Algebraic products of tensor products

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Abstract

Given a discrete semigroup $(S, \cdot)$, there is a natural operation on the
Stone-Čech compactification βS of S which extends the operation of S and
makes $(\beta S, \cdot)$ a compact right topological semigroup with S contained in
its topological center. If S and T are discrete semigroups, $p \in \beta S$, and
$q \in \beta T$, then the *tensor product* $p \otimes q$ is a member of $\beta(S \times T)$. It is
known that tensor products are both algebraically and topologically rare
in $\beta(S \times T)$. We investigate when the algebraic product of two tensor
products is again a tensor product. We get a simple characterization for
a large class of semigroups. The characterization is in terms of a notion
of cancellation. We investigate where that notion sits among standard
cancellation] notions.

1   Introduction

Given a discrete space S we take the Stone-Čech compactification βS of S to
consist of the ultrafilters on S, with the points of S identified with the principal
ultrafilters. Given $A \subseteq S$ we let $\overline{A} = c\ell_{\beta S} A = \{p \in \beta S : A \in p\}$ and let
$A^* = \overline{A} \setminus A$.

Definition 1.1. Let S and T be discrete spaces, let $p \in \beta S$, and let $q \in \beta T$.
Then the *tensor product of p and q* is defined by

$$p \otimes q = \{A \subseteq S \times T : \{x \in S : \{y \in T : (x, y) \in A\} \in q\} \in p\} .$$

Tensor products were apparently first introduced in the proof of [9, Theorem
6.7]. They are also a special case of the notion of sums of ultrafilters introduced
by Frolík in [5] which were used to provide the first ZFC proof that $\beta\mathbb{N}$ is not
homogeneous, where $\mathbb{N}$ is the set of positive integers.

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Tensor products have been used extensively in model theory. (See [4] and the already cited [9].) Several results about tensor products were included in [2] and used in [3, Chapter 7].

Tensor products can be characterized in terms of limits as follows. If $p \in \beta S$ and $q \in \beta T$, then $p \otimes q = \lim_{s \to p} \lim_{t \to q} (s, t)$, where s denotes a member of S and t denotes a member of T. An alternate notation involves the notion of p-limit. If $p \in \beta S$, $A \in p$, and and f is a function from S to a compact Hausdorff space X, then $p\text{-}\lim_{s \in A} f(s) = y$ if and only if for every neighborhood U of y, $\{s \in A : f(s) \in U\} \in p$. Then given any $A \in p$ and any $B \in q$, $p \otimes q = p\text{-}\lim_{s \in A} q\text{-}\lim_{t \in B} (s, t)$. Observe that the statement $p\text{-}\lim_{s \in A} f(s) = y$ is equivalent to the statement that $\lim_{s \to p} f(s) = y$ where s denotes a member of A; i.e. it is equivalent to the statement that $\widetilde{f}(p) = y$ where $\widetilde{f} : \beta S \to X$ denotes the continuous extension of f. See [7, Section 3.5] for more information about p-limits.

We shall frequently use the following basic facts. Given $q \in \beta T$, the function $R_q : \beta S \to \beta(S \times T)$ defined by $R_q(p) = p \otimes q$ is continuous. And, given $s \in S$, the function $L_s : \beta T \to \beta(S \times T)$ defined by $L_s(q) = s \otimes q$ is continuous.

If $\cdot$ is a binary operation on S, then the operation extends uniquely to βS so that, for every $q \in \beta S$, the function $\rho_q : \beta S \to \beta S$ is continuous, and for each $s \in S$, the function $\lambda_s : \beta S \to \beta S$ is continuous, where $\rho_q(p) = p \cdot q$ and $\lambda_s(q) = s \cdot q$. This extended operation has the property that $p \cdot q = \lim_{s \to p} \lim_{t \to q} (s \cdot t)$ for every $p, q \in \beta S$, where s and t denote elements of S. If $p, q \in \beta S$ and $A \subseteq S$, then $A \in p \cdot q$ if and only if $\{s \in S : s^{-1} A \in q\} \in p$ where $s^{-1} A = \{t \in S : st \in A\}$. If the operation on S is associative, then so is the extended operation. The reader is referred to [7, Chapter 4] for basic information about the extended operation.

This property of binary operations has an immediate application to Ramsey Theory. Let $\cdot$ be any binary operation on a set S. Let $\mathcal{F}$ be a family of finite subsets of F and let $\mathcal{G}$ be a family of arbitrary subsets of S. Suppose that, given any finite coloring of S, there exists $F \in \mathcal{F}$ and $G \in \mathcal{G}$ which are monochromatic. Then, given any finite coloring of S, there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ for which $F \cdot G$ is monochromatic. This was proved in [1] as Theorem 1.1 (b). But it seems worth pointing out in this paper that it is an immediate consequence of the property of binary operations that we have just stated. We can choose ultrafilters p and q in βS such that every member of p contains a set $F \in \mathcal{F}$ and every member of q contains a set $G \in \mathcal{G}$ by [7, Theorem 5.7]. To see that every member of $p \cdot q$ contains a set of the form $F \cdot G$, where $F \in \mathcal{F}$ and $G \in \mathcal{G}$, let $A \in p \cdot q$. Then $\{s \in S : s^{-1} A \in q\} \in p$ so pick $F \in \mathcal{F}$ such that for each $s \in F$, $s^{-1} A \in q$. Since F is finite, $\bigcap_{s \in F} s^{-1} A \in q$ so one may pick $G \in \mathcal{G}$ such that $G \subseteq \bigcap_{s \in F} s^{-1} A \in q$.

Let $\cdot$ be any binary operation on a set S and let $g : S \times S \to S$ be defined by $g(s, t) = s \cdot t$. Then g extends to a continuous function $\widetilde{g} : \beta(S \times S) \to \beta S$. For every $p \in \beta S$ and $q \in \beta T$, $\widetilde{g}(p \otimes q) = \lim_{s \to p} \lim_{t \to q} g(s, t) = \lim_{s \to p} \lim_{t \to q} (s \cdot t) = p \cdot q$, where s and t denote elements of S.

In [8] we showed that the set of tensor products in $\beta(S \times S)$ is both algebraically and topologically thin. For example, by [8, Corollary 2.5], $\beta S \otimes \beta S$ is not a Borel subset of $\beta(S \times S)$. And it is a consequence of [8, Corollary 2.9] that if S is a countable and cancellative semigroup, then $\beta S \otimes \beta S$ does not meet the smallest ideal of $\beta(S \times S)$.

The main question that we address in this paper is the following.

Question 1.2. *Let $(S, \cdot)$ and $(T, \cdot)$ be infinite semigroups and let $p, r \in \beta S$ and $q, w \in \beta T$. Under what conditions is $(p \otimes q) \cdot (r \otimes w)$ a tensor product?*

The question can be rephrased as *When is the algebraic product of tensor products a tensor products?* The answer which we will develop in Section 2 is this: *Assuming that S and T are countably infinite and suitably civilized, $p, r \in \beta S$, and $q, w \in \beta T$, then $(p \otimes q) \cdot (r \otimes w)$ is a tensor product if and only if either $r \in S$ or $q \in T$, in which case $(p \otimes q) \cdot (r \otimes w) = (p \cdot r) \otimes (q \cdot w)$.* What "suitably civilized" means involves some weak notions of cancellation.

Definition 1.3. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(a) The set A is a *left solution set* iff there exist $w, z \in S$ such that $A = \{s \in S : w \cdot s = z\}$.

(b) The set A is a *right solution set* iff there exist $w, z \in S$ such that $A = \{s \in S : s \cdot w = z\}$.

Definition 1.4. Let $(S \cdot)$ be a semigroup.

(a) S is *weakly left cancellative* iff every left solution set in S is finite.

(b) S is *weakly right cancellative* iff every right solution set in S is finite.

(c) S is *quasi cancellative* iff for every $w, z \in \beta S$, $\{s \in S : s \cdot w = z\}$ is finite.

(d) A subset A of S is a *QC-set* iff there exist $w, z \in \beta S$, $s \cdot w = z$ for all $s \in A$.

Thus a semigroup is quasi cancellative if it does not contain any infinite QC-sets. Quasi cancellative is a one sided notion, and we would call it "quasi right cancellative" or "right quasi cancellative", but either of those names suggest that any right cancellative semigroup is quasi cancellative, which we shall see is not true. Note that if S is quasi cancellative, then S^* is a right ideal of βS. (Suppose $p \in S^*$, $q \in \beta S$, and $p \cdot q = z \in S$. Then $\{s \in S : sq = z\} = \{s \in S : s^{-1}\{z\} \in q\}$ is a member of p, so is infinite.)

If S is cancellative, the QC-sets in S are the singletons ([7, Corollary 8.2]). However, if S is only weakly cancellative, S itself could be a QC-set. This is the case, for example, if $S = (\mathbb{N}, \max)$ or if $S = (\mathcal{P}_f(X), \cup)$, where $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of the set X.

In Section 2 we shall show that "S and T are suitably civilized" means that S is weakly left cancellative and T is quasi cancellative.

In Section 3 we shall investigate where the notion of quasi cancellative sits among the standard notions of cancellativity, as well as a couple of other hybrid notions.

2 When the algebraic product of tensor products is a tensor product

The following very simple result tells us that if the algebraic product of two tensor products is a tensor product, we know what tensor product it is.

Theorem 2.1. *Let $(S, \cdot)$ and $(T, \cdot)$ be semigroups, let $p, r \in \beta S$, and let $q, w \in \beta T$. If $u \in \beta S$, $v \in \beta T$, and $(p \otimes q) \cdot (r \otimes w) = u \otimes v$, then $u = pr$ and $v = qw$.*

Proof. Let π_1 and π_2 denote the projections from $S \times T$ to S and T respectively and let $\widetilde{\pi}_1 : \beta(S \times T) \to \beta S$ and $\widetilde{\pi}_2 : \beta(S \times T) \to \beta T$ be their continuous extensions. It is routine to show that if $x \in \beta S$ and $y \in \beta T$, then $\widetilde{\pi}_1(x \otimes y) = x$ and $\widetilde{\pi}_2(x \otimes y) = y$.

Now π_1 and π_2 are homomorphisms and therefore by [7, Corollary 4.22] so are $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$. Therefore $\widetilde{\pi}_1\big((p \otimes q) \cdot (r \otimes w)\big) = \big(\widetilde{\pi}_1(p \otimes q) \cdot \widetilde{\pi}_1(r \otimes w)\big) = pr$ and $\widetilde{\pi}_2\big((p \otimes q) \cdot (r \otimes w)\big) = \big(\widetilde{\pi}_2(p \otimes q) \cdot \widetilde{\pi}_2(r \otimes w)\big) = qw$. $\qquad\square$

Lemma 2.2. *Let $(S, \cdot)$ and $(T, \cdot)$ be infinite semigroups and assume that $w, z \in \beta T$ and $Q = \{t \in T : tw = z\}$ is infinite. Let $q \in Q^*$ and let $p, r \in \beta S$. Then $(p \otimes q) \cdot (r \otimes w) = (pr) \otimes (qw)$.*

Proof. It is routine to establish from the definition that for $s \in S$ and $t \in T$, $(s, t) \cdot (r \otimes w) = (sr) \otimes (tw)$. Note also that, since for all $t \in Q$, $tw = z$, $qw = z$ and for any $s \in S$, $q\text{-}\lim\limits_{t \in Q}\big((sr) \otimes (tw)\big) = q\text{-}\lim\limits_{t \in Q}\big((sr) \otimes z\big) = \big((sr) \otimes z\big)$. Thus we have

$$
\begin{aligned}
(p \otimes q) \cdot (r \otimes w) &= \big(p\text{-}\lim_{s \in S} q\text{-}\lim_{t \in Q}(s, t)\big) \cdot (r \otimes w) \\
&= p\text{-}\lim_{s \in S} q\text{-}\lim_{t \in Q} \big((s, t) \cdot (r \otimes w)\big) \quad \text{since } \rho_{r \otimes w} \text{ is continuous} \\
&= p\text{-}\lim_{s \in S} q\text{-}\lim_{t \in Q} \big((sr) \otimes (tw)\big) \\
&= p\text{-}\lim_{s \in S} \big((sr) \otimes z\big) \\
&= \big(p\text{-}\lim_{s \in S}(sr)\big) \otimes z \quad \text{since } R_z \text{ is continuous} \\
&= (pr) \otimes (qw)
\end{aligned}
$$

where the last equality holds because ρ_r is continuous and $qw = z$. $\qquad\square$

Lemma 2.3. *Let $(S, \cdot)$ and $(T, \cdot)$ be semigroups, let $p, r \in \beta S$, and let $q, w \in \beta T$. If either $r \in S$ or $q \in T$, then $(p \otimes q) \cdot (r \otimes w) = (pr) \otimes (qw)$.*

Proof. Assume first that $r \in S$. Using successively the facts that $\rho_{r \otimes w} : \beta(S \times T) \to \beta(S \times T)$ is continuous and for $(s, t) \in S \times T$, $\lambda_{(s,t)} : \beta(S \times T) \to \beta(S \times T)$ is continuous one sees that

$$(p \otimes q) \cdot (r \otimes w) = p\text{-}\lim_{s \in S} q\text{-}\lim_{t \in T} w\text{-}\lim_{v \in T} \big((s, t) \cdot (r, v)\big).$$

Using successively the facts that $R_{qw} : \beta S \to \beta(S \times T)$ is continuous and for $s \in S$, $L_{sr} : \beta T \to \beta(S \times T)$ is continuous, one sees that

$$(pr) \otimes (qw) = p\text{-}\lim_{s \in S} q\text{-}\lim_{t \in T} w\text{-}\lim_{v \in T} \big((sr, tv)\big)$$

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so the conclusion holds.

Now assume that $q \in T$. Using the facts that $\rho_{r \otimes w} : \beta(S \times T) \to \beta(S \times T)$ is continuous and for $s \in S$, $\lambda_{(s,q)} : \beta(S \times T) \to \beta(S \times T)$ is continuous one sees that $(p \otimes q) \cdot (r \otimes w) = p\text{-}\lim_{s \in S} r\text{-}\lim_{u \in S} w\text{-}\lim_{v \in T} \big((s,q) \cdot (u,v)\big)$. Using successively the facts that $R_{qw} : \beta S \to \beta(S \times T)$ is continuous and for $s, u \in S$, $L_{su} : \beta T \to \beta(S \times T)$ is continous, one sees that $(pr) \otimes (qw) = p\text{-}\lim_{s \in S} r\text{-}\lim_{u \in S} w\text{-}\lim_{v \in T} \big((su, qv)\big)$ so the conclusion holds. $\square$

Lemma 2.4. *Let $(S, \cdot)$ and $(T, \cdot)$ be infinite semigroups. If S is not weakly left cancellative, then there exist $r \in S^*$ and $p \in S$ such that for all q and w in βT, $(p \otimes q) \cdot (r \otimes w) = (pr) \otimes (qw)$*

Proof. Assume that S is not weakly left cancellative. Pick p and y in S such that $H = \{u \in S : pu = y\}$ is infinite and pick $r \in H^*$. Let q and w in βT be given.

Using the facts that $\rho_{r \otimes w} : \beta(S \times T) \to \beta(S \times T)$ is continuous and for any $t \in T$, $\lambda_{(p,t)} : \beta(S \times T) \to \beta(S \times T)$ is continuous we see that $(p \otimes q) \cdot (r \otimes w) = q\text{-}\lim_{t \in T} r\text{-}\lim_{u \in H} w\text{-}\lim_{v \in T}(pu, tv)$.

Using the facts that $R_{qw} : \beta S \to \beta(S \times T)$ is continuous and for any $u \in S$, $L_{pu} : \beta T \to \beta(S \times T)$ is continuous, we see that $(pr) \otimes (qw) = r\text{-}\lim_{u \in H} q\text{-}\lim_{t \in T} w\text{-}\lim_{v \in T}(pu, tv)$. Since for all $u \in H$. $pu = y$, we have that

$$
\begin{aligned}
q\text{-}\lim_{t \in T} r\text{-}\lim_{u \in H} w\text{-}\lim_{v \in T}(pu, tv) &= q\text{-}\lim_{t \in T} w\text{-}\lim_{v \in T}(y, tv) \\
&= r\text{-}\lim_{u \in H} q\text{-}\lim_{t \in T} w\text{-}\lim_{v \in T}(pu, tv) \,.
\end{aligned}
$$

$\square$

The following is the main result of the section.

Theorem 2.5. *Let $(S, \cdot)$ and $(T, \cdot)$ be infinite semigroups. Statements (2), (3), and (4) are equivalent and imply statement (1). If S and T are countable, then all four statements are equivalent.*

(1) *S is weakly left cancellative and T is quasi cancellative.*

(2) $(\forall q \in T^*)(\forall r \in S^*)(\forall p \in \beta S)(\forall w \in \beta T)$
   $\big((p \otimes q) \cdot (r \otimes w) \neq (pr) \otimes (qw)\big)$.

(3) *For all $p, r \in \beta S$ and all $q, w \in \beta T$, $(p \otimes q) \cdot (r \otimes w) = (pr) \otimes (qw) \Leftrightarrow r \in S$ or $q \in T$.*

(4) *For all $p, r \in \beta S$ and all $q, w \in \beta T$, $(p \otimes q) \cdot (r \otimes w) \in \beta S \otimes \beta T \Leftrightarrow r \in S$ or $q \in T$.*

Proof. (2) implies (3). Assume statement (2) holds, let $p, r \in \beta S$, and let $q, w \in \beta T$. The sufficiency holds by Lemma 2.3. The necessity is an immediate consequence of statement (2).

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Using Theorem 2.1, it is immediate that statements (3) and (4) are equivalent.

It is trivial that statement (3) implies statement (2).

To see that statement (2) implies statement (1) assume that statement (2) holds. By Lemma 2.4, S is weakly left cancellative and by Lemma 2.2, T is quasi cancellative.

Finally, assume that S and T are countable and statement (1) holds. Let $q \in T^*$, $r \in S^*$, $p \in \beta S$, and $w \in \beta T$ be given and suppose that $(p \otimes q) \cdot (r \otimes w) = (pr) \otimes (qw)$. Let $Q = \{t \in T : tw = qw\}$. Since T is quasi cancellative, Q is finite. Let $T' = T \setminus Q$ and note that $T' \in q$. We claim that $w \notin Q$. If so, we would have $ww \in T$ while, as we have noted. T^* is a right ideal of βT so $qw \notin T$.

Enumerate S and T' as $\langle s_n \rangle_{n=1}^\infty$ and $\langle t_n \rangle_{n=1}^\infty$ respectively. Write $s \prec s'$ if s preceeds s' and $t \prec t'$ if t preceeds t' in these orderings. Define $f : S \to T'$ by, for each $n \in \mathbb{N}$, $f(s_n) = t_n$.

Given $b \in T'$ we have $bw \neq qw$ so pick $Y_b \in qw$ and $Z_b \in bw$ such that $Y_b \cap Z_b = \emptyset$. For $t \in T'$, let $X_t = \bigcap_{b \preceq t} Y_b$ and let $W_t = \bigcap_{b \preceq t} b^{-1} Z_b$. Then for each $t \in T'$, $X_t \in qw$, $W_t \in w$, and whenever $b \preceq t$, $bW_t \cap X_t = \emptyset$.

Let $A = \bigcup_{a \in S}(\{a\} \times X_{f(a)})$. Then $A \in (pr) \otimes (qw)$ so $A \in (p \otimes q) \cdot (r \otimes w)$. Pick $(s,t) \in S \times T'$ such that $(s,t)^{-1}A \in r \otimes w$.

Let $D = \{(u,v) \in S \times T' : t \prec f(su) \text{ and } v \in W_{f(su)}\}$ We claim that $D \in r \otimes w$. First, since S is weakly left cancellative, $\{u \in S : f(su) \preceq t\}$ is finite so, letting $E = \{u \in S : t \prec f(su)\}$, we have that $E \in r$. We claim that $E \subseteq \{u \in S : \{v \in T : (u,v) \in D\} \in w\}$, so let $u \in E$. Then $\{v \in T : (u,v) \in D\} = W_{f(su)} \in w$ as required.

Since $D \in r \otimes w$, pick $(u,v) \in D \cap (s,t)^{-1}A$. Then $(su, tv) \in A$ so $tv \in X_{f(su)}$. Also, $(u,v) \in D$ so $tv \in tW_{f(su)}$. Thus $X_{f(su)} \cap tW_{f(su)} \neq \emptyset$, while $t \prec f(su)$, a contradiction. $\square$

Corollary 2.6. *If S and T are countably infinite cancellative semigroups, no element of $S^* \otimes T^*$ can be idempotent.*

We now show that, if S is any infinite discrete semigroup which can be embedded in the direct sum of a family of countable cancellative semigroups, each of which has an identity, then no element of $S^* \otimes S^*$ can be idempotent. We observe that all discrete abelian groups have this property by [6, Theorems 19.1 and 20.1].

Lemma 2.7. *Let S, S', T, T' be arbitrary semigroups, let $f : S \to S'$, let $g : T \to T'$, and let $\widetilde{f} : \beta S \to \beta S'$ and $\widetilde{g} : \beta T \to \beta T'$ be their continuous extensions. Assume that $p \in \beta S$, $q \in \beta T$, and that $p \otimes q$ is an idempotent in $\beta(S \otimes T)$. Assume also that there exist $P \in p$ and $Q \in q$ such that for every $s \in P$ and every $t \in Q$, $P_s = \{u \in S : f(su) = f(s)f(u)\} \in p$ and $Q_t = \{v \in T : g(tv) = g(t)g(v)\} \in q$. Then $\widetilde{f}(p) \otimes \widetilde{g}(q)$ is an idempotent in $\beta(S' \times T')$.*

Proof. Let $h : S \times T \to S' \times T'$ be defined by $h(s,t) = \big(f(s), g(t)\big)$ and let $\widetilde{h} : \beta(S \times T) \to \beta(S' \times T')$ be its continuous extension. It is routine to verify

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from the definitions that $\widetilde{h}(p \otimes q) = \widetilde{f}(p) \otimes \widetilde{g}(q)$. Further

$$
\begin{aligned}
p \otimes q &= (p \otimes q) \cdot (p \otimes q) \\
&= \rho_{p \otimes q}\Big(p\text{-}\lim_{s \in P} q\text{-}\lim_{t \in Q}(s,t)\Big) \\
&= p\text{-}\lim_{s \in P} q\text{-}\lim_{t \in Q}\Big((s,t) \cdot p\text{-}\lim_{u \in P_s} q\text{-}\lim_{v \in Q_t}(u,v)\Big) \\
&= p\text{-}\lim_{s \in P} q\text{-}\lim_{t \in Q} p\text{-}\lim_{u \in P_s} q\text{-}\lim_{v \in Q_t}\big((s,t)(u,v)\big) .
\end{aligned}
$$

Let $s \in P, u \in P_s, t \in Q$ and $v \in Q_t$. Then

$$
h\big((s,t)(u,v)\big) = \big(f(su), g(tv)\big) = \big(f(s)f(u), g(t)g(v)\big) = \big(h(s,t)h(u,v)\big) .
$$

Therefore

$$
\begin{aligned}
\widetilde{h}(p \otimes q) &= \widetilde{h}\Big(p\text{-}\lim_{s \in P} q\text{-}\lim_{t \in Q} p\text{-}\lim_{u \in P_s} q\text{-}\lim_{v \in Q_t}(s,t)(u,v)\Big) \\
&= p\text{-}\lim_{s \in P} q\text{-}\lim_{t \in Q} p\text{-}\lim_{u \in P_s} q\text{-}\lim_{v \in Q_t} h\big((s,t)(u,v)\big) \\
&= p\text{-}\lim_{s \in P} q\text{-}\lim_{t \in Q} p\text{-}\lim_{u \in P_s} q\text{-}\lim_{v \in Q_t} \big(h(s,t)h(u,v)\big) \\
&= p\text{-}\lim_{s \in P} q\text{-}\lim_{t \in Q} h(s,t) \cdot \widetilde{h}\Big(p\text{-}\lim_{u \in P_s} q\text{-}\lim_{v \in Q_t}(u,v)\Big) \\
&= \Big(\widetilde{h}(p\text{-}\lim_{s \in P} q\text{-}\lim_{t \in Q}(s,t))\Big) \cdot \widetilde{h}(p \otimes q) \\
&= \widetilde{h}(p \otimes q) \cdot \widetilde{h}(p \otimes q) .
\end{aligned}
$$

Since $\widetilde{h}(p \otimes q) = \widetilde{f}(p) \otimes \widetilde{g}(q)$, we are done. $\qquad\square$

Theorem 2.8. *Let $\langle S_\alpha \rangle_{\alpha \in A}$ be a family of countable cancellative semigroups. Assume also that, for each $\alpha \in A$, S_α has an identity e_α. If S is a subsemigroup of $\bigoplus_{\alpha \in A} S_\alpha$, no member of $S^* \otimes S^*$ can be an idempotent.*

Proof. Let $p, q \in S^*$. Assume that $p \otimes q$ is an idempotent in $\beta(S \times S)$. As we saw in the proof of Theorem 2.1, p and q are homomorphic images of $p \otimes q$ so are idempotents.

Let e denote the identity of $\bigoplus_{\alpha \in A} S_\alpha$. For each finite non-empty subset F of A, let S_F denote $\bigoplus_{\alpha \in F} S_\alpha$, let π_F denote the natural projection of $\bigoplus_{\alpha \in A} S_\alpha$ onto S_F, and let $\widetilde{\pi}_F : \beta(\bigoplus_{\alpha \in A} S_\alpha) \to \beta S_F$ denote its continuous extension. Let e_F denote the identity of S_F. For $s \in A$, we put $\operatorname{supp}(s) = \{\alpha \in A : \pi_\alpha(s) \neq e_\alpha\}$.

For every finite non-empty subset F of A, $\widetilde{\pi}_F(p) \otimes \widetilde{\pi}_F(q)$ is an idempotent in $\beta(S_F \times S_F)$, by Lemma 2.7. By Corollary 2.6, $\widetilde{\pi}_F(p) \otimes \widetilde{\pi}_F(q) \notin S_F^* \otimes S_F^*$ so either $\widetilde{\pi}_F(p) \in S_F$ or $\widetilde{\pi}_F(q) \in S_F$. Since $\widetilde{\pi}_F$ is a homomorphism, if $\widetilde{\pi}_F(p) \in S_F$, then $\widetilde{\pi}_F(p) = e_F$ and if $\widetilde{\pi}_F(q) \in S_F$, then $\widetilde{\pi}_F(q) = e_F$. So either (1) $\widetilde{\pi}_F(p) = e_F$ for every finite non-empty subset F of A, or (2) $\widetilde{\pi}_F(q) = e_F$ for every finite non-empty subset F of A. (If $\widetilde{\pi}_F(p) \neq e_F$ and $\widetilde{\pi}_G(q) \neq e_G$, then $\widetilde{\pi}_{F \cup G}(p) \neq e_{F \cup G}$ and $\widetilde{\pi}_{F \cup G}(q) \neq e_{F \cup G}$.)

Assume that (1) holds. Define $\phi : S \to \omega = \mathbb{N} \cup \{0\}$ by $\phi(s) = |\operatorname{supp}(s)|$ and let $\widetilde{\phi} : \beta S \to \beta \omega$ be its continuous extension. Then, for every $s \in S$, $\phi(su) = \phi(s) + \phi(u)$ if $\pi_{\operatorname{supp}(s)}(u) = e_{\operatorname{supp}(s)}$ so, since $\widetilde{\pi}_{\operatorname{supp}(s)}(p) = e_{\operatorname{supp}(s)}$, $\{u \in S : \phi(su) = \phi(s) + \phi(u)\} \in p$.

Since $p, q \in S^*$ and 0 is isolated in $\beta\omega$, we know that $\widetilde{\phi}(p) \neq 0$ and $\widetilde{\phi}(q) \neq 0$. Let F be an arbitrary finite nonempty subset of A. By Lemma 2.7, $\widetilde{\phi}(p) \otimes \widetilde{\pi}_F(q)$ is an idempotent in $\beta(\omega \times S_F)$ and thus $\widetilde{\phi}(p)$ is an idempotent in $\beta\omega$ and $\widetilde{\pi}_F(q)$ is an idempotent in βS_F. Since $\widetilde{\phi}(p) \neq 0$, $\widetilde{\phi}(p) \in \omega^*$ so by Corollary 2.6, $\widetilde{\pi}_F(q) \notin S_F^*$. We thus have that for every finite nonempty subset F of A, $\widetilde{\pi}_F(q) = e_F$ so that for each $s \in S$, $\{u \in S : \phi(su) = \phi(s) + \phi(u)\} \in q$. Therefore by Lemma 2.7, $\widetilde{\phi}(p) \otimes \widetilde{\phi}(q)$ is an idempotent in $\beta(\omega \times \omega)$. Since $\widetilde{\phi}(p) \neq 0$ and $\widetilde{\phi}(q) \neq 0$, we again get a contradiction to Corollary 2.6.

Similarly, we can refute the assumption that (2) holds. $\qquad\square$

3   The notion of quasi cancellative

In this section we show where the notion of quasi cancellative fits among several other cancellation notions.

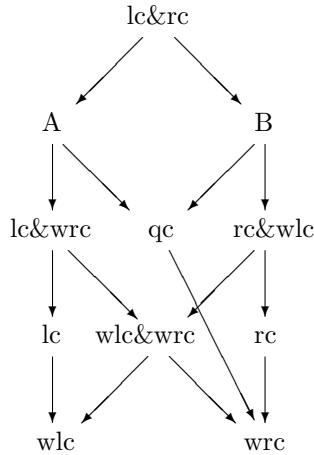

Figure 1: Diagram of Implications

Except for A and B, the abbreviations used in Figure 1 should be obvious. For example "wlc" abbreviates "weakly left cancellative". "A" abbreviates the statement "S is left cancellative and there is a finite bound on the size of right solution sets." "B" abbreviates the statement "S is right cancellative and, if $M = \{s \in S :$ there is a finite bound on the size of sets of the form $\{x \in S : sx = y\}$ for $y \in S\}$, then M is cofinite."

We shall show that all of the indicated implications in Figure 1 are valid, and with one exception, none of the missing implications can be added. The one exception is that we do not know whether right cancellative and weakly left cancellative implies quasi cancellative.

All of the indicated implications are obvious except that A implies quasi cancellative, B implies weakly left cancellative, and B implies quasi cancellative. We set out to verify those implications now.

Theorem 3.1. *Let $(S, \cdot)$ be an infinite left cancellative semigroup and assume that there is a finite bound on the size of right solution sets in S. Then S is quasi cancellative.*

Proof. Pick $b \in \mathbb{N}$ such that every right solution set has fewer than b elements. Let $w, z \in \beta S$ and let $A = \{s \in S : sw = z\}$. We shall show that A has fewer than b elements. If $w \in S$ and $A \neq \emptyset$, then A is a right solution set so has fewer than b elements. So assume that $w \in S^*$ and suppose that $|A| \geq b$. For distinct s and t in A, let $B_{s,t} = \{u \in S : su = tu\}$ and note that by [7, Lemma 8.5], $B_{s,t} \in w$. Pick distinct $s_1, s_2, \ldots s_b \in A$ and pick $u \in \bigcap_{j=2}^{b} B_{s_1,s_j}$ and let $x = s_1 u$. Then $\{s_1, s_2, \ldots, s_b\} \subseteq \{y \in S : yu = x\}$. $\square$

Theorem 3.2. *Let $(S, \cdot)$ be an infinite semigroup and assume that S is right cancellative. Let $M = \big\{s \in S : \text{there is a finite bound on the size of sets of the form } \{x \in S : sx = y\} \text{ for } y \in S\big\}$ and assume that M is cofinite. Then S is weakly left cancellative and quasi cancellative.*

Proof. To see that S is weakly left cancellative, suppose instead that we have $a, b \in S$ such that $B = \{x \in S : ax = b\}$ is infinite. Since S is right cancellative, $\{sa : s \in S\}$ is infinite so pick $s \in S$ such that $sa \in M$. Then (with $y = sb$), $\{x \in S : sax = sb\}$ is finite, while $B \subseteq \{x \in S : sax = sb\}$.

To see that S is quasi cancellative, let $w, z \in \beta S$ and suppose that $\{s \in S : sw = z\}$ is infinite. Pick $s \in M$ such that $sw = z$. Pick $t \neq s$ such that $tw = z$. Let $b = \max\{|\{x \in S : sx = y\}| : y \in S\}$. Let $E = \{u \in S : su = tu\}$. We shall show that $E \in w$. This will be a contradiction since S is right cancellative so $E = \emptyset$.

Suppose that $E \notin w$. Define an equivalence relation $\approx$ on $S \setminus E$ by $u \approx v$ iff $su = sv$. Let $\mathcal{F}$ be the set of equivalence classes. For each $F \in \mathcal{F}$, write $F = \{u_{F,1}, u_{F,2}, \ldots, u_{F,b}\}$, with repetition if necessary. Pick $i \in \{1, 2, \ldots, b\}$ such that $\{u_{F,i} : F \in \mathcal{F}\} \in w$ and let $U = \{u_{F,i} : F \in \mathcal{F}\}$

Define $f : S \to S$ with no fixed points as follows. For $u \in S \setminus E$, pick $F \in \mathcal{F}$ such that $u \in F$ and define $f(su) = tu_{F,i}$. (This is well defined because, if $su = sv$, then $v \in F$ as well.) Pick an element $z \in \{su : u \in S \setminus E\}$. For $x \in S \setminus \{su : u \in S \setminus E\}$ let $f(x) = z$. Let $\widetilde{f} : \beta S \to \beta S$ be the continuous extension of f. By [7, Theorem 3.34] $\widetilde{f}$ has no fixed points so $\widetilde{f}(sw) \neq sw$. But for $u \in U$, $f(su) = tu$ so $\widetilde{f} \circ \lambda_s$ agrees with λ_t on U so $\widetilde{f}(sw) = tw = sw$, a contradiction. $\square$

We now set out to show that, with one possible exception, none of the missing implications in Figure 1 can be added. We begin by posing that question.

Question 3.3. *If $(S, \cdot)$ is an infinite semigroup which is right cancellative and weakly left cancellative, must S be quasi cancellative?*

Now we present some very easy examples. Recall that a semigroup $(S, \cdot)$ is *right zero* iff $ab = b$ for all $a, b \in S$ and is *left zero* iff $ab = a$ for all $a, b \in S$.

Theorem 3.4. *(a) An infinite right zero semigroup is left cancellative and not weakly right cancellative.*

(b) An infinite left zero semigroup is quasi cancellative and right cancellative and not weakly left cancellative.

(c) If $(S, \cdot)$ is an infinite cancellative semigroup and R is a two element right zero semigroup, then $S \times R$ satisfies statement A and is not right cancellative.

(d) If $(S, \cdot)$ is an infinite cancellative semigroup and L is a two element left zero semigroup, then $S \times L$ satisfies statement B and is not left cancellative.

(e) For $n, m \in \mathbb{N}$, let $n \vee m = \max\{m, n\}$. Then $(\mathbb{N}, \vee)$ is weakly left cancellative and weakly right cancellative and is not quasi cancellative nor left cancellative nor right cancellative.

Proof. All conclusions are immediate except possibly the fact that $(\mathbb{N}, \vee)$ is not quasi cancellative. For this note that if $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, then since $n \vee m = m$ for all $m > n$, one has that $n \vee p = p$. $\qquad\square$

The remaining three needed results all utilize some special constructions.

Theorem 3.5. *There is a countably infinite semigroup which is left cancellative and weakly right cancellative but is not quasi cancellative.*

Proof. Let $D = \{s_n : n \in \mathbb{N}\} \cup \{u_n : n \in \mathbb{N}\} \cup \{w_n : n \in \mathbb{N}\}$ be an alphabet of distinct letters. Let S be the set of words over D that do not have any occurrences of $s_k u_m$ with $k \leq m$. Given $\alpha, \delta \in S$, let $\alpha \cdot \delta = \alpha ^\frown \delta$, the ordinary concatenation of words, unless there exist $\gamma, \mu \in S \cup \{\emptyset\}$ and $k \leq m$ in $\mathbb{N}$ such that $\alpha = \gamma s_k$ and $\delta = u_m \mu$ in which case $\alpha \cdot \delta = \gamma w_m \mu$.

It is routine to verify that the operation is associative, S is left cancellative, and S is weakly right cancellative.

To complete the proof, choose $w \in S^*$ with $\{u_n : n \in \mathbb{N}\} \in w$. Given $k < n$ in $\mathbb{N}$, λ_{s_k} and λ_{s_n} agree on $\{u_m : m \geq n\}$ so $s_k \cdot w = s_n \cdot w$. $\qquad\square$

Theorem 3.6. *There is a countably infinite semigroup which is right cancellative and weakly left cancellative but does not satisfy statement B.*

Proof. Let $D = \{s\} \cup \{x_{l,t} : l, t \in \mathbb{N} \text{ and } l \leq t\} \cup \{y_t : t \in \mathbb{N}\}$ be an alphabet of distinct letters. Let S be the set of words over D that do not have any occurrences of $s x_{l,t}$ with $l \leq t$. Given $\alpha, \delta \in S$, let $\alpha \cdot \delta = \alpha ^\frown \delta$ unless there exist $\gamma, \mu \in S \cup \{\emptyset\}$ and $l \leq t$ in $\mathbb{N}$ such that $\alpha = \gamma s$ and $\delta = x_{l,t} \mu$ in which case $\alpha \cdot \delta = \gamma y_t \mu$.

It is routine to verify that the operation is associative, S is right cancellative, and S is weakly left cancellative. To see that S does not satisfy statement B, note that if $t \in \mathbb{N}$, then $|\{x \in S : s \cdot x = y_t\}| = t$. $\qquad\square$

Theorem 3.7. *There is a countably infinite semigroup which is right cancellative but not quasi cancellative.*

Proof. Let S be the semigroup of [7, Example 8.3]. That is, $S = \{f : f$ is an injective function from $\mathbb{N}$ to $\mathbb{N}$ and there exist $m, r \in \mathbb{N}$ such that $(\forall n \geq m)(f(n) = 2^r n)\}$. For $f, g \in S$, define $f * g = g \circ f$. It is routine to verify that $(S, \circ)$ is countable and left cancellative, so that $(S, *)$ is right cancellative.

For $i \in \mathbb{N}$, define $g_i \in T$ by $g_i(1) = 2i - 1$ and for $n > 1$, $g_i(n) = 2n$. To complete the proof we shall need the following lemma.

Lemma 3.8. *For every finite partition $\mathcal{F}$ of S and every $m \in \mathbb{N}$, there exists $A \in \mathcal{F}$ such that $\bigcap_{i=1}^m A \circ g_i \neq \emptyset$.*

Proof. Let a finite partition $\mathcal{F}$ of S and $m \in \mathbb{N}$ be given. Let

$$E = \{k \in S : k(1) < k(3) < \ldots < k(2m - 1) \text{ and}$$
$$(\forall i \in \{1, 3, \ldots, 2m - 1\})(k(i) \text{ is odd}) \text{ and}$$
$$(\forall n \in \mathbb{N} \setminus \{1, 3, 5, \ldots, 2m - 1\})(k(n) = 2n)\}.$$

Given $D \in [2\mathbb{N} - 1]^m$, define $\psi(D) = k \in E$ as follows. List the elements of D in order as $\{a_1, a_2, \ldots, a_m\}$. For $i \in \{1, 2, \ldots, m\}$, let $k(2i - 1) = a_i$ and for $n \in \mathbb{N} \setminus \{1, 3, 5, \ldots, 2m - 1\}$, let $k(n) = 2n$. Define $\varphi : [2\mathbb{N} - 1]^m \to \mathcal{F}$ by $\varphi(D) = A$ if and only if $\psi(D) \in A$. By Ramsey's Theorem there exists an infinite subset X of $2\mathbb{N} - 1$ such that φ is constant on $[X]^m$. Pick $C \in [X]^{2m-1}$ and $A \in \mathcal{F}$ such that for all $D \in [C]^m$, $\varphi(D) = A$.

List the elements of C in order as $\{a_1, a_2, \ldots, a_{2m-1}\}$. For $i \in \{1, 2, \ldots, m\}$, let $k_{m-i+1} = \psi(\{a_i, a_{i+1}, \ldots, a_{i+m-1}\})$. So for $i, j \in \{1, 2, \ldots, m\}$, $k_{m-i+1}(2j-1) = a_{i+j-1}$ and $k_i(2j-1) = a_{m-i+j}$. Further, each $k_i \in A$. Define $h \in S$ by $h(1) = a_m$ and for $n > 1$, $h(n) = 4n$, Then for each $i \in \{1, 2, \ldots, m\}$, $h = k_i \circ g_i$ so $h \in \bigcap_{i=1}^m A \circ g_i$ as required. $\qquad \square$

By Lemma 3.8 and [7, Theorem 5.7], for each $m \in \mathbb{N}$, pick $p_m \in \beta S$ such that for each $A \in p_m$, $\bigcap_{i=1}^m A \circ g_i \neq \emptyset$. Let w be a cluster point in βS of the sequence $\langle p_m \rangle_{m=1}^\infty$. Let $z = g_1 * w$, We shall show that for all $j \in \mathbb{N}$, $z = g_j * w$, which will establish that $(S, *)$ is not quasi cancellative. Suppose instead we have some $j \in \mathbb{N}$ such that $g_j * w \neq z$. Pick $B \in z$ such that $S \setminus B \in q_j * w$. Pick $C \in w$ such that $g_1 * \overline{C} \subseteq \overline{B}$ and $g_j * \overline{C} \subseteq \overline{S \setminus B}$. Pick $m > j$ such that $p_m \in \overline{C}$. Pick $k_1, k_j \in C$ such that $k_1 \circ g_1 = k_j \circ g_j$. Then $g_1 * k_1 = k_1 \circ g_1 \in B$ while $g_j * k_j = k_j \circ g_j \in S \setminus B$, a contradiction. $\qquad \square$

We leave it to the reader to verify that, with the one noted exception, none of the missing implications in Figure 1 are valid. For example, to see that quasi cancellative does not imply left cancellative use either Theorem 3.4(d) or Theorem 3.4(b) and to see that quasi cancellative does not imply right cancellative, use Theorem 3.4(c).

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