SMALL SETS SATISFYING THE CENTRAL SETS THEOREM

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Received: , Accepted: , Published:

Abstract

The *Central Sets Theorem* is a powerful theorem, one of whose consequences is that any *central* set in $\mathbb{N}$ contains solutions to any partition regular system of homogeneous linear equations. Since at least one set in any finite partition of $\mathbb{N}$ must be central, any of the consequences of the Central Sets Theorem must be valid for any partition of $\mathbb{N}$. It is a result of Beiglböck, Bergelson, Downarowicz, and Fish that if p is an idempotent in $(\beta\mathbb{N}, +)$ with the property that any member of p has positive Banach density, then any member of p satisfies the conclusion of the Central Sets Theorem. Since all central sets are members of such idempotents, the question naturally arises whether any set satisfying the conclusion of the Central Sets Theorem must have positive Banach density. We answer this question here in the negative.

1. Introduction

In [6] H. Furstenberg introduced the notion of *central* subsets of $\mathbb{N}$ in terms of notions from topological dynamics. He showed that one cell of any finite partition of a $\mathbb{N}$ must contain a central set and proved the original Central Sets Theorem. (Given a set X, we denote by $\mathcal{P}_f(X)$ the set of finite nonempty subsets of X.)

1.1 Theorem. *Let C be a central subset of $\mathbb{N}$. Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let f_i be a sequence in $\mathbb{Z}$. Then there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*
(1) for all n, $\max H_n < \min H_{n+1}$ and
(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n\in F} \big(a_n + \sum_{t\in H_n} f_i(t)\big) \in C$.

Proof. [6, Proposition 8.21]. □

Furstenberg used central sets to prove Rado's Theorem [10] by showing that any central subset of $\mathbb{N}$ contains solutions to all partition regular systems of homogeneous linear equations.

[1] The author acknowledges support received from the National Science Foundation via Grant DMS-0554803.

Based on an idea of V. Bergelson, central sets in $\mathbb{N}$ were characterized quite simply [4] as members of minimal idempotents of $(\beta\mathbb{N}, +)$, and this characterization extended naturally to define central subsets of an arbitrary discrete semigroup S.

What is currently the most general version of the Central Sets Theorem (for commutative semigroups) is the following.

1.2 Theorem. *Let $(S,+)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$, the set of sequences in S. Let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

(2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in C$.

Proof. [5, Theorem 2.2]. □

To derive Theorem 1.1 from Theorem 1.2, note that one may assume that the sequences $f_1, f_2, \ldots, f_l$ in the statement of Theorem 1.1 are distinct. Choose additionally distinct sequences f_k for $k > l$ and let for each $n \in \mathbb{N}$, $G_n = \{f_1, f_2, \ldots, f_n\}$. For $n \in \mathbb{N}$, let $a_n = \alpha(G_n)$ and let $H_n = H(G_n)$.

For some of the motivating results that we will present, it is necessary to describe briefly the algebraic structure of the Stone-Čech compactification. If the reader is willing to accept that the question of whether every subset of $\mathbb{N}$ which satisfies the conclusion of Theorem 1.2 must have positive Banach density is interesting, she may proceed directly to Section 2 where that question is answered.

Given a discrete semigroup $(S,+)$, the Stone-Čech compactification βS of S is the set of ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$, $c\ell A = \overline{A} = \{p \in \beta S : A \in p\}$. The family $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (and a basis for the closed sets) of βS. The operation $+$ extends to βS so that $(\beta S, +)$ is a right topological semigroup (meaning that for each $p \in \beta S$ the funtion $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q + p$ is continuous) with S contained in its topological center (meaning that for each $x \in S$ the funtion $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x + q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, one has that $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$.

As is true of any compact Hausdorff right topological semigroup, βS has a smallest two sided ideal $K(\beta S)$ and there are idempotents in $K(\beta S)$. Such idempotents are said to be *minimal*, and a subset C of S is central if and only if it is a member of a minimal idempotent. The reader is referred to [8] for an elementary introduction to the algebra of βS.

The following notion was originally introduced by Polya in [9], but it is commonly referred to as "Banach density".

1.3 Definition. Let $A \subseteq \mathbb{N}$. Then

$$d^*(A) = \sup\{\alpha \in \mathbb{R} : (\forall k \in \mathbb{N})(\exists n \geq k)(\exists a \in \mathbb{N})(|A \cap \{a+1, a+2, \ldots, a+n\}| \geq \alpha \cdot n)\} .$$

$$\Delta^* = \{p \in \beta\mathbb{N} : (\forall A \in p)(d^*(A) > 0)\}.$$

Since Δ^* is a two sided ideal of $\beta\mathbb{N}$, one has that $K(\beta\mathbb{N}) \subseteq \Delta^*$, and in particular, if C is a central subset of $\mathbb{N}$, then $d^*(C) > 0$. The following result of Beiglböck, Bergelson, Downarowicz, and Fish establishes that a weaker assumption than central yields the conclusion of the original Central Sets Theorem.

1.4 Theorem. *Let $C \subseteq \mathbb{N}$ and assume that C is a member of an idempotent in Δ^*. Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let f_i be a sequence in $\mathbb{Z}$. Then there exist sequences $\langle a_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that*
(1) for all n, $\max H_n < \min H_{n+1}$ and
(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n \in F} \big(a_n + \sum_{t \in H_n} f_i(t)\big) \in C$.

Proof. [2, Theorem 10]. □

In fact, the proof of [2, Theorem 10] is easily modified to show that any member of an idempotent in Δ^* satisfies the conclusion of Thoerem 1.2. It is a result of C. Adams [1, Theorem 2.21] that there is a set C which is a member of an idempotent in Δ^* but $\overline{C}$ misses the closure of the smallest ideal of $\beta\mathbb{N}$ and in particular, C is not central.

One is naturally led by the above results to ask whether any subset of $\mathbb{N}$ which satisfies the conclusion of Theorem 1.2 must in fact have positive Banach density. We show in Section 2 that this is not the case.

We close this introduction with an interesting contrast between members of idempotents in Δ^* and central sets, that is members of idempotents in $K(\beta\mathbb{N})$. Those sets $A \subseteq \mathbb{N}$ such that $\overline{A} \cap K(\beta\mathbb{N}) \neq \emptyset$ are exactly the *piecewise syndetic* subsets of $\mathbb{N}$ by [8, Theorem 4.40] while a set $A \subseteq \mathbb{N}$ has $\overline{A} \cap \Delta^* \neq \emptyset$ if and only if $d^*(A) > 0$ by [8, Theorem 3.11]. If A is piecewise syndetic, then by [8, Theorem 4.43] there is some $x \in \mathbb{N}$ such that $-x + A$ is central. On the other hand, it is a result of Ernst Straus that there exist sets $A \subseteq \mathbb{N}$ with asymptotic density arbitrarily close to 1 (and thus $d^*(A)$ arbitrarily close to 1) such that no translate of A is a member of any idempotent. (See [3, Theorem 2.20].)

2. A small subset of $\mathbb{N}$ satisfying the conclusion of the Central Sets Theorem

We produce in this section a subset of $\mathbb{N}$ with zero Banach density which satisfies the conclusion of Theorem 1.2 applied to the group $(\mathbb{Z},+)$. The construction is based on that of [7, Lemma 5.2]. For $x \in \mathbb{N}$ we denote by $\mathrm{supp}(x)$ the subset of $\omega = \mathbb{N} \cup \{0\}$ such that $x = \sum_{t \in \mathrm{supp}(x)} 2^t$.

2.1 Theorem. *Let $\mathcal{T} = {}^{\mathbb{N}}\mathbb{Z}$, the set of sequences in $\mathbb{Z}$. There is a subset A of $\mathbb{N}$ such that $d^*(A) = 0$ and there exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to \mathbb{N}$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

(2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has $\sum_{i=1}^m \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in A$.

Proof. For $n \in \mathbb{N}$, let $a_n = \min\{t \in \mathbb{N} : (\frac{2^n-1}{2^n})^t \leq \frac{1}{2}\}$ and let $s_n = \sum_{i=1}^n a_i$. (So $s_1 = 1$ and $s_2 = 4$.) Let $b_0 = 0$, let $b_1 = 1$, and for $n \in \mathbb{N}$ and $t \in \{s_n, s_n+1, s_n+2, \ldots, s_{n+1}-1\}$, let $b_{t+1} = b_t + n + 1$. For $k \in \omega$, let $B_k = \{b_k, b_k+1, b_k+2, \ldots, b_{k+1}-1\}$. Let

$$A = \{x \in \mathbb{N} : (\forall k \in \omega)(B_k \setminus \mathrm{supp}(x) \neq \emptyset)\}$$

and let $A' = \{x \in \omega : (\forall k \in \omega)(B_k \setminus \mathrm{supp}(x) \neq \emptyset)\}$ (so $A' = A \cup \{0\}$).

We show first that $d^*(A) = 0$. Notice that for any x and m in $\mathbb{N}$,

$$|A \cap \{x, x+1, x+2, \ldots, x+2^m-1\}| \leq |A' \cap \{0, 1, 2, \ldots 2^m-1\}|\,.$$

Indeed, given any $y \in \{0, 1, 2, \ldots 2^m-1\} \setminus A'$, there is some k with $b_{k+1} \leq m$ such that $B_k \subseteq \mathrm{supp}(y)$ and there is a unique $z(y) \in \{x, x+1, x+2, \ldots, x+2^m-1\}$ such that the rightmost m bits in the binary representation of $z(y)$ are equal to those of y and so $B_k \subseteq \mathrm{supp}\big(z(y)\big)$. Further, if $y \neq y'$, then $z(y) \neq z(y')$.

Let $x, m \in \mathbb{N}$, let $k = s_{m+1}$ and let $l \geq 2^{b_k}$. We shall show that

$$\frac{|A \cap \{x, x+1, x+2, \ldots, x+l-1\}|}{l} < \left(\frac{1}{2}\right)^m.$$

Pick $r \in \mathbb{N}$ such that $2^{r-1} \leq l < 2^r$. Then

$$|A \cap \{x, x+1, \ldots, x+l-1\}| \leq |A \cap \{x, x+1, \ldots, x+2^r-1\}| \leq |A' \cap \{0, 1, \ldots, 2^r-1\}|$$

so

$$\frac{|A \cap \{x, x+1, x+2, \ldots, x+l\}|}{l} \leq \frac{|A' \cap \{0, 1, 2, \ldots, 2^r-1\}|}{2^{r-1}}\,.$$

Now

$$\begin{aligned} |A' \cap \{0, 1, 2, \ldots, 2^r-1\}| &= \textstyle\sum_{t=0}^{2^{r-b_k}-1} |A' \cap \{t2^{b_k}, t2^{b_k}+1, \ldots, (t+1)2^{b_k}-1\}| \\ &\leq \textstyle\sum_{t=0}^{2^{r-b_k}-1} |A' \cap \{0, 1, \ldots, 2^{b_k}-1\}| \\ &= 2^{r-b_k} \cdot |A' \cap \{0, 1, \ldots, 2^{b_k}-1\}| \end{aligned}$$

so

$$\frac{|A' \cap \{0,1,2,\ldots,2^r-1\}|}{2^{r-1}} \leq \frac{2^{r-b_k}\cdot|A'\cap\{0,1,\ldots,2^{b_k}-1\}|}{2^{r-1}} = \frac{|A'\cap\{0,1,\ldots,2^{b_k}-1\}|}{2^{b_k-1}}.$$

We have that $|A'\cap\{0,1,\ldots,2^{b_k}-1\}| = \prod_{t=0}^{k-1}(2^{b_{t+1}-b_t}-1)$ and $2^{b_k-1} = \frac{1}{2}\prod_{t=0}^{k-1}2^{b_{t+1}-b_t}$ so

$$\begin{aligned}\frac{|A'\cap\{0,1,\ldots,2^{b_k}-1\}|}{2^{b_k-1}} &= 2\cdot\prod_{t=0}^{k-1}\left(\frac{2^{b_{t+1}-b_t}-1}{2^{b_{t+1}-b_t}}\right)\\ &= 2\cdot\frac{2^1-1}{2^1}\cdot\prod_{n=1}^{m}\prod_{t=s_n}^{s_{n+1}-1}\left(\frac{2^{b_{t+1}-b_t}-1}{2^{b_{t+1}-b_t}}\right)\\ &= \prod_{n=1}^{m}\left(\frac{2^{n+1}-1}{2^{n+1}}\right)^{a_{n+1}}\\ &\leq \left(\frac{1}{2}\right)^m.\end{aligned}$$

Now we show that A satisfies the conclusion of Theorem 1.2. First note that if $n,k\in\mathbb{N}$ and and $b_{k+1}-b_k>n$, then whenever $z_1,z_2,\ldots,z_n\in\mathbb{N}$, there must exist $r\in B_k$ such that for all $t\in\{1,2,\ldots,n\}$, $B_k\setminus\mathrm{supp}(2^r+z_t)\neq\emptyset$. Indeed, if $r\in B_k$, $z\in\mathbb{N}$, and $B_k\subseteq\mathrm{supp}(2^r+z)$ then $\mathrm{supp}(z)\cap B_k=B_k\setminus\{r\}$. Consequently

$$|\{r\in B_k:\ \text{there is some } i\in\{1,2,\ldots,n\} \text{ with } B_k\subseteq\mathrm{supp}(2^r+z_i)\}|\leq n\,.$$

Now we claim that

$(*)$ for each $n,m\in\mathbb{N}$ and each $F\in\mathcal{P}_f(\mathcal{T})$, there exist $d\in\mathbb{N}$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that $\min H>m$ and for all $f\in F$, $d+\sum_{t\in H}f(t)\in A\cap\mathbb{N}2^n$.

To see this, let $r=|F|$ and pick k such that $b_{k+1}-b_k>r$ and $b_k>n$. Pick $H\in\mathcal{P}_f(\mathbb{N})$ such that $\min H>m$ and for all $f\in F$, $\sum_{t\in H}f(t)\in\mathbb{Z}2^{b_k}$. (Choose an infinite subset C of $\mathbb{N}$ such that for all $s,t\in C$ and all $f\in F$, $f(s)\equiv f(t) \pmod{2^{b_k}}$. Then pick $H\subseteq C$ such that $\min H>m$ and $|H|=2^{b_k}$.) Pick $c\in\mathbb{N}2^{b_k}$ such that for all $f\in F$, $c+\sum_{t\in H}f(t)>0$.

Let $l=\max\bigcup\{\mathrm{supp}\big(c+\sum_{t\in H}f(t)\big):f\in F\}$ and pick j such that $l<b_j$. Pick $r_0\in B_k$ such that $B_k\setminus\mathrm{supp}\big(2^{r_0}+c+\sum_{t\in H}f(t)\big)\neq\emptyset$ for each $f\in F$. Inductively for $i\in\{1,2,\ldots,j-k\}$, pick $r_i\in B_{k+i}$ such that $B_{k+i}\setminus\mathrm{supp}\big(2^{r_i}+\sum_{t=0}^{i-1}2^{r_t}+c+\sum_{t\in H}f(t)\big)\neq\emptyset$ for each $f\in F$. Let $d=c+\sum_{i=0}^{j-k}2^{r_i}$. Then $(*)$ is established.

Now we define $\alpha(F)\in\mathbb{N}$ and $H(F)\in\mathcal{P}_f(\mathbb{N})$ for $F\in\mathcal{P}_f(\mathcal{T})$ inductively on $|F|$. If $F=\{f\}$, pick $\alpha(F)\in\mathbb{N}$ and $H(F)\in\mathcal{P}_f(\mathbb{N})$ by $(*)$ such that $\alpha(F)+\sum_{t\in H(F)}f(t)\in A$. Now let $F\in\mathcal{P}_f(\mathcal{T})$ with $|F|>1$ and assume that we have defined $\alpha(G)$ and $H(G)$ for all G such that $\emptyset\neq G\subsetneq F$ so that

(1) $\alpha(G)+\sum_{t\in H(G)} f(t)\in A$ for each $f\in G$ and

(2) if $K\subsetneq G$, then

(a) $\max H(K)<\min H(G)$ and

(b) there exists $k\in\mathbb{N}$ such that for all $f\in K$ and all $g\in G$,

$$\max\operatorname{supp}\big(\alpha(K)+\textstyle\sum_{t\in H(K)} f(t)\big)<b_k<\min\operatorname{supp}\big(\alpha(G)+\sum_{t\in H(G)} g(t)\big)\,.$$

Let $m=\max\bigcup\{H(G):\emptyset\neq G\subsetneq F\}$ and pick $k\in\mathbb{N}$ such that for all $G\in\mathcal{P}_f(\mathcal{T})$ with $G\subsetneq F$ and all $f\in G$, $\max\operatorname{supp}\big(\alpha(G)+\sum_{t\in H(G)} f(t)\big)<b_k$. Pick by $(*)$ some $H(F)\in\mathcal{P}_f(\mathbb{N})$ and $\alpha(F)\in\mathbb{N}$ such that $\min H(F)>m$ and for all $f\in F$, $\alpha(F)+\sum_{t\in H(F)} f(t)\in A\cap\mathbb{N}2^{b_k+1}$.

To verify that α and H are as required for Theorem 1, let $m\in\mathbb{N}$, let

$$G_1,G_2,\ldots,G_m\in\mathcal{P}_f(\mathcal{T})\,,$$

and assume that $G_1\subsetneq G_2\subsetneq\ \ldots\ \subsetneq G_m$, and for each $i\in\{1,2,\ldots,m\}$, $f_i\in G_i$. We claim that $\sum_{i=1}^m\big(\alpha(G_i)+\sum_{t\in H(G_i)} f_i(t)\big)\in A$. Suppose instead one has some $k\in\mathbb{N}$ such that $B_k\subseteq \operatorname{supp}\Big(\sum_{i=1}^m\big(\alpha(G_i)+\sum_{t\in H(G_i)} f_i(t)\big)\Big)$. Then there is some i such that $B_k\subseteq\operatorname{supp}\big(\alpha(G_i)+\sum_{t\in H(G_i)} f_i(t)\big)$, contradicting hypothesis (1) of the construction. □

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