) = ua_{\sigma(m)}\phi(u)$, then $m = n$ and $z = u$.)

Let $B_i = \{za_n\phi(z) : n \in \mathbb{N}, z \in F_n, \text{ and } \phi(z) = a_i\}$, for $i \in \{1, 2\}$. We claim that B_i is not strongly right piecewise syndetic. To see this, let $F = \{a_{3-i}\}$ and suppose we have $H \in \mathcal{P}_f(S)$ and $w \in S$ such that $Fw \subseteq \bigcup_{t \in H} B_i t^{-1}$. Pick $t \in H$ such that $a_{3-i}wt \in B_i$. Then $a_{3-i}wt = za_n\phi(z)$ for some $n \in \mathbb{N}$ and $z \in F_n$ such that $\phi(z) = a_i$. But z begins with a_{3-i}. So $\phi(z) = a_{3-i}$, a contradiction. □

Theorem 5.3. *In the free semigroup on the alphabet $\{a_n : n \in \mathbb{N}\}$ the notion of right weak Q set is not partition regular.*

Proof. Let $A = \{a_n : n \in \mathbb{N}\} \cup \{a_n a_{n+1} : n \in \mathbb{N}\}$. Then A is a right weak Q set. Let $B_0 = \{a_{2n} : n \in \mathbb{N}\} \cup \{a_{2n}a_{2n+1} : n \in \mathbb{N}\}$ and

$$B_1 = \{a_{2n-1} : n \in \mathbb{N}\} \cup \{a_{2n-1}a_{2n} : n \in \mathbb{N}\}.$$

Then $A = B_0 \cup B_1$ and neither B_0 nor B_1 is right weak Q. To see this, suppose we have $i \in \{0, 1\}$ and $\langle y_n \rangle_{n=1}^{\infty}$ in B_i with $\{y_n y_{n+1} : n \in \mathbb{N}\} \subseteq B_i$. Since for $w \in A$, $l(w) \leq 2$, each y_n is $a_{2r(n)-i}$ for some $r(n) \in \mathbb{N}$ and no product of two such elements can belong to A because words of length two in A have consecutive indices. □

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