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The Research of Thirteen Students at Howard University☆

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Abstract

Results from, and problems related to, the dissertations of my Ph.D. students are discussed.

Key words:

1. Introduction

I am deeply honored by the existence of this conference and those who chose to attend. Whenever I give a lecture and there are young people present I repeat the following advice, which has done wonders for me throughout my lengthy career. That is to find people who are smarter than you are and get them to put your name on their papers. All of the invited speakers at this conference fit that description, and before I get to the main topic, I would like to record my gratitude to each of them.

Dona Strauss has collaborated with me on forty three papers and a book since I first met her in 1990. She informs me that I was polite at the time, but I confess that I was not impressed at our initial meeting when I was introduced to her by John Pym. She had just learned about the topic of algebra in the Stone-Čech compactification, and told me some things I already knew. Soon thereafter, we began a correspondence – initially pen and paper, envelopes, and stamps – and I soon discovered that she could prove circles around me.

☆This is an expanded version of an address given July 27, 2008 at the *Conference on Ramsey Theory and Topological Algebra* at Miami University.

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I was introduced to Vitaly Bergelson in a letter from Bruce Rothschild who was visiting in Jerusalem where Vitaly was a student. Since then Vitaly and I have collaborated on twenty two papers. And I shall be eternally grateful to Vitaly for introducing me to central sets, a topic to which I will return later.

Another topic to which I will return later is image partition regularity of matrices. A matrix A is *kernel partition regular* over $\mathbb{N}$ if and only if whenever $\mathbb{N}$ is partitioned into finitely many cells (or "finitely colored") there exists a vector $\vec{x}$, all of whose entries are in the same cell (or "are monochromatic") such that $A\vec{x} = \vec{0}$. The matrix A is *image partition regular* over $\mathbb{N}$ if and only if whenever $\mathbb{N}$ is finitely colored there exists a vector $\vec{x}$ with entries from $\mathbb{N}$ such that the entries of $A\vec{x}$ are monochromatic. (The terms "kernel" and "image" both refer to the linear transformation $\vec{x} \mapsto A\vec{x}$.) Finite kernel partition regular matrices were completely characterized by R. Rado in 1932 [29]. Rado called a subset of $\mathbb{N}$ *large* provided it contained solutions to all kernel partition regular matrices and conjectured that whenever a large set was partitioned into finitely many pieces, one of those pieces must be large. This conjecture was proved by W. Deuber in 1973 who used certain image partition regular matrices in his proof. Especially since image partition regular matrices are naturally associated with many of the classic theorems of Ramsey Theory, I was surprised to discover in the late 1980's that there was no known characterization of finite image partition regular matrices. I worked on the problem and only succeeded in characterizing *weakly image partition regular* matrices. (The definition is the same as for image partition regular matrices except that the entries of $\vec{x}$ are allowed to come from $\mathbb{Z}$.) I wrote to Imre Leader with my solution and he succeeded in coming up with the first characterization of image partition regular matrices. And of course, in keeping with my advice above, my name is on the paper [18]. Imre and I have collaborated on a total of thirteen papers.

I collaborated with Randall McCutcheon on five papers over a period of five years, including the time he had a post doctoral fellowship at the University of Maryland, which is just down the road from my house. Randall has an inventive mind, and a talent for making difficult concepts easy to understand.

I have only three joint papers with Andreas Blass, but that significantly understates his value to me. The web site for the conference in honor of his 60^{th} birthday at the Fields Institute in Toronto referred to his "legendary patience", and I have been foremost among the beneficiaries of that patience. Whenever I have a question about any of his many areas of expertise, I send him some email and will usually have a response by the next day. A year and a half ago I sent him email asking whether it was consistent that ultrafilters with a certain property exist, and he wrote back patiently explaining that the answer could be found in a paper by Blass and Hindman.

Finally, I have only two joint papers with my dissertation advisor, Wis Comfort. But I owe him an unpayable debt. He taught me how to prove theorems. He taught me how to teach. He taught me how to deal fairly and honestly with everyone – but not so honestly as to cause unneeded hurt. And he told me about a question of Fred Galvin's which led eventually to what is widely known as "Hindman's Theorem".

Theorem 1.1. *Theorem. Let* $r \in \mathbb{N}$ *and let* $\mathbb{N} = \bigcup_{i=1}^{r} A_i$*. There exist* $i \in \{1, 2, \ldots, r\}$ *and a sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *such that for every finite nonempty subset* F *of* $\mathbb{N}$, $\sum_{t \in F} x_t \in A_i$.

Unfortunately, I have lost the letter from Wis, but I recall that in that letter he told me that Erdős had asked him whether there existed an "almost translation invariant ultrafilter" on $\mathbb{N}$, that is, an ultrafilter p on $\mathbb{N}$ such that for all $A \in p$, $\{x \in \mathbb{N} : x + A \in p\} \in p$.

I showed that no such ultrafilter could exist, found out that the question originated with Galvin, and told him the answer. He said something like "that's nice, but I wanted a *downward* almost translation invariant ultrafilter." That is, for all $A \in p$, $\{x \in \mathbb{N} : -x + A \in p\} \in p$, where $-x + A = \{y \in \mathbb{N} : x + y \in A\}$. The reason he wanted such an ultrafilter is that he knew it would provide a simple proof of Theorem 1.1.

A few years later, Galvin ran into Steven Glazer and found out that a downward almost translation invariant ultrafilter was simply an idempotent in the compact right topological semigroup $(\beta\mathbb{N}, +)$, and every compact (Hausdorff) right topological semigroup has idempotents. Consequently, Theorem 1.1, which had been very difficult to prove, now became a triviality. And my long love affair with the algebra of the Stone-Čech compactification of a discrete semigroup and its applications to Ramsey Theory began.

All of my Ph.D. students have written dissertations on Ramsey Theory, the algebra of βS, applications of one of these areas to the other, or some combination of these topics. In this paper, I shall group the dissertations by subject matter, discussing some of the questions answered and some of the questions remaining. I apologize in advance to each of my students because I will necessarily have to omit mention of many of the results in their dissertations and even of some of the broad topics covered. Many of the theses could be featured in more than one of the sections that follow. My guiding principle in choosing material to present was to try to find among their results those which are reasonably easy to describe without introducing a lot of notation.

Section 2 will present background material which is necessary to understand the problems addressed and solved in the dissertations. For a reader interested in some but not all of the dissertations, I would suggest temporarily skipping Section 2 and referring back to it as needed.

2. Preliminaries

In this section we present a summary of background material needed to understand the rest of the paper. For an elementary derivation of these facts, the reader is referred to [21].

Given a discrete semigroup $(S, \cdot)$, we take the points of βS to be the ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S$. Given $A \subseteq S$, $c\ell(A) = \overline{A} = \{p \in \beta S : A \in p\}$. The operation on S can be extended to βS so that βS is *right topological* meaning that for each $p \in \beta S$, ρ_p is continuous, where $\rho_p(q) = q \cdot p$, with S contained in its topological center, meaning that for each $x \in S$, λ_x is continuous, where $\lambda_x(q) = x \cdot q$. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$ where $x^{-1}A = \{y \in S : xy \in A\}$. If the operation is written additively, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$ where $-x + A = \{y \in S : x + y \in A\}$.

Any compact right topological semigroup T has a smallest two sided ideal, $K(T)$ which is the union of all minimal right ideals of T and is also the union of all minimal left ideals of T. Given any minimal left ideal L and any minimal right ideal R, $L \cap R$ is a group, and any two such groups are isomorphic.

Given any idempotent p in a semigroup S, we let $H(p)$ be the union of all subgroups of S with p as identity. Then $H(p)$ is the *maximal group associated with* p.

There are several notions of size in a semigroup which arise in some of the studies below. All of these except *IP-set* have their origins in topological dynamics, and all of them are one-sided notions. We refer to these as the "right" versions to correspond to our choice of βS as a right topological semigroup. The use of the term without the right or left modifier always means the right version.

If X is a set, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X. If $(S, \cdot)$ is a semigroup and $\langle x_n \rangle_{n=1}^\infty$ is a sequence in S, then $FP(\langle x_n \rangle_{n=1}^\infty) = \{\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$, where $\prod_{n \in F} x_n$ is computed in increasing order of indices. (For the "left" version, the product would be computed in decreasing order of indices.) If the operation on S is denoted by $+$, we write $FS(\langle x_n \rangle_{n=1}^\infty) = \{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$. Similarly if $\langle H_n \rangle_{n=1}^\infty$ is a sequence in $\mathcal{P}_f(\omega)$, we write $FU(\langle H_n \rangle_{n=1}^\infty) = \{\bigcup_{n \in F} H_n : F \in \mathcal{P}_f(\mathbb{N})\}$.

Definition 2.1. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(a) The set A is *(right) syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} t^{-1}A$.

(b) The set A is *(right) thick* if and only if for every $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq A$.

(c) The set A is a *(right) IP-set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A$.

(d) The set A is *(right) piecewise syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} t^{-1}A$ is (right) thick.

(e) The set A is a *(right) IP*-set* if and only if, whenever $\langle x_n \rangle_{n=1}^\infty$ is a sequence in S, $A \cap FP(\langle x_n \rangle_{n=1}^\infty) \neq \emptyset$.

In $(\mathbb{N}, +)$ a set is syndetic if and only if it has bounded gaps and a set is thick if and only if it contains arbitrarily long blocks.

Another very important notion of size is *central*. This notion, originally defined by Furstenberg [17] in terms of the dynamical notions of *proximal* and *uniformly recurrent*, has a simple algebraic characterization which we take as the definition. (It also has a very complicated elementary characterization.)

Definition 2.2. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set A is *(right) central* if and only if there exists an idempotent $p \in K(\beta S) \cap \overline{A}$.

Each of the notions defined in Definition 2.1 has a simple algebraic characterization.

Theorem 2.3. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.*

(a) The set A is syndetic if and only if for every left ideal L of βS, $L \cap \overline{A} \neq \emptyset$.

(b) The set A is thick if and only if there exists a left ideal L of βS such that $L \subseteq \overline{A}$.

(c) The set A is an IP-set if and only if there exists an idempotent $p \in \overline{A}$.

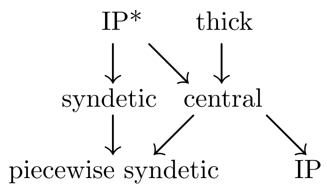
(d) The set A is piecewise syndetic if and only if $K(\beta S) \cap \overline{A} \neq \emptyset$.

(e) The set A is an IP-set if and only if $\{p \in \beta S : p \cdot p = p\} \subseteq \overline{A}$.*

Proof. (a) [8, Theorem 2.9(d)].
(b) [8, Theorem 2.9(c)].
(c) [21, Theorem 5.12].
(d) [21, Theorem 4.40] .
(e) This is an immediate consequence of (c) and the fact that A is an IP*-set if and only if $S \setminus A$ is not an IP-set. □

Notice that *IP*, *central*, and *piecewise syndetic* are partition regular properties, in the sense that if the finite union of sets has the named property, one of those sets must have that property. Notice also that the intersection of any two IP*-sets is an IP*-set.

It is clear from Definition 2.2 and Theorem 2.3 that the following pattern of implications holds. A table presented in [7] shows that none of the missing implications is valid in $(\mathbb{N}, +)$.



Central sets are important because on the one hand they are partition regular (meaning that if the finite union of sets is central, one of them is central) and they have remarkably strong combinatorial properties, which are consequences of the *Central Sets Theorem*. For example, If C is a central set in $(\mathbb{N}, +)$, A is a $u \times v$ kernel partition regular matrix with rational entries, and B is a $u \times v$ image partition regular matrix with rational entries, then there exist $\vec{x} \in C^v$ and $\vec{y} \in \mathbb{N}^v$ such that $A\vec{x} = \vec{0}$ and $B\vec{y} \in C^u$.

The original Central Sets Theorem is [17, Proposition 8.21], which applied to central subsets of $(\mathbb{N},+)$. Following is what is currently the strongest version of the Central Sets Theorem for commutative semigroups. There is also a version for arbitrary semigroups, but that version is much more complicated to state.

Theorem 2.4 (Central Sets Theorem). *Let $(S,+)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$, the set of sequences in S. Let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

(2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1,2,\ldots,m\}$, $\langle y_{i,n}\rangle_{n=1}^{\infty} \in G_i$, one has $\sum_{i=1}^{m}(\alpha(G_i) + \sum_{t\in H(G_i)} y_{i,t}) \in C$.

Proof. [14, Theorem 2.2]. □

Definition 2.5. Let $(S,+)$ be a semigroup and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in S. The sequence $\langle y_n\rangle_{n=1}^{\infty}$ is a *sum subsystem* of $\langle x_n\rangle_{n=1}^{\infty}$ if and only if there exists a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $y_n = \sum_{t\in H_n} x_t$ and $\max H_n < \min H_{n+1}$.

Both parts of the following theorem are consequences of Theorem 1.1.

Theorem 2.6. *(a) Let $r \in \mathbb{N}$ and let $\mathcal{P}_f(\mathbb{N}) = \bigcup_{i=1}^{r} \mathcal{A}_i$. There exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle H_n\rangle_{n=1}^{\infty}) \subseteq \mathcal{A}_i$ and for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$.*

(b) Let $(S,+)$ be a semigroup, let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in S, let $r \in \mathbb{N}$, and let $FS(\langle x_n\rangle_{n=1}^{\infty}) = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1,2,\ldots,r\}$ and a sum subsystem $\langle y_n\rangle_{n=1}^{\infty}$ of $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A_i$.

Proof. [21, Corollaries 5.15 and 5.17]. □

3. Algebraic Structure of βS

Even the simplest of semigroups S can have surprisingly rich algebraic structure in βS. For example, $(\mathbb{N},+)$ is the granddaddy of all semigroups. And it has been known for some time [20] that the maximal groups in the smallest ideal of $(\beta\mathbb{N},+)$ all contain a copy of the free group on 2^c generators, where $c = |\mathbb{R}|$. And many questions remain. For example, it is not known whether there is any nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$.

When I was first introduced to the algebra of βS, I took $(\beta S,\cdot)$ to be left topological, and therefore my early students also took that approach. When I cite their results in what follows, I will convert them to the right topological viewpoint. (This only matters if the reader decides to consult the original sources.)

Dennis Davenport

As is well known, given any discrete space X, the compact subsets of βX correspond exactly to the filters on X.

Definition 3.1. Let X be a discrete space and let $\mathcal{A}$ be a filter on X. Then $\overline{\mathcal{A}} = \{p \in \beta X : \mathcal{A} \subseteq p\}$.

Given any filter $\mathcal{A}$ on X, $\overline{\mathcal{A}}$ is a closed subset of βX. Further, if T is a closed subset of βX and $\mathcal{A} = \bigcap T$, then $T = \overline{\mathcal{A}}$. If S is a discrete semigroup and $\mathcal{A}$ is a filter on S, one is naturally interested in knowing whether $\overline{\mathcal{A}}$ is a subsemigroup of βS, and if so, is it a right ideal or a left ideal? Furthermore, if $\overline{\mathcal{A}}$ is a subsemigroup of βS, it is then a compact right topological semigroup, and so is guaranteed to have a smallest two sided ideal. One is naturally interested in knowing which ultrafilters in $\overline{\mathcal{A}}$ are members of the smallest ideal. Davenport solved these problems in [12]. (The main results are also published in [13].)

Theorem 3.2. *Let $(S, \cdot)$ be a semigroup and let $\mathcal{A}$ be a filter on S.*

(a) The set $\overline{\mathcal{A}}$ is a subsemigroup of βS if and only if for each $A \in \mathcal{A}$ and each $B \subseteq S$, if $S \setminus B \notin \mathcal{A}$, then there exists $F \in \mathcal{P}_f(B)$ such that $\bigcup_{x \in F} x^{-1}A \in \mathcal{A}$.

(b) The set $\overline{\mathcal{A}}$ is a left ideal of βS if and only if for each $A \in \mathcal{A}$ and for each $x \in S$, $x^{-1}A \in \mathcal{A}$.

(c) The set $\overline{\mathcal{A}}$ is a right ideal of βS if and only if for each $A \in \mathcal{A}$ and each $B \subseteq S$, if $S \setminus B \notin \mathcal{A}$, then there exists $F \in \mathcal{P}_f(B)$ such that $S = \bigcup_{x \in F} x^{-1}A$.

In [12] Davenport also characterized the minimal left ideals of $\overline{\mathcal{A}}$, the minimal right ideals of $\overline{\mathcal{A}}$, $K(\overline{\mathcal{A}})$, and, with certain additional assumptions, the closure of $K(\overline{\mathcal{A}})$. He showed that the additional assumptions are not necessary, and the main unanswered question is to find a characterization of the closure of $K(\overline{\mathcal{A}})$ without special assumptions. In particular, while the closure of a right ideal in $\overline{\mathcal{A}}$ is necessarily a right ideal, it is not known whether the closure of $K(\overline{\mathcal{A}})$ is a left ideal of $\overline{\mathcal{A}}$.

Hanson Umoh

If $\beta S \setminus S$ is an ideal of βS (as holds if S is cancellative), then $S^* \cdot S^*$ is an ideal of βS, and so $K(\beta S) \subseteq S^* \cdot S^*$. The question naturally arises as to whether $S^* \cdot S^*$ contains the closure of $K(\beta S)$. In [34], part of which was published earlier in [35], Umoh determined a class of countable left cancellative semigroups, which he called *inflatable* and proved the following theorem.

Theorem 3.3. *Let $(S, \cdot)$ be an inflatable semigroup. Then $c\ell K(\beta S) \setminus (S^* \cdot S^*) \neq \emptyset$.*

In [36], Umoh proved that any countable cancellative semigroup is inflatable, and established that Theorem 3.3 holds for a strictly wider class than the inflatable semigroups.

Lakeshia Legette

We have seen that maximal groups in βS can be large. In fact if S is cancellative and $|S| = \kappa$, then there exists an idempotent $p \in \beta S$ such that $H(p)$ contains a copy of the free group on 2^{2^κ} generators. In [22], Legette showed that it is consistent that maximal groups in such semigroups are as small as possible.

Theorem 3.4. *Let S and G be respectively the free semigroup and the free group on a countably infinite set of generators. For an idempotent $p \in \beta S$, let $H_S(p)$ and $H_G(p)$ be the maximal groups associated with p in βS and βG respectively. Assume Martin's Axiom. Then there is an idempotent $p \in \beta S$ such that $H_S(p) = H_G(p) = \{p\}$.*

The ultrafilters which Legette produces for the proof of Theorem 3.4 are essentially equivalent to *ordered union ultrafilters* introduced in [9], and the existence of ordered union ultrafilters is known to be independent of ZFC. However, we do not know whether it can be proved in ZFC that there are trivial maximal groups in βS for the free semigroup on two or countably many generators, or on any cancellative semigroup, for that matter.

4. The Right Continuous and Left Continuous Operations on βS

As we remarked earlier, the choice of continuity for $(\beta S, \cdot)$ is arbitrary, and in fact, I used to customarily take $(\beta S, \cdot)$ to be left topological. For the present section denote by $\odot$ the operation on βS which extends the operation on S with respect to which λ_p is continuous for each $p \in \beta S$ and ρ_x is continuous for each $x \in S$. If S is commutative, then for any $p, q \in \beta S$ one has $p \cdot q = q \odot p$. In particular, subsemigroups of $(\beta S, \cdot)$ are subsemigroups of $(\beta S, \odot)$ and vice versa; left ideals of $(\beta S, \cdot)$ are right ideals of $(\beta S, \odot)$ and vice versa; and $K(\beta S, \cdot) = K(\beta S, \odot)$. In [16], El-Mabhou, Pym, and Strauss showed that if S is the free semigroup on a countably infinite set of generators, then there is a subsemigroup H of $(\beta S, \cdot)$ with the property that $H \cap (\beta S \odot \beta S) = \emptyset$. This semigroup resided far away from the smallest ideals of either $(\beta S, \cdot)$ or $(\beta S, \odot)$. The dissertations discussed in this section addressed the question of how different $K(\beta S, \cdot)$ and $K(\beta S, \odot)$

Patty Anthony

In [3], also published in [4], Anthony established the following two theorems.

Theorem 4.1. *Let S be the free semigroup on two generators. Then $K(\beta S, \cdot) \setminus c\ell K(\beta S, \odot) \neq \emptyset$ and $K(\beta S, \odot) \setminus c\ell K(\beta S, \cdot) \neq \emptyset$.*

Theorem 4.2. *Let S be any semigroup. Then $K(\beta S, \cdot) \cap c\ell K(\beta S, \odot) \neq \emptyset$ and $K(\beta S, \odot) \cap c\ell K(\beta S, \cdot) \neq \emptyset$.*

The following corollary is of combinatorial interest since piecewise syndetic sets are translates of central sets, so any translation invariant structure which is guaranteed to be present in a central set is also guaranteed to be present in a piecewise syndetic set.

Theorem 4.3. *Let S be any semigroup, let $r \in \mathbb{N}$, and let $S = \bigcup_{i=1}^{r} A_i$. There exists $i \in \{1, 2, \ldots, r\}$ such that A_i is both left piecewise syndetic and right piecewise syndetic.*

Proof. Pick $p \in K(\beta S, \cdot) \cap c\ell K(\beta S, \odot)$ and pick $i \in \{1, 2, \ldots, r\}$ such that $A_i \in p$. By Theorem 2.3(d), A_i is both left piecewise syndetic and right piecewise syndetic. □

Shea Burns

In [10], also published in [11], Burns extended Theorem 4.1.

Theorem 4.4. *Let S be either the free semigroup or the free group on 2 generators. Then $K(\beta S, \cdot) \cap K(\beta S, \odot) = \emptyset$.*

As with all of the dissertations I am discussing, there is material in [3] and [10] that I have not mentioned. However, neither of these dissertations come close to characterizing those semigroups for which $K(\beta S, \cdot)$ and $K(\beta S, \odot)$ are different or those semigroups for which they are disjoint. (Lack of commutativity is not enough, nor is an empty center enough. For example, if S is a left zero semigroup – that is $ab = a$ for all $a \in S$ – then $(\beta S, \cdot)$ and $(\beta S, \odot)$ are both also left zero semigroups.)

5. Sums and Products

One of the first results proved after the discovery of the Galvin-Glazer proof of the Finite Sums Theorem was the following, first proved in 1975 (though not published until 1979).

Theorem 5.1. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1, 2, \ldots, r\}$ and sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

$$FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A_i \,.$$

Proof. See [21, Corollary 5.22]. □

For a few years, it remained an open question as to whether one could choose the sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ to be the same. The answer was "no". In fact there is a finite partition of $\mathbb{N}$ so that no cell contains all of the pairwise sums and pairwise products from some infinite sequence. (See [21, Theorem 17.16].)

I customarily refer to the following conjecture as a “fact”, while acknowledging that I cannot prove it. And I was tempted to write it that way below, but I am afraid that someone browsing through this as a published paper would not notice the disclaimers.

Conjecture 5.2. *Let $r, m \in \mathbb{N}$. Whenever $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and a finite sequence $\langle x_n \rangle_{n=1}^{m}$ in $\mathbb{N}$ such that*

$$FS(\langle x_n \rangle_{n=1}^{m}) \cup FP(\langle x_n \rangle_{n=1}^{m}) \subseteq A_i \,.$$

This conjecture has only been proved to be true for $m = r = 2$. That proof was done by computer by R. Graham who showed that if $\{1, 2, \ldots, 252\}$ is two colored there exist x, y such that $\{x, y, x + y, xy\}$ is monochromatic (and 252 is the best possible).

Gregory Smith

Smith considered the sums of a fixed number of products from a given sequence.

Definition 5.3. Let $m \in \mathbb{N}$ and let $\langle x_t \rangle_{t=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then

$$\begin{aligned} SP_m(\langle x_t \rangle_{t=1}^{\infty}) = \ & \{\textstyle\sum_{k=1}^{m} \prod_{t \in F_k} x_t : F_1, F_2, \ldots, F_m \in \mathcal{P}_f(\mathbb{N}) \text{ and for each} \\ & k \in \{1, 2, \ldots, m-1\}\,,\, \max F_k < \min F_{k+1}\}\,. \end{aligned}$$

Using strongly the algebraic structure of $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$, Smith proved the following theorem in [30], also published in [31].

Theorem 5.4. *Let $m, r \in \mathbb{N}$ and let $\langle x_t \rangle_{t=1}^{\infty}$ be a sequence in $\mathbb{N}$. Assume that $SP_m(\langle x_t \rangle_{t=1}^{\infty}) = \bigcup_{i=1}^{r} A_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle y_t \rangle_{t=1}^{\infty}$ such that $SP_m(\langle y_t \rangle_{t=1}^{\infty}) \subseteq A_i$.*

By purely combinatorial reasoning, he also showed that the cell guaranteed by Theorem 5.4 strongly depends on the choice of m.

Theorem 5.5. *Let $m, n \in \mathbb{N}$. There exist A_1, A_2 such that $\mathbb{N} = A_1 \cup A_2$ and if $\langle x_t \rangle_{t=1}^{\infty}$ is any sequence in $\mathbb{N}$, then $SP_m(\langle x_t \rangle_{t=1}^{\infty})$ is not contained in A_1 and $SP_n(\langle x_t \rangle_{t=1}^{\infty})$ is not contained in A_2.*

Dan Tang

Tang began a direct computer based attack on the $r = 3$ case of Conjecture 5.2. One should note that coloring by three colors is vastly more complicated than coloring by two. (If one is trying to avoid a configuration in color # 1 and x would complete the forbidden configuration, if one is two coloring one knows x must go to color # 2, while one has no such information if one is three coloring.)

Specifically, the question Tang investigated was the following:

Question 5.6. *Let $m \in \mathbb{N}$. Does there exist $n \in \mathbb{N}$ such that whenever $\{m, m+1, m+2, \ldots, n\} = A_1 \cup A_2 \cup A_3$, there must be some $i \in \{1,2,3\}$ and some $x, y \in \mathbb{N}$ such that $\{x+y, xy\} \subseteq A_i$?*

That is, Tang investigated Conjecture 5.2 without the requirement that x and y be in the specified color. (It is another result of Graham's that if $\mathbb{N}$ is two colored, then for each $m \in \mathbb{N}$, there exist $x, y \in \mathbb{N}$ such that $\min\{x, y\} \geq m$ and $\{x+y, xy\}$ is monochromatic.) Tang established an affirmative answer to Question 5.6 for each $m \leq 42$, finding the exact least value of n. (The minimum value for $m = 42$ is 435.)

If the above question is modified to require that $x \neq y$ then one of course expects the value of n to increase. (In Schur's Theorem, where $\{x, y, x+y\}$ is supposed to be contained in one cell of the partition, if x is required to be distinct from y, the bound almost exactly doubles.) Tang proved that if the bound for the $x \neq y$ version of the question is sufficiently small (no more than $\lceil \frac{m+1}{2} \rceil^2$), then that bound is exactly the same as the bound when $x = y$ is allowed. Further, his computer results establish that the bound is sufficiently small for $30 \leq m \leq 42$.

Elaine Terry

We have seen that it is not true that whenever $\mathbb{N}$ is finitely colored, there must exist one cell with a sequence whose finite sums and finite products are monochromatic. However, any IP*-set in $(\mathbb{N}, +)$ must have substantial multiplicative structure.

Theorem 5.7. *Let A be an IP*-set in $(\mathbb{N}, +)$ and let $\langle y_n \rangle_{n=1}^{\infty}$ be any sequence in $\mathbb{N}$. There exists a sum subsystem $\langle x_n \rangle_{n=1}^{\infty}$ of $\langle y_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. [6, Theorem 2.6]. □

In her dissertation Terry significantly extended Theorem 5.7 to *weak rings*.

Definition 5.8. A *weak ring* is a triple $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups and for all $a, b, c \in S$, $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(a+b) \cdot c = (a \cdot c) + (b \cdot c)$.

Notice that neither $(S, +)$ nor $(S, \cdot)$ are required to be commutative. As an example, let $(S, +)$ be any commutative semigroup and let $Hom(S)$ be the set of homomorphisms from S to S. Then $(Hom(S), +, \circ)$ is a weak ring and it is unlikely that $(Hom(S), \circ)$ is commutative.

Recall that in the definition of $FP(\langle x_n \rangle_{n=1}^{\infty})$, one required that the products be taken in increasing order of indices. Restricting to a finite sequence, one has that $FP(\langle x_n \rangle_{n=1}^{3} = \{x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\}$. We define $AP(\langle x_n \rangle_{n=1}^{\infty})$ to be the set of all finite products of distinct terms in any order. Again restricting to a finite sequence, we have that $AP(\langle x_n \rangle_{n=1}^{3}) = \{x_1, x_2, x_3, x_1x_2, x_2x_1, x_1x_3, x_3x_1, x_2x_3, x_3x_2, x_1x_2x_3, x_1x_3x_2, x_2x_1x_3, x_2x_3x_1, x_3x_1x_2, x_3x_2x_1\}$. The main theorem of [33] is the following. (See [21, Theorem 17.16] for a proof.)

Theorem 5.9. *Let $(S,+,\cdot)$ be a weak ring, let A be an IP*-set in $(S,+)$, and let $\langle y_n\rangle_{n=1}^\infty$ be any sequence in $\mathbb{N}$. There exists a sum subsystem $\langle x_n\rangle_{n=1}^\infty$ of $\langle y_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty)\cup AP(\langle x_n\rangle_{n=1}^\infty)\subseteq A$.*

6. Size in Partial Semigroups

A *partial semigroup* is a set S together with an operation $\cdot$ which is defined on a subset of $S\times S$ and is associative where it is defined, in the sense that for all $a,b,c\in S$, if either of $a\cdot(b\cdot c)$ or $(a\cdot b)\cdot c$ is defined, then so is the other and they are equal. Such semigroups were introduced in [5] and used to prove partition theorems about spaces of variable words.

The utility of partial semigroups arises out of being able to concentrate on cases where an operation either has a natural definition, or where a naturally defined operation is well behaved. For example, if for $F,G\in\mathcal{P}_f(\mathbb{N})$, one defines $F*G=F\cup G$ if $F\cap G=\emptyset$ and leaves $F*G$ undefined otherwise, then $f:\mathcal{P}_f(\mathbb{N})\to\mathbb{N}$ defined by $f(F)=|F|$ is a *partial semigroup homomorphism* (defined in the obvious way).

Definition 6.1. Let $(S,\cdot)$ be a partial semigroup.

(a) For $x\in S$, $\varphi(x)=\{y\in S: x\cdot y \text{ is defined}\}$.

(b) The partial semigroup S is *adequate* if and only if whenever $F\in\mathcal{P}_f(S)$, $\bigcap_{x\in F}\varphi(x)\neq\emptyset$.

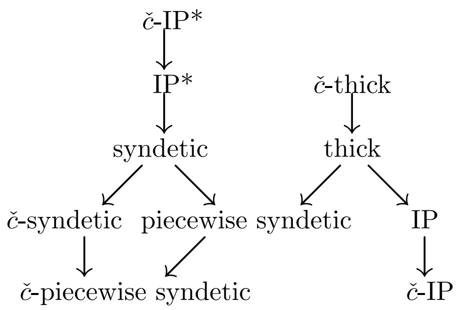
(c) If S is adequate, $\delta S=\bigcap_{x\in S} c\ell_{\beta S}\varphi(x)$.

If S is adequate, (which is precisely what is required for δS to be non-empty), then the operation extends naturally to δS in such a way that δS is a compact right topological semigroup, and so has all of the structure guaranteed to such objects.

Jillian McLeod

Let $(S,\cdot)$ be a partial semigroup. All of the notions of size defined in Definition 2.1 have obvious analogues of their algebraic characterizations in Theorem 2.3 in terms of δS, and more-or-less obvious analogues of their combinatorial definitions. For example, one defines a set $A\subseteq S$ to be algebraically thick if and only if there is a left ideal L of δS with $L\subseteq\overline{A}$. And A is combinatorially thick if and only if for each $F\in\mathcal{P}_f(S)$, there exists $y\in\bigcap_{x\in F}\varphi(x)$ such that $F\cdot y\subseteq A$.

McLeod denoted the algebraic analogues by the same name as used for semigroups and prefixed the combinatorial characterizations by "$\check{c}$". In [25] (also published in [26]) she showed that all of the implications in the following table hold among these notions and produced examples of partial semigroups showing that none of the missing implications is valid in general. (Her diagram was larger than this because she considered several other notions that we have not mentioned.)



7. Partition Regularity of Affine Transformations

We have already mentioned that in his 1933 paper [29] Rado characterized the kernel partition regularity of linear transformations. In that same paper he also characterized the kernel partition regularity of affine transformations. These characterizations are not as well known as his linear characterizations, probably because, with the exception of Theorem 7.1(b)(ii), the answer is that the affine transformation is kernel partition regular if and only if it is trivially so, that is it has a constant solution. (Given a number k we write $\overline{k}$ for a vector with all terms equal to k.)

Theorem 7.1. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$.*

(a) Whenever $\mathbb{Z}$ is finitely colored, there exists a monochromatic $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} + \vec{b} = \overline{0}$ if and only if there exists $k \in \mathbb{Z}$ such that $A\overline{k} + \vec{b} = \overline{0}$.

(b) Whenever $\mathbb{N}$ is finitely colored, there exists a monochromatic $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} + \vec{b} = \vec{0}$ if and only if either

(i) there exists $k \in \mathbb{N}$ such that $A\overline{k} + \vec{b} = \overline{0}$ or

(ii) there exists $k \in \mathbb{Z}$ such that $A\overline{k} + \vec{b} = \overline{0}$ and the linear mapping $\vec{x} \mapsto A\vec{x}$ is kernel partition regular.

Proof. (a) [29, Satz VIII].
(b) [29, Satz V]. □

While on the subject of partition regularity of matrices, I should point out that, while there are several partial results known, we are a long way from characterization of either image or kernel partition regularity of infinite matrices.

Irene Moshesh

In [28] (also published in [19]), Moshesh considered several notions of image partition regularity of affine transformations. She characterized image partition regularity of an affine transformation over $\mathbb{Z}$ in a fashion nearly identical to Rado's characterization of kernel partition regularity.

Theorem 7.2. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. Whenever $\mathbb{Z}$ is finitely colored, there exists $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic if and only if there exist $\vec{x} \in \mathbb{Z}^v$ and $k \in \mathbb{Z}$ such that $A\vec{x} + \vec{b} = \overline{k}$.*

The characterization in the following is significantly more interesting. (Note in particular the appearance of central sets.)

Theorem 7.3. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. Whenever $\mathbb{N}$ is finitely colored there exists $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic if and only if either*

(i) there exists $k \in \mathbb{N}$ and $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} + \vec{b} = \overline{k}$ or,

(ii) there exists $k \in \mathbb{Z}$ and $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} + \vec{b} = \overline{k}$ and for every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} \in C^u$.

8. The smallest ideal of $\beta\mathbb{N}$

As we have observed, the smallest ideal $K(\beta\mathbb{N})$ of $(\beta\mathbb{N}, +)$ is known to have substantial algebraic structure. It contains 2^c minimal left ideals and 2^c minimal right ideals, and we have already mentioned the fact that the intersection of any minimal left ideal with any minimal right ideal contains a copy of the free group on 2^c generators.

Amha Lisan

Much of the structure of $K(\beta\mathbb{N})$, including the copies of the free semigroup on 2^c generators mentioned earlier, lie in $K(\beta\mathbb{N}) \cap \mathbb{H}$, where $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}2^n)$. In [23], also published in [24], Lisan published the following When we write that two subsets of βS are algebraically and topologically isomorphic, we mean that there is a single function which is simultaneously an isomorphism and a homeomorphism.

Theorem 8.1. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^{n} x_t$. Then $\bigcap_{m=1}^{\infty} c\ell_{\beta\mathbb{N}} FS(\langle x_n \rangle_{n=m}^{\infty})$ is topologically and algebraically isomorphic to $\mathbb{H}$.*

In fact, Lisan's proof with no substantive modification establishes the following theorem. When we say that a sequence $\langle x_n \rangle_{n=1}^{\infty}$ *satisfies uniqueness of finite products* we mean that whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ and $\prod_{t \in F} x_t = \prod_{t \in H} x_t$, one must have $F = H$.

Theorem 8.2. *Let $(S, \cdot)$ be a semigroup and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S which satisfies uniqueness of finite products. Then $\bigcap_{m=1}^{\infty} c\ell_{\beta S} FP(\langle x_n \rangle_{n=m}^{\infty})$ is topologically and algebraically isomorphic to $\mathbb{H}$.*

Since it is easy to construct sequences $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^\infty)$ is not piecewise syndetic, one has that all of the algebraic structure found in $\mathbb{H}$ can be found in parts of $\beta\mathbb{N}$ which miss the smallest ideal.

Gugu Moche

The identity function $\iota : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N} \subseteq \beta\mathbb{N}\times\beta\mathbb{N}$ has a continuous extension $\widetilde{\iota} : \beta(\mathbb{N}\times\mathbb{N}) \to \beta\mathbb{N}\times\beta\mathbb{N}$, and $\widetilde{\iota}\big[K\big(\beta(\mathbb{N}\times\mathbb{N})\big)\big] = K(\beta\mathbb{N})\times K(\beta\mathbb{N}) = K(\beta\mathbb{N}\times\beta\mathbb{N})$. It has been known since the early 1970's that there are points $(p,q) \in \beta\mathbb{N}\times\beta\mathbb{N}$ such that $|\widetilde{\iota}^{-1}[\{(p,q)\}]| = 2^c$ and that it follows from the Continuum Hypothesis that there are points $(p,q) \in \beta\mathbb{N}\times\beta\mathbb{N}$ such that $|\widetilde{\iota}^{-1}[\{(p,q)\}]| = 2$.

In [27], Moche proved the following theorem.

Theorem 8.3. *Let $(p,q) \in K(\beta\mathbb{N})\times K(\beta\mathbb{N})$. Then*

$$\{r \in K\big(\beta(\mathbb{N}\times\mathbb{N})\big) : \widetilde{\iota}(r) = (p,q)\}$$

is infinite.

It is almost an axiom that all interesting subsets of $\beta\mathbb{N}$, or in this case $\beta(\mathbb{N}\times\mathbb{N})$, have as many points as $\beta\mathbb{N}$, namely 2^c. And it is certainly a fact that all closed infinite subsets of $\beta\mathbb{N}$ have 2^c points. However, $K\big(\beta(\mathbb{N}\times\mathbb{N})\big)$ is not closed, so the following question remains.

Question 8.4. *Let $(p,q) \in K(\beta\mathbb{N})\times K(\beta\mathbb{N})$. Must*

$$|\{r \in K\big(\beta(\mathbb{N}\times\mathbb{N})\big) : \widetilde{\iota}(r) = (p,q)\}| = 2^c \text{ ?}$$

Chase Adams, III

We saw in Theorem 8.1 that if $\langle x_n\rangle_{n=1}^\infty$ is a sequence in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$, then $\bigcap_{m=1}^\infty c\ell_{\beta S} FS(\langle x_n\rangle_{n=m}^\infty)$ contains much of the known algebraic structure of $K(\beta\mathbb{N})$. Adams proved in [1], also published in [2], that several notions of size are equivalent for such nicely behaved sequences in $\mathbb{N}$.

Theorem 8.5. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$. The following statements are equivalent:*

(a) For all $m \in \mathbb{N}$, $FS(\langle x_n\rangle_{n=m}^\infty)$ is central.

(b) $FS(\langle x_n\rangle_{n=1}^\infty)$ is central.

(c) For all $m \in \mathbb{N}$, $FS(\langle x_n\rangle_{n=m}^\infty)$ is piecewise syndetic.

(d) $FS(\langle x_n\rangle_{n=1}^\infty)$ is piecewise syndetic.

(e) $\{x_{n+1} - \sum_{t=1}^n x_t : n \in \mathbb{N}\}$ is bounded.

(f) $FS(\langle x_n\rangle_{n=1}^\infty)$ is syndetic.

(g) For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is syndetic.

(h) $\bigcap_{m=1}^{\infty} c\ell_{\beta S} FS(\langle x_n \rangle_{n=m}^{\infty}) \cap K(\beta\mathbb{N}) \neq \emptyset$.

In [1], given $\epsilon > 0$, Adams constructed a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^{n} x_t$, $\{x_{n+1} - \sum_{t=1}^{n} x_t : n \in \mathbb{N}\}$ is unbounded, and the density $d\big(FS(\langle x_n \rangle_{n=1}^{\infty})\big) > 1 - \epsilon$. As a consequence one has much of the algebraic structure of $K(\beta\mathbb{N})$, specifically all of the structure of $K(\mathbb{H})$, close to, but disjoint from, $K(\beta\mathbb{N})$.

9. Conclusion

People often accuse me of working. I steadfastly deny that, saying that I teach and do mathematics – neither of which can be construed as work. (Well, I do admit that grading exams is not exactly fun.) I would like to take this opportunity to thank all of my collaborators, and especially my Ph.D. students, for participating with me in this marvelous venture.

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