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Notions of size in a semigroup – an update from a historical perspective

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Abstract

Previous papers have investigated relationships among several notions of largeness in a semigroup, some of which have their origins in topological dynamics, others with pure combinatorial roots, and still others based on the algebraic structure of the Stone-Ćech compactification of a discrete semigroup. Here we consider 52 distinct notions of largeness, giving to the extent possible a description of the origins and why the notions are of interest. We establish implications that must hold among these notions. In the event the semigroup is commutative, these reduce to 24 distinct notions. We give examples in $(\mathbb{N}, +)$ showing that the notions satisfy only the implications which we have established for commutative semigroups.

1 Introduction

We shall be concerned with several notions of largeness for subsets of a semigroup (S, \cdot) . All of these notions are closed under passage to supersets. Given a property R which is closed under passage to supersets, there is a dual notion R^* defined as follows: A subset A of S has property R^* if and only if it has nonempty intersection with every subset B of S which has property R . Equivalently, A has property R^* if and only if $S \setminus A$ does not have property R . Notice that if property R implies property K , then property K^* implies property R^* and that property R^{**} is the same as property R .

For motivation of most of the notions, and for the definitions of some of them, we need to refer to the Stone-Ćech compactification βS of S and its algebraic structure. Accordingly, we give a very brief introduction to that structure. For much more detail, see [27, Part I].

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Let (S, \cdot) be a discrete semigroup. We take the Stone-Ćech compactification βS of S to consist of the set of ultrafilters on S , identifying the principal ultrafilters with the points of S so that we take S to be a subset of βS . Given $A \subseteq S$, $\bar{A} = \{p \in \beta S : A \in p\}$. Then $\{\bar{A} : A \subseteq S\}$ forms a basis for the open sets and a basis for the closed sets of βS and \bar{A} is the closure of A with respect to this topology.

The operation \cdot extends to βS so that $(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for each $p \in \beta S$, ρ_p is continuous where $\rho_p(q) = q \cdot p$) with S contained in its topological center (meaning that for each $x \in S$, λ_x is continuous where $\lambda_x(q) = x \cdot q$). As a compact Hausdorff right topological semigroup, $(\beta S, \cdot)$ has a smallest two sided ideal, $K(\beta S, \cdot)$ which is the union of all of the minimal left ideals and is also the union of all of the minimal right ideals. The intersection of any minimal left ideal and any minimal right ideal is a group. In particular, $(\beta S, \cdot)$ has idempotents. Given any semigroup T , we let $E(T)$ be the set of idempotents in T .

The operation can also be extended to an operation \odot on βS so that $(\beta S, \odot)$ becomes a left topological semigroup. If the operation on S is commutative, then for all p and q in βS , $p \cdot q = q \odot p$, so that minimal left ideals of $(\beta S, \cdot)$ are minimal right ideals of $(\beta S, \odot)$ and $K(\beta S, \cdot) = K(\beta S, \odot)$. As a consequence for all of the notions that we will consider, the right version (corresponding to $(\beta S, \cdot)$) is identical to the left version (corresponding to $(\beta S, \odot)$). However, if S is not commutative, the two structures can be quite different. For example, by [11, Corollary 2.2], if S is the free semigroup on two generators, then $K(\beta S, \cdot) \cap K(\beta S, \odot) = \emptyset$. All but two of the notions that we consider have distinct left and right versions. When we are multiplying members x and y of S , we will often simply denote the product as xy . But if we are multiplying p and q which might be in $\beta S \setminus S$, we will always specify whether the product is $p \cdot q$ or $p \odot q$. We will, however, rely on the context to tell whether $\lambda_p(q)$ is $p \cdot q$ or $p \odot q$.

Many of the notions we will consider are *partition regular* meaning that, if the union of two sets has the property, then one of them does. An important fact is that property R is partition regular if and only if there is an ultrafilter every member of which has property R .

2 The origin of the notions

The earliest of the notions that we will consider arises from van der Waerden's Theorem [36] published in 1927. That is, whenever \mathbb{N} is partitioned into finitely many cells (or *finitely colored*), one cell must contain arbitrarily long arithmetic progressions (or is *monochromatic*). A straightforward translation of the notion of a length k arithmetic progression into multiplicative notation would be $\{ad^t : t \in \{1, 2, \dots, k\}\}$. This would make the notion of a progression a one sided notion, and it does not seem that it should be inherently one sided, so we adjust that slightly.

Definition 2.1. Let (S, \cdot) be a semigroup.

- (1) Given $k \in \mathbb{N}$, a set $B \subseteq S$ is a *length k progression* if and only if there exist $a \in S^2$ and $d \in S$ such that $B = \{a(1)d^t a(2) : t \in \{1, 2, \dots, k\}\}$.
- (2) Given $k \in \mathbb{N}$, a set $B \subseteq S$ is a *length k weak progression* if and only if there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $d \in S$ such that

$$B = \{a(1)d^t a(2)d^t \cdots a(m)d^t a(m+1) : t \in \{1, 2, \dots, k\}\}.$$

- (3) A set $A \subseteq S$ is a P-set if and only if for each $k \in \mathbb{N}$, A contains a length k progression.
- (4) A set $A \subseteq S$ is a WP-set if and only if for each $k \in \mathbb{N}$, A contains a length k weak progression.
- (5) $\text{Prog}(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a P-set})\}$.
- (6) $\text{WProg}(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a WP-set})\}$.

Notice that in $(\mathbb{N}, +)$ any length k progression is a length k arithmetic progression. The converse is not quite true since, for example, $\{1+2, 1+4, 1+6, 1+8\}$ is not a length 4 progression, but it does contain $\{1+2+2, 1+4+2, 1+6+2\}$ which is a length 3 progression. A subset of \mathbb{N} is a P-set if and only if it contains arbitrarily long arithmetic progressions.

Since the motivation for considering P and WP is the partition regularity of arithmetic progressions, we want to verify that these are partition regular properties.

Lemma 2.2. *Let (S, \cdot) be a semigroup. The properties P and WP are partition regular. Consequently if A is a P-set (respectively a WP-set), then $\overline{A} \cap \text{Prog}(S) \neq \emptyset$ (respectively $\overline{A} \cap \text{WProg}(S) \neq \emptyset$). $\text{Prog}(S)$ and $\text{wProg}(S)$ are compact two sided ideals of $(\beta S, \cdot)$ and of $(\beta S, \odot)$.*

Proof. We do the proofs for P . The proofs for wP are similar, though notationally more cumbersome. To see that P is a partition regular property, let $A_1 \cup A_2$ be a P-set. It suffices to show that for each $k \in \mathbb{N}$, there exists $i(k) \in \{1, 2\}$ such that A_i contains a length k progression. (Then pick i such that $i(k) = i$ for infinitely many values of k .) So let $k \in \mathbb{N}$. By van der Waerden's Theorem, pick n such that whenever $\{1, 2, \dots, n\}$ is 2-colored, there is a monochromatic length k arithmetic progression. Pick $c \in S^2$ and $d \in S$ such that $\{c(1)d^t c(2) : t \in \{1, 2, \dots, n\}\} \subseteq A_1 \cup A_2$. For $i \in \{1, 2\}$, let $B_i = \{t \in \{1, 2, \dots, n\} : c(1)d^t c(2) \in A_i\}$. Pick $i(k) \in \{1, 2\}$ and $a, b \in \mathbb{N}$ such that $\{a + sb : s \in \{1, 2, \dots, k\}\} \subseteq B_{i(k)}$. Then for each $s \in \{1, 2, \dots, k\}$, $c(1)d^{a+sb} c(2) \in A_{i(k)}$. Let $e(1) = c(1)d^a$, let $f = d^b$, and let $e(2) = c(2)$. Then $\{e(1)f^s e(2) : s \in \{1, 2, \dots, k\}\} \subseteq A_{i(k)}$.

The fact that $\overline{A} \cap \text{Prog}(S) \neq \emptyset$ whenever A is a P-set now follows from [27, Theorem 3.11].

Now we show that $\text{Prog}(S)$ is a compact two sided ideal of $(\beta S, \odot)$. It is trivially compact. Since S is a P-set, we have that $\text{Prog}(S) \neq \emptyset$. Since the definition of $\text{Prog}(S)$ is completely symmetrical, the corresponding proof for $(\beta S, \cdot)$ follows by left-right switches. Let $p \in \text{Prog}(S)$ and let $q \in \beta S$. To see that $p \odot q \in \text{Prog}(S)$, let $A \in p \odot q$ and let $k \in \mathbb{N}$. We show that A contains a length k progression. Now $A \in \lambda_p(q)$ and λ_p is continuous with respect to \odot so pick $B \in q$ such that $\lambda_p[\overline{B}] \subseteq \overline{A}$. Pick $x \in B$. Then $\rho_x(p) = p \odot x \in \overline{A}$. Pick $C \in p$ such that $\rho_x[\overline{C}] \subseteq \overline{A}$. Pick $a \in S^2$ and $d \in S$ such that $\{a(1)d^t a(2) : t \in \{1, 2, \dots, k\}\} \subseteq C$. Then $\{a(1)d^t a(2)x : t \in \{1, 2, \dots, k\}\} \subseteq A$.

To see that $q \odot p \in \text{Prog}(S)$, let $A \in q \odot p$ and let $k \in \mathbb{N}$. We show that A contains a length k progression. Now $A \in \lambda_q(p)$ and λ_q is continuous with respect to \odot so pick $B \in p$ such that $\lambda_q[\overline{B}] \subseteq \overline{A}$. Pick $a \in S^2$ and $d \in S$ such that $\{a(1)d^t a(2) : t \in \{1, 2, \dots, k\}\} \subseteq B$. For each $t \in \{1, 2, \dots, k\}$, let $x(t) = a(1)d^t a(2)$ and note that $\rho_{x(t)}(q) \in \overline{A}$ so that we may pick $C_t \in q$ such that $\rho_{x(t)}[\overline{C_t}] \subseteq \overline{A}$. Pick $z \in \bigcap_{t=1}^k C_t$. Then $\{za(1)d^t a(2) : t \in \{1, 2, \dots, k\}\} \subseteq A$. \square

All of the remaining notions we consider have both right and left versions. The choice of which to call “right” and which to call “left” is based on whether they relate most naturally to $(\beta S, \cdot)$ or $(\beta S, \odot)$.

The second notion which is of interest to us is a notion of density, introduced for subsets of \mathbb{N} by G. Polya in [33], published in 1929. To avoid proliferation of terminology, we go along with Furstenberg in [15, Definition 3.7] in calling it *Banach density*.

Definition 2.3. Let $A \subseteq \mathbb{N}$. The *Banach density* of A is

$$d^*(A) = \sup\{\alpha \in [0, 1] : (\forall m \in \mathbb{N})(\exists n \geq m)(\exists x \in \mathbb{N})(|A \cap \{x+1, x+2, \dots, x+n\}| \geq \alpha \cdot n)\}.$$

Definition 2.4. A set $A \subseteq \mathbb{N}$ is a *B-set* if and only if $d^*(A) > 0$.

In [35], E. Szemerédi proved a longstanding conjecture of Erdős which generalized van der Waerden’s Theorem, showing that any B-set in \mathbb{N} contains arbitrarily long arithmetic progressions. Recently in [30], J. Moreira, F. Richter, and D. Robertson proved another longstanding conjecture of Erdős showing that for any B-set A , there exist infinite B and C such that $B + C = \{b + c : b \in B \text{ and } c \in C\} \subseteq A$.

In [9], V. Bergelson and A. Leibman used ergodic theory to provide strong generalizations of Szemerédi’s Theorem. A special case of one of those generalizations is the following. If S is a B-set, $k \in \mathbb{N}$, p_1, p_2, \dots, p_k are polynomials with rational coefficients taking on integer values on the integers and satisfying

$p_i(0) = 0$ for each $i \in \{1, 2, \dots, k\}$, then for any $v_1, v_2, \dots, v_k \in \mathbb{Z}$, there exist n and u in \mathbb{N} such that $u + p_i(n)v_i \in S$ for each $i \in \{1, 2, \dots, k\}$.

We will introduce right and left versions of B-sets in Section 3 after developing what we believe is an appropriate extension of the notion of Banach density.

The next notion which is of interest to us is that of a *difference set*. The fact that this notion is partition regular is a consequence of Ramsey's Theorem [34], published in 1930. In its simplest nontrivial form, Ramsey's Theorem says that whenever $\{\{m, n\} : m, n \in \mathbb{N} \text{ and } m \neq n\} = B_1 \cup B_2$, there exist $i \in \{1, 2\}$ and an infinite set $M \subseteq \mathbb{N}$ such that $\{\{m, n\} : m, n \in M \text{ and } m \neq n\} \subseteq B_i$.

- Definition 2.5.** (1) A set $A \subseteq \mathbb{N}$ is a *difference set* if and only if there is a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} such that $\{x_n - x_m : m < n\} \subseteq A$.
- (2) Let (S, \cdot) be a semigroup and let $A \subseteq S$. Then A is an *rQ-set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that whenever $m < n$ in \mathbb{N} , $x_n \in x_m A$.
- (3) Let (S, \cdot) be a semigroup and let $A \subseteq S$. Then A is an *ℓQ-set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that whenever $m < n$ in \mathbb{N} , $x_n \in A x_m$.

If S is commutative, we refer simply to Q-sets.

The letter Q is intended to represent “quotient”. Notice that if S is embeddable in a group, then $A \subseteq S$ is an rQ-set if and only if there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $\{x_m^{-1} x_n : m < n\} \subseteq A$. In particular a subset of $(\mathbb{N}, +)$ is a difference set if and only if it is a Q-set.

Lemma 2.6. *Let (S, \cdot) be a semigroup. The properties rQ and ℓQ are partition regular.*

Proof. It suffices to establish the assertion for rQ. Assume that A_1 and A_2 are subsets of S and $A_1 \cup A_2$ is an rQ-set. Pick a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that whenever $m < n$ in \mathbb{N} , $x_n \in x_m(A_1 \cup A_2)$. For $i \in \{1, 2\}$, let $B_i = \{\{m, n\} : m, n \in \mathbb{N}, m < n \text{ and } x_n \in x_m A_i\}$. By Ramsey's Theorem, pick $i \in \{1, 2\}$ and an infinite set $M \subseteq \mathbb{N}$ such that $\{\{m, n\} : m, n \in M \text{ and } m \neq n\} \subseteq B_i$. Enumerate B_i in increasing order as $\langle t(n) \rangle_{n=1}^\infty$. For $n \in \mathbb{N}$, let $y_n = x_{t(n)}$. Then whenever $m < n$ in \mathbb{N} , $y_n \in y_m A_i$. \square

The historically next notions that we deal with are *syndetic* and *thick*. These are notions from topological dynamics which go back at least to 1955. They appear in [18] (where “thick” was called “replete”). Neither one is partition regular but both are quite useful in describing the algebraic structure of βS .

Given a set X we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X . Given x in a semigroup S and $A \subseteq S$, $x^{-1}A = \{y \in S : xy \in A\}$ and $Ax^{-1} = \{y \in S : yx \in A\}$.

Definition 2.7. Let (S, \cdot) be a semigroup.

- (1) A set $A \subseteq S$ is *right syndetic* (abbreviated rSynd) if and only if there exists $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} t^{-1}A$.
- (2) A set $A \subseteq S$ is *left syndetic* (abbreviated ℓSynd) if and only if there exists $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} At^{-1}$.
- (3) A set $A \subseteq S$ is *right thick* (abbreviated rThick) if and only if for every $F \in \mathcal{P}_f(S)$, there exists $x \in S$ such that $Fx \subseteq A$.
- (4) A set $A \subseteq S$ is *left thick* (abbreviated ℓThick) if and only if for every $F \in \mathcal{P}_f(S)$, there exists $x \in S$ such that $xF \subseteq A$.

Note that A is right thick if and only if $S \setminus A$ is not right syndetic. That is $\text{rThick} = \text{rSynd}^*$.

Lemma 2.8. Let (S, \cdot) be a semigroup.

- (a) A set $A \subseteq S$ is right syndetic if and only if for every left ideal L of $(\beta S, \cdot)$, $L \cap \overline{A} \neq \emptyset$.
- (b) A set $A \subseteq S$ is left syndetic if and only if for every right ideal R of $(\beta S, \odot)$, $R \cap \overline{A} \neq \emptyset$.
- (c) A set $A \subseteq S$ is right thick if and only if there exists a left ideal L of $(\beta S, \cdot)$ such that $L \subseteq \overline{A}$.
- (d) A set $A \subseteq S$ is left thick if and only if there exists a right ideal R of $(\beta S, \odot)$ such that $R \subseteq \overline{A}$.

Proof. Statements (a) and (c) are [27, Theorem 4.48]. □

By now it should be clear how to convert a “right” statement into the corresponding “left” statement, so we will cease to specifically state the left versions.

Next in historical order is the notion of *piecewise syndetic*. Without giving the property a name, T. Brown in [10, Lemma 1], published in 1971, showed that if \mathbb{N} is partitioned into finitely many pieces, one of them is piecewise syndetic.

In [19, Definition 2.3], published in 1973, this author introduced a notion for subsets of \mathbb{N} that he called *property S*, which is the negation of piecewise syndetic, and stated without proof as Lemma 2.4 that if A and B have property S , then so does $A \cup B$.

In [15, Definition 1.11], published in 1981, the notion, again applied only to subsets of \mathbb{N} or \mathbb{Z} , was given the name *piecewise syndetic* and defined as the intersection of a syndetic set with a thick set. Further, in [15, Theorem 1.24] it was proved that if a piecewise syndetic subset of \mathbb{N} or \mathbb{Z} is finitely colored, then there is a monochromatic piecewise syndetic subset.

The earliest instance that I have found of the notion being defined for an arbitrary semigroup is in [27, Definition 4.38], the first edition of which was published in 1998.

Definition 2.9. Let (S, \cdot) be a semigroup. A set $A \subseteq S$ is *right piecewise syndetic* (abbreviated rPS) if and only if there exists $H \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ such that $Fx \subseteq \bigcup_{t \in H} t^{-1}A$.

Lemma 2.10. Let (S, \cdot) be a semigroup and let $A \subseteq S$. Then A is right piecewise syndetic if and only if $\overline{A} \cap K(\beta S, \cdot) \neq \emptyset$.

Proof. [27, Theorem 4.40]. □

Next in historical order is the notion of an *IP-set* which stems from the proof as [20, Theorem 3.1], published in 1974, of the Finite Sums Theorem and from [20, Corollary 3.3] which easily allows the Finite Sums Theorem to apply to an arbitrary semigroup. Given a sequence $\langle x_n \rangle_{n=1}^\infty$, in a semigroup (S, \cdot) , let $FP(\langle x_n \rangle_{n=1}^\infty) = \{\prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$ where the products are taken in increasing order of indices.

Theorem 2.11. Let (S, \cdot) be a semigroup, let $r \in \mathbb{N}$ and let $S = \bigcup_{i=1}^r A_i$. There exist $i \in \{1, 2, \dots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i$.

Proof. Pick any sequence $\langle y_n \rangle_{n=1}^\infty$ in S . For $i \in \{1, 2, \dots, r\}$, let $\mathcal{B}_i = \{F \in \mathcal{P}_f(\mathbb{N}) : \prod_{t \in F} y_t \in A_i\}$. By the proof of [20, Corollary 3.3], pick $i \in \{1, 2, \dots, r\}$ and a sequence $\langle F_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max F_n < \min F_{n+1}$ for each $n \in \mathbb{N}$ and $\{\bigcup_{t \in H} F_t : H \in \mathcal{P}_f(\mathbb{N})\} \subseteq \mathcal{B}_i$. For $n \in \mathbb{N}$, let $x_n = \prod_{t \in F_n} y_t$. Given $H \in \mathcal{P}_f(\mathbb{N})$, if $G = \bigcup_{n \in H} F_n$, then $\prod_{n \in H} x_n = \prod_{t \in G} y_t$ because $\max F_n < \min F_{n+1}$ for each n so $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i$. □

The term IP-set (applied only to subsets of \mathbb{N}) originated in [17, Definition 2.2], published in 1978.

Definition 2.12. Let (S, \cdot) be a semigroup. A set $A \subseteq S$ is a *right IP-set* (abbreviated rIP) if and only if there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A$.

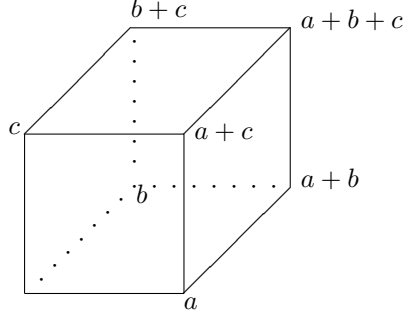
The following lemma is due to F. Galvin in personal communications. It was not published by him.

Lemma 2.13. Let (S, \cdot) be a semigroup. The set $A \subseteq S$ is a right IP-set if and only if there is an idempotent $p \in (\beta S, \cdot)$ with $A \in p$.

Proof. [27, Theorem 5.12]. □

For left IP-sets, the products are taken in decreasing order of indices and the idempotent of Lemma 2.13 is an idempotent of $(\beta S, \odot)$.

Furstenberg [15, page 53] says that IP-sets were so named because of the relationship to idempotents. But he considered a finite-dimensional paralleloiped, such as the 3-dimensional one indicated below and pointed out that “an IP-set might be thought of as an Infinite-dimensional Paralleloiped”.



Next in historical order is the notion of *central* sets, which were introduced for subsets of \mathbb{N} by H. Furstenberg in [15, Definition 8.3], published in 1981. The original definition was in terms of a dynamical system and involved the notions of uniform recurrence and proximality. See [27, Section 19.3] for a proof that this definition is equivalent to the one we are using here and see the notes to [27, Chapter 14 and Chapter 19] for historical information about the evolution of the Central Sets Theorem.

Definition 2.14. Let (S, \cdot) be a semigroup. A set $A \subseteq S$ is *right central* (abbreviated rCtrl) if and only if there is an idempotent $p \in \overline{A} \cap K(\beta S, \cdot)$.

The original Central Sets Theorem was [15, Proposition 8.21] and applied to subsets of \mathbb{N} and finitely many sequences in \mathbb{Z} . What is currently the strongest version follows. It is due to D. De, D. Strauss, and this author in [12] and to J. Johnson in [28]. We write ${}^{\mathbb{N}}S$ for the set of sequences in S .

Definition 2.15. Let $m \in \mathbb{N}$. Then $\mathcal{J}_m = \{(t(1), t(2), \dots, t(m)) \in \mathbb{N}^m : t(1) < t(2) < \dots < t(m)\}$.

Theorem 2.16. Let (S, \cdot) be a semigroup and let C be a right central subset of S . There exist

$$m : \mathcal{P}_f({}^{\mathbb{N}}S) \rightarrow \mathbb{N}, \alpha \in \times_{F \in \mathcal{P}_f({}^{\mathbb{N}}S)} S^{m(F)+1}, \text{ and } \tau \in \times_{F \in \mathcal{P}_f({}^{\mathbb{N}}S)} \mathcal{J}_m(F)$$

such that

- (a) if $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and
- (b) whenever $n \in \mathbb{N}$, $G_1, G_2, \dots, G_n \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n$, and for each $i \in \{1, 2, \dots, n\}$, $f_i \in G_i$, one has

$$\prod_{i=1}^n \left(\left(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j) \cdot f_i(\tau(G_i)(j)) \right) \cdot \alpha(G_i)(m(G_i) + 1) \right) \in A.$$

Proof. [28, Corollary 3.3]. □

By [27, Theorem 4.44], for any semigroup S , $\text{cl}K(\beta S, \cdot)$ is a two sided ideal of $(\beta S, \cdot)$. The following notion was introduced in [25], published in 1996.

Definition 2.17. Let (S, \cdot) be a semigroup. A set $A \subseteq S$ is a *right quasi-central set* (abbreviated rQC) if and only if there is an idempotent $p \in \overline{A} \cap \text{cl}K(\beta S, \cdot)$.

It is not obvious that there are quasi-central sets that are not central. It was shown in [25, Theorem 4.4] that there is a quasi-central subset of \mathbb{N} which is not central.

The following notion was introduced in [7, Definition 2.11], published in 1998. Its definition is a cross between the definitions of right piecewise syndetic and left piecewise syndetic. We shall see that it sits between the notions of left syndetic and right piecewise syndetic.

Definition 2.18. Let (S, \cdot) be a semigroup. A set $A \subseteq S$ is *right strongly piecewise syndetic* (abbreviated rSPS) if and only if there exists $H \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ such that $Fx \subseteq \bigcup_{t \in H} At^{-1}$.

Next come the notions of C-set and J-set, introduced in [12, Definition 3.3] which was published in 2008. (C-sets were called there *strongly rich* sets.) The main reason one is interested in central sets is because they satisfy the Central Sets Theorem (Theorem 2.16). And C-sets are precisely those sets which satisfy the Central Sets Theorem. Because of this fact, C-sets have many (but not all) of the desirable properties of central sets. For example, it follows from [27, Theorem 15.5.2 and (a) \Rightarrow (f) of Theorem 15.24] that given any C-set C contained in \mathbb{N} and any finite image partition regular matrix A with rational entries, there is an image of A with all of its entries in C .

Definition 2.19. Let (S, \cdot) be a semigroup.

- (1) A set $A \subseteq S$ is a *right J-set* (abbreviated rJ) if and only if for each $F \in \mathcal{P}_f(\mathbb{N}S)$, there exist $m \in \mathbb{N}$, $\alpha \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F$, $\left(\prod_{j=1}^m \alpha(j)f(t(j)) \right) \alpha(m+1) \in A$.
- (2) $\text{rJ}(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a right J-set})\}$.
- (3) A set $A \subseteq S$ is a *right C-set* (abbreviated rC) if and only if there exist

$$m : \mathcal{P}_f(\mathbb{N}S) \rightarrow \mathbb{N}, \alpha \in \times_{F \in \mathcal{P}_f(\mathbb{N}S)} S^{m(F)+1}, \text{ and } \tau \in \times_{F \in \mathcal{P}_f(\mathbb{N}S)} \mathcal{J}_{m(F)}$$

such that

- (a) if $F, G \in \mathcal{P}_f(\mathbb{N}S)$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

- (b) whenever $n \in \mathbb{N}$, $G_1, G_2, \dots, G_n \in \mathcal{P}_f(\mathbb{N}S)$, $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n$, and for each $i \in \{1, 2, \dots, n\}$, $f_i \in G_i$, one has

$$\prod_{i=1}^n \left(\left(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j) \cdot f_i(\tau(G_i)(j)) \right) \cdot \alpha(G_i)(m(G_i) + 1) \right) \in A.$$

Lemma 2.20. *Let (S, \cdot) be a semigroup.*

- (a) $rJ(S)$ is a compact two sided ideal of $(\beta S, \cdot)$ and of $(\beta S, \odot)$.
- (b) A set $A \subseteq S$ is a right C-set if and only if there is an idempotent $p \in \overline{A} \cap rJ(S)$.

Proof. The first part of conclusion (a) is [27, Theorem 14.14.4] except that we neglected to show that $rJ(S) \neq \emptyset$, so it should have been placed after [27, Lemma 14.14.6]. The second part is [24, Lemma 3.1].

Conclusion (b) is [27, Theorem 14.15.1]. □

The historically next to last notion that we will consider is D-set which was introduced for subsets of \mathbb{N} by V. Bergelson and T. Downarowicz in [5, Definition 1.2], published in 2008.

Definition 2.21. A set $A \subseteq \mathbb{N}$ is a *D-set* if and only if there is an idempotent $p \in \overline{A}$ such that every $B \in p$ has $d^*(B) > 0$.

M. Beiglböck, V. Bergelson, T. Downarowicz, and A. Fish in [4, Theorem 11] proved that any D-set in \mathbb{N} satisfies the conclusion of the original Central Sets Theorem [15, Proposition 8.21] and remarked that in fact one could show that a D-set in \mathbb{N} satisfies the full Central Sets Theorem; that is that D-sets in \mathbb{N} are C-sets.

As with B-sets, introduction of right and left versions of D-sets await the introduction of an appropriate generalization of Banach density in the next section.

The historically last of the notions that we are considering is *strongly central* which was introduced by V. Bergelson, D. Strauss, and this author in [8, Definition 2.1], published in 2012.

Definition 2.22. Let (S, \cdot) be a semigroup. A set $A \subseteq S$ is *right strongly central* (abbreviated rSC) if and only if for every minimal left ideal L of $(\beta S, \cdot)$, there is an idempotent in $\overline{A} \cap L$.

Note that since every left ideal of $(\beta S, \cdot)$ contains a minimal left ideal, one can equivalently require that there is an idempotent in $\overline{A} \cap L$ for every left ideal L of $(\beta S, \cdot)$. Note that A is rSC* if and only if there is a left ideal L of $(\beta S, \cdot)$ with $E(L) \subseteq \overline{A}$.

To the best of my knowledge, there are five previous papers that considered the relationships among some of the notions that we have introduced in this section.

- In [7] both right and left versions of syndetic, thick, piecewise syndetic, and strongly piecewise syndetic were considered, defined in an arbitrary semigroup.
- In [6] Q, Q^*, IP, IP^* , central, central*, PS, PS^* , syndetic, and syndetic* were considered, defined in an arbitrary semigroup, but only the “right” versions were considered.
- In [5] IP, IP^*, D, D^* , central, and central* were considered, all defined only for subsets of \mathbb{Z} . They also considered infinite and infinite* (i.e. cofinite) and shift invariant extensions of all of these.
- In [23] $Q, Q^*, IP, IP^*, C, C^*, J, J^*$, central, central*, syndetic, syndetic*, PS, PS^*, P, P^* , and SC (but not SC^*) were considered, all defined only for subsets of \mathbb{N} .
- In [24] both right and left versions of Q, IP, C, J , central, PS, SPS, P, WP , syndetic, syndetic*, SC , and SC^* were considered, defined for arbitrary semigroups. They also considered another notion, called a *weak Q-set* (about which, more below).

As we noted, in [5] “infinite” was included among the things studied, as well as translation invariant extensions of all of the notions they studied. It is certainly reasonable to think that “large” sets ought to be infinite. But since we are dealing with arbitrary semigroups, one cannot arrange this without strongly modifying the definitions. For example, $\{0\}$ is an IP-set in \mathbb{Z} . (In [5], this was excluded by fiat.)

There is no obvious reason not to study the translation invariant extensions of the notions, except that I feel that results in an unmanageable number of notions. (The innocent reader can be excused if she thinks I already have an unmanageable number.)

In [24] a *right weak Q-set* was defined as a set A for which there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in A such that $x_n x_{n+1} \in A$ for each $n \in \mathbb{N}$. I decided that this is not an interesting notion. In $(\mathbb{N}, +)$, $\{x, 2x\}$ satisfies that definition for each $x \in \mathbb{N}$. (At least in $(\mathbb{N}, +)$ any set satisfying any of the notions that we study is infinite.)

3 Følner Density, B-sets, and D-sets

In this section we argue that semigroups satisfying (left or right versions of) the *Strong Følner Condition* are a natural context for extending the notion of Banach density and therefore for studying right and left B-sets and D-sets.

A semigroup (S, \cdot) is *left amenable* if and only if there exists a left invariant mean for S . See [32] for the definitions. We don’t need to deal with means here.

E. Følner established in [13] that a group (S, \cdot) is left amenable if and only if it satisfies what is now known as the *Følner condition*:

$$(\forall H \in \mathcal{P}_f(S))(\forall \epsilon > 0)(\exists K \in \mathcal{P}_f(S))(\forall s \in H)(|sK \setminus K| < \epsilon|K|).$$

We will denote this condition as rFC. If “ $|sK \setminus K|$ ” is replaced by “ $|Ks \setminus K|$ ” we will refer to the condition as ℓFC.

In [14], A. Frey showed that any left amenable semigroup satisfies rFC. A more easily accessible proof was given by I. Namioka in [31].

Using an argument from [32, Section 4.22] we see that if (S, \cdot) is a semigroup, $K \in \mathcal{P}_f(S)$, and $s \in S$, then $|K \cap sK| + |K \setminus sK| = |K| \geq |sK| = |sK \cap K| + |sK \setminus K|$ so that $|K \setminus sK| \geq |sK \setminus K|$ and equality holds if s is left cancelable. Consequently the Følner condition follows from the *strong Følner condition*

$$(\forall H \in \mathcal{P}_f(S))(\forall \epsilon > 0)(\exists K \in \mathcal{P}_f(S))(\forall s \in H)(|K \setminus sK| < \epsilon|K|)$$

which we will denote by rSFC. Further, if S is left cancellative, then rFC and rSFC are equivalent and in particular any left cancellative left amenable semigroup satisfies rSFC. (If the reader wonders about the right versus left terminology, we will see that for certain semigroups rSFC is associated with an ideal of $(\beta S, \cdot)$, which is the structure that we have been associating with “right” terminology.)

L. Argabright and C. Wilde showed in [3, Theorem 1] that any semigroup satisfying the strong Følner condition is left amenable and in [3, Theorem 4] that any commutative semigroup satisfies the strong Følner condition. (For a simple elementary proof of the latter fact, see [26, Section 7].) In [29] M. Klawe proved that there exists a right cancellative left amenable semigroup which does not satisfy the strong Følner condition. An isomorphic copy of this semigroup is presented below in Theorem 6.1. (She did not mention this fact, but her example is in fact weakly left cancellative, meaning that for any $x, y \in S$, $\{z \in S : xz = y\}$ is finite.)

Given any semigroup satisfying rSFC, there is a natural notion of density associated, which we will argue is an appropriate generalization of Banach density.

Definition 3.1. Let (S, \cdot) be a semigroup which satisfies rSFC.

- (1) For $A \subseteq S$, the *right Følner density* of A is defined by
$$d_r(A) = \sup\{\alpha \in [0, 1] : (\forall H \in \mathcal{P}_f(S))(\forall \epsilon > 0)(\exists K \in \mathcal{P}_f(S))$$

$$((\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|) \text{ and } |A \cap K| \geq \alpha \cdot |K|)\}.$$
- (b) $\Delta_r^*(S) = \{p \in \beta S : (\forall A \in p)(d_r(A) > 0)\}.$

We now note some important facts about Følner density from [26]. We first observe that Følner density does generalize Banach density.

Lemma 3.2. *Let $A \subseteq \mathbb{N}$. Then $d_r(A) = d^*(A)$.*

Proof. [26, Theorem 1.9]. □

A desirable property of a notion of density is left invariance.

Lemma 3.3. *Let (S, \cdot) be a semigroup satisfying rSFC, let $A \subseteq S$, and let $t \in S$. Then $d_r(t^{-1}A) = d_r(A)$.*

Proof. [26, Theorem 6.3]. □

Another desirable property of a notion of density is subadditivity.

Lemma 3.4. *Let (S, \cdot) be a semigroup satisfying rSFC and let A and B be subsets of S . Then $d_r(A \cup B) \leq d_r(A) + d_r(B)$.*

Proof. Suppose that we have $\delta > 0$ such that $d_r(A \cup B) > d_r(A) + d_r(B) + \delta$. Let $a = d_r(A)$ and $b = d_r(B)$. Since $d_r(A) < a + \frac{\delta}{3}$ pick $H_1 \in \mathcal{P}_f(S)$ and $\epsilon_1 > 0$ such that

$$(\forall K \in \mathcal{P}_f(S))((\forall s \in H_1)(|K \setminus sK| < \epsilon_1|K|) \Rightarrow |A \cap K| < (a + \frac{\delta}{3})|K|).$$

Since $d_r(B) < b + \frac{\delta}{3}$ pick $H_2 \in \mathcal{P}_f(S)$ and $\epsilon_2 > 0$ such that

$$(\forall K \in \mathcal{P}_f(S))((\forall s \in H_2)(|K \setminus sK| < \epsilon_2|K|) \Rightarrow |A \cap K| < (a + \frac{\delta}{3})|K|).$$

Let $H = H_1 \cup H_2$ and let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Pick $K \in \mathcal{P}_f(S)$ such that for all $s \in H$, $|K \setminus sK| < \epsilon|K|$ and $|(A \cup B) \cap K| \geq (a + b + \delta)|K|$. This is a contradiction. □

Lemma 3.5. *Let S be a semigroup satisfying rSFC and let $A \subseteq S$. Then $\overline{A} \cap \Delta_r^*(S) \neq \emptyset$ if and only if $d_r(A) > 0$.*

Proof. This is an immediate consequence of Lemma 3.4 and [27, Theorem 3.11]. □

Definition 3.6. Let (S, \cdot) be a semigroup satisfying rSFC.

- (1) The set $A \subseteq S$ is a *right B-set* (abbreviated rB) if and only if $d_r(A) > 0$.
- (2) The set $A \subseteq S$ is a *right D-set* (abbreviated rD) if and only if A is a member of an idempotent in $\Delta_r^*(S)$.

Lemma 3.7. *Let (S, \cdot) be a semigroup satisfying rSFC. Then $\Delta_r^*(S)$ is a left ideal of $(\beta S, \cdot)$. If S is left cancellative or there exists $b \in \mathbb{N}$ such that for each $x \in S$, ρ_x is at most b -to-1, then Δ_r^* is a compact two sided ideal of $(\beta S, \cdot)$.*

Proof. [26, Corollary 6.6]. □

A strong argument in favor of the appropriateness of the notion of Følner density is the fact that we extend the main result of [4].

Theorem 3.8. *Let (S, \cdot) be a commutative semigroup.*

- (a) *If A is an rB -set in S , then A is an rJ -set in S .*
- (b) *If A is an rD -set in S , then A is an rC -set in S .*

Proof. Conclusion (a) follows from [26, Lemma 2.2 and Theorem 6.10]. By conclusion (a), $\Delta_r^*(S) \subseteq rJ(S)$ so if A is an rD -set, then A is a member of an idempotent in $rJ(S)$ so that [27, Theorem 14.15.1] applies. \square

We would like to get rid of the commutativity assumption in Theorem 3.8. However, the proof of [26, Theorem 6.10] relies heavily on [16, Theorem A] which is a deep theorem about commuting transformations. So we do not have high hopes of success.

The following simple fact will be useful later.

Lemma 3.9. *Assume that (S, \cdot) is a semigroup satisfying $rSFC$ and there exists $b \in \mathbb{N}$ such that for each $x \in S$, ρ_x is at most b -to-1. If A is a right thick subset of S , then $d_r(A) = 1$. If A is a right piecewise syndetic subset of S , then there exists $H \in \mathcal{P}_f(S)$ such that $d_r(\bigcup_{t \in H} t^{-1}A) = 1$.*

Proof. Assume that A is right thick. Let $H \in \mathcal{P}_f(S)$ and $\epsilon > 0$ be given. Pick $K \in \mathcal{P}_f(S)$ such that for all $s \in H$, $|K \setminus sK| < \frac{\epsilon}{b}|K|$. Pick $x \in S$ such that $Kx \subseteq A$. Then for each $s \in H$, $Kx \setminus sKx \subseteq \rho_x[K \setminus sK]$ so $|Kx \setminus sKx| \leq |K \setminus sK| < \frac{\epsilon}{b}|K| \leq \epsilon|Kx|$ and $|A \cap Kx| = |Kx|$. If A is right piecewise syndetic, then there is some $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} t^{-1}A$ is right thick. \square

There is a density notion naturally associated with semigroups satisfying rFC . We could define:

$$d'(A) = \sup\{\alpha \in [0, 1] : (\forall H \in \mathcal{P}_f(S))(\forall \epsilon > 0)(\exists K \in \mathcal{P}_f(S)) \\ ((\forall s \in H)(|sK \setminus K| < \epsilon \cdot |K|) \text{ and } |A \cap K| \geq \alpha \cdot |K|)\}.$$

However, this notion is not particularly well behaved. Let S be an infinite left zero semigroup – that is, $xy = x$ for all $x, y \in S$. Then S is not left amenable, but it does satisfy rFC . And it is an easy exercise to show that for $A \subseteq S$, $d'(A) = 1$ if A is infinite and $d'(A) = 0$ if A is finite. So if A is an infinite subset of S and $t \in S \setminus A$, then $d'(A) = 1$ while $d'(t^{-1}A) = 0$. Then if one defines $\Delta'(S) = \{p \in \beta S : (\forall A \in p)(d'(A) > 0)\}$, one has $\Delta'(S) = \beta S \setminus S$ so that $\Delta'(S)$ is not a left ideal of $(\beta S, \cdot)$.

It is possible that d' is a respectable notion for semigroups that are left amenable; we do not know. Certainly, if S is left cancellative, then $d'(A) = d_r(A)$ for all $A \subseteq S$ so we do have that d' is a decent notion.

An infinite left zero semigroup S does satisfy ℓ SFC, and is, in particular, right amenable. For $A \subseteq S$, $d_\ell(A) = 1$ if and only if $A \neq \emptyset$ and given $t \in S$, $At^{-1} = A$.

4 Implications among the notions

In Figure 1 we have diagrammed the implications that we know hold among the 52 notions that we have introduced.

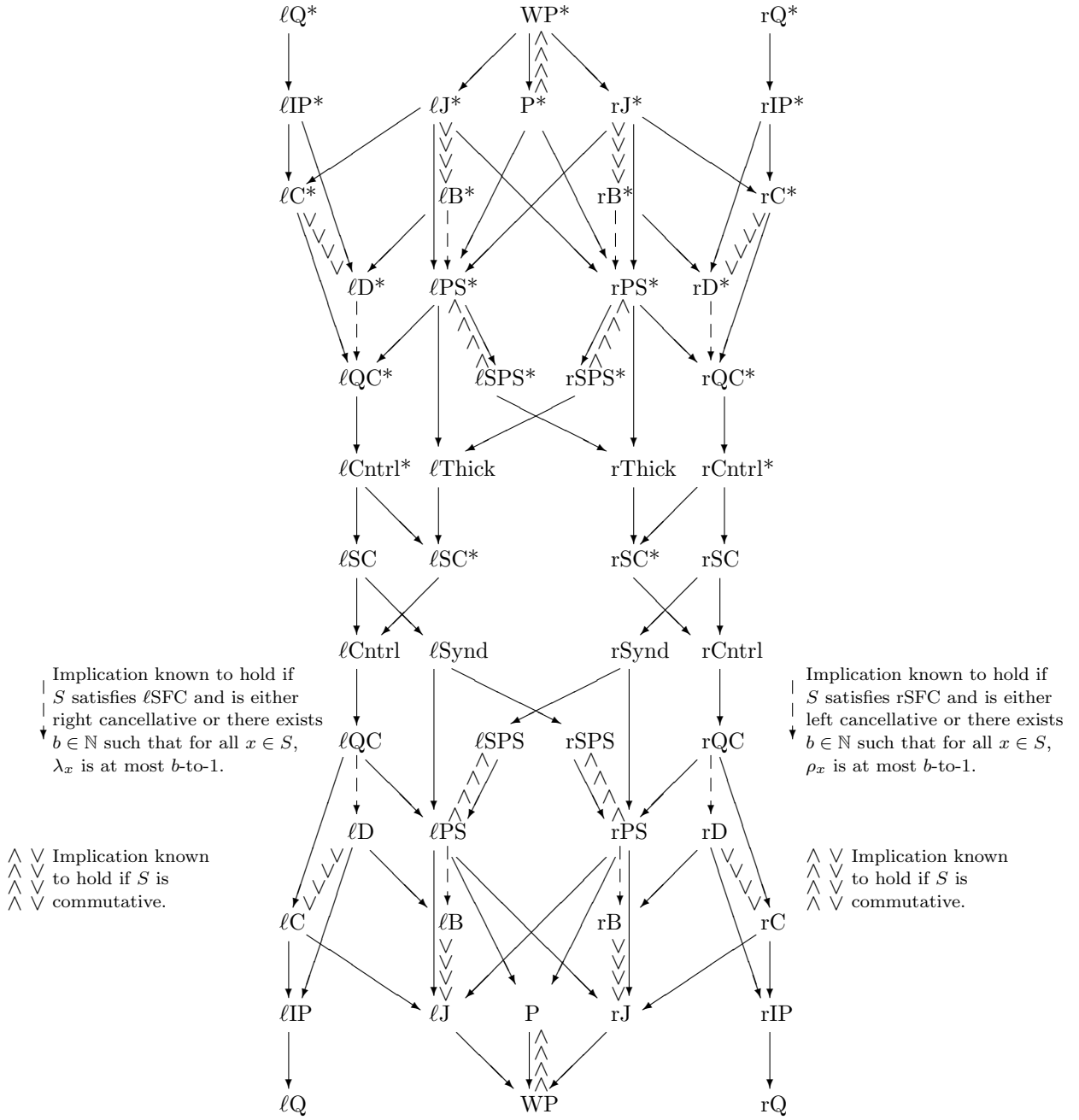


Figure 1: Implications for arbitrary S

piecewise syndetic, then $\overline{A} \cap K(\beta S, \odot) \neq \emptyset$. Consequently $\text{rPS} \Rightarrow \text{P}$, $\text{rPS} \Rightarrow \text{rJ}$, and $\ell\text{PS} \Rightarrow \text{rJ}$. By a left-right switch, $\text{rPS} \Rightarrow \ell\text{J}$. Also, if p is an idempotent in $\text{cl}K(\beta S, \cdot)$, then $p \in \text{rJ}(S)$ so by Lemma 2.20, any member of p is an rC -set and consequently $\text{rQC} \Rightarrow \text{rC}$.

By Lemma 2.13 a set $A \subseteq S$ is an rIP -set if and only if A is a member of an idempotent in $(\beta S, \cdot)$. Therefore $\text{rD} \Rightarrow \text{rIP}$ and by Lemma 2.20, $\text{rC} \Rightarrow \text{rIP}$. The fact that $\text{rC} \Rightarrow \text{rJ}$ is a direct consequence of Lemma 2.20.

To see that $\text{rIP} \Rightarrow \text{rQ}$, let A be an rIP -set and pick a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A$. For $n \in \mathbb{N}$, let $y_n = \prod_{t=1}^n x_t$. Then if $m < n$, we have $y_n = y_m \prod_{t=m+1}^n x_t \in y_m A$.

To see that $\text{rJ} \Rightarrow \text{WP}$, let $k \in \mathbb{N}$ be given and fix $d \in S$. For $t \in \{1, 2, \dots, k\}$, let f_t be the sequence in S constantly equal to d^t . Let A be an rJ -set. Pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $s \in \mathcal{J}_m$ such that for each $t \in \{1, 2, \dots, k\}$, $a(1)f_t(s(1))a(2) \cdots a(m)f_t(s(m))a(m+1) \in A$. Then

$$\{a(1)d^t a(2) \cdots a(m)d^t a(m+1) : t \in \{1, 2, \dots, k\}\}$$

is a length k weak progression contained in A .

To see that $\text{rSPS} \Rightarrow \text{rPS}$, let A be a right strongly piecewise syndetic subset of S . Pick $H \in \mathcal{P}_f(S)$ such that for each $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq \bigcup_{t \in H} At^{-1}$. Then $\bigcup_{t \in H} At^{-1}$ is right thick so by Lemma 2.8, pick a minimal left ideal L of $(\beta S, \cdot)$ such that $L \subseteq \bigcup_{t \in H} At^{-1}$. Pick $p \in L$. Then $\bigcup_{t \in H} At^{-1} \in p$ so pick $t \in H$ such that $At^{-1} \in p$. Then $A \in pt$ and $pt \in K(\beta S, \cdot)$ so $\overline{A} \cap K(\beta S, \cdot) \neq \emptyset$. Thus by Lemma 2.10, A is right piecewise syndetic.

It remains for us to show that if S satisfies rSFC and is either left cancellative or there exists $b \in \mathbb{N}$ such that for each $x \in S$, ρ_x is at most b -to-1, then $\text{rPS} \Rightarrow \text{rB}$ and $\text{rQC} \Rightarrow \text{rD}$. So assume that S satisfies rSFC and is either left cancellative or there exists $b \in \mathbb{N}$ such that for each $x \in S$, ρ_x is at most b -to-1. Then by Lemma 3.7, $\Delta_r^*(S)$ is a compact two sided ideal of $(\beta S, \cdot)$ and consequently $\text{cl}K(\beta S, \cdot) \subseteq \Delta_r^*(S)$. Thus, if p is an idempotent in $\text{cl}K(\beta S, \cdot)$ we have that p is an idempotent in $\Delta_r^*(S)$ and so $\text{rQC} \Rightarrow \text{rD}$. If A is right piecewise syndetic then by Lemma 2.10, $\overline{A} \cap K(\beta S, \cdot) \neq \emptyset$ and hence $\overline{A} \cap \Delta_r^*(S) \neq \emptyset$. Thus $\text{rPS} \Rightarrow \text{rB}$.

It would be nice to know whether any of the missing implications hold in general. We had partial success in [24] where it was shown that the only implications that hold in general among the notions rThick , rSC^* , rSynd , rSPS , rCntrl , rPS , rC , rJ , rIP , rQ , ℓThick , ℓSC^* , ℓSynd , ℓSPS , ℓCntrl , ℓPS , ℓC , ℓJ , ℓIP , ℓQ , P , and WP are those that follow from the implications in Figure 1. What little information we have to add to this will be presented in Section 6

5 Commutative semigroups

In the event that S is commutative, Figure 1 collapses to Figure 2. All of the implications in Figure 2 hold because they follow from implications in Figure 1. In this section, we will show that none of the missing implications in Figure 2 hold in general. In fact, we will show that for each of the 24 properties listed, there is a subset of $(\mathbb{N}, +)$ having that property and only those other properties that hold as a consequence of the implications in the diagram. This is stronger than simply showing that in $(\mathbb{N}, +)$ none of the missing implications is valid. For example, to see that central does not satisfy any missing implication, it suffices to find a central set A which is not syndetic and a central set B which is not SC^* . Then $A \cap B$ will be neither syndetic nor SC^* , but may not be central. We shall show that there is a central set which is neither syndetic nor SC^* .

Theorem 5.1. *For each of the properties in Figure 2, there is a subset of \mathbb{N} with that property and only those other properties that it must have as a consequence of the implications in Figure 2.*

Proof. (1) Let $A = \{2^{2n} - 2^{2m} : m, n \in \mathbb{N} \text{ and } m < n\}$. Then A is a Q-set which is neither a P-set nor an IP-set.

Trivially A is a Q-set and is not an IP-set. We claim A contains no length 3 progression. Suppose instead one has $a, d \in \mathbb{N}$ with $a > 1$ such that $\{a, a + d, a + 2d\} \subseteq A$. Pick $m, n, r, s, l, t \in \mathbb{N}$ such that $a = 2^{2n} - 2^{2m}$, $a + d = 2^{2r} - 2^{2s}$, and $a + 2d = 2^{2l} - 2^{2t}$. Note that $n > m$, $r > s$, $l > t$, and $l \geq r \geq n$. Then $d = 2^{2r} - 2^{2s} - 2^{2n} + 2^{2m} = 2^{2l} - 2^{2t} - 2^{2r} + 2^{2s}$ so $2^{2r+1} + 2^{2m} + 2^{2t} = 2^{2l} + 2^{2s+1} + 2^{2n}$. A little checking of cases shows that this is impossible.

(2) Let $A = \{2^{2n} + m2^n + 1 : m, n \in \mathbb{N} \text{ and } m < n\}$. Then A is a P-set which is neither a Q-set nor a J-set.

Trivially A is a P-set. Since $A \subseteq 2\mathbb{N} + 1$, A is not a Q-set. By [23, Lemma 4.3], A is not a J-set.

(3) Let $A = \{\sum_{n \in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$. Then A is an IP-set which is not P-set.

To see that A is not a P-set, we show that A does not contain a length 3 progression. Suppose instead one has $a, d \in \mathbb{N}$ with $a > 1$ such that $\{a, a + d, a + 2d\} \subseteq A$. Pick $F, G, H \in \mathcal{P}_f(\mathbb{N})$ such that $a = \sum_{n \in F} 2^{2n}$, $a + d = \sum_{n \in G} 2^{2n}$, and $a + 2d = \sum_{n \in H} 2^{2n}$. Note that $G \setminus F \neq \emptyset$. Then $d = \sum_{n \in G \setminus F} 2^{2n} - \sum_{n \in F \setminus G} 2^{2n}$ and $d = \sum_{n \in H \setminus G} 2^{2n} - \sum_{n \in G \setminus H} 2^{2n}$ so that $x = \sum_{n \in G \setminus F} 2^{2n} + \sum_{n \in G \setminus H} 2^{2n} = \sum_{n \in H \setminus G} 2^{2n} + \sum_{n \in F \setminus G} 2^{2n}$. We write $\text{supp}(x)$ for the binary support of x , that is the powers of 2 occurring in the binary expansion of x . Pick $t \in G \setminus F$. If $t \notin H$, then $2t + 1 \in \text{supp}(x)$. If $t \in H$, then $2t \in \text{supp}(x)$. But $\{2t, 2t + 1\} \cap \text{supp}(\sum_{n \in H \setminus G} 2^{2n} + \sum_{n \in F \setminus G} 2^{2n}) = \emptyset$, a contradiction.

(4) In [22, Theorem 2.1] a subset of \mathbb{N} is produced which is a C-set and not a B-set.

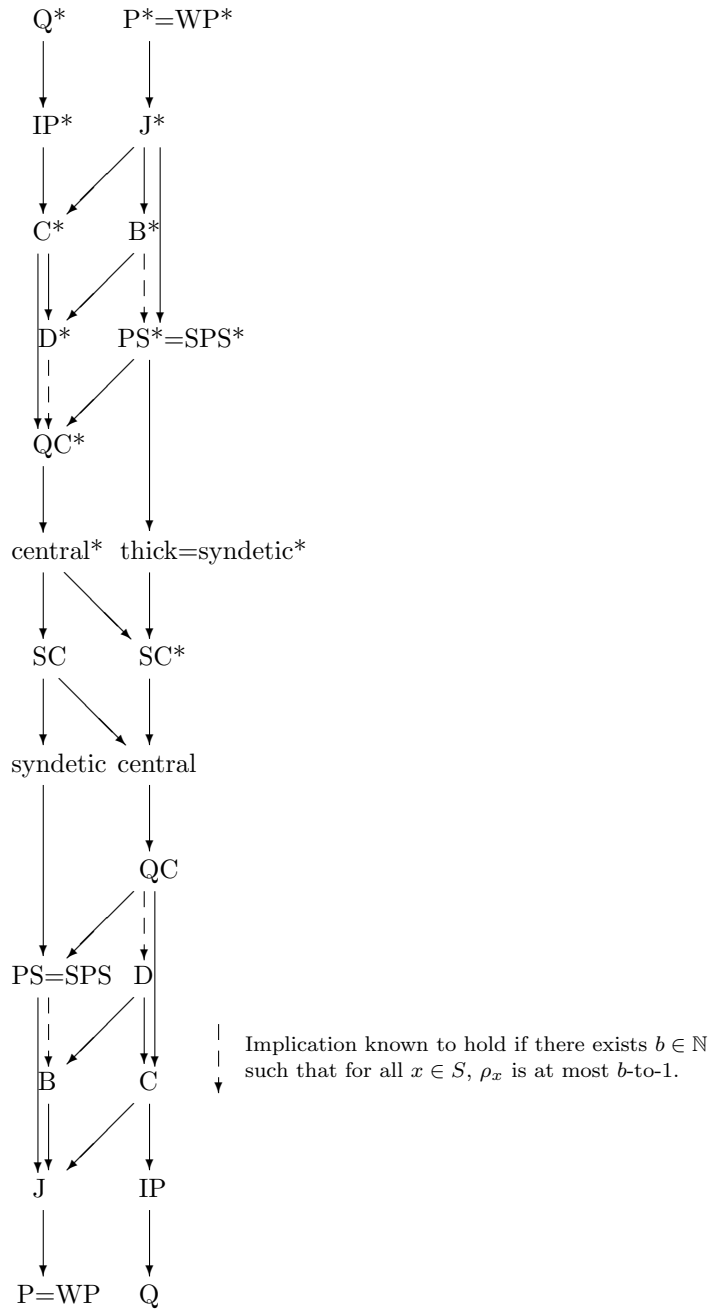


Figure 2: Implications for S commutative.

(5) Let A be the set produced in [22, Theorem 2.1] and let $B = A + 1$. Then B is a J-set which is not a Q-set and not a B-set.

Since A is a C-set, it is also a J-set so B is a J-set. Since $d(A) = 0$, $d(B) = 0$. Finally, we claim that $A \subseteq 2\mathbb{N}$, so that $B \subseteq 2\mathbb{N} + 1$ and therefore B is not a Q-set. To see that $A \subseteq 2\mathbb{N}$, note that in the proof of [22, Theorem 2.1], $B_0 = \{0\}$ and for all $x \in A$, $B_0 \setminus \text{supp}(x) \neq \emptyset$. That is, $0 \notin \text{supp}(x)$.

(6) By [21, Theorem 3.1] pick a set $B \subseteq \mathbb{N}$ such that $d_r(B) = \frac{3}{4}$ and for all $b \in \mathbb{N}$, $d_r(\bigcup_{t=1}^b -t + B) < 1$. Let $A = B \cap (2\mathbb{N} + 1)$. Then A is a B-set which is not piecewise syndetic and is not a Q-set.

By Lemma 3.9, B is not piecewise syndetic so A is not piecewise syndetic. Since $A \subseteq 2\mathbb{N} + 1$, A is not a Q-set. Also $\frac{3}{4} = d_r(B) \leq d_r(A \cup (2\mathbb{N} + 1)) \leq d_r(A) + d_r(2\mathbb{N} + 1) = d_r(A) + \frac{1}{2}$, so A is a B-set.

(7) Let $x_1 = 1$, for $n \in \mathbb{N}$, let $x_{n+1} = \sum_{t=1}^n x_t + n$, and let $A = FS(\langle x_n \rangle_{n=1}^\infty)$. Then A is a D-set which is not piecewise syndetic.

By [1, Theorem 2.21], $\Delta_r^*(\mathbb{N}) \cap \bigcap_{m=1}^\infty \overline{FS(\langle x_n \rangle_{n=m}^\infty)}$ is a semigroup so has an idempotent. Consequently A is a D-set. By [2, Corollary 4.2], A is not piecewise syndetic.

(8) Let $A = \{2^n + 2m - 1 : n, m \in \mathbb{N} \text{ and } m < n\}$. It is immediate that A is piecewise syndetic and not syndetic. Since $A \subseteq 2\mathbb{N} + 1$, it is not a Q-set.

(9) Let $X = \{2^{2^n(2b+1)} + a2^{2^n} + 2^{2^n-2} : a, b, n \in \mathbb{N} \text{ and } a \leq b\}$ and let $A = \{\sum_{i=1}^m x_i : l \in \mathbb{N}, \text{ for each } i \in \{1, 2, \dots, m\}, x_i \in X, \text{ and if } 1 \leq i < m, \text{ then } \max \text{supp}(x_i) < \min \text{supp}(x_{i+1})\}$. Then A is quasi-central and is neither central nor syndetic.

By [25, Theorem 4.4] A is quasi-central and not central. If $y \in A$, then there exist $b, n \in \mathbb{N}$ such that $\max \text{supp}(y) = 2^n(2b + 1) \in 2\mathbb{N}$ so $A \cap \{2^{2^n-1} + m : m, n \in \mathbb{N} \text{ and } m < n\} = \emptyset$. Therefore A is not syndetic.

(10) $2\mathbb{N} + 1$ is syndetic and not a Q-set.

(11) Define $f : \mathbb{N} \rightarrow [-\frac{1}{2}, \frac{1}{2}]$ by $f(x) = \sqrt{x} - \lfloor \sqrt{2x} + \frac{1}{2} \rfloor$. Let $B = \{x \in \mathbb{N} : f(x) \in (0, \frac{1}{2})\}$ and let $A = \{2^n + 2m : n, m \in \mathbb{N} \text{ and } m < n\}$. Then $A \cap B$ is central, not syndetic, and not SC^* .

Pick $p \in \mathbb{N}^*$ such that $\{2^n : n \in \mathbb{N}\} \in p$. We claim that $E(\beta\mathbb{N} + p) \subseteq \overline{A}$. To see this, let $C = \{2^n + m : n, m \in \mathbb{N} \text{ and } m < 2n\}$. It suffices to show that $\beta\mathbb{N} + p \subseteq \overline{C}$, since $2\mathbb{N}$ is a member of any idempotent in $\beta\mathbb{N}$. So let $q \in \beta\mathbb{N}$. Given $m \in \mathbb{N}$, $\{2^n : 2n > m\} \subseteq -m + C$ so $C \in q + p$. Pick a minimal left ideal L of $\beta\mathbb{N}$ such that $L \subseteq \beta\mathbb{N} + p$.

Let $\tilde{f} : \beta\mathbb{N} \rightarrow [-\frac{1}{2}, \frac{1}{2}]$ be the continuous extension of f , let $U = \{q \in \beta\mathbb{N} : \{x \in \mathbb{N} : \tilde{f}(q) < f(x)\} \in q\}$, and let $D = \{q \in \beta\mathbb{N} : \{x \in \mathbb{N} : \tilde{f}(q) > f(x)\} \in q\}$. By [27, Theorem 10.8], U and D are right ideals of $\beta\mathbb{N}$. Pick a minimal right ideal R of $\beta\mathbb{N}$ such that $R \subseteq U$. Let q be the identity of $R \cap L$.

Let π be the projection from \mathbb{R} to the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let $h = \pi \circ f$.

Then by [27, Lemma 10.3], $\tilde{h} : \beta\mathbb{N} \rightarrow \mathbb{T}$ is a homomorphism so $\tilde{f}(q) = 0$. Therefore $B \in q$. Since $L \subseteq \overline{A}$, $A \in q$. Therefore $A \cap B$ is central. Since A is not syndetic, $A \cap B$ is not syndetic.

Suppose that $A \cap B$ is SC^* and pick a minimal left ideal M of $\beta\mathbb{N}$ such that $E(M) \subseteq \overline{A \cap B}$. Pick an idempotent $r \in M \cap D$. Since $\tilde{f}(r) = 0$, and $r \in D$, $B \notin r$, a contradiction.

(12) As in (11), define $f : \mathbb{N} \rightarrow [-\frac{1}{2}, \frac{1}{2}]$ by $f(x) = \sqrt{x} - \lfloor \sqrt{2x} + \frac{1}{2} \rfloor$. Let $A = \{x \in \mathbb{N} : f(x) \in \bigcup_{n=1}^{\infty} (\frac{1}{2n+1}, \frac{1}{2n})\}$. Then A is strongly central, not SC^* .

By [8, Theorem 3.1] with $k = l = 1$ and $\mu_1 = \sqrt{2}$, both A and $\mathbb{N} \setminus A$ are strongly central.

(13) Let $A = \{2^n + 2m : n, m \in \mathbb{N} \text{ and } m < n\}$. Then A is SC^* , not thick, and not syndetic.

Trivially A is neither thick nor syndetic. If $p \in \mathbb{N}^*$ such that $\{2^n : n \in \mathbb{N}\} \in p$, then as in the proof of (11), $E(\beta\mathbb{N} + p) \subseteq \overline{A}$ so A is SC^* .

(14) Let $A = \{2^n + m : n, m \in \mathbb{N} \text{ and } m < n\}$. Then A is thick and not syndetic.

(15) Let A be as in (9) and let $B = 2\mathbb{N} \setminus A$. Then B is central*, not QC^* , and not thick.

We have that A is QC , not central. Since $2\mathbb{N}$ is central*, in fact IP^* , and A is not central, B is central*. Since $2\mathbb{N}$ is not thick, neither is B . Since A is QC , B is not QC^* .

(16) As in (7), let $x_1 = 1$, for $n \in \mathbb{N}$, let $x_{n+1} = \sum_{t=1}^n x_t + n$, and let $A = FS(\langle x_n \rangle_{n=1}^{\infty})$. Then $2\mathbb{N} \setminus A$ is QC^* , not thick, and not D^* .

In the proof of (7) we saw that A is a D -set and not piecewise syndetic. Since A is not piecewise syndetic, $K(\beta\mathbb{N}) \subseteq \overline{\mathbb{N} \setminus A}$ so $\text{cl}K(\beta\mathbb{N}) \subseteq \overline{\mathbb{N} \setminus A}$ and thus $E(\text{cl}K(\beta\mathbb{N})) \subseteq \overline{2\mathbb{N} \setminus A}$. Consequently, $2\mathbb{N} \setminus A$ is QC^* . Since $2\mathbb{N}$ is not thick, neither is $2\mathbb{N} \setminus A$. Since A is a D -set, $2\mathbb{N} \setminus A$ is not D^* .

(17) As in (7), let $x_1 = 1$, for $n \in \mathbb{N}$, let $x_{n+1} = \sum_{t=1}^n x_t + n$, and let $A = FS(\langle x_n \rangle_{n=1}^{\infty})$. Then $\mathbb{N} \setminus A$ is PS^* and not D^* .

In the proof of (7) we saw that A is a D -set and not piecewise syndetic. Since A is not piecewise syndetic, $K(\beta\mathbb{N}) \subseteq \overline{\mathbb{N} \setminus A}$ so $\mathbb{N} \setminus A$ is PS^* . Since A is a D -set, $\mathbb{N} \setminus A$ is not D^* .

(18) Let A be the set produced in [22, Theorem 2.1] which is a C -set and not a B -set. Then $2\mathbb{N} \setminus A$ is D^* , not C^* , and not thick.

Since A is not a B -set, $\Delta_r^*(\mathbb{N}) \subseteq \overline{\mathbb{N} \setminus A}$ so $E(\Delta_r^*(\mathbb{N})) \subseteq \overline{2\mathbb{N} \setminus A}$ and so $2\mathbb{N} \setminus A$ is D^* . Since A is a C -set, $2\mathbb{N} \setminus A$ is not C^* . Since $2\mathbb{N}$ is not thick, $2\mathbb{N} \setminus A$ is not thick.

(19) Let A be the set produced in [22, Theorem 2.1] which is a C -set and not a B -set. Then $\mathbb{N} \setminus A$ is B^* and not C^* .

(20) Let $A = \{\sum_{n \in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$. Then $2\mathbb{N} \setminus A$ is C^* , not IP^* , and not thick.

Since $2\mathbb{N}$ is not thick, neither is $2\mathbb{N} \setminus A$ and $2\mathbb{N} \setminus A$ is trivially not IP^* . We saw in (3) that A is not a P -set and therefore not a J -set so that $J(\mathbb{N}) \subseteq \overline{\mathbb{N} \setminus A}$ and thus $E(J(\mathbb{N})) \subseteq \overline{2\mathbb{N} \setminus A}$. Therefore $2\mathbb{N} \setminus A$ is C^* .

(21) Let $A = \{\sum_{n \in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$ and let $B = \{2^{2n} + m2^n + 1 : m, n \in \mathbb{N} \text{ and } m < n\}$. Then $\mathbb{N} \setminus (A \cup B)$ is J^* , not IP^* , and not P^* .

Trivially $\mathbb{N} \setminus (A \cup B)$ is not IP^* . Since B is a P -set, $\mathbb{N} \setminus (A \cup B)$ is not P^* . We saw in (2) that B is not a J -set. We saw in (3) that A is not a J -set. Therefore by [27, Lemma 14.14.6], $A \cup B$ is not a J -set. Therefore $\mathbb{N} \setminus (A \cup B)$ is J^* .

(22) Let $A = \{2^n - 2^m : n, m \in \mathbb{N} \text{ and } m < n\}$. Then $2\mathbb{N} \setminus A$ is IP^* , not thick, and not Q^* .

Since $2\mathbb{N}$ is not thick neither is $2\mathbb{N} \setminus A$. Since A is a Q -set, $2\mathbb{N} \setminus A$ is not Q^* . It is easy to see that A is not an IP -set, so $2\mathbb{N} \setminus A$ is IP^* .

(23) $2\mathbb{N}$ is Q^* and not thick.

(24) Let $A = \{\sum_{n \in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$. We saw in (3) that A is an IP -set which is not a P -set. So $\mathbb{N} \setminus A$ is P^* and not IP^* .

□

6 The missing implications in Figure 1

We know quite a bit about notions that do not imply any of the other notions except those that follow from the implications in Figure 1. And there is quite a lot more that we do not know.

Each of the right and left versions of Thick, SC^* , Synd, Cntrl, SPS, QC, PS, C, J, IP, and Q, as well as the two sided notions of P and WP does not imply any of the other 52 notions we are considering, unless that implication is forced by the implications in Figure 1. This can be established in each case by considering the examples in Section 5 and [24, Theorems 4.13 and 4.14]. For example, in Section 5, there is a subset of \mathbb{N} which is quasi-central and neither syndetic nor central, so neither rQC nor ℓQC implies any of the properties that lie above it in Figure 1. Further rQC does imply each of rPS , rC , ℓJ , P , rJ , rIP , WP , and rQ . The example of [24, Theorem 4.14(1)] is rSC , hence rQC , and is not $rSPS$. The example of [24, Theorem 4.13(1)] is $rThick$, hence rQC , and is neither ℓQ nor ℓPS , and thus has none of the properties that imply these. That leaves only the question of whether rQC implies any or all of ℓB , rB , or rD . The examples of [24, Theorem 4.13(1) and Theorem 4.14(1)] are in free semigroups, which are not left or right amenable, so do not satisfy ℓSFC nor $rSFC$, so do not satisfy ℓB nor rB .

That almost ends the good news. We do not know whether ℓSC implies any

or all of rCntrl, rQC, rC, rIP, or rQ. A similar statement holds for rSC. All of the properties in Figure 1 above the level of rSC and ℓ SC except ℓD^* , ℓB^* , rD^* , and rB^* imply either rSC or ℓ SC. Thus, for example, if we knew that there is a rC^* set which is not ℓ Cntrl, we would know that there is a rSC set which is not ℓ Cntrl.

We do know that WP^* , rJ^* , and ℓJ^* do not imply any of the notions except those that are forced by the implications in Figure 1 and possibly any or all of ℓB^* , rB^* , ℓD^* , and rD^* . For WP^* , this fact is a consequence of the fact that there is a subset of \mathbb{N} which is WP^* and not IP^* , as well as the fact that if S is any semigroup which is neither left nor right amenable, it will not satisfy rSFC nor ℓ SFC so that S is WP^* but not ℓB , not rB , not ℓD , and not rD . For rJ^* , one needs to note that by [24, Theorems 4.8 and 4.9], there is a subset of the free semigroup on countably many generators which is ℓC and not rJ , so its complement is rJ^* and not ℓC^* .

If S is any semigroup not satisfying rSFC, then any subset of S is rB^* and rD^* (because its complement is not rB nor rD). This fact complicates the search for examples of sets which are, say, rJ^* but not rB^* .

We conclude by considering what implications must hold from the notions of ℓB , rB , ℓD , and rD . Since there are semigroups satisfying rSFC and not ℓ SFC, we have that rB does not imply either ℓB or ℓD . Since there is a subset of \mathbb{N} which is a B set but not a Q set, we see that the only properties that might be implied by rB and do not follow from the implications in Figure 1 are ℓJ , rJ , P , and WP .

The reader is invited to work out what we know about things that must be implied by ℓD . Among the things that we don't know at this stage is whether ℓD implies rIP . We conclude the paper with a proof that it does not. To do this we need to construct a semigroup S and an idempotent in $\Delta_\ell^*(S)$ which has a member which is not an rIP set. The construction is based on an example due to M. Klawe [29, Counterexample 3.5]. Here $\omega = \mathbb{N} \cup \{0\}$.

Theorem 6.1. *Let $S = (\bigoplus_{i=0}^{\infty} \omega) \times \mathbb{N}$. For $\vec{x} = ((x_0, x_1, \dots), m)$ and $\vec{y} = ((y_0, y_1, \dots), n)$, let $\vec{x} + \vec{y} = ((x_0 + \sum_{i=0}^m y_i, x_1 + y_{m+1}, x_2 + y_{m+2}, \dots), m + n)$. Then $(S, +)$ is left and right amenable, right cancellative, not left cancellative, and does not satisfy rSFC.*

Proof. It is easy to verify that S is isomorphic to the semigroup produced in [29, Counterexample 3.5] where all of the assertions of the theorem are proved. \square

It is not hard to show that the semigroup of Theorem 6.1 satisfies ℓ SFC. (We will not need this fact.)

Definition 6.2. Let $T = \mathbb{N} \times \mathbb{N}$ and for $\vec{x} = (x_0, x_1)$ and $\vec{y} = (y_0, y_1)$, define $\vec{x} + \vec{y} = (x_0 + y_0 + y_1, x_1)$. For $m \in \mathbb{N}$, let $K_m = \{1, 2, \dots, 2^{m+1}\} \times \{2^{m+1}\}$.

Theorem 6.3. $(T, +)$ is a semigroup, right cancellative, not left cancellative, does not satisfy rSFC, but does satisfy ℓSFC. In fact, if $r, k \in \mathbb{N}$, $H \subseteq \{1, 2, \dots, k\} \times \{1, 2, \dots, k\}$, $\epsilon > 0$, and $2^r > \frac{k}{\epsilon}$, then for all $\vec{x} \in H$, $|K_r \setminus (K_r + \vec{x})| < \epsilon |K_r|$.

Proof. Given $(x_0, x_1), (y_0, y_1), (z_0, z_1) \in T$, $(x_0, x_1) + (y_0, y_1) + (z_0, z_1) = (x_0 + y_0 + y_1 + z_0 + z_1, x_1)$ computed in either order, so T is associative. Right cancellation is easy to check. And $(1, 1) + (2, 1) = (4, 1) = (1, 1) + (1, 2)$, so left cancellation fails.

In [32, Section 4.22], it is shown that any semigroup which is right cancellative and not left cancellative does not satisfy rSFC. (Paterson assumes that the semigroup is left amenable, but does not use the assumption.)

To verify that T satisfies ℓSFC, it suffices to verify the final assertion. So assume that $r, k \in \mathbb{N}$, $H \subseteq \{1, 2, \dots, k\} \times \{1, 2, \dots, k\}$, $\epsilon > 0$, and $2^r > \frac{k}{\epsilon}$. Let $\vec{x} \in H$. Then $K_r \setminus (K_r + \vec{x}) = \{\vec{y} \in K_r : y_0 \leq x_0 + x_1\}$ so $K_r \setminus (K_r + \vec{x}) = \{1, 2, \dots, x_0 + x_1\} \times \{2^{r+1}\}$ and thus $|K_r \setminus (K_r + \vec{x})| = x_0 + x_1 \leq 2k < \epsilon 2^{r+1} = \epsilon |K_r|$. \square

If one cares, it is easy to check that T is weakly left cancellative, i.e., for all $\vec{x}, \vec{y} \in T$, $\{\vec{z} \in T : \vec{x} + \vec{z} = \vec{y}\}$ is finite. It is probably worth pointing out that T does not have the weak cancellation property referred to on the left side of Figure 1. That is, there does not exist $b \in \mathbb{N}$ such that for all $\vec{x} \in T$, $\lambda_{\vec{x}}$ is at most b -to-1. Indeed, there is no $b \in \mathbb{N}$ such that $\lambda_{(1,1)}$ is b -to-1.

Definition 6.4. For $m \in \mathbb{N}$ and $u \in \{1, 2, \dots, 2^m - 1\}$, $B_{m,u} = \{2^m, 2^m + 1, 2^m + 2, \dots, 2^{m+1} - u\} \times \{2^{m+1}\}$ and $A_{m,u} = \bigcup_{n=m}^{\infty} B_{n,u}$.

Lemma 6.5. Let $m \in \mathbb{N}$ and $u \in \{1, 2, \dots, 2^m - 1\}$. Then $d_\ell(A) \geq \frac{1}{2}$.

Proof. $d_\ell(A)$ is defined by replacing $K \setminus sK$ in the definition of d_r in Definition 3.1 with $K \setminus Ks$. In the current context this means we must show that for $0 < \alpha < \frac{1}{2}$, $H \in \mathcal{P}_f(T)$, and $\epsilon > 0$, there exists $K \in \mathcal{P}_f(T)$ such that for all $\vec{x} \in H$, $|K \setminus (K + \vec{x})| < \epsilon |K|$ and $|A_{m,u} \cap K| \geq \alpha |K|$. So let $0 < \alpha < \frac{1}{2}$, $H \in \mathcal{P}_f(T)$, and $\epsilon > 0$ be given. Pick $k \in \mathbb{N}$ such that $H \subseteq \{1, 2, \dots, k\} \times \{1, 2, \dots, k\}$ and pick $r \in \mathbb{N}$ such that $2^r > \frac{k}{\epsilon}$, $\frac{u}{2^{r+1}} < \frac{1}{2} - \alpha$, and $r \geq m$.

By Theorem 6.3, for all $\vec{x} \in H$, $|K_r \setminus (K_r + \vec{x})| < \epsilon |K_r|$ so we only need to show that $|A_{m,u} \cap K_r| \geq \alpha |K_r|$. We have that $B_{r,u} \subseteq A_{m,u} \cap K_r$ and $|B_{r,u}| = 2^r - u + 1$ so

$$\frac{|B_{r,u}|}{|K_r|} = \frac{2^r - u + 1}{2^{r+1}} > \frac{1}{2} - \frac{u}{2^{r+1}} > \alpha.$$

\square

Lemma 6.6. Let $M = \bigcap_{m=1}^{\infty} \bigcap_{u=1}^{2^m-1} \overline{A_{m,u}}$. Then $M \cap \Delta_\ell^*(T) \neq \emptyset$.

Proof. It suffices to show that

$$\{\overline{A_{m,u}} \cap \Delta_\ell^*(T) : m \in \mathbb{N} \text{ and } u \in \{1, 2, \dots, 2^m - 1\}\}$$

has the finite intersection property. Let $F \in \mathcal{P}_f(\mathbb{N})$ and for $m \in F$, let $\emptyset \neq G_m \subseteq \{1, 2, \dots, 2^m - 1\}$. Let $k = \max F$ and let $w = 2^k - 1$. Then $A_{k,w} \subseteq \bigcap_{m \in F} \bigcap_{u \in G_m} A_{m,u}$ so it suffices to show that $\overline{A_{k,w}} \cap \Delta_\ell^* \neq \emptyset$. By Lemma 6.5, $d_\ell(A_{k,w}) \geq \frac{1}{2}$ so by (the left-right switch of) Lemma 3.5, $\overline{A_{k,w}} \cap \Delta_\ell^* \neq \emptyset$. \square

Theorem 6.7. *Let $M = \bigcap_{m=1}^\infty \bigcap_{u=1}^{2^m-1} \overline{A_{m,u}}$. Then $M \cap \Delta_\ell^*(T)$ is a compact subsemigroup of $(\beta T, \oplus)$, the left topological extension of the operation on T with T contained in its topological center.*

Proof. By the left-right switch of Lemma 3.7, Δ_ℓ^* is a right ideal of $(\beta T, \oplus)$ and is therefore a semigroup. By Lemma 6.6, it suffices to show that M is a subsemigroup of $(\beta T, \oplus)$. So let $p, q \in M$. Let $m \in \mathbb{N}$ and $u \in \{1, 2, \dots, 2^m - 1\}$. We will show that $A_{m,u} \in p \oplus q$ by showing that $A_{m,u} \subseteq \{\vec{x} \in T : A_{m,u} - \vec{x} \in p\}$ so that $\{\vec{x} \in T : A_{m,u} - \vec{x} \in p\} \in q$. (Here $A_{m,u} - \vec{x} = \{\vec{y} \in T : \vec{y} + \vec{x} \in A_{m,u}\}$.) Let $\vec{x} \in A_{m,u}$ and pick $n \geq m$ such that $\vec{x} \in B_{n,u}$. Pick $k \in \{0, 1, \dots, 2^n - u\}$ such that $\vec{x} = (2^n + k, 2^{n+1})$. Let $v = 2^{n+2}$ and let $r = n + 3$. Then $v \in \{1, 2, \dots, 2^r - 1\}$ so $A_{r,v} \in p$.

We claim that $A_{r,v} \subseteq A_{m,u} - \vec{x}$. Let $\vec{y} \in A_{r,v}$ and pick $s \geq r$ such that $\vec{y} \in B_{s,v}$. Pick $t \in \{0, 1, \dots, 2^s - v\}$ such that $\vec{y} = (2^s + t, 2^{s+1})$. Then $\vec{y} + \vec{x} = (2^s + t + 2^n + k + 2^{n+1}, 2^{s+1})$. Now $t + 2^n + k + 2^{n+1} \leq 2^s - v + 2^n + 2^n - u + 2^{n+1} = 2^s - u$ so $\vec{y} + \vec{x} \in B_{s,u} \subseteq A_{m,u}$. \square

Theorem 6.8. *$A_{1,1}$ is an ℓD set which is not a rIP set (and thus not a rC set).*

Proof. By Theorem 6.7 we may pick $p = p \oplus p \in M \cap \Delta_\ell^*(T)$.

Now suppose there is a sequence with all finite products in increasing order of indices in $A_{1,1}$. In particular, there exist $m \leq n$, $\vec{x} \in B_{m,1}$ and $\vec{y} \in B_{n,1}$ such that $\vec{x} + \vec{y} \in A_{1,1}$. Pick $t \in \{0, 1, \dots, 2^m - 1\}$ and $k \in \{0, 1, \dots, 2^n - 1\}$ such that $\vec{x} = (2^m + t, 2^{m+1})$ and $\vec{y} = (2^n + k, 2^{n+1})$. Then $\vec{x} + \vec{y} = (2^m + t + 2^n + k + 2^{n+1}, 2^{m+1})$. This is a contradiction because whenever $(z_0, z_1) \in A_{1,1}$, one has that $z_0 < z_1$. \square

As a consequence of Theorem 6.8 we have that ℓD does not imply either rC or rIP (and by the left-right switch, rD does not imply either ℓC or ℓIP). We still do not know whether ℓD implies any or all of ℓC , ℓJ , P, WP, rJ, or rQ. We do know that $A_{1,1}$ is a rQ set. To see this, for $n \in \mathbb{N}$, let $\vec{x}_n = (\sum_{t=0}^{2^n-2} 2^t, 1)$. If $m < n$, then $\vec{x}_n = \vec{x}_m + (\sum_{t=2^{m-1}}^{2^n-3} 2^t, 2^{2n-2})$ and $(\sum_{t=2^{m-1}}^{2^n-3} 2^t, 2^{2n-2}) \in B_{2n-3,1}$.

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