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Sets satisfying the Central Sets Theorem

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Abstract. *Central* subsets of a discrete semigroup S have very strong combinatorial properties which are a consequence of the *Central Sets Theorem*. We investigate here the class of semigroups that have a subset with zero Følner density which satisfies the conclusion of the Central Sets Theorem. We show that this class includes any direct sum of countably many finite abelian groups as well as any subsemigroup of $(\mathbb{R},+)$ which contains $\mathbb{Z}$. We also show that if S and T are in this class and either both are left cancellative or T has a left identity, then $S\times T$ is in this class. We also extend a theorem proved in [3], which states that, if p is an idempotent in $\beta\mathbb{N}$ whose members have positive density, then every member of p satisfies the Central Sets Theorem. We show that this holds for all commutative semigroups. Finally, we provide a simple elementary proof of the fact that any commutative semigroup satisfies the Strong Følner Condition.

1. Introduction

Given a discrete semigroup $(S,\cdot)$, the operation can be extended to the Stone-Čech compactification βS of S so that $(\beta S,\cdot)$ is a right topological semigroup (meaning that for any $p\in\beta S$, the function $\rho_p:\beta S\to\beta S$ defined by $\rho_p(q)=q\cdot p$ is continuous) with S contained in its topological center (meaning that for any $x\in S$, the function $\lambda_x:\beta S\to\beta S$ defined by $\lambda_x(q)=x\cdot q$ is continuous). Any compact right topological semigroup T has a smallest two sided ideal denoted $K(T)$ and there are idempotents in $K(T)$. We shall present a brief introduction to the algebraic structure of $(\beta S,\cdot)$ in Section 2.

1.1 Definition. Let S be a discrete semigroup and let $C\subseteq S$. The set C is *central* if and only if there is an idempotent $p\in K(\beta S)\cap c\ell C$.

The original Central Sets Theorem was proved by Furstenberg in [10] (using a different but equivalent definition of *central*). Given a set X we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X. We let $\mathbb{N}$ be the set of positive integers.

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1.2 Theorem (Furstenberg). *Let C be a central subset of $\mathbb{N}$. Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let f_i be a sequence in $\mathbb{Z}$. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) for all n, $\max H_n < \min H_{n+1}$ and

(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n \in F} \big(a_n + \sum_{t \in H_n} f_i(t)\big) \in C$.

Proof. [10, Proposition 8.21]. □

This version of the Central Sets Theorem was already strong enough to derive several combinatorial consequences such as Rado's Theorem [23]. Subsequently, several incremental strengthenings were found. (See [6] for a listing of these.) What is currently the most general version of the Central Sets Theorem (for commutative semigroups) is the following. (There is also a version for noncommutative semigroups. See [6].)

1.3 Theorem. *Let $(S, +)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$, the set of sequences in S. Let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

(2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in C$.

Proof. [6, Theorem 2.2]. □

We introduce a name for sets satisfying the conclusion of the Central Sets Theorem.

1.4 Definition. Let $(S, +)$ be a commutative semigroup, let $C \subseteq S$, and let $\mathcal{T} = {}^{\mathbb{N}}S$. The set C is a *C-set* if and only if there exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that

(1) if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

(2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in C$.

Central sets are easy to work with because, for example, if p is an idempotent in the smallest ideal of the right topological semigroup T, U is a semigroup, and $\varphi : T \to U$ is a surjective homomorphism, then U has a smallest ideal and $\varphi(p)$ is an idempotent in that smallest ideal. However, from a combinatorial viewpoint, C-sets are the objects that matter. They are sets which, in $\mathbb{N}$ and $\mathbb{Z}$ and many other commutative semigroups, contain solutions to all non-trivial partition regular systems of homogeneous equations as well as having the other myriads of properties that are a consequence of the Central Sets Theorem. (See Theorem 2.8 for example.)

We shall be interested in showing that the existence of C-sets which are not central is common. In the process, we shall be concerned with a generalization of the following version of density. While this notion was introduced by Polya in [22], it is commonly called "Banach density".

1.5 Definition. Let $A \subseteq \mathbb{N}$. Then

$$d^*(A) = \sup\{\alpha \in [0,1] : (\forall k \in \mathbb{N})(\exists n \geq k)(\exists a \in \mathbb{N})(|A \cap \{a, a+1, \ldots, a+n-1\}| \geq \alpha \cdot n)\},$$

$$\text{and } \Delta^* = \{p \in \beta\mathbb{N} : (\forall A \subseteq \mathbb{N})(p \in c\ell(A) \Rightarrow d^*(A) > 0)\}.$$

Since Δ^* is a two sided ideal of $(\beta\mathbb{N}, +)$, one has that if C is a central subset of $\mathbb{N}$, then $d^*(C) > 0$. And the following result establishes that a set need not be central in order to satisfy the conclusion of the original Central Sets Theorem. (It is a consequence of [1, Theorem 2.21], due to C. Adams, that there are idempotents in $\Delta^* \setminus c\ell\big(K(\beta\mathbb{N})\big)$.)

1.6 Theorem (Beiglböck, Bergelson, Downarowicz, and Fish). *Let $C \subseteq \mathbb{N}$ and assume that there is an idempotent in $\Delta^* \cap c\ell(C)$. Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let f_i be a sequence in $\mathbb{Z}$. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) for all n, $\max H_n < \min H_{n+1}$ and

(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n \in F} \big(a_n + \sum_{t \in H_n} f_i(t)\big) \in C$.

Proof. [3, Theorem 11]. $\square$

The question then naturally arose as to whether any subset C of $\mathbb{N}$ which satisfies the conclusion of Theorem 1.2 must satisfy $d^*(C) > 0$. This question was answered in the negative in [13, Theorem 2.1], where it was shown that there is a C-set $C \subseteq \mathbb{N}$ with $d^*(C) = 0$.

We will be interested in this paper in seeing how widespread such a phenomenon is. In order to even ask this question, we need an appropriate generalization of the notion of Banach density. We believe that the notion of *Følner density*, which is defined for every semigroup satisfying the *Strong Følner Condition* provides such a generalization. (By [2, Theorem 4], every commutative semigroup satisfies the Strong Følner Condtition. See Section 7 for an elementary proof of this fact.)

1.7 Definition. A semigroup $(S,\cdot)$ satisfies the *Strong Følner Condition* (SFC) if and only if $\big(\forall H\in\mathcal{P}_f(S)\big)\big(\forall\epsilon>0\big)\big(\exists K\in\mathcal{P}_f(S)\big)\big(\forall s\in H\big)\big(|K\bigtriangleup sK|<\epsilon\cdot|K|\big)$.

The semigroup $(S,\cdot)$ satisfies the *Følner Condition* (FC) if and only if $\big(\forall H\in\mathcal{P}_f(S)\big)\big(\forall\epsilon>0\big)\big(\exists K\in\mathcal{P}_f(S)\big)\big(\forall s\in H\big)\big(|sK\setminus K|<\epsilon\cdot|K|\big)$.

Notice that (using an argument from [21]) for any $K\in\mathcal{P}_f(S)$ and any $s\in S$, $|K\setminus sK|+|K\cap sK|=|K|\geq|sK|=|sK\setminus K|+|K\cap sK|$ so $|K\setminus sK|\geq|sK\setminus K|$ and equality holds if s is left cancelable. Thus, one has that SFC implies FC and is equivalent to the apparently weaker statement

$$\big(\forall H\in\mathcal{P}_f(S)\big)\big(\forall\epsilon>0\big)\big(\exists K\in\mathcal{P}_f(S)\big)\big(\forall s\in H\big)\big(|K\setminus sK|<\epsilon\cdot|K|\big)\,.$$

(The converse fails. Any finite left zero semigroup, that is a semigroup S in which $ab=a$ for all a in S, satisfies FC but not SFC. See [21, Section 4.22] for a description of the relationship among the notions of FC, SFC, and left amenability.)

We will follow in this paper the custom of writing arbitrary (not necessarily commutative) semigroups multiplicatively and semigroups that are assumed to be commutative additively.

1.8 Definition. Let $(S,\cdot)$ be a semigroup satisfying SFC.

(a) For $A\subseteq S$, $d(A)=\sup\{\alpha\in[0,1]:\big(\forall H\in\mathcal{P}_f(S)\big)(\forall\epsilon>0)\big(\exists K\in\mathcal{P}_f(S)\big)$ $\big((\forall s\in H)(|K\setminus sK|<\epsilon\cdot|K|)\text{ and }|A\cap K|\geq\alpha\cdot|K|\big)\}$.

(b) $\Delta^*(S)=\{p\in\beta S:(\forall A\subseteq S)(p\in c\ell(A)\Rightarrow d(A)>0)\}$.

It may be that A is contained in two relevant semigroups. In such an event, we will write $d_S(A)$ instead of $d(A)$ to emphasize that the density is computed in terms of S.

We shall refer to $d(A)$ as the *Følner density* of A. (In [15, Section 4], where we were dealing with several different notions of density, it was denoted by $d_{\text{Fø}}(A)$.) We observe now that for subsets of $\mathbb{N}$, the Følner density of a set is equal to its Banach density.

1.9 Theorem. *Let $A\subseteq\mathbb{N}$. Then $d(A)=d^*(A)$.*

Proof. Let $\delta=d^*(A)$ and let $\mu=d(A)$. To see that $\mu\geq\delta$, let $\alpha\in[0,1]$ such that

$$(\forall k\in\mathbb{N})(\exists n\geq k)(\exists a\in\mathbb{N})(|A\cap\{a,a+1,\ldots,a+n-1\}|\geq\alpha\cdot n)\,.$$

Let $H\in\mathcal{P}_f(\mathbb{N})$ and let $\epsilon>0$. Let $l=\max H$ and pick $n>\frac{l}{\epsilon}$ and $a\in\mathbb{N}$ such that $|A\cap\{a,a+1,\ldots,a+n-1\}|\geq\alpha\cdot n$. Let $K=\{a,a+1,\ldots,a+n-1\}$. Given $s\in H$, $K\setminus(s+K)=\{a,a+1,\ldots,a+s-1\}$ so $|K\setminus(s+K)|=s<\epsilon\cdot|K|$. Thus $\alpha\leq\mu$.

Now suppose that $\mu > \delta$ and pick α and γ such that $\mu > \alpha > \gamma > \delta$. Since $\delta < \gamma$, pick $k \in \mathbb{N}$ such that for all $n \geq k$ and all $a \in \mathbb{N}$, $|A \cap \{a, a+1, \ldots, a+n-1\}| < \gamma \cdot n$. Let $H = \{1, 2, \ldots, k\}$ and let $\epsilon = (\alpha - \gamma)/2k$. Pick $K \in \mathcal{P}_f(\mathbb{N})$ such that for all $s \in H$, $|K \setminus (s+K)| < \epsilon \cdot |K|$ and $|A \cap K| \geq \alpha \cdot |K|$. For $s \in \mathbb{N}$, let $-s+K = \{t \in \mathbb{N} : s+t \in K\}$. Notice that, since $(\mathbb{N}, +)$ is cancellative, one has that for any $s \in H$, $|K \setminus (-s+K)| \leq |(s+K) \setminus K|$ and we have already seen that $|(s+K) \setminus K| \leq |K \setminus (s+K)|$.

Let $L = \{x \in K : (\exists a \in \mathbb{N})(x \in \{a, a+1, \ldots, a+k-1\} \subseteq K)\}$. Since L is the union of blocks of length at least k, we have that $|A \cap L| < \gamma \cdot |L| \leq \gamma \cdot |K|$. Also, $K \setminus L \subseteq \bigcup_{s=1}^{k-1} \big(K \setminus (-s+K)\big)$ so $|K \setminus L| \leq (k-1) \cdot \epsilon \cdot |K| < \frac{\alpha-\gamma}{2} \cdot |K|$. Thus $\alpha \cdot |K| \leq |A \cap K| \leq |A \cap L| + |K \setminus L| < \gamma \cdot |K| + \frac{\alpha-\gamma}{2} \cdot |K| < \alpha \cdot |K|$, a contradiction. □

The following generalization of Theorem 1.6 provides motivation for our search for small C-sets, that is C sets with Følner density equal to 0. We shall present the proof of this theorem in Section 6.

1.10 Theorem.

(a) Let $(S, \cdot)$ be a left cancellative semigroup which satisfies SFC. Then $\Delta^*(S)$ is a two sided ideal of $(\beta S, \cdot)$ so if C is a central subset of S, then $d(C) > 0$.

(b) If $(S, +)$ is any commutative semigroup and if E is a subset of S for which $\Delta^*(S) \cap c\ell(E)$ contains an idempotent, then E is a C-set.

In Section 3 we shall show that if S and T are infinite left cancellative semigroups satisfying SFC, A is a C-set in S, B is a C-set in T, and either $d(A) = 0$, or $d(B) = 0$, then $A \times B$ is a C-set in $S \times T$ with $d(A \times B) = 0$.

In Section 4 we shall show that if S is any subsemigroup of $(\mathbb{R}, +)$ containing $\mathbb{Z}$, then there is a C-set in S which has zero Følner density. In Section 5 we shall show that the same conclusion applies to any direct sum of countably many finite abelian groups. In Section 7 we shall provide a simple elementary proof that any commutative semigroup satisfies SFC.

2. Preliminaries

We begin with our promised presentation of some details about the algebraic structure of βS.

Given a discrete semigroup $(S, \cdot)$, we take the points of βS to be the ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S$. Given $A \subseteq S$, $c\ell(A) = \overline{A} = \{p \in \beta S : A \in p\}$. Thus $A \subseteq S$ is central if and only if there is an idempotent $p \in K(\beta S)$ such that $A \in p$. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$ where $x^{-1}A = \{y \in S : xy \in A\}$. If the operation is written additively, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$ where $-x + A = \{y \in S : x + y \in A\}$. Notice that, while in this case we write the operation on βS additively, $(\beta S, +)$ is very unlikely to be commutative. See [14] for an elementary introduction to the algebra of βS and for any unfamiliar details.

The concept of density is closely related to that of left invariant means.

2.1 Definition. Let S be a discrete semigroup. Then $l_\infty(S)$ will denote the real Banach space of bounded real-valued functions from S to $\mathbb{R}$ with the uniform norm. A *mean* on S is an element $\mu \in l_\infty(S)^*$ satisfying $\|\mu\| = 1$ and $\mu(f) \geq 0$ for every $f \geq 0$ in $l_\infty(S)$. A *left invariant mean* on S is a mean μ on S with the property that $\mu(f \circ \lambda_s) = \mu(f)$ for every $f \in l_\infty(S)$ and every $s \in S$. We shall denote the set of left invariant means on S by $LIM(S)$.

Now $l_\infty(S)$ can be identified with the Banach space of continuous real-valued functions defined on βS. Hence, by the Riesz Representation Theorem, $l_\infty(S)^*$ can be identified with the space of real-valued regular Borel measures defined on βS. (If $\eta \in l_\infty(S)^*$ corresponds to the real-valued regular Borel measure μ on βS and $A \subseteq S$, then $\eta(\chi_A) = \mu(\overline{A})$.) So an element $\mu \in LIM(S)$ can be regarded as a regular Borel probability measure defined on βS with the property that $\int f(st)d\mu(t) = \int f(t)d\mu(t)$ for every continuous $f : \beta S \to \mathbb{R}$ and every $s \in S$. This is easily seen to be equivalent to the condition that $\mu(\overline{s^{-1}A}) = \mu(\overline{A})$ for every $A \subseteq S$ and every $s \in S$. In this paper, we shall regard left invariant means as measures of this kind. S is said to be *left amenable* if $LIM(S) \neq \emptyset$. In [2] Argabright and Wilde showed that any semigroup satisfying SFC is left amenable.

2.2 Lemma. *Let S be a semigroup which satisfies SFC. For every $A \subseteq S$, there exists $\mu \in LIM(S)$ such that $d(A) = \mu(\overline{A})$.*

Proof. This was shown in the proof of [16, Theorem 2.14]. □

We now introduce some notation from [6].

2.3 Definition. Let $(S, +)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$.

(a) A set $A \subseteq S$ is a *J-set* if and only if whenever $F \in \mathcal{P}_f(\mathcal{T})$, there exist $d \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $d + \sum_{t \in H} f(t) \in A$.

(b) $J(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a } J\text{-set})\}$.

2.4 Lemma. *Let $(S,+)$ be a commutative semigroup, let $\mathcal{T} = {}^{\mathbb{N}}S$, and let A be a J-set in S. Then for all $F \in \mathcal{P}_f(\mathcal{T})$ and all $r \in \mathbb{N}$, there exist $d \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > r$ and for all $f \in F$, $d + \sum_{t\in H} f(t) \in A$.*

Proof. Let $F \in \mathcal{P}_f(\mathcal{T})$ and $r \in \mathbb{N}$. For $f \in F$, define $g_f \in \mathcal{T}$ by $g_f(t) = f(r+t)$ for all $t \in \mathbb{N}$. Pick $d \in S$ and $K \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in F$, $d + \sum_{t\in K} g_f(t) \in A$. Let $H = r + K$. $\square$

The following is a consequence of [6, Theorem 3.8]. We present the proof because the commutative case is much simpler than the general version established there.

2.5 Theorem. *Let $(S,+)$ be a commutative semigroup, let $\mathcal{T} = {}^{\mathbb{N}}S$, and let $A \subseteq S$. Then A is a C-set if and only if there is an idempotent $p \in \overline{A} \cap J(S)$.*

Proof. Sufficiency. Let $A^\star = \{x \in A : -x + A \in p\}$. By [14, Lemma 4.14] if $x \in A^\star$, then $-x + A^\star \in p$.

We define $\alpha(F)$ and $H(F)$ inductively on $|F|$ such that

(1) if $\emptyset \neq G \subsetneq F$, then $\max H(G) < \min H(F)$ and

(2) if $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m = F$, and $(\forall i \in \{1,2,\ldots,m\})(f_i \in G_i)$, then $\sum_{i=1}^m \big(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)\big) \in A^\star$.

If $F = \{f\}$, since $A^\star$ is a J-set, pick $\alpha(F) \in S$ and $H(F) \in \mathcal{P}_f(\mathbb{N})$ such that $\alpha(F) + \sum_{t\in H(F)} f(t) \in A^\star$.

Now assume that $|F| > 1$ and $\alpha(K)$ and $H(K)$ have been chosen for all K with $\emptyset \neq K \subsetneq F$. Let $r = \max \bigcup\{H(K) : \emptyset \neq K \subsetneq F\}$. Let

$$\begin{aligned} B = A^\star \cap \bigcap \Big\{ -\Big(\textstyle\sum_{i=1}^m \big(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)\big)\Big) + A^\star : m \in \mathbb{N}\,, \\ G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})\,,\ G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m \subsetneq F, \text{ and} \\ (\forall i \in \{1,2,\ldots,m\})(f_i \in G_i)\Big\}\,. \end{aligned}$$

Then $B \in p$ so pick by Lemma 2.4, $\alpha(F) \in S$ and $H(F) \in \mathcal{P}_f(\mathbb{N})$ such that $\min H(F) > r$ and for each $f \in F$, $\alpha(F) + \sum_{t\in H(F)} f(t) \in B$.

Necessity. Let $\mathcal{T} = {}^{\mathbb{N}}S$. Pick α and H as guaranteed by the definition of C-set. For $F \in \mathcal{P}_f(\mathcal{T})$ let

$$\begin{aligned} T_F = \Big\{ \textstyle\sum_{i=1}^m \big(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)\big) : m \in \mathbb{N}\,, G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})\,, \\ F \subsetneq G_1 \subsetneq \ldots \subsetneq G_m, \text{ and} \\ (\forall i \in \{1,2,\ldots,m\})(f_i \in G_i)\Big\}\,. \end{aligned}$$

Then each $T_F \neq \emptyset$. Let $\mathbf{Q} = \bigcap_{F\in\mathcal{P}_f(\mathcal{T})} \overline{T_F}$. We show first that $\mathbf{Q}$ is a semigroup.

Given $F, G \in \mathcal{P}_f(\mathcal{T})$, $T_{F\cup G} \subseteq T_F \cap D_G$ and so $\mathbf{Q} \neq \emptyset$. By [14, Theorem 4.20], to see that $\mathbf{Q}$ is a semigroup, it suffices to show that

$$\big(\forall F \in \mathcal{P}_f(\mathcal{T})\big)(\forall x \in T_F)\big(\exists K \in \mathcal{P}_f(\mathcal{T})\big)(T_K \subseteq -x + T_F)$$

so let $F \in \mathcal{P}_f(\mathcal{T})$ and let $x \in T_F$. Pick m, $G_1, G_2, \ldots, G_m$, and $f_1, f_2, \ldots, f_m$ as in the definition of T_F so that $x = \sum_{i=1}^m \big(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)\big)$. Then $T_{G_m} \subseteq -x + T_F$.

To complete the proof we show, using an argument due to Furstenberg and Katznelson in [12], that $K(\mathbf{Q}) \subseteq \overline{A} \cap J(S)$, so that any idempotent in $K(\mathbf{Q})$ establishes the result. We have that each $T_F \subseteq A$ so $\mathbf{Q} \subseteq \overline{A}$. Now let $p \in K(\mathbf{Q})$ and let $B \in p$. We need to show that B is a J-set, so let $F \in \mathcal{P}_f(\mathcal{T})$ be given. Let $k = |F|$ and write $F = \{f_1, f_2, \ldots, f_k\}$. Let $\mathcal{D} = \{G \in \mathcal{P}_f(\mathcal{T}) : F \subseteq G\}$ and note that $\mathbf{Q} = \bigcap_{G\in\mathcal{D}} \overline{T_G}$.

Let $Y = \times_{i=1}^k \beta S$. By [14, Theorem 2.22] Y is a right topological semigroup and if $\vec{x} \in \times_{i=1}^k S$, then $\lambda_{\vec{x}}$ is continuous. For $G \in \mathcal{D}$ let

$$I_G = \big\{\vec{x} \in \times_{i=1}^k T_G : (\exists d \in S)\big(\exists L \in \mathcal{P}_f(\mathbb{N})\big) \\ \big(\vec{x} = \big(d + \textstyle\sum_{t\in L} f_1(t), \ldots, d + \sum_{t\in L} f_k(t)\big)\big)\big\}$$

and let $E_G = I_G \cup \{(d, d, \ldots, d) : d \in T_G\}$. Let $I = \bigcap_{G\in\mathcal{D}} c\ell_Y I_G$ and let $E = \bigcap_{G\in\mathcal{D}} c\ell_Y E_G$. We claim that E is a subsemigroup of $\times_{i=1}^l \mathbf{Q}$ and that I is an ideal of E.

Trivially $E \subseteq \times_{i=1}^l \mathbf{Q}$. Given $G_1, G_2 \in \mathcal{D}$ we have that $I_{G_1\cup G_2} \subseteq I_{G_1} \cap I_{G_2}$ so to see that $I \neq \emptyset$ it suffices to let $G \in \mathcal{D}$ and show that $I_G \neq \emptyset$. Pick $G_1, G_2 \in \mathcal{D}$ such that $G \subsetneq G_1 \subsetneq G_2$. Let $L = H(G_2)$. Let $d = \alpha(G_1) + \sum_{t\in H(G_1)} f_1(t) + \alpha(G_2)$. Then $\big(d + \sum_{t\in L} f_1(t), \ldots, d + \sum_{t\in L} f_k(t)\big) \in I_G$.

Now let $\vec{q}, \vec{r} \in E$. We show that $\vec{q} + \vec{r} \in E$ and, if either $\vec{q} \in I$ or $\vec{r} \in I$, then $\vec{q} + \vec{r} \in I$. To this end, let $G \in \mathcal{D}$ and let U be an open neighborhood of $\vec{q} + \vec{r}$. Pick a neighborhood V of $\vec{q}$ such that $V + \vec{r} \subseteq U$. Pick $\vec{x} \in V \cap E_G$, with $\vec{x} \in I_G$ if $\vec{q} \in I$. For each $i \in \{1, 2, \ldots, k\}$, we have that $x_i \in T_G$ so pick $K_i \in \mathcal{D}$ such that $T_{K_i} \subseteq -x_i + T_G$ and let $K = \bigcup_{i=1}^k K_i$. Pick a neighborhood W of $\vec{r}$ such that $\vec{x} + W \subseteq \vec{U}$. Pick $\vec{y} \in W \cap E_K$ with $\vec{y} \in I_K$ if $\vec{r} \in I$. Then $\vec{x} + \vec{y} \in E_G$ and, if $\vec{x} \in I_G$ or $\vec{y} \in I_K$, then $\vec{x} + \vec{y} \in I_G$.

Recall that we have chosen $p \in K(\mathbf{Q})$ and $B \in p$. We claim that $\overline{p} = (p, p, \ldots, p) \in E$. To see this let $G \in \mathcal{D}$ be given and let U be a neighborhood of $\overline{p}$ in Y. Pick $C \in p$ such that $\times_{i=1}^k \overline{C} \subseteq U$ and pick $d \in C \cap T_G$. Then $(d, d, \ldots, d) \in U \cap E_G$. By [14, Theorem 2.23], we have that $K(\times_{i=1}^k \mathbf{Q}) = \times_{i=1}^k K(\mathbf{Q})$ so $\overline{p} \in E \cap K(\times_{i=1}^k \mathbf{Q})$. Therefore by [14, Theorem 1.65] we have that $\overline{p} \in K(E)$ and, since I is an ideal of E, we have that $\overline{p} \in I$.

Since $\times_{i=1}^{k} \overline{B}$ is a neighborhood of $\overline{p}$, we have some $\vec{x} \in \times_{i=1}^{k} \overline{B} \cap \times_{i=1}^{k} T_F$. Thus B is a J-set as required. $\square$

We conclude this section with a verification that C-sets in commutative semigroups have rich combinatorial properties.

2.6 Definition. Let A be a matrix over ω. A is said to be a *first entries matrix* if no row of A is identically zero and if the first non-zero entries in any two rows of A are equal if they occur in the same column. The first non-zero entry in any row of A is called a *first entry* of A.

2.7 Definition. Let $(S,+)$ be a semigroup with identity 0, let $u, v \in \mathbb{N}$, and let A be a $u \times v$ matrix with entries from ω. Then A is *image partition regular over S* if and only if whenever $r \in \mathbb{N}$ and $S = \bigcup_{i=1}^{r} E_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in (S \backslash \{0\})^v$ such that $A\vec{x} \in {E_i}^u$.

It is a fact due to Deuber [7] that first entries matrices are image partition regular over $\mathbb{N}$. Some of the classical results in Ramsey Theory are naturally stated in terms of the image partition regularity of first entries matrices. For example Schur's Theorem is the assertion that the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is image partition regular over $\mathbb{N}$ and van der Waerden's Theorem [25] is the assertion that for each $k \in \mathbb{N}$, the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & k \end{pmatrix}$$

is image partition regular over $\mathbb{N}$.

It is known [14, Theorem 15.5] that first entries matrices have images contained in any central set. We see now, via minor modifications of the proof of that theorem, that the same conclusion holds for C-sets.

2.8 Theorem. *Let $(S,+)$ be an infinite commutative semigroup with identity 0, let $u, v \in \mathbb{N}$, and let A be a $u \times v$ first entries matrix with entries from ω. Let p be an idempotent in $J(S)$, assume that for every first entry c of A, $cS \in p$, and let $C \in p$. There exist sequences $\langle x_{1,n} \rangle_{n=1}^{\infty}$, $\langle x_{2,n} \rangle_{n=1}^{\infty}$, $\ldots$, $\langle x_{v,n} \rangle_{n=1}^{\infty}$ in S such that for every $F \in$*

$\mathcal{P}_f(\mathbb{N})$, $\vec{x}_F \in (S \setminus \{0\})^v$ *and* $A\vec{x}_F \in C^u$, *where*

$$\vec{x}_F = \begin{pmatrix} \sum_{n\in F} x_{1,n} \\ \sum_{n\in F} x_{2,n} \\ \vdots \\ \sum_{n\in F} x_{v,n} \end{pmatrix} .$$

Proof. If 0 were a minimal idempotent, then $\beta S = 0+\beta S = \beta S+0$ would be a minimal left ideal and a minimal right ideal, hence a group by [14, Theorem 1.61]. In particular, S would be cancellative so by [14, Corollary 4.33], S^* would be a left ideal properly contained in βS, a contradiction. Thus we may presume that $0 \notin C$.

We proceed by induction on v. Assume first $v = 1$. We can assume A has no repeated rows, so in this case we have $A = (c)$ for some $c \in \mathbb{N}$ such that $cS \in p$. Then $C\cap cS \in p$ so pick by [14, Theorem 5.8] a sequence $\langle k_n\rangle_{n=1}^\infty$ with $FS(\langle k_n\rangle_{n=1}^\infty) \subseteq C\cap cS$. For each $n \in \mathbb{N}$ pick some $x_{1,n} \in S$ such that $k_n = cx_{1,n}$. The sequence $\langle x_{1,n}\rangle_{n=1}^\infty$ is as required.

Now let $v \in \mathbb{N}$ and assume the theorem is true for v. Let A be a $u \times (v+1)$ first entries matrix with entries from ω and assume that whenever c is a first entry of A, $cS \in p$. By rearranging the rows of A and adding additional rows to A if need be, we may assume that we have some $r \in \{1, 2, \ldots, u-1\}$ and some $d \in \mathbb{N}$ such that

$$a_{i,1} = \begin{cases} 0 & \text{if } i \in \{1, 2, \ldots, r\} \\ d & \text{if } i \in \{r+1, r+2, \ldots, u\} \end{cases} .$$

Let B be the $r \times v$ matrix with entries $b_{i,j} = a_{i,j+1}$. Pick sequences $\langle z_{1,n}\rangle_{n=1}^\infty$, $\langle z_{2,n}\rangle_{n=1}^\infty$, $\ldots$,$\langle z_{v,n}\rangle_{n=1}^\infty$ in S as guaranteed by the induction hypothesis for the matrix B. For each $i \in \{r+1, r+2, \ldots, u\}$ and each $n \in \mathbb{N}$, let $f_i(n) = \sum_{j=2}^{v+1} a_{i,j} \cdot z_{j-1,n}$ and let $f_r(n) = 0$ for all $n \in \mathbb{N}$.

Since $C\cap dS \in p$, it is a C-set so pick functions α and H as guaranteed by Definition 1.4. Choose a sequence $\langle g_n\rangle_{n=1}^\infty$ of distinct members of ${}^{\mathbb{N}}S \setminus \{f_r, f_{r+1}, \ldots, f_u\}$. For each $n \in \mathbb{N}$, let $G_n = \{f_r, f_{r+1}, \ldots, f_u\} \cup \{g_1, g_2, \ldots, g_n\}$, let $k_n = \alpha(G_n)$ and let $H_n = H(G_n)$.

Then $\max H_n < \min H_{n+1}$ for each n and for each $i \in \{r, r+1, \ldots, u\}$,

$$FS(\langle k_n + \textstyle\sum_{t\in H_n} f_i(t)\rangle_{n=1}^\infty) \subseteq C \cap dS\,.$$

Note in particular that each $k_n = k_n + \sum_{t\in H_n} f_r(t) \in C \cap dS$, so pick $x_{1,n} \in S$ such that $k_n = dx_{1,n}$. For $j \in \{2, 3, \ldots, v+1\}$, let $x_{j,n} = \sum_{t\in H_n} z_{j-1,t}$.

We claim that the sequences $\langle x_{j,n}\rangle_{n=1}^\infty$ are as required. To see this, let $F \in \mathcal{P}_f(\mathbb{N})$ be given. We need to show that for each $j \in \{1, 2, \ldots, v+1\}$, $\sum_{n\in F} x_{j,n} \neq 0$ and for each $i \in \{1,2,\ldots,u\}$, $\sum_{j=1}^{v+1} a_{i,j} \sum_{n\in F} x_{j,n} \in C$.

For the first assertion note that if $j > 1$, then $\sum_{n\in F} x_{j,n} = \sum_{t\in G} z_{j-1,t}$ where $G = \bigcup_{n\in F} H_n$. If $j = 1$, then $d\sum_{n\in F} x_{1,n} = \sum_{n\in F}\big(k_n + \sum_{t\in H_n} f_r(t)\big) \in C$.

To establish the second assertion, let $i \in \{1,2,\ldots,u\}$ be given.

Case 1. $i \leq r$. Then

$$\begin{aligned}\textstyle\sum_{j=1}^{v+1} a_{i,j} \sum_{n\in F} x_{j,n} &= \textstyle\sum_{j=2}^{v+1} a_{i,j} \sum_{n\in F} \sum_{t\in H_n} z_{j-1,t}\\ &= \textstyle\sum_{j=1}^{v} b_{i,j} \sum_{t\in G} z_{j,t} \in C\end{aligned}$$

where $G = \bigcup_{n\in F} H_n$.

Case 2. $i > r$. Then

$$\begin{aligned}\textstyle\sum_{j=1}^{v+1} a_{i,j} \sum_{n\in F} x_{j,n} &= \textstyle d\sum_{n\in F} x_{1,n} + \sum_{j=2}^{v+1} a_{i,j} \sum_{n\in F} x_{j,n}\\ &= \textstyle\sum_{n\in F} dx_{1,n} + \sum_{n\in F} \sum_{j=2}^{v+1} a_{i,j} \sum_{t\in H_n} z_{j-1,t}\\ &= \textstyle\sum_{n\in F} dx_{1,n} + \sum_{n\in F} \sum_{t\in H_n} \sum_{j=2}^{v+1} a_{i,j} z_{j-1,t}\\ &= \textstyle\sum_{n\in F}\big(k_n + \sum_{t\in H_n} f_i(t)\big) \in C\,. \end{aligned}$$

□

There are many familiar commutative semigroups S, such as $(\omega,+)$, $(\mathbb{Q},+)$, or $(\mathbb{R},+)$, in which cS is a member of every idempotent in βS for every $c \in \mathbb{N}$.

2.9 Corollary. *Let $(S,+)$ be an infinite commutative semigroup with identity 0, let $u,v \in \mathbb{N}$, let A be a $u \times v$ first entries matrix with entries from ω, and let C be a C-set in S. If for every first entry c of A and every idempotent $p \in \beta S$, $cS \in p$, then there exists $\vec{x} \in (S\setminus\{0\})^v$ such that all entries of $A\vec{x}$ are in C. In particular, if 1 is the only first entry of A, then there exists $\vec{x} \in (S\setminus\{0\})^v$ such that all entries of $A\vec{x}$ are in C.*

Proof. Pick by Theorem 2.5 an idempotent $p \in \overline{C} \cap J(S)$ and apply Theorem 2.8. □

3. Products of semigroups with small C-sets

In this section we investigate the class of semigroups which contain a C-set with Følner density zero, deriving sufficient conditions for the product of two members of that class to remain in the class.

3.1 Theorem. *Let $(S,\cdot)$ and $(T,\cdot)$ be infinite semigroups, let A be a C-set in S, and let B be a C-set in T. Then $A \times B$ is a C-set in T.*

Proof. [17, Theorem 2.16]. □

3.2 Theorem. *Let $(S,\cdot)$ and $(T,\cdot)$ be left cancellative semigroups which satisfy SFC, let $A \subseteq S$, and let $B \subseteq T$. Then $d(A \times B) = d(A) \cdot d(B)$.*

Proof. [16, Theorems 2.12 and 3.4]. $\square$

The conclusion of Theorem 3.4 is much weaker than that of Theorem 3.2, but except for the requirement that T have a left identity, the assumptions are much weaker as well, and the conclusion is enough to guarantee that $d(A \times T) = 0$ whenever $d(A) = 0$.

3.3 Lemma. *Let $(S, \cdot)$ be a semigroup and let $(T, \cdot)$ be a semigroup with left identity e. Let $G \in \mathcal{P}_f(S)$ and let $\epsilon > 0$. Assume that $M \subseteq S \times T$ and*

$$(\forall s \in G)(|M \setminus (s,e)M| < \frac{\epsilon}{|G|} \cdot |M|)\,.$$

Then there is some $b \in \pi_2[M]$ such that, if $H = \pi_1[M \cap (S \times \{b\})]$, then

$$(\forall s \in G)(|H \setminus sH| < \epsilon \cdot |H|)\,.$$

Proof. Let $C = \pi_2[M]$. For $b \in C$, let $H_b = \pi_1[M \cap (S \times \{b\})]$. For $s \in G$, let $U_s = \{b \in C : |H_b \setminus sH_b| \geq \epsilon \cdot |H_b|\}$. We show that $C \setminus \bigcup_{s \in G} U_s \neq \emptyset$. Suppose instead that $C = \bigcup_{s \in G} U_s$.

Let $s \in G$ be given. Then $(M \setminus (s,e)M) = \bigcup_{b \in C} \big((H_b \setminus sH_b) \times \{b\}\big)$. Thus

$$\begin{aligned}
\tfrac{\epsilon}{|G|} \cdot \textstyle\sum_{b \in C} |H_b| &= \tfrac{\epsilon}{|G|} \cdot |M| \\
&> |M \setminus (s,e)M| \\
&= \textstyle\sum_{b \in C} |H_b \setminus sH_b| \\
&\geq \textstyle\sum_{b \in U_s} |H_b \setminus sH_b| \\
&\geq \textstyle\sum_{b \in U_s} \epsilon \cdot |H_b|\,.
\end{aligned}$$

Thus $\epsilon \cdot \sum_{b \in C} |H_b| = \sum_{s \in G} \frac{\epsilon}{|G|} \cdot \sum_{b \in C} |H_b| > \epsilon \cdot \sum_{s \in G} \sum_{b \in U_s} |H_b|$. Since $C = \bigcup_{s \in G} U_s$, we have that $\sum_{b \in C} |H_b| \leq \sum_{s \in G} \sum_{b \in U_s} |H_b|$, so $\epsilon > \epsilon$, a contradiction. $\square$

3.4 Theorem. *Let $(S, \cdot)$ be a semigroup and let $(T, \cdot)$ be a semigroup with left identity e. Assume that S and T satisfy SFC and let $A \subseteq S$. Then $d(A) = d(A \times T)$.*

Proof. By [16, Lemma 3.1] $S \times T$ satisfies SFC and $d(A \times T) \geq d(A) \cdot d(T)$ and trivially $d(T) = 1$.

Suppose $d(A) < d(A \times T)$ and pick α and δ such that $d(A) < \alpha < \delta < d(A \times T)$. Pick $G \in \mathcal{P}_f(S)$ and $\epsilon > 0$ such that

$$\big(\forall H \in \mathcal{P}_f(S)\big)\big((\forall s \in G)(|H \setminus sH| < \epsilon \cdot |H|) \Rightarrow |A \cap H| < \alpha \cdot |H|\big)\,.$$

Let $\gamma = \dfrac{\epsilon}{|G|} \cdot (\delta - \alpha + 1)$. Pick $K \in \mathcal{P}_f(S \times T)$ such that $(\forall s \in G)(|K \setminus (s,e)K| < \gamma \cdot |K|)$ and $|(A \times T) \cap K| \geq \delta \cdot |K|$.

Let $C = \pi_2[K]$. For $b \in C$, let $H_b = \pi_1[K \cap (S \times \{b\})]$. Then $K = \bigcup_{b \in C} (H_b \times \{b\})$. Let $L = \{b \in C : |H_b \cap A| \geq \alpha \cdot |H_b|\}$ and let $M = \bigcup_{b \in L} (H_b \times \{b\})$. We shall show that $(\forall s \in G)(|M \setminus (s,e)M| < \frac{\epsilon}{|G|} \cdot |M|)$. Then by Lemma 3.3 we will have some $b \in L$ such that $(\forall s \in G)(|H_b \setminus sH_b| < \epsilon \cdot |H_b|)$. Since also $|H_b \cap A| \geq \alpha \cdot |H_b|$, this will be a contradiction.

So let $s \in G$ and suppose that $|M \setminus (s,e)M| \geq \frac{\epsilon}{|G|} \cdot |M|$. For $b \in C$, let $x_b = |H_b \setminus sH_b|$, let $y_b = |H_b|$, and let $z_b = |H_b \cap A|$. Then $L = \{b \in C : z_b \geq \alpha \cdot y_b\}$. Now

(1) $C \setminus L = \{b \in C : z_b < \alpha \cdot y_b\}$ so $\sum_{b \in C \setminus L} z_b < \alpha \cdot \sum_{b \in C \setminus L} y_b$;

(2) $|(A \times T) \cap K| \geq \delta \cdot |K|$ so $\sum_{b \in C} z_b \geq \delta \cdot \sum_{b \in C} y_b$;

(3) $|M \setminus (s,e)M| \geq \frac{\epsilon}{|G|} \cdot |M|$ so $\sum_{b \in L} x_b \geq \frac{\epsilon}{|G|} \cdot \sum_{b \in L} y_b$; and

(4) $|K \setminus (s,e)K| < \gamma \cdot |K|$ so $\sum_{b \in C} x_b < \gamma \cdot \sum_{b \in C} y_b$.

From (1) and (2) we have

$$\begin{aligned}\delta \cdot \textstyle\sum_{b \in L} y_b + \delta \cdot \sum_{b \in C \setminus L} y_b = \delta \cdot \sum_{b \in C} y_b & \\ \leq \textstyle\sum_{b \in L} z_b + \sum_{b \in C \setminus L} z_b & \\ < \textstyle\sum_{b \in L} y_b + \alpha \cdot \sum_{b \in C \setminus L} y_b &\end{aligned}$$

so

$$(*) \qquad \textstyle\sum_{b \in C \setminus L} y_b < \left(\frac{1-\delta}{\delta-\alpha}\right) \cdot \sum_{b \in L} y_b \, .$$

From (3) and (4) we have $\frac{\epsilon}{|G|} \cdot \sum_{b \in L} y_b \leq \sum_{b \in L} x_b$ and $\sum_{b \in L} x_b \leq \sum_{b \in C} x_b < \gamma \cdot \sum_{b \in C} y_b$ so

$$(**) \qquad \left(\tfrac{\epsilon}{\gamma \cdot |G|} - 1\right) \cdot \textstyle\sum_{b \in L} y_b < \sum_{b \in C \setminus L} y_b \, .$$

Combining $(*)$ and $(**)$, we conclude that $\gamma > \dfrac{\epsilon}{|G|} \cdot (\delta - \alpha + 1)$, a contradiction.$\square$

Now we see that under appropriate hypotheses, the existence of small C-sets is preserved under products.

3.5 Theorem. *Let $(S,\cdot)$ and $(T,\cdot)$ be infinite semigroups which satisfy SFC and assume that either S and T are both left cancellative or T has a left identity. If A is a C-set in S with $d(A) = 0$ and B is a C-set in T, then $A \times B$ is a C-set in $S \times T$ and $d(A \times B) = 0$.*

Proof. By Theorem 3.1, $A \times B$ is a C-set in $S \times T$. By [16, Lemma 3.1] $S \times T$ satisfies SFC. By Theorem 3.2 or Theorem 3.4, $d(A \times B) = 0$. $\square$

4. Small C-sets in subsemigroups of $(\mathbb{R},+)$

Throughout this section S will denote a subsemigroup of $(\mathbb{R},+)$ with $\mathbb{Z} \subseteq S$ and we will let $\mathcal{T} = {}^{\mathbb{N}}S$. We shall show that there is a set $A \subseteq S$ such that A is a C-set in S and $d(A) = 0$.

We shall denote by S_d, the set S with the discrete topology. We represent $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as $\left[-\frac{1}{2}, \frac{1}{2}\right)$. Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S, we let

$$FS(\langle x_n \rangle_{n=1}^{\infty}) = \left\{\textstyle\sum_{t\in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\right\}.$$

4.1 Definition. Define $h : S \to \mathbb{Z}$ by $h(x) = \left\lfloor x + \frac{1}{2} \right\rfloor$ for $x \in S$, and let $\pi : S \to \mathbb{T}$ be the natural projection. (So that, for $x \in S$, $\pi(x) = x - h(x)$.) Let $\widetilde{h} : \beta S_d \to \beta\mathbb{Z}$ and $\widetilde{\pi} : \beta S_d \to \mathbb{T}$ be the continuous extensions of h and π respectively.

4.2 Lemma. *Let $l \in \mathbb{N}$ and let $f_1, f_2, \ldots, f_l \in \mathcal{T}$. There exists a sequence $\langle L_m \rangle_{m=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for all $m \in \mathbb{N}$, $\max L_m < \min L_{m+1}$ and for all $i \in \{1, 2, \ldots, l\}$ and all $m \in \mathbb{N}$, $\pi\left(\sum_{t\in L_m} f_i(t)\right) \in \left(-\frac{1}{2^m}, \frac{1}{2^m}\right)$.*

Proof. We proceed by induction on l, so assume first that $l = 1$. Pick by [14, Lemma 5.11] an idempotent $p \in \beta S_d$ such that for all $m \in \mathbb{N}$, $FS(\langle f_1(t) \rangle_{t=m}^{\infty}) \in p$. Since $p = p + p$ and by [14, Corollary 4.22] (due originally to P. Milnes in [19]), $\widetilde{\pi}$ is a homomorphism, we have that $\widetilde{\pi}(p) = 0$ and so for each $m \in \mathbb{N}$, $\pi^{-1}\left[\left(-\frac{1}{2^m}, \frac{1}{2^m}\right)\right] \in p$.

Choose $L_1 \in \mathcal{P}_f(\mathbb{N})$ such that $\sum_{t\in L_1} f_1(t) \in \pi^{-1}\left[\left(-\frac{1}{2}, \frac{1}{2}\right)\right]$. Now assume that $m \in \mathbb{N}$ and L_m has been chosen. Let $k = \max L_m + 1$ and pick

$$x \in FS(\langle f_1(t) \rangle_{t=k}^{\infty}) \cap \pi^{-1}\left[\left(-\tfrac{1}{2^{m+1}}, \tfrac{1}{2^{m+1}}\right)\right].$$

Pick $L_{m+1} \in \mathcal{P}_f(\mathbb{N})$ with $\min L_{m+1} \geq k$ such that $x = \sum_{t\in L_{m+1}} f_1(t)$.

Now let $l \in \mathbb{N}$ and assume that the lemma is valid for l. Let $f_1, f_2, \ldots, f_{l+1} \in \mathcal{T}$. Pick a sequence $\langle F_m \rangle_{m=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for all $m \in \mathbb{N}$, $\max F_m < \min F_{m+1}$ and for all $i \in \{1, 2, \ldots, l\}$ and all $m \in \mathbb{N}$, $\pi\left(\sum_{t\in F_m} f_i(t)\right) \in \left(-\frac{1}{2^{m+1}}, \frac{1}{2^{m+1}}\right)$. (One may do this by deleting the first term of the sequence guaranteed by the induction hypothesis.)

For each $m \in \mathbb{N}$, define $g(m) = \sum_{t\in F_m} f_{l+1}(t)$. Pick a sequence $\langle K_m \rangle_{m=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for all $m \in \mathbb{N}$, $\max K_m < \min K_{m+1}$ and $\pi\left(\sum_{n\in K_m} g(n)\right) \in \left(-\frac{1}{2^m}, \frac{1}{2^m}\right)$. For each $m \in \mathbb{N}$, let $L_m = \bigcup_{n\in K_m} F_n$. Then for each $m \in \mathbb{N}$, $\sum_{t\in L_m} f_{l+1}(t) = \sum_{n\in K_m} \sum_{t\in F_n} f_{l+1}(t) = \sum_{n\in K_m} g(n) \in \pi^{-1}\left[\left(-\frac{1}{2^m}, \frac{1}{2^m}\right)\right]$.

For $i \in \{1, 2, \ldots, l\}$ and $m \in \mathbb{N}$,

$$\pi\left(\textstyle\sum_{t\in L_m} f_i(t)\right) = \pi\left(\sum_{n\in K_m} \sum_{t\in F_n} f_i(t)\right) = \sum_{n\in K_m} \pi\left(\sum_{t\in F_n} f_i(t)\right).$$

Given $m \in \mathbb{N}$ and $n \in K_m$, we have $\pi\left(\sum_{t\in F_n} f_i(t)\right) \in \left(-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right)$ and if $v = \min K_m$, then $\sum_{n\in K_m} \frac{1}{2^{n+1}} < \frac{1}{2^v} \le \frac{1}{2^m}$ and so $\sum_{n\in K_m} \pi\left(\sum_{t\in F_n} f_i(t)\right) \in \left(-\frac{1}{2^m}, \frac{1}{2^m}\right)$. $\square$

The following lemma provides the basis for the main result of the section.

4.3 Lemma. *Let $p = p+p \in J(\mathbb{Z})$ and let $T = \bigcap_{n=2}^{\infty} \overline{\pi^{-1}\left[\left(-\frac{1}{n}, \frac{1}{n}\right)\right]}$. Then $T \cap \widetilde{h}^{-1}[\{p\}]$ is a subsemigroup of βS_d and if $q+q = q \in K(T \cap \widetilde{h}^{-1}[\{p\}])$, then $q \in J(S)$.*

Proof. We show first that T is a semigroup. Since $\mathbb{Z} \subseteq T$, we know that $T \neq \emptyset$. Let $r, s \in T$ and let $n \in \mathbb{N} \setminus \{1\}$. Then given any $x \in \left[\left(-\frac{1}{2n}, \frac{1}{2n}\right)\right]$, $\pi^{-1}\left[\left(-\frac{1}{2n}, \frac{1}{2n}\right)\right] \subseteq -x + \pi^{-1}\left[\left(-\frac{1}{n}, \frac{1}{n}\right)\right]$ so $\pi^{-1}\left[\left(-\frac{1}{2n}, \frac{1}{2n}\right)\right] \subseteq \{x \in S : -x + \pi^{-1}\left[\left(-\frac{1}{n}, \frac{1}{n}\right)\right] \in s\}$ and thus $\pi^{-1}\left[\left(-\frac{1}{n}, \frac{1}{n}\right)\right] \in r+s$. Next note that (viewing $\beta\mathbb{Z}$ as a subset of βS_d) $p \in T \cap \widetilde{h}^{-1}[\{p\}]$ so $T \cap \widetilde{h}^{-1}[\{p\}] \neq \emptyset$. Given $x, y \in \pi^{-1}\left[\left(-\frac{1}{4}, \frac{1}{4}\right)\right]$, $h(x+y) = h(x) + h(y)$ so by [14, Theorem 4.21] $\widetilde{h}$ is a homomorphism on T so $T \cap \widetilde{h}^{-1}[\{p\}]$ is a semigroup.

Now let $q+q = q \in K(T \cap \widetilde{h}^{-1}[\{p\}])$. We need to show that every element of q is a J-set. So let $C \in q$, let $l \in \mathbb{N}$, and let $f_1, f_2, \ldots, f_l \in {}^{\mathbb{N}}S$. We shall eventually show that there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $i \in \{1, 2, \ldots, l\}$, $a + \sum_{t\in H} f_i(t) \in C$.

For each $A \in p$, let $A^\star = \{x \in A : -x + A \in p\}$ and recall that by [14, Lemma 4.14], whenever $x \in A^\star$, one has $-x + A^\star \in p$.

By Lemma 4.2, we may choose a sequence $\langle L_t \rangle_{t=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ with the property that for each $t \in \mathbb{N}$, $\max L_t < \min L_{t+1}$ and $\pi\left(\sum_{u\in L_t} f_i(u)\right) \in \left(-\frac{1}{2^{t+1}}, \frac{1}{2^{t+1}}\right)$ for each $i \in \{1, 2, \ldots, l\}$. For $t \in \mathbb{N}$ and $i \in \{1, 2, \ldots, l\}$, let $g_i(t) = \sum_{u\in L_t} f_i(u)$. Then for every $a \in \pi^{-1}[(-\frac{1}{4}, \frac{1}{4})]$, every $H \in \mathcal{P}_f(\mathbb{N})$, and every $i \in \{1, 2, \ldots, l\}$, $h\big(a + \sum_{t\in H} g_i(t)\big) = h(a) + \sum_{t\in H} h\big(g_i(t)\big)$.

Let $Y = \times_{i=1}^{l} \beta S$. For $A \in p$ and $n \in \mathbb{N}$, let

$$\begin{aligned} I_{A,n} = \big\{\langle a + \textstyle\sum_{t\in H} g_i(t)\rangle_{i=1}^{l} : {} & H \in \mathcal{P}_f(\mathbb{N}),\, \min H > n\,, \\ & a \in \pi^{-1}\big[\big(-\tfrac{1}{2^{n+2}}, \tfrac{1}{2^{n+2}}\big)\big] \cap h^{-1}[A^\star]\,, \\ & \text{and for } i \in \{1, 2, \ldots, l\}\,,\ h\big(a + \textstyle\sum_{t\in H} g_i(t)\big) \in A^\star\big\} \end{aligned}$$

and let

$$E_{A,n} = I_{A,n} \cup \big\{\langle a, a, \ldots, a\rangle : a \in \pi^{-1}\big[\big(-\tfrac{1}{2^{n+2}}, \tfrac{1}{2^{n+2}}\big)\big] \cap h^{-1}[A^\star]\big\}\,.$$

Let $I = \bigcap\{c\ell_Y I_{A,n} : A \in p \text{ and } n \in \mathbb{N}\}$, and let $E = \bigcap\{c\ell_Y E_{A,n} : A \in p \text{ and } n \in \mathbb{N}\}$. We claim that for each $A \in p$ and each $n \in \mathbb{N}$, $I_{A,n} \neq \emptyset$, and consequently $I \neq \emptyset$. So let $A \in p$ and $n \in \mathbb{N}$. Since $A^\star$ is a J-set in $\mathbb{Z}$, pick by Lemma 2.4 applied to the functions $\overline{0}, h \circ g_1, h \circ g_2, \ldots, h \circ g_l$, some $a \in A^\star$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > n$ and for each $i \in \{1, 2, \ldots, l\}$, $a + \sum_{t\in H} h\big(g_i(t)\big) \in A^\star$. Then $\langle a + \sum_{t\in H} g_i(t)\rangle_{i=1}^{l} \in I_{A,n}$. (Given $i \in \{1, 2, \ldots, l\}$, $h\big(a + \sum_{t\in H} g_i(t)\big) = h(a) + \sum_{t\in H} h\big(g_i(t)\big) = a + \sum_{t\in H} h\big(g_i(t)\big) \in A^\star$.)

Now we show that E is a subsemigroup of Y and I is an ideal of E. To this end, let $\vec{r},\vec{s}\in E$. We shall show that $\vec{r}+\vec{s}\in E$ and if $\vec{r}\in I$ or $\vec{s}\in I$, then $\vec{r}+\vec{s}\in I$.

We shall use the convention that $\sum_{t\in\emptyset} f(t)=0$. Let U be an open neighborhood of $\vec{r}+\vec{s}$, let $A\in p$, and let $n\in\mathbb{N}$. We shall show that $U\cap E_{A,n}\neq\emptyset$ and if either $\vec{r}\in I$ or $\vec{s}\in I$, then $U\cap I_{A,n}\neq\emptyset$. By [14, Theorem 2.22] we have that Y is a right topological semigroup and if $\vec{x}\in \times_{i=1}^{l} S$, then $\lambda_{\vec{x}}$ is continuous. Pick a neighborhood V of $\vec{r}$ such that $V+\vec{s}\subseteq U$. Pick $\vec{x}\in V\cap E_{A,n}$ with $\vec{x}\in I_{A,n}$ if $\vec{r}\in I$. If $\vec{x}\in I_{A,n}$, pick a and H as in the definition of $I_{A,n}$ such that $\vec{x}=\langle a+\sum_{t\in H} g_i(t)\rangle_{i=1}^{l}$. If $\vec{x}\notin I_{A,n}$, pick $a\in\pi^{-1}\big[\big(-\frac{1}{2^{n+2}},\frac{1}{2^{n+2}}\big)\big]\cap h^{-1}[A^\star]$ such that $\vec{x}=\langle a,a,\ldots,a\rangle$ and let $H=\emptyset$.

Pick a neighborhood W of $\vec{s}$ such that $\vec{x}+\vec{W}\subseteq U$. Pick $m\in\mathbb{N}$ such that $\big(\pi(a)-\frac{1}{2^{m+2}},\pi(a)+\frac{1}{2^{m+2}}\big)\subseteq\big(-\frac{1}{2^{n+2}},\frac{1}{2^{n+2}}\big)$ and note that $m\geq n$. If $H\neq\emptyset$ require also that $\max H\leq m$. Let $B=(-h(a)+A^\star)\cap\bigcap_{i=1}^{l}\big(-h\big(a+\sum_{t\in H} g_i(t)\big)+A^\star\big)$. Then $B\in p$. Pick $\vec{y}\in W\cap E_{B,m}$ with $\vec{y}\in I_{B,m}$ if $\vec{s}\in I$. If $\vec{y}\in I_{B,m}$, pick

$$b\in\pi^{-1}\big[\big(-\tfrac{1}{2^{m+2}},\tfrac{1}{2^{m+2}}\big)\big]\cap h^{-1}[B^\star]$$

and $G\in\mathcal{P}_f(\mathbb{N})$ with $\min G>m$ such that $\vec{y}=\langle b+\sum_{t\in G} g_i(t)\rangle_{i=1}^{l}$ and for $i\in\{1,2,\ldots,l\}$, $h\big(b+\sum_{t\in G} g_i(t)\big)\in B^\star$. If $\vec{y}\notin I_{B,m}$, pick $b\in\pi^{-1}\big[\big(-\frac{1}{2^{m+2}},\frac{1}{2^{m+2}}\big)\big]\cap h^{-1}[B^\star]$ such that $\vec{y}=\langle b,b,\ldots,b\rangle$ and let $G=\emptyset$. Then $\pi(a+b)\in\big(-\frac{1}{2^{n+2}},\frac{1}{2^{n+2}}\big)$ and $h(a+b)=h(a)+h(b)\in A^\star$. Further, given $i\in\{1,2,\ldots,l\}$, $\pi\big(a+\sum_{t\in H} g_i(t)\big)\in\big(-\frac{1}{2^{n+2}},\frac{1}{2^{n+2}}\big)$ and $\pi\big(b+\sum_{t\in G} g_i(t)\big)\in\big(-\frac{1}{2^{n+2}},\frac{1}{2^{n+2}}\big)$ so

$$h\big(a+b+\textstyle\sum_{t\in H\cup G} g_i(t)\big)=h\big(a+\sum_{t\in H} g_i(t)\big)+h\big(b+\sum_{t\in G} g_i(t)\big)\in A^\star\,.$$

Consequently $\vec{x}+\vec{y}\in U\cap E_{A,n}$ and if $\vec{r}\in I$ or $\vec{s}\in I$,then $\vec{x}+\vec{y}\in U\cap I_{A,n}$.

We thus have that E is a subsemigroup of I and I is an ideal of E. Let $\overline{q}=\langle q,q,\ldots,q\rangle\in Y$ and let $X=\times_{i=1}^{l}\widetilde{h}^{-1}[\{p\}]$. We claim that $\overline{q}\in E$. To this end, let U be a neighborhood of $\overline{q}$, let $A\in p$, and let $n\in\mathbb{N}$. Pick $B\in q$ such that $\times_{i=1}^{l}\overline{B}\subseteq U$. Since $q\in T$, $\pi^{-1}\left[\big(-\frac{1}{2^{n+2}},\frac{1}{2^{n+2}}\big)\right]\in q$ and since $\widetilde{h}(q)=p$, $\widetilde{h}^{-1}[A^\star]\in q$. Pick $a\in B\cap\pi^{-1}\left[\big(-\frac{1}{2^{n+2}},\frac{1}{2^{n+2}}\big)\right]\cap\widetilde{h}^{-1}[A^\star]$. Then $\langle a,a,\ldots,a\rangle\in U\cap E_{A,n}$. Now

$$K(T\cap\widetilde{h}^{-1}[\{p\}])=(T\cap\widetilde{h}^{-1}[\{p\}])\cap K(\widetilde{h}^{-1}[\{p\}])$$

by [14, Theorem 1.65] so $q\in K(\widetilde{h}^{-1}[\{p\}])$. Also $K(X)=\times_{i=1}^{l}K(\widetilde{h}^{-1}[\{p\}])$ by [14, Theorem 2.23] so $\overline{q}\in K(X)$.

Next we claim that $E\subseteq X$. To see this, let $\vec{r}\in E$, let $i\in\{1,2,\ldots,l\}$, and suppose that $r_i\notin\widetilde{h}^{-1}[\{p\}]$. Pick $A\in p\setminus\widetilde{h}(r_i)$ and pick $B\in r_i$ such that $\widetilde{h}[\,\overline{B}\,]\cap\overline{A}=\emptyset$. Then $\{\vec{s}\in Y: s_i\in\overline{B}\}$ is a neighborhood of $\vec{r}$ missing $E_{A,1}$, a contradiction.

Thus we have that $\overline{q} \in E \cap K(X)$ so, again using [14, Theorem 1.65], $\overline{q} \in K(E) \subseteq I$. Now $\times_{i=1}^{l} \overline{C}$ is a neighborhood of $\overline{q}$ so pick $\vec{x} = \langle a + \sum_{t \in H} g_i(t) \rangle_{i=1}^{l} \in (\times_{i=1}^{l} \overline{C}\,) \cap I_{S,1}$. Let $K = \bigcup_{t \in H} L_t$. Then for $i \in \{1, 2, \ldots, l\}$, $a + \sum_{u \in K} f_i(u) = a + \sum_{t \in H} g_i(t) \in C$ as required. $\square$

Recall that we have fixed a subsemigroup S of $(\mathbb{R}, +)$ with $\mathbb{Z} \subseteq S$.

4.4 Lemma. *Let $A \subseteq \mathbb{N}$. Then $d_S(h^{-1}[A]) \leq d_{\mathbb{N}}(A)$.*

Proof. Suppose instead that $d_{\mathbb{N}}(A) < d_S(h^{-1}[A])$ and pick γ and α such that $d_{\mathbb{N}}(A) < \gamma < \alpha < d_S(h^{-1}[A])$. Recall that by Theorem 1.9, $d_{\mathbb{N}}(A)$ is the Banach density of A. Pick $l \in \mathbb{N}$ such that for all $a \in \mathbb{N}$, $|A \cap \{a, a+1, \ldots, a+l\}| < \gamma \cdot (l+1)$. Then in fact for all $a \in \mathbb{Z}$, $|A \cap \{a, a+1, \ldots, a+l\}| < \gamma \cdot (l+1)$. (If $a \leq 0$ then $A \cap \{a, a+1, \ldots, a+l\} \subseteq A \cap \{1, 2, \ldots, 1+l\}$.) Let $\epsilon = \frac{\alpha - \gamma}{l \cdot (l+1)}$.

Let $H = \{-1, -2, \ldots, -l\}$. Then $H \in \mathcal{P}_f(S)$ so pick $K \in \mathcal{P}_f(S)$ such that for all $s \in H$, $|K \setminus (s + K)| < \epsilon \cdot |K|$ and $|h^{-1}[A] \cap K| \geq \alpha \cdot |K|$.

Let $a_0 = \min K$. Having chosen $a_0, a_1, \ldots, a_t$, if $K \subseteq \bigcup_{i=0}^{t} \{a_i, a_i + 1, \ldots, a_i + l\}$, let $k = t + 1$. Otherwise, let $a_{t+1} = \min(K \setminus \bigcup_{i=0}^{t} \{a_i, a_i + 1, \ldots, a_i + l\})$.

Note that if $i, j \in \{0, 1, \ldots, k-1\}$ and $i \neq j$, then

$$\{a_i, a_i + 1, \ldots, a_i + l\} \cap \{a_j, a_j + 1, \ldots, a_j + l\} = \emptyset\,.$$

Let $B = \{i \in \{0, 1, \ldots, k-1\} : \{a_i + 1, a_i + 2, \ldots, a_i + l\} \setminus K \neq \emptyset\}$. If $i \in B$, then $a_i \in \bigcup_{s \in H} \big(K \setminus (s + K)\big)$ so $|B| \leq l \cdot \epsilon \cdot |K|$.

Now $\bigcup \big\{\{a_i, a_i + 1, \ldots, a_i + l\} : i \in \{0, 1, \ldots, k-1\} \setminus B\big\} \subseteq K$ so

$$|K| \geq (k - |B|) \cdot (l + 1) > k \cdot (l + 1) - l \cdot (l + 1) \cdot \epsilon \cdot |K|$$

and thus $|K| > \frac{k \cdot (l+1)}{l \cdot (l+1) \cdot \epsilon + 1}$.

Next observe that for any $a \in S$,

$$|h^{-1}[A] \cap \{a, a + 1, \ldots, a + l\}| = |A \cap \{h(a), h(a) + 1, \ldots, h(a) + l\}|$$

because for each $a \in S$ and each $s \in \{1, 2, \ldots, l\}$, $h(a + s) = h(a) + s$. Consequently

$$\begin{aligned}
\alpha \cdot \frac{k \cdot (l + 1)}{l \cdot (l + 1) \cdot \epsilon + 1} &< \alpha \cdot |K| \\
&\leq |h^{-1}[A] \cap K| \\
&\leq \textstyle\sum_{i=0}^{k-1} |h^{-1}[A] \cap \{a_i, a_i + 1, \ldots, a_i + l\}| \\
&= \textstyle\sum_{i=0}^{k-1} |A \cap \{h(a_i), h(a_i) + 1, \ldots, h(a_i) + l\}| \\
&< \gamma \cdot k \cdot (l + 1)\,.
\end{aligned}$$

Thus $\alpha < \gamma \cdot l \cdot (l+1) \cdot \epsilon + \gamma < l \cdot (l+1) \cdot \epsilon + \gamma$ and so $\epsilon > \frac{\alpha-\gamma}{l\cdot(l+1)}$, a contradiction. $\square$

In the following theorem we restate our standing hypothesis for this section.

4.5 Theorem. *Let S be a subsemigroup of $(\mathbb{R},+)$ such that $\mathbb{Z} \subseteq S$. There is a C-set B contained in S such that $d_S(B) = 0$.*

Proof. By [13, Theorem 2.1] pick a set $A \subseteq \mathbb{N}$ such that A is a C-set in $\mathbb{Z}$ and $d_{\mathbb{N}}(A) = 0$. Let $B = h^{-1}[A]$. By Lemma 4.4, $d_S(B) = 0$. Pick by Theorem 2.5 an idempotent $p \in \overline{A} \cap J(\mathbb{Z})$.

Let $T = \bigcap_{n=2}^{\infty} \overline{\pi^{-1}\left[\left(-\frac{1}{n}, \frac{1}{n}\right)\right]}$. By Lemma 4.3, there is an idempotent $q \in K(T \cap \widetilde{h}^{-1}[\{p\}])$ and $q \in J(S)$. Since $B \in q$, we have by Theorem 2.5 that B is a C-set. $\square$

5. Small C-sets in the direct sum of finite abelian groups

In this section we show that if G is the direct sum of countably many finite abelian groups, then there is a C-set A in G with $d(A) = 0$. Since each finite abelian group is the direct sum of cyclic groups, we shall assume throughout this section that $G = \bigoplus_{n=0}^{\infty} \mathbb{Z}_{\nu(n)}$ where each $\nu(n) \in \mathbb{N} \setminus \{1\}$. As usual, we let $\omega = \mathbb{N} \cup \{0\}$.

5.1 Definition. For $x \in G$, $\mathrm{supp}(x) = \{t \in \omega : x(t) \neq 0\}$. For $n \in \mathbb{N}$, $K_n = \big\{x \in G : \mathrm{supp}(x) \subseteq \{0, 1, \ldots, n\}\big\}$.

5.2 Lemma. *Let $A \subseteq G$. Then*

$$d(A) = \sup\{\alpha \in \mathbb{R} : (\forall k \in \mathbb{N})(\exists n > k)(\exists d \in G)(|A \cap (d + K_n)| \geq \alpha \cdot |K_n|)\}\,.$$

Proof. Let $\delta = \sup\{\alpha \in \mathbb{R} : (\forall k \in \mathbb{N})(\exists n > k)(\exists d \in G)(|A \cap (d + K_n)| \geq \alpha \cdot |K_n|)\}$. To see that $d(A) \geq \delta$, suppose instead we have some α such that $\delta > \alpha > d(A)$. Pick $H \in \mathcal{P}_f(G)$ and $\epsilon > 0$ such that

$$\big(\forall L \in \mathcal{P}_f(G)\big)\big((\forall s \in H)(|L \setminus (s + L)| < \epsilon \cdot |L| \Rightarrow |A \cap L| < \alpha \cdot |L|\big)\,.$$

Pick $k \in \mathbb{N}$ such that for all $s \in H$, $\mathrm{supp}(s) \subseteq \{0, 1, \ldots, k\}$. Pick $n > k$ and $d \in G$ such that $|A \cap (d + K_n)| \geq \alpha \cdot |K_n|$. Let $L = d + K_n$. Given $s \in H$, $s + K_n = K_n$ so $|L \setminus (s + L)| = 0 < \epsilon \cdot |L|$, a contradiction.

Now suppose that we have α and γ such that $\delta < \gamma < \alpha < d(A)$. Pick $l \in \mathbb{N}$ such that for all $n \geq l$ and all $a \in G$, $|A \cap (a + K_n)| < \gamma \cdot |K_n|$. Let $\epsilon = \dfrac{\alpha - \gamma}{|K_l|}$. Pick $M \in \mathcal{P}_f(G)$ such that $(\forall s \in K_l)(|M \setminus (s + M)| < \epsilon \cdot |M|)$ and $|A \cap M| \geq \alpha \cdot |M|$. Let $L = \bigcup\{a + K_l : a \in G \text{ and } a + K_l \subseteq M\}$. Since each $a + K_l$ is a coset of K_l, if

$(a+K_l)\cap(b+K_l)\neq\emptyset$, then $a+K_l=b+K_l$. Pick D such that $L=\bigcup_{a\in D}(a+K_l)$ and if $a,b\in D$ and $a\neq b$, then $(a+K_l)\cap(b+K_l)=\emptyset$. Thus

$$|A\cap L|=\textstyle\sum_{a\in D}|A\cap(a+K_l)|<\sum_{a\in D}\gamma\cdot|K_l|=\gamma\cdot|L|\leq\gamma\cdot|M|\,.$$

Also $M\setminus L\subseteq\bigcup_{s\in K_l}\big(M\setminus(s+M)$ so $|M\setminus L|\leq|K_l|\cdot\epsilon\cdot|M|$. Thus $|A\cap M|\leq|A\cap L|+|M\setminus L|<\gamma\cdot|M|+|K_l|\cdot\epsilon\cdot|M|=\alpha\cdot|M|\leq|A\cap M|$, a contradiction. $\square$

5.3 Theorem. *There is a set $A\subsetneq G$ such that $d(A)=0$ and A is a C-set in G,*

Proof. For $n\in\mathbb{N}$, let $a_n=\min\{t\in\mathbb{N}:(\frac{2^n-1}{2^n})^t\leq\frac{1}{2}\}$ and let $s_n=\sum_{i=1}^n a_i$. (So $s_1=1$ and $s_2=4$.) Let $b_0=0$, let $b_1=1$, and for $n\in\mathbb{N}$ and $t\in\{s_n,s_n+1,s_n+2,\ldots,s_{n+1}-1\}$, let $b_{t+1}=b_t+n+1$. For $k\in\omega$, let $B_k=\{b_k,b_k+1,b_k+2,\ldots,b_{k+1}-1\}$. Let

$$A=\{x\in G:(\forall k\in\omega)(B_k\setminus\mathrm{supp}(x)\neq\emptyset)\}\,.$$

We show first that $d(A)=0$. We claim that for any $d\in G$ and $l\in\mathbb{N}$,

$$|A\cap(d+K_l)|\leq|A\cap K_l|\,.$$

To see this, note that we may presume that $\mathrm{supp}(d)\cap\{0,1,\ldots,l\}=\emptyset$, since if

$$d'(i)=\begin{cases}d(i) & \text{if } i>l\\ 0 & \text{if } i\leq l\,,\end{cases}$$

then $d'+K_l=d+K_l$. It follows that $d+(K_l\setminus A)\subseteq(d+K_l)\setminus A$. (If $y\in K_l\setminus A$, then there is some k with $b_{k+1}-1\leq l$ such that $B_k\subseteq\mathrm{supp}(y)$ and so $B_k\subseteq\mathrm{supp}(d+y)$.) Thus $|A\cap(d+K_l)|+|K_l\setminus A|\leq|A\cap(d+K_l)|+|(d+K_l)\setminus A|=|d+K_l|=|K_l|=|A\cap K_l|+|K_l\setminus A|$ so $|A\cap(d+K_l)|\leq|A\cap K_l|$ as required.

Now let $d\in G$, $m\in\mathbb{N}$, $k=s_{m+1}$, and $l\geq b_k$. We shall show that $|A\cap(d+K_l)|\leq(\frac{1}{2})^m\cdot|K_l|$ for which it suffices that $|A\cap K_l|\leq(\frac{1}{2})^m\cdot|K_l|$. To this end, we first note that for any $i\in\omega$, $\frac{\nu(i)}{\nu(i)-1}\leq 2$ so for $t\in\mathbb{N}$, $\prod_{i=b_t}^{b_t+n}\Big(\frac{\nu(i)}{\nu(i)-1}\Big)\leq 2^{n+1}$ and so

$$\frac{\prod_{i=b_t}^{b_t+n}\nu(i)-\prod_{i=b_t}^{b_t+n}(\nu(i)-1)}{\prod_{i=b_t}^{b_t+n}\nu(i)}\leq\left(\frac{2^{n+1}-1}{2^{n+1}}\right)\,.$$

Now let $T=\big\{x\in G:\mathrm{supp}(x)\subseteq\{b_k,b_k+1,\ldots,l\}\big\}$. Then

$$|A\cap K_l|=\textstyle\sum_{x\in T}|A\cap(x+K_{b_k-1})|\leq|T|\cdot|A\cap K_{b_k-1}|$$

and $K_l=|T|\cdot|K_{b_k-1}|$ so it suffices to show that $|A\cap K_{b_k-1}|\leq\big(\frac{1}{2}\big)^m\cdot|K_{b_k-1}|$. Now $|A\cap K_{b_k-1}|=\prod_{t=0}^{k-1}|\{x\in G:\mathrm{supp}(x)\subsetneq B_t\}|$ and $\{x\in G:\mathrm{supp}(x)\subsetneq B_0\}=\{0\}$ so

$$\begin{aligned}|A\cap K_{b_k-1}| &= \textstyle\prod_{t=1}^{k-1}|\{x\in G:\mathrm{supp}(x)\subsetneq B_t\}|\\ &= \textstyle\prod_{n=1}^{m}\prod_{t=s_n}^{s_{n+1}-1}|\{x\in G:\mathrm{supp}(x)\subsetneq B_t\}|\\ &= \textstyle\prod_{n=1}^{m}\prod_{t=s_n}^{s_{n+1}-1}\big(\prod_{i=b_t}^{b_t+n}\nu(i)-\prod_{i=b_t}^{b_t+n}(\nu(i)-1)\big)\end{aligned}$$

and $|K_{b_k-1}| = \prod_{n=1}^m \prod_{t=s_n}^{s_{n+1}-1} \prod_{i=b_t}^{b_t+n} \nu(i)$ so

$$\begin{array}{rcl} \dfrac{|A\cap K_{b_k-1}|}{|K_{b_k-1}|} & = & \prod_{n=1}^m \prod_{t=s_n}^{s_{n+1}-1} \left(\dfrac{\prod_{i=b_t}^{b_t+n}\nu(i) - \prod_{i=b_t}^{b_t+n}(\nu(i)-1)}{\prod_{i=b_t}^{b_t+n}\nu(i)}\right) \\ & \leq & \prod_{n=1}^m \prod_{t=s_n}^{s_{n+1}-1} \left(\frac{2^{n+1}-1}{2^{n+1}}\right) \\ & = & \prod_{n=1}^m \left(\frac{2^{n+1}-1}{2^{n+1}}\right)^{a_{n+1}} \\ & \leq & \left(\frac{1}{2}\right)^m . \end{array}$$

We have thus shown that $d(A) = 0$.

Now we claim that if $n, k \in \mathbb{N}$, $b_{k+1} - b_k \geq n$, and $z_0, z_1, \ldots, z_{n-1} \in G$, then there is some $x \in G$ such that $\mathrm{supp}(x) \subseteq B_k$ and for all $t \in \{0, 1, \ldots, n-1\}$, $B_k \setminus \mathrm{supp}(x+z_t) \neq \emptyset$. To see this, define x by, for $i \in \{0, 1, \ldots, n-1\}$, $x(b_k+i) = \nu(b_k+i) - z_i(b_k+i)$ and $x(t) = 0$ otherwise.

For $n \in \omega$, let $C_n = \{x \in G : \min \mathrm{supp}(x) \geq b_n \text{ and } (\forall k \in \omega)(B_k \setminus \mathrm{supp}(x) \neq \emptyset)\}$. Then $C_0 = A$.

We claim that

(i) for all $n \in \mathbb{N}$ and all $x \in C_n$, there exists $m \in \mathbb{N}$ such that $C_m \subseteq x^{-1}C_n$ and

(ii) for all $n \in \mathbb{N}$, C_n is a J-set.

so that by [18, Theorem 2.6], A is a C-set. To see this, let $n \in \mathbb{N}$ and let $x \in C_n$. Pick $m \in \mathbb{N}$ such that $b_m > \max \mathrm{supp}(x)$. Then $C_m \subseteq -x + C_n$.

Next let $n \in \mathbb{N}$. We claim that C_n is a J-set, so let $F \in \mathcal{P}_f(\mathcal{T})$. Let $r = |F|$ and pick $k \geq n$ such that $b_{k+1} - b_k > r$. First choose an infinite subset M of $\mathbb{N}$ such that for all $t, s \in M$, all $f \in F$, and all $i \in \{0, 1, \ldots, b_k - 1\}$, $f(t)(i) = f(s)(i)$. Then pick $H \in \mathcal{P}_f(\mathbb{N})$ such that $\nu(i)$ divides $|H|$ for each $i \in \{0, 1, \ldots, b_k - 1\}$. Then for each $f \in F$, $\min \mathrm{supp}\big(\sum_{t\in H} f(t)\big) \geq b_k$. Pick $s \in \mathbb{N}$ such that for all $f \in F$, $\max \mathrm{supp}\big(\sum_{t\in H} f(t)\big) < b_{s+1}$. For $l \in \{k, k+1, \ldots, s\}$ pick x_l with $\mathrm{supp}(x_l) \subseteq B_l$ such that for each $f \in F$, $B_l \setminus \mathrm{supp}\big(x_l + \sum_{t\in H} f(t)\big) \neq \emptyset$. Let $d = \sum_{l=k}^s x_l$. □

5.4 Corollary. *There is an idempotent in $J(G) \setminus \Delta^*(G)$.*

Proof. Pick A as guaranteed by Theorem 5.3. By Theorem 2.5, A is a member of an idempotent in $J(G)$ and $\overline{A} \cap \Delta^*(G) = \emptyset$. □

6. Large subsets of commutative semigroups

In this section we generalize Theorem 1.6 by proving Theorem 1.10 (as Theorem 6.12). We first establish some basic facts about density and $\Delta^*(S)$ for arbitrary semigroups which satisfy SFC. The following lemma establishes that if $K \setminus sK$ is small, then λ_s is nearly one-to-one on $K \cap s^{-1}K$.

6.1 Lemma. *Let $(S, \cdot)$ be a semigroup, let $\delta > 0$, let $s \in S$, and let $K \in \mathcal{P}_f(S)$ such that $|K \setminus sK| < \delta \cdot |K|$. Let $G = \{x \in K :$ there is a unique $t \in K$ such that $x = st\}$. Then $|G| \geq (1 - 2\delta) \cdot |K|$.*

Proof. Let $n = |K|$, let $m = |G|$, let $H = \{x \in K :$ there exist $t \neq t'$ in K such that $x = st = st'\}$, and let $k = |H|$. We first note that $n \geq 2k + m$. To see this, for each $x \in G$, pick $t_x \in K$ such that $x = st_x$ and for $x \in H$ pick $u_x \neq v_x$ in K such that $x = su_x = sv_x$. Then $\{t_x : x \in G\} \cup \{u_x : x \in H\} \cup \{v_x : x \in H\}$ is a subset of K with $2k + m$ elements.

Now $K \setminus H = (K \setminus sK) \cup G$ so $n - k = |K \setminus H| \leq \delta \cdot |K| + m$. Thus $2k + m - k \leq n - k \leq \delta \cdot |K| + m$ so $k \leq \delta \cdot |K|$ and thus $m + \delta \cdot |K| \geq n - k \geq n - \delta \cdot |K|$. □

6.2 Lemma. *Let $(S, \cdot)$ be a semigroup, let $A \subseteq S$, let $s \in S$, let $\delta > 0$, and let $K \in \mathcal{P}_f(S)$ such that $|K \setminus sK| < \delta \cdot |K|$. Then $\big||s^{-1}A \cap K| - |A \cap K|\big| < 2\delta \cdot |K|$.*

Proof. Let $G = \{x \in K :$ there is a unique $t \in K$ such that $x = st\}$. By Lemma 6.1, $|G| \geq (1 - 2\delta) \cdot |K|$.

Notice that $\lambda_s : s^{-1}G \cap K \xrightarrow[\text{onto}]{\text{1-1}} G$ and so also $\lambda_s : s^{-1}A \cap s^{-1}G \cap K \xrightarrow[\text{onto}]{\text{1-1}} A \cap G$. Now

$$\begin{aligned} |K \setminus s^{-1}G| &= |K| - |s^{-1}G \cap K| \\ &= |K| - |G| \\ &\leq |K| - (1 - 2\delta) \cdot |K| \\ &= 2\delta \cdot |K| \end{aligned}$$

and $|K \setminus G| = |K| - |G| \leq 2\delta \cdot |K|$. Thus

$$\begin{aligned} |s^{-1}A \cap K| &\leq |s^{-1}A \cap s^{-1}G \cap K| + |K \setminus s^{-1}G| \\ &= |A \cap G| + |K \setminus s^{-1}G| \\ &\leq |A \cap K| + 2\delta \cdot |K| \end{aligned}$$

and

$$\begin{aligned} |A \cap K| &\leq |A \cap G| + |K \setminus G| \\ &= |s^{-1}A \cap s^{-1}G \cap K| + |K \setminus G| \\ &\leq |s^{-1}A \cap K| + 2\delta|K|\,. \end{aligned}$$

□

We know from [15, Theorem 4.17] that if S is left cancellative and satisfies SFC, then for all $t \in S$ and $A \subseteq S$, $d(A) = d(t^{-1}A) = d(tA)$. We see now that the first of these equalities holds in an arbitrary semigroup satisfying SFC.

6.3 Theorem. *Let $(S, \cdot)$ be a semigroup satisfying SFC, let $A \subseteq S$, and let $t \in S$. Then $d(t^{-1}A) = d(A)$.*

Proof. Suppose first that $d(t^{-1}A) < d(A)$ and pick $\alpha > 0$ and $\delta > 0$ such that $d(t^{-1}A) < \alpha - \delta < \alpha + \delta < d(A)$. Pick $H \in \mathcal{P}_f(S)$ and $\epsilon > 0$ such that for all $K \in \mathcal{P}_f(S)$, if $(\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|)$, then $|t^{-1}A \cap K| < (\alpha - \delta) \cdot |K|$. Let $H' = H \cup \{t\}$ and pick $K \in \mathcal{P}_f(S)$ such that $(\forall s \in H')(|K \setminus sK| < \min\{\epsilon, \delta\} \cdot |K|)$ and $|A \cap K| > (\alpha + \delta) \cdot |K|$. Then, by Lemma 6.2,

$$2\delta \cdot |K| = (\alpha + \delta) \cdot |K| - (\alpha - \delta) \cdot |K| < |A \cap K| - |t^{-1}A \cap K| < 2\delta \cdot |K| \,,$$

a contradiction.

The proof that $d(t^{-1}A) \leq d(A)$ is essentially identical. $\square$

We do not need the following result, but feel that it is worth noting.

6.4 Theorem. *Let $(S\cdot)$ be a semigroup satisfying SFC, let $A \subseteq S$, and let $t \in S$. Then $d(tA) \geq d(A)$. There exist a countable commutative semigroup $(S, +)$ satisfying SFC and a subset A of S such that $d(A) = 0$ but for each $t \in S$, λ_t is exactly two-to-one and $d(t + A) = 1$.*

Proof. For the first assertion, note that $A \subseteq t^{-1}(tA)$ so that $d(A) \leq d\big(t^{-1}(tA)\big) = d(tA)$ by Theorem 6.3.

For the second assertion, let $S = (\{0, 1\}, +) \times (\mathbb{N}, +)$ where $0 + 0 = 0 + 1 = 1 + 0 = 1 + 1 = 0$, let $A = \{1\} \times \mathbb{N}$, and let $t \in S$. Since S is commutative, we have that S satisfies SFC by Theorem 7.2. (Or see the verification below that $d(t + A) = 1$ which establishes also that S satisfies SFC.)

To see that $d(A) = 0$, suppose instead one has some $\epsilon > 0$ such that $d(A) > \epsilon$. Pick $K \in \mathcal{P}_f(S)$ such that $|K \setminus (t+K)| < \epsilon \cdot |K|$ and $|A \cap K| \geq \epsilon \cdot |K|$. Then $A \cap K \subseteq K \setminus (t+K)$ so $\epsilon \cdot |K| < \epsilon \cdot |K|$, a contradiction.

To see that $d(t + A) = 1$, let $H \in \mathcal{P}_f(S)$ and let $\epsilon > 0$. Let $F = \pi_2[H]$. Pick $L \in \mathcal{P}_f(\mathbb{N})$ such that $\min L > \pi_2(t)$ (so that $L \subseteq \pi_2(t) + \mathbb{N}$) and for all $s \in F$, $|L \setminus s + L| < \epsilon \cdot |L|$. (Any sufficiently long interval begining after $\pi_2(t)$ will do.) Let $K = \{0\} \times L$. Then for any $s \in H$, $|K \setminus s + K| = |L \setminus s + L| < \epsilon \cdot |L| = |L \setminus s + L| < \epsilon \cdot |K|$, and $K \subseteq t + A$. $\square$

6.5 Theorem. *Let $(S,\cdot)$ be a semigroup which satisfies SFC. Then $\Delta^*(S)$ is a left ideal of $(\beta S,\cdot)$. If there exists $b\in\mathbb{N}$ such that for all $x\in S$, ρ_x is at most b-to-1, then $\Delta^*(S)$ is a right ideal of $(\beta S,\cdot)$.*

Proof. It is routine to verify that if A and B are subsets of S and $d(A\cup B)>0$, then $d(A)>0$ or $d(B)>0$ and so, by [14, Theorem 3.11], $\Delta^*(S)\neq\emptyset$.

To see that $\Delta^*(S)$ is a left ideal, let $p\in\Delta^*(S)$, let $q\in\beta S$, and let $A\in q\cdot p$. Pick $x\in S$ such that $x^{-1}A\in p$. Then $d(A)=d(x^{-1}A)>0$ by Theorem 6.3.

Now assume that we have $b\in\mathbb{N}$ such that for all $x\in S$, ρ_x is at most b-to-1. Let $p\in\Delta^*(S)$, let $q\in\beta S$, and let $A\in p\cdot q$. Let $B=\{x\in S: x^{-1}A\in q\}$. Then $B\in p$ so pick $\alpha>0$ such that $d(B)>\alpha$. We claim that $d(A)\geq\frac{\alpha}{b}$. Suppose instead that $d(A)<\frac{\alpha}{b}$. Pick $H\in\mathcal{P}_f(S)$ and $\epsilon>0$ such that for all $K\in\mathcal{P}_f(S)$, if $(\forall s\in H)(|K\setminus sK|<\epsilon\cdot|K|)$, then $|K\cap A|<\frac{\alpha}{b}\cdot|K|$.

Pick $L\in\mathcal{P}_f(S)$ such that $(\forall s\in H)(|L\setminus sL|<\frac{\epsilon}{b}\cdot|L|)$ and $|B\cap L|>\alpha\cdot|L|$. Then $\bigcap_{x\in B\cap L}x^{-1}A\in q$ so pick $y\in\bigcap_{x\in B\cap L}x^{-1}A$. Let $K=Ly$. Now, given $s\in H$, $K\setminus sK\subseteq\rho_y[L\setminus sL]$ so

$$|K\setminus sK|\leq|L\setminus sL|<\frac{\epsilon}{b}\cdot|L|\leq\epsilon\cdot|K|\,.$$

Now $\rho_y: B\cap L\to A\cap K$ and ρ_y is at most b-to-1, so

$$\alpha\cdot|L|<|B\cap L|\leq b\cdot|A\cap K|<\alpha\cdot|K|\leq\alpha\cdot|L|\,,$$

a contradiction. □

6.6 Corollary. *Let $(S,\cdot)$ be a semigroup which satisfies SFC. Then $\Delta^*(S)$ is a left ideal of $(\beta S,\cdot)$. If S is left cancellative or there exists $b\in\mathbb{N}$ such that for all $x\in S$, ρ_x is at most b-to-1, then $\Delta^*(S)$ is a right ideal of $(\beta S,\cdot)$.*

Proof. Assume that S is left cancellative. By [16, Theorems 2.12, 2.14, and 5.9], Δ^* is a right ideal of $(\beta S,\cdot)$. The rest of the corollary follows from Theorem 6.5. □

6.7 Definition. Let S be a semigroup. Then

$$\Delta^m(S)=\{x\in\beta S:(\forall A\in x)\big(\exists\mu\in LIM(S)\big)(\mu(\,\overline{A}\,)>0)\}\,.$$

6.8 Theorem. *Let S be a left amenable semigroup. Then $\Delta^m(S)$ is a closed ideal of βS, $\Delta^*(S)\subseteq\Delta^m(S)$, and if S is left cancellative, $\Delta^*(S)=\Delta^m(S)$.*

Proof. It is clear that $\Delta^m(S)$ is closed. To see that it is a left ideal, let $x\in\Delta^*(S)$, let $y\in\beta S$, and let $A\in yx$. Pick $s\in S$ such that $s^{-1}A\in x$ and pick $\mu\in LIM(S)$ such that $\mu(\,\overline{s^{-1}A}\,)>0$. Then $\mu(\,\overline{A}\,)=\mu(\,\overline{s^{-1}A}\,)>0$.

To see that $\Delta^m(S)$ is a right ideal, let $x \in \Delta^*(S)$, let $y \in \beta S$, and let $B \in xy$. Then $\rho_y^{-1}[\,\overline{B}\,]$ is a neighborhood of x in βS and therefore $\rho_y^{-1}[\,\overline{B}\,] \cap S \in x$. So $\nu(\rho_y^{-1}[\,\overline{B}\,]) > 0$ for some $\nu \in LIM(S)$. We can define a Borel probability measure σ on βS by putting $\sigma(E) = \nu(\rho_y^{-1}[E])$ for every Borel subset E of βS. Since $\rho_y^{-1}[s^{-1}E] = s^{-1}\rho_y^{-1}[E]$ for every $E \subseteq \beta S$, σ is left invariant. We claim that σ is regular. To see this, let E be a Borel subset of βS and let $\varepsilon > 0$. We can choose a compact subset C of $\rho_y^{-1}[E]$ such that $\nu(C) > \nu(\rho_y^{-1}[E]) - \varepsilon$. Now $\rho_y[C]$ is a compact subset of E. Since $C \subseteq \rho_y^{-1}\rho_y[C]$, we have $\sigma(\rho_y[C]) = \nu(\rho_y^{-1}\rho_y[C]) \geq \nu(C) > \nu(\rho_y^{-1}[E]) - \varepsilon = \sigma(E) - \varepsilon$. So σ is regular and hence $\sigma \in LIM(S)$. Since $\sigma(\overline{B}) > 0$, it follows that $xy \in \Delta^m(S)$. Thus $\Delta^m(S)$ is a right ideal.

That $\Delta^*(S) \subseteq \Delta^m(S)$ follows from Lemma 2.2. Finally, assume that S is left cancellative and let $p \in \Delta^m(S)$. To see that $p \in \Delta^*(S)$, let $A \in p$. Pick $\mu \in LIM(S)$ such that $\mu(\,\overline{A}\,) > 0$. By [16, Theorems 2.12 and 2.14] we have that $d(A) \geq \mu(\,\overline{A}\,)$. $\square$

We now concentrate on a proof of the second assertion of Theorem 1.10. For this proof we shall need (as did the authors of [3]) the following strong result of Furstenberg and Katznelson.

6.9 Theorem. *Let F be a finite set, let X be a compact metric space, let $\mathcal{E}$ be a σ-algebra of subsets of X, let ν be a nonnegative countably additive measure on $\mathcal{E}$ with $\nu(X) = 1$, and for each $n \in \mathbb{N}$ and $f \in F$, let $R_n^f : X \to X$ be a continuous transformation such that for each $E \in \mathcal{E}$, $(R_n^f)^{-1}[E] \in \mathcal{E}$ and $\nu\big((R_n^i)^{-1}[E]\big) = \nu(E)$. Assume further that if $n, m \in \mathbb{N}$ and $f, g \in F$, then $R_n^f \circ R_m^g = R_m^g \circ R_n^f$. If $E \in \mathcal{E}$ and $\nu(E) > 0$, then there exists $k \in \mathbb{N}$ and $n_1 < n_2 < \ldots < n_k$ in $\mathbb{N}$ such that $\nu\big(E \cap \bigcap_{f \in F}(R_{n_1}^f \circ R_{n_2}^f \circ \ldots \circ R_{n_k}^f)^{-1}[E]\big) > 0$.*

Proof. [11, Theorem A]. $\square$

Previous applications of Theorem 6.9 to semigroups have relied on Furstenberg's Correspondence Principle, and as such have been restricted to countable semigroups. By using a metrizable quotient of βS we avoid that restriction in the following theorem.

6.10 Theorem. *Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$ be such that $\mu(\,\overline{A}\,) > 0$ for some $\mu \in LIM(S)$. Then A is a J-set.*

Proof. Let $\mathcal{T} = {}^{\mathbb{N}}S$ and let $F \in \mathcal{P}_f(\mathcal{T})$. Let M denote the subsemigroup of S generated by $\{f(n) : f \in F \text{ and } n \in \mathbb{N}\}$ and let $\mathcal{B}$ denote the countable Boolean algebra of subsets of S generated by $\{A\} \cup \{-t + A : t \in M\}$. We define an equivalence relation $\sim$ on βS by stating that $x \sim y$ if and only if $(\forall B \in \mathcal{B})(B \in x \Leftrightarrow B \in y)$.

Let $X = \beta S/\sim$ and let $\pi : \beta S \to X$ be the projection map. Note that for $B \in \mathcal{B}$, $\pi^{-1}\big[\pi[\,\overline{B}\,]\big] = \overline{B}$. Consequently X is Hausdorff and, as the continuous image of a compact space, is compact. Also, if $B \in \mathcal{B}$, then $\pi[\,\overline{B}\,]$ is open (in fact clopen) so the topology with basis $\{\pi[\,\overline{B}\,] : B \in \mathcal{B}\}$ is a Hausdorff topology contained in the (compact Hausdorff) quotient topology. These topologies must therefore be equal. That is, the quotient topology has a countable base, so by the Urysohn Metrization Theorem, X is metrizable.

For each $t \in M$, define $T_t : X \to X$ by $T_t\big(\pi(x)\big) = \pi(t + x)$. To see that T_t is well defined, assume that $x \sim y$ and let $B \in \mathcal{B}$. Then

$$\begin{aligned} B \in (t + x) &\Leftrightarrow (-t + B) \in x \\ &\Leftrightarrow (-t + B) \in y \\ &\Leftrightarrow B \in (t + y)\,. \end{aligned}$$

Since for each $B \in \mathcal{B}$, $T_t^{-1}\big[\pi[\,\overline{B}\,]\big] = \pi[\,\overline{-t + B}\,]$ we have that T_t is continuous.

Let ν be the image measure of μ defined by $\nu(E) = \mu(\pi^{-1}[E])$ for every Borel subset E of X. Now let $t \in M$ and let $B \in \mathcal{B}$. Then

$$\begin{aligned} \nu(\pi[\,\overline{B}\,]) &= \mu\big(\pi^{-1}\big[\pi[\,\overline{B}\,]\big]\big) \\ &= \mu[\,\overline{B}\,] \\ &= \mu[\,\overline{-t + B}\,] \\ &= \mu\big(\pi^{-1}\big[\pi[\,\overline{-t + B}\,]\big]\big) \\ &= \nu(\pi[\,\overline{-t + B}\,]) \\ &= \nu\big(T_t^{-1}\big[\pi[\,\overline{B}\,]\big]\big)\,. \end{aligned}$$

Since ν and $\nu \circ T_t^{-1}$ agree on a countable basis for the topology for X, they agree on every Borel subset of X. That is, if E is a Borel subset of X and $t \in M$, then $\nu(E) = \nu(T_t^{-1}[E])$. Note that for any $t, s \in M$, $T_t \circ T_s = T_{t+s} = T_{s+t} = T_s \circ T_t$.

For $f \in F$ and $n \in \mathbb{N}$ we put $R_n^f = T_{f(n)}$. The hypotheses of Theorem 6.9 are satisfied with $\mathcal{E}$ as the set of Borel subsets of X. Since $\nu(\pi[\,\overline{A}\,]) = \nu[\,\overline{A}\,]$, it follows that there exist $k \in \mathbb{N}$ and $n_1 < n_2 < \ldots < n_k$ in $\mathbb{N}$ such that

$$\nu\big(\pi[\,\overline{A}\,] \cap \textstyle\bigcap_{f \in F} (R_{n_1}^f \circ R_{n_2}^f \circ \ldots \circ R_{n_k}^f)^{-1}\big[\pi[\,\overline{A}\,]\big]\big) > 0\,.$$

Pick $x \in \beta S$ such that $\pi(x) \in \pi[\,\overline{A}\,] \cap \bigcap_{f \in F} (R_{n_1}^f \circ R_{n_2}^f \circ \ldots \circ R_{n_k}^f)^{-1}\big[\pi[\,\overline{A}\,]\big]$. Let $H = \{n_1, n_2, \ldots, n_k\}$. Then $A \cap \bigcap_{f \in F} (-\sum_{n \in H} f(n) + A) \in x$. Pick

$$d \in A \cap \textstyle\bigcap_{f \in F} (-\sum_{n \in H} f(n) + A)\,.$$

Then $d + \sum_{n \in H} f(n) \in A$ for every $f \in F$. So A is a J-set. □

6.11 Theorem. *Let S be a commutative semigroup and let E be a subset of S. If $\Delta^m(S) \cap c\ell(E)$ contains an idempotent, then E is a C-set.*

Proof. Theorems 6.10 and 2.5. □

We now restate Theorem 1.10.

6.12 Theorem.

(a) Let $(S, \cdot)$ be a left cancellative semigroup which satisfies SFC. Then $\Delta^*(S)$ is a two sided ideal of $(\beta S, \cdot)$ so if C is a central subset of S, then $d(C) > 0$.

(b) If $(S, +)$ is any commutative semigroup and if E is a subset of S for which $\Delta^*(S) \cap c\ell(E)$ contains an idempotent, then E is a C-set.

Proof. By Corollary 6.6, $\Delta^*(S)$ is a two sided ideal of βS and therefore $K(\beta S) \subseteq \Delta^*(S)$.

Now assume that S is commutative, that $E \subseteq S$, and that there is an idempotent in $\Delta^*(S) \cap c\ell(E)$. By Theorem 6.10 and Lemma 2.2, $\Delta^*(S) \subseteq J(S)$ so Theorem 2.5 applies. □

7. A simple elementary proof that commutative semigroups satisfy SFC

Argabright and Wilde [2, Theorem 4] established that all commutative semigroups satisfy SFC, using the fact that all commutative cancellative semigroups satisfy SFC. (We present a version of their proof in Theorem 7.2 below.) However, showing this latter fact involved the following chain of reasoning.

First, any commutative semigroup is amenable. This fact is stated in [5]. (The review in *Mathematical Reviews* incorrectly says that it is proved in [5]. In fact in [5] one is simply referred to [4] for the proof. And the result is not explicitly stated in [4] – the term "amenable" occurs nowhere in that paper.)

Second, any left amenable semigroup satisfies FC. This fact is due to A. Frey [9] and is based on the proof by E. Følner [8] for groups. A simplified proof is given by Namioka [20, Theorem 3.5].

Third, any left cancellative semigroup which satisfies FC also satisfies SFC. This is an easy elementary fact since then $|(s + K) \setminus K| = |K \setminus (s + K)|$ for any $K \in \mathcal{P}_f(S)$ and any $s \in S$.

We felt that, since the assumption and conclusion were both algebraic and elementary, there should be an elementary proof of the fact that all commutative semgroups satisfy SFC. We present such a proof now.

7.1 Theorem. *Let $(S,+)$ be a commutative cancellative semigroup. Then S satisfies FC (and therefore satisfies SFC).*

Proof. Let $F \in \mathcal{P}_f(S)$ and let $\epsilon > 0$. Pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Let

$$K = \left\{ \textstyle\sum_{s \in F} m_s s : \text{for each } s \in F\,,\, m_s \in \{1,2,\ldots,n\}\right\}.$$

Now let $t \in F$ be given. We shall show that $|(t+K) \setminus K| \leq \frac{1}{n} \cdot |K|$. Let

$$M_t = \left\{ \textstyle\sum_{s \in F \setminus \{t\}} m_s s : \text{for each } s \in F \setminus \{t\}\,,\, m_s \in \{1,2,\ldots,n\}\right\}.$$

If $t + K \subseteq K$, we are done so assume that $(t+K) \setminus K \neq \emptyset$ and define

$$\psi : \big((t+K) \setminus K\big) \times \{1,2,\ldots,n\} \to K$$

as follows. Given $x \in (t+K) \setminus K$ we have, since $x \notin K$, that $x = (n+1)t + u$ for some $u \in M_t$. Notice that, since S is cancellative, u is uniquely determined (even though the choice of the m_s's need not be). Define $\psi(x,k) = kt + u$.

We claim that ψ is injective so that $|(t+K) \setminus K\big)| \leq \frac{1}{n} \cdot |K|$ as required. To this end let (x,k) and (y,l) be in $\big((t+K) \setminus K\big) \times \{1,2,\ldots,n\}$ and assume that $\psi(x,k) = \psi(y,l)$. Pick $u,v \in M_t$ such that $x = (n+1)t + u$ and $y = (n+1)t + v$. Then $kt + u = lt + v$. If $k = l$, then $u = v$ so that $(x,k) = (y,l)$ as required. So suppose without loss of generality that $k < l$. Then $(k+n+1-l)t + u = (n+1)t + v$ so that $y \in K$, a contradiction. $\square$

The following proof is a simplification of the proof of [2, Theorem 4].

7.2 Theorem. *Let $(S,+)$ be a commutative semigroup. Then S satisfies SFC.*

Proof. Define a relation R on S by $xRy \Leftrightarrow (\exists u \in S)(x+u = y+u)$. Then R is an equivalence relation on S. For $x \in S$, let $[x]$ denote the R-equivalence class of x. The operation $[x]+[y] = [x+y]$ is well defined and makes S/R into a cancellative commutative semigroup, which satisfies SFC by Theorem 7.1.

To see that S satisfies SFC, let $F \in \mathcal{P}_f(S)$ and $\epsilon > 0$ be given. Pick $B \in \mathcal{P}_f(S/R)$ such that for all $x \in F$, $|B \setminus ([x]+B)| < \epsilon \cdot |B|$. Choose A as a set of representatives for B. That is, $B = \{[x] : x \in A\}$ and if $x,y \in A$ and $[x] = [y]$, then $x = y$. Let $C = \{(x,a,b) : x \in F\,,\, a,b \in A \text{ and } [x+a] = [b]\}$.

Given $(x,a,b) \in C$, $\{u \in S : x+a+u = b+u\}$ is an ideal of S and the finite intersection of ideals is an ideal so we may pick $u \in S$ such that for all $(x,a,b) \in C$, $x+a+u = b+u$. Let $D = \{a+u : a \in A\}$.

Define $\varphi : D \to B$ by $\varphi(a+u) = [a]$ and observe that φ is well defined, one-to-one, and onto B, and so $|D| = |B|$. Now let $x \in F$. We claim that $\varphi[D \setminus (x+D)] \subseteq B \setminus ([x]+B)$. (Actually equality holds, but this is all we need.) To this end, let $b \in A$ such that $b+u \in D \setminus (x+D)$. We claim that $[b] \in B \setminus ([x]+B)$ so suppose instead that we have some $a \in A$ such that $[b] = [x+a]$. Then $(x,a,b) \in C$ so $x+a+u = b+u$ and therefore $b+u \in x+D$, a contradiction.

We thus have that $|D \setminus (x+D)| \leq |B \setminus ([x]+B)| < \epsilon \cdot |B| = \epsilon \cdot |D|$ as required. $\square$

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