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Thick sets are exactly the sets with Følner density 1

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Abstract

Følner density is a very natural notion of density which is defined on any semigroup satisfying the Strong Følner Condition (SFC). (These include all commutative semigroups and all left cancellative left amenable semigroups.) *Piecewise syndetic* and *thick* are notions of largeness arising from topological dynamics. It has been known that if S satisfies SFC and is either left cancellative or satisfies a weak right cancellation requirement, then every thick subset has density 1. We show here that in any semigroup S satisfying SFC a subset of S is thick if and only if it has density 1. As a consequence, every piecewise syndetic set has positive density.

1 Introduction

We begin by introducing the notions that we are concerned with here. Given a set X, we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X. If $(S,\cdot)$ is a semigroup, $A\subseteq S$, and $x\in S$, then $x^{-1}A=\{y\in S: xy\in A\}$.

Definition 1.1. Let $(S,\cdot)$ be a semigroup.

(a) The semigroup S satisfies the *Følner Condition* (FC) if and only if $\big(\forall H\in\mathcal{P}_f(S)\big)(\forall\epsilon>0)\big(\exists K\in\mathcal{P}_f(S)\big)(\forall s\in H)(|sK\setminus K|<\epsilon\cdot|K|)$.

(b) The semigroup S satisfies the *Strong Følner Condition* (SFC) if and only if $\big(\forall H\in\mathcal{P}_f(S)\big)(\forall\epsilon>0)\big(\exists K\in\mathcal{P}_f(S)\big)(\forall s\in H)(|K\setminus sK|<\epsilon\cdot|K|)$.

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Note that for $K \in \mathcal{P}_f(S)$, $s \in S$, and $\epsilon > 0$, the statements $|K \setminus sK| < \epsilon \cdot |K|$ and $|K \cap sK| > (1-\epsilon) \cdot |K|$ are equivalent.

Definition 1.2. Let $(S, \cdot)$ be a semigroup which satisfies SFC and let $A \subseteq S$. The *Følner density of* A is defined by
$d(A) = \sup\{\alpha \in [0,1] : \big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists K \in \mathcal{P}_f(S)\big)$
$\big((\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|) \text{ and } |A \cap K| \geq \alpha \cdot |K|\big)\}$.

Definition 1.3. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(a) The set A is *thick* if and only if for every $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq A$.

(b) The set A is *piecewise syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in G} t^{-1}A$ is thick.

In the semigroup $(\mathbb{N}, +)$ of positive integers, a subset is thick if and only if it contains arbitrarily long blocks of integers. And a subset is piecewise syndetic if and only if there exists a bound b and arbitrarily long blocks that have no gaps longer than b.

Our motivation for the study of thick and piecewise syndetic sets comes from the algebraic structure of the Stone-Čech compactification βS of the discrete semigroup S. We mention now their characterizations in terms of that structure, but one does not need to know anything about that algebraic structure to follow the proofs in this paper. We do expect familiarity with the product topology and the notion of a net in a topological space.

A subset A of S is thick if and only if $c\ell_{\beta S} A$ contains a left ideal of $(\beta S, \cdot)$ and A is piecewise syndetic if and only if $c\ell_{\beta S} A$ has nonempty intersection with the smallest ideal of $(\beta S, \cdot)$. The curious reader is referred to [9] for the derivation of these facts.

It has been known for some time that if $(S, \cdot)$ satisfies SFC and is left cancellative, then every thick subset of S has density 1. More recently it was shown in [6, Lemma 3.9] that the same conclusion holds if there exists $b \in \mathbb{N}$ such that for every $x \in S$, ρ_x is at most b-to-1, where $\rho_x(y) = yx$. (This is the weak right cancellation requirement mentioned in the abstract.)

Definition 1.4. Let $(S, \cdot)$ be a semigroup. A *Følner net* in $\mathcal{P}_f(S)$ is a net $\langle F_\alpha \rangle_{\alpha \in D}$ such that for each $s \in S$, $\lim_{\alpha \in D} \dfrac{|F_\alpha \setminus sF_\alpha|}{|F_\alpha|} = 0$.

It is immediate that if there exists a Følner net in $\mathcal{P}_f(S)$, then S satisfies SFC. Since trivially $d(S) = 1$, the converse to that assertion is part of the content of the following lemma.

Lemma 1.5. *Let $(S, \cdot)$ be a semigroup satisfying SFC and let A be a subset of S with $d(A) = 1$. There is a Følner net $\langle F_\alpha \rangle_{\alpha \in D}$ in $\mathcal{P}_f(S)$ such that the net* $\left\langle \dfrac{|F_\alpha \cap A|}{|F_\alpha|} \right\rangle_{\alpha \in D}$ *converges to* 1.

Proof. Let $D = \mathcal{P}_f(S) \times \mathbb{N}$ and direct D by $(H, n) \leq (K, m)$ if and only if $H \subseteq K$ and $n \leq m$. For $\alpha = (H, n) \in D$, pick $F_\alpha \in \mathcal{P}_f(S)$ such that $(\forall s \in H)(|F_\alpha \setminus sF_\alpha| < \frac{1}{n} \cdot |F_\alpha|$ and $|F_\alpha \cap A| > \frac{n-1}{n} \cdot |F_\alpha|$. □

Let $l_\infty(S)$ be the set of bounded real valued functions on S with the supremum norm, denoted by $\| \ \|_\infty$. A *mean* on S is a continuous real valued linear functional μ on $l_\infty(S)$ such that $||\mu||_\infty = 1$ and whenever $g \in l_\infty(S)$ and for all $s \in S$ $g(s) \geq 0$, one has that $\mu(g) \geq 0$. A *left invariant mean* on S is a mean μ such that for all $s \in S$ and all $g \in l_\infty(S)$, $\mu(g \circ \lambda_s) = \mu(g)$, where $\lambda_s : S \to S$ is defined by $\lambda_s(t) = st$. The semigroup S is defined to be *left amenable* if and only if there exists a left invariant mean on S.

The reader is referred to [13, Section 4.22] for a readable discussion of the relationship among FC, SFC, and left amenability.

In the next lemma we present an elementary proof of a well known fact.

Lemma 1.6. *Let $(S, \cdot)$ and $(T, \cdot)$ be semigroups. If S is left amenable and there exists a surjective homomorphism $h : S \to T$, then T is left amenable.*

Proof. Pick a left invariant mean μ on S and let $h : S \to T$ be a surjective homomorphism. Define $\nu : l_\infty(T) \to \mathbb{R}$ by, for $f \in l_\infty(T)$, $\nu(f) = \mu(f \circ h)$. It is routine to verify that ν is a mean on T. To see that ν is left invariant, let $t \in T$ and $f \in l_\infty(T)$ and pick $s \in S$ such that $h(s) = t$. Then $\lambda_t \circ h = h \circ \lambda_s$ so $\nu(f \circ \lambda_t) = \mu(f \circ \lambda_t \circ h) = \mu(f \circ h \circ \lambda_s) = \mu(f \circ h) = \nu(f)$. □

2 Sets with density 1 are thick

We provide now an elementary proof that subsets of S with density 1 are thick.

Lemma 2.1. *Let $(S, \cdot)$ be a semigroup satisfying SFC, let $F \in \mathcal{P}_f(S)$, and define $\mu : l_\infty(S) \to \mathbb{R}$ by, for $f \in l_\infty(S)$, $\mu(f) = \frac{1}{|F|}\sum_{t \in F} f(t)$. Let $f \in l_\infty(S)$, let $s \in S$, let $\epsilon > 0$, and assume that $|F \setminus sF| < \epsilon \cdot |F|$. Then $|\mu(f) - \mu(f \circ \lambda_s)| < 2\epsilon \cdot ||f||_\infty$.*

Proof. Let $X = F \cap sF$ and let $Y = F \setminus sF$. Then $X \cup Y = F$ and $X \cap Y = \emptyset$. For $x \in X$ pick $x' \in F$ such that $x = sx'$, let $X' = \{x' : x \in X\}$, and let $Y' = F \setminus X'$. Then $X' \cup Y' = F$ and $X' \cap Y' = \emptyset$. Now $|Y'| = |F| - |X'| = |F| - |X| = |Y|$ so pick a bijection τ from Y onto Y'. For $y \in Y$, let $y' = \tau(y)$. We shall show that $|\sum_{x \in F}\big(f(x) - f(sx)\big)| < 2\epsilon \cdot ||f||_\infty \cdot |F|$, so $|\mu(f) - \mu(f \circ \lambda_s)| < 2\epsilon \cdot ||f||_\infty$ as required. Now

$$\begin{aligned}
\textstyle\sum_{x \in F} f(x) &= \textstyle\sum_{x \in X} f(x) + \sum_{y \in Y} f(y) \text{ and} \\
\textstyle\sum_{x \in F} f(sx) &= \textstyle\sum_{t \in X'} f(st) + \sum_{t \in Y'} f(st) \\
&= \textstyle\sum_{x \in X} f(sx') + \sum_{y \in Y} f(sy') \text{ so} \\
\textstyle\sum_{x \in F}\big(f(x) - f(sx)\big) &= \textstyle\sum_{x \in X}\big(f(x) - f(sx')\big) + \sum_{y \in Y}\big(f(y) - f(sy')\big) \\
&= \textstyle\sum_{y \in Y}\big(f(y) - f(sy')\big), \text{ so} \\
|\textstyle\sum_{x \in F}\big(f(x) - f(sx)\big)| &= |\textstyle\sum_{y \in Y}\big(f(y) - f(sy')\big)|
\end{aligned}$$

$$\begin{aligned} &\leq 2 \cdot ||f||_\infty \cdot |Y| \\ &< 2\epsilon \cdot ||f||_\infty \cdot |F|\,. \end{aligned}$$

□

Lemma 2.2. *Let $(S, \cdot)$ be a semigroup satisfying SFC and let $\langle F_\alpha \rangle_{\alpha \in D}$ be a Følner net in $\mathcal{P}_f(S)$. For $\alpha \in D$ define $\mu_\alpha \in \times_{f \in l_\infty(S)}[-||f||_\infty, ||f||_\infty]$ by $\mu_\alpha(f) = \frac{1}{|F_\alpha|}\sum_{t \in F_\alpha} f(t)$. Let ν be a cluster point of the net $\langle \mu_\alpha \rangle_{\alpha \in D}$ in*

$$\times_{f \in l_\infty(S)}[-||f||_\infty, ||f||_\infty]\,.$$

Then ν is a left invariant mean on S.

Proof. The verification that ν is a mean on S is routine, if somewhat tedious. We shall verify that ν is left invariant. Let $s \in S$ and let $f \in l_\infty(S)$. Given $\epsilon > 0$, there exists $\alpha_0 \in D$ such that $|F_\alpha \setminus sF_\alpha| < \epsilon \cdot |F_\alpha|$ whenever $\alpha_0 \prec \alpha$. So by Lemma 2.1, $|\mu_\alpha(f) - \mu_\alpha(f \circ \lambda_s)| < 2\epsilon \cdot ||f||_\infty$ whenever $\alpha_0 \prec \alpha$. It follows that $\nu(f) = \nu(f \circ \lambda_s)$. □

As a consequence of Lemmas 1.5 and 2.2 we have the fact, due to Argabright and Wilde [1], that any semigroup satisfying SFC is left amenable.

We omit the routine proof of the following lemma. If S is a semigroup, $B \subseteq S$, and ν is a mean on S, we write $\nu(B)$ for $\nu(\chi_B)$ where χ_B is the characteristic function of B.

Lemma 2.3. *Let $(S, \cdot)$ be a left amenable semigroup and let ν be a left invariant mean on S.*

(1) If B and C are disjoint subsets of S, then $\nu(B \cup C) = \nu(B) + \nu(C)$.

(2) If B and C are subsets of S and $\nu(B) = \nu(C) = 1$, then $\nu(B \cap C) = 1$.

(3) If $B \subseteq S$ and $s \in S$, then $\nu(B) = \nu(s^{-1}B)$.

Theorem 2.4. *Let $(S, \cdot)$ be a semigroup satisfying SFC and let A be a subset of S with $d(A) = 1$. Then A is thick.*

Proof. By Lemma 1.5, there is a Følner net $\langle F_\alpha \rangle_{\alpha \in D}$ such that $\left\langle \dfrac{|F_\alpha \cap A|}{|F_\alpha|} \right\rangle_{\alpha \in D}$ converges to 1. Let ν be a limit point of the net $\langle \mu_\alpha \rangle_{\alpha \in D}$ where $\mu_\alpha(f) = \frac{1}{|F_\alpha|}\sum_{t \in F_\alpha} f(t)$. By Lemma 2.2, ν is a left invariant mean on S. Since for each $\alpha \in D$, $\mu_\alpha(\chi_A) = \frac{|F_\alpha \cap A|}{|F_\alpha|}$, we have immediately that $\nu(A) = 1$

Let $F \in \mathcal{P}_f(S)$. By Lemma 2.3(3), $\nu(s^{-1}A) = 1$ for every $s \in F$. By Lemma 2.3(2), $\nu(\bigcap_{s \in F} s^{-1}A) = 1$. If x is in this set, $sx \in A$ for every $s \in F$. □

It has been known for some time that any left amenable semigroup S has the property that any pair of right ideals of S meet. (Klawe mentions this fact on the first page of [10].) We have not been able to find the origin of this fact. A proof can be found in [13, Proposition 1.23]. The following special case has a very easy proof.

Corollary 2.5. *Let $(S,\cdot)$ be a semigroup satisfying SFC and let $\mathcal{R}$ be a finite set of right ideals of S. Then $\bigcap \mathcal{R} \neq \emptyset$.*

Proof. Pick ν as in the proof of Theorem 2.4. Let R be a right ideal in S and let $s \in R$. Since $s^{-1}R = S$, $\nu(R) = \nu(s^{-1}R) = \nu(S) = 1$. We can now apply Lemma 2.3(2). $\square$

The following more general result does not demand that S satisfy SFC (though, of course, the Følner density is not defined if it doesn't).

Theorem 2.6. *Let $(S,\cdot)$ be a left amenable semigroup, let ν be a left invariant mean on S, let $A \subseteq S$, and assume that $\nu(A) = 1$. Then A is thick.*

Proof. Let $H \in \mathcal{P}_f(S)$. For each $s \in S$, $\nu(s^{-1}A) = 1$ by Lemma 2.3(3) so by Lemma 2.3(2), $\nu(\bigcap_{s\in H} s^{-1}A) = 1$. Pick $x \in \bigcap_{s\in H} s^{-1}A$. Then $Hx \subseteq A$. $\square$

3 Thick sets have density 1

In this section we establish the converse to Theorem 2.4. With the exception of the following lemma, this proof is also entirely elementary and included in this paper.

Lemma 3.1. *Let $(S,\cdot)$ be a left cancellatiive and left amenable semigroup. Then S satifsfies SFC.*

Proof. Since S is left amenable, S satisfies FC by the left-right switch of [12, Theorem 3.5]. Since S is left cancellative, for each $s \in S$ and each $K \in \mathcal{P}_f(S)$, $|sK \setminus K| = |K \setminus sK|$, so S satisfies SFC. $\square$

The proof of the following lemma is based on the proofs of [3, Lemma 2 and Remark 3] and [10, Theorem 2.2].

Lemma 3.2. *Let $(S,\cdot)$ be a semigroup satisfying SFC and define a relation $\sim$ on S by, for $a, b \in S$, $a \sim b$ if and only if there exists $x \in S$ such that $ax = bx$. Then $\sim$ is an equivalence relation on S and the quotient $T = S/\!\sim$ is a cancellative semigroup which satisfies SFC.*

Proof. We first show that $\sim$ is an equivalence relation; it is obviously reflexive and symmetric. So assume that $a \sim b$ and $b \sim c$ and pick x and y in S such that $ax = bx$ and $by = cy$. By Corollary 2.5, $xS \cap yS \neq \emptyset$ so pick z and w in S such that $xz = yw$. Then $axz = bxz = byw = cyw = cxz$.

For $x \in S$, let $[x] = \{y \in S : x \sim y\}$, so that $T = \{[x] : x \in S\}$. Let $h : S \to T$ be the quotient map, so that for $x \in S$, $h(x) = [x]$.

Now we claim that if $a, b, c \in S$ and $a \sim b$, then $ca \sim cb$ and $ac \sim bc$. The first conclusion is trivial. So assume that $a, b, c \in S$ and $a \sim b$. Pick $x \in S$ such that $ax = bx$. By Corollary 2.5, $xS \cap cS \neq \emptyset$ so pick z and w in S such that $xz = cw$. Then $acw = axz = bxz = bcw$.

Define an operation on T by $[a]\cdot[b]=[ab]$. We show that the operation is well defined so assume $a,b,c,d\in S$, $a\sim c$, and $b\sim d$. We need to show that $ab\sim cd$. This holds since $ab\sim cb\sim cd$.

To see that T is right cancellative, let $a,b,c\in S$ and assume that $ab\sim cb$. Pick $x\in S$ such that $abx=cbx$. Then $a\sim c$.

To see that T is left cancellative, suppose instead that we have $r,u,v\in S$ such that $ru\sim rv$ but $u\not\sim v$. Pick $y\in S$ such that $ruy=rvy$ and let $s=uy$ and $t=vy$. Then $rs=rt$. We claim that for all $x\in S$, $sx\neq tx$. Suppose that we have $x\in S$ such that $sx=tx$. Then $uy\sim vy$ so by right cancellation in T, $u\sim v$, a contradiction. We shall derive a contradiction from the existence of r, s, and t.

Let $\epsilon=\frac{1}{5}$ and pick $K\in\mathcal{P}_f(S)$ such that $|K\cap xK|>(1-\epsilon)|K|$ for every $x\in\{r,s,t\}$. Let $x\in\{r,s,t\}$. For each $k\in K\cap xK$, pick $k'\in K$ such that $xk'=k$. Observe that the map $k\mapsto k'$ defined on $K\cap xK$ is injective. So, if $E_x=\{k':k\in K\cap xK\}$, $|E_x|=|K\cap xK|>(1-\epsilon)|K|$. Furthermore, $xE_x\subseteq K$ and λ_x is injective on E_x.

Let $E=E_r\cap E_s\cap E_t$. Then $|E|>(1-3\epsilon)|K|$. Since λ_s is injective on E, $|E\cap s^{-1}(K\setminus E_r)|\leq|K\setminus E_r|=|K|-|E_r|=|K|-|K\cap rK|<|K|-(1-\epsilon)|K|=\epsilon|K|$ and $|E\cap t^{-1}(K\setminus E_r)|<\epsilon|K|$. Thus there exists $k\in E\cap s^{-1}E_r\cap t^{-1}E_r$. Then sk and tk are distinct elements of E_r. This is a contradiction, because $rsk=rtk$ and λ_r is injective on E_r.

Since h is a surjective homomorphism and S is left amenable, we have by Lemma 1.6 that T is left amenable. Since T is left cancellative we have by Lemma 3.1 that T satisfies SFC. □

It seems worth mentioning the following corollary of Lemma 3.2. (This consequence was noted by Klawe in [10, Corollary 2.3].)

Corollary 3.3. *A semigroup which satisfies SFC is cancellative if it is right cancellative.*

Proof. Let S be a semigroup which satisfies SFC. If S is right cancellative, then S is isomorphic to T. □

Theorem 3.4. *Let $(S,\cdot)$ be a semigroup satisfying SFC and let A be a thick subset of S. Then $d(A)=1$.*

Proof. Let $\sim$ be as in Lemma 3.2. Let $g:T\to S$ be a choice function for the $\sim$ equivalence classes. Note that $h\circ g$ is the identity on T.

To see that $d(A)=1$, let $0<\alpha<1$. We shall show that for every $H\in\mathcal{P}_f(S)$ and every $\epsilon>0$, there exists $G\in\mathcal{P}_f(S)$ such that

(1) for every $s\in H$, $|(sG)\cap G|>(1-\epsilon)\cdot|G|$ and

(2) $|G\cap A|>\alpha\cdot|G|$.

Let $H\in\mathcal{P}_f(S)$ and $\epsilon>0$ be given. Using the fact that $h[H]\in\mathcal{P}_f(T)$ and, by Lemma 3.2, T satisfies SFC, pick $F\in\mathcal{P}_f(T)$ such that

(3) for every $s \in H$, $|F \cap (h(s)F)| > (1-\epsilon) \cdot |F|$.

For each $s \in H$ and each $t \in F \cap (h(s)F)$, pick $r(s,t) \in F$ such that $t = h(s)r(s,t)$ and let $R_{s,t} = \big\{y \in S : g(t)y = sg\big(r(s,t)\big)y\big\}$. Since $h\big(g(t)\big) = t = h(s)r(s,t) = h\big(sg\big(r(s,t)\big)\big)$, there exists $y \in S$ such that $g(t)y = sg\big(r(s,t)\big)y$; that is, $R_{s,t} \neq \emptyset$, so $R_{s,t}$ is a right ideal of S. By Corollary 2.5,

$$\bigcap\{R_{s,t} : s \in H \text{ and } t \in F \cap (h(s)F)\} \neq \emptyset\,.$$

Pick $a \in \bigcap\{R_{s,t} : s \in H \text{ and } t \in F \cap (h(s)F)\}$. Now $g[F]a \in \mathcal{P}_f(S)$ and A is thick so pick $b \in S$ such that $g[F]ab \subseteq A$. Let $u = ab$, note that

$$u \in \bigcap\{R_{s,t} : s \in H \text{ and } t \in F \cap (h(s)F)\}\,,$$

and let $G = g[F]u$. We shall show that G satisfies statements (1) and (2) above.

We first claim that $|G| = |F|$. For this we show that h is injective on G. Having shown this, $h[G] = h\big[g[F]\big]h(u) = Fh(u)$ and since T is right cancellative, $|h[G]| = |F|$ so that $|G| = |F|$. So assume that $x, y \in G$ and $h(x) = h(y)$. Pick z and w in F such that $x = g(z)u$ and $y = g(w)u$. Then $zh(u) = h(x) = h(y) = wh(u)$ so $z = w$ by right cancellation in T.

Next we establish statement (1). Equivalently, since $|G| = |F|$, we show that for every $s \in H$, $|(sg[F]u) \cap (g[F]u)| > (1-\epsilon) \cdot |F|$. For this, we let $s \in H$ be given, define $\Psi : F \cap (h(s)F) \to (sg[F]u) \cap (g[F]u)$ by $\Psi(t) = g(t)u$, and show that Ψ is a bijection.

To see that $\Psi[F \cap (h(s)F)] \subseteq (sg[F]u) \cap (g[F]u)$, let $t \in F \cap (h(s)F)$. Then $u \in R_{s,t}$ so $g(t)u = sg\big(r(s,t)\big)u$ so $\Psi(t) \in (sg[F]u) \cap (g[F]u)$.

To see that Ψ is onto $(sg[F]u) \cap (g[F]u)$, let $x \in (sg[F]u) \cap (g[F]u)$ and pick $t, r \in F$ such that $x = sg(t)u = g(r)u$. Then $h(s)th(u) = rh(u)$ so $r = h(s)t \in F \cap (h(s)F)$ and $\Psi(r) = g(r)u = x$.

To see that Ψ is one-to-one, assume that $t, x \in F \cap (h(s)F)$ and $\Psi(t) = \Psi(x)$. Then $g(t)u = g(x)u$ so $th(u) = xh(u)$ so $t = x$.

Statement (2) trivially holds since $G \subseteq A$ so $G \cap A = G$. □

Corollary 3.5. *Let $(S,\cdot)$ be a semigroup satisfying SFC and let $A \subseteq S$.*

(1) The set A is thick if and only if $d(A) = 1$.

(2) If A is piecewise syndetic, then $d(A) > 0$.

Proof. Statement (1) follows from Theorems 2.4 and 3.4.

To verify (2), pick $G \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in G} t^{-1}A$ is thick. Then by Theorem 3.4, $d(\bigcup_{t \in G} t^{-1}A) = 1$. By [6, Lemma 3.4], $d(\bigcup_{t \in G} t^{-1}A) \leq \sum_{t \in G} d(t^{-1}A)$ so pick $t \in G$ such that $d(t^{-1}A) > 0$. By [8, Theorem 6.3], $d(t^{-1}A) = d(A)$. □

The converse to Corollary 3.5(2) is not valid. By [8, Theorem 1.9] Banach density and Følner density agree on $(\mathbb{N}, +)$. (The Banach density of a set $A \subseteq \mathbb{N}$ is $\sup\{\alpha \in [0,1] : (\forall k \in \mathbb{N})(\exists n \geq k)(\exists a \in \mathbb{N})(|A \cap \{a, a+1, \ldots, a+n-1\}| \geq \alpha \cdot n)$.) In [5, Theorem 3.1] it was shown that for any $\gamma \in (0,1)$, there is a set $B \subseteq \mathbb{N}$ whose Banach density is γ but for all $b \in \mathbb{N}$, the Banach density of $\bigcup_{t=1}^{b}(-t+B)$ is less than 1 so by Theorem 3.4 $\bigcup_{t=1}^{b}(-t+B)$ is not thick, so B is not piecewise syndetic.

4 Some closing remarks

Thick sets have been studied at least since 1949. They were introduced for subsemigroups of abelian groups by Gottschalk and Hedlund [2] under the name *replete*. In [11] and [4], they are called *left thick*. (When we need to distinguish between the left and right versions, we have called them *right thick*.) In [14, Lemma 5.1] it is proved that, for a subset A of an arbitrary semigroup $(S,\cdot)$, $c\ell_{\beta S}(A)$ contains a left ideal of βS if and only if A is thick, but the notion is not given a name there.

We used strongly the fact that two right ideals of a semigroup S satisfying SFC have nonempty intersection. The following result of Gray and Kambites [4, Proposition 2.1] establishes a strong relationship between this property and the notion of thick.

Theorem 4.1. *Let $(S,\cdot)$ be a semigroup. The following statements are equivalent.*

(1) Whenever R and T are right ideals of S, $R\cap T\neq\emptyset$.

(2) Every right ideal of S is thick.

Proof. To see that (1) implies (2), let R be a right ideal of S and let $F\in\mathcal{P}_f(S)$. Enumerate F as $\langle f_i\rangle_{i=1}^n$. We shall show that there exist $\langle t_i\rangle_{i=1}^n$ in S such that for each $i\in\{1,2,\ldots,n\}$, $f_it_1t_2\cdots t_i\in R$. For $i=1$ this follows from the fact that $f_1S\cap R\neq\emptyset$. If $i\in\{1,2,\ldots,n-1\}$ and $\langle t_j\rangle_{j=1}^i$ have been chosen, one has $f_{i+1}t_1t_2\cdots t_iS\cap R\neq\emptyset$ so we can choose t_{i+1} as required. Let $x=t_1t_2\cdots t_n$. Then for each $i\in\{1,2,\ldots,n\}$, $f_ix\in R$. That is $Fx\subseteq R$.

To see that (2) implies (1), let R and T be right ideals of S and pick $a\in R$ and $b\in T$. Then $\{b\}\in\mathcal{P}_f(S)$ so pick $x\in S$ such that $\{b\}x\subseteq R$. Then $bx\in R\cap T$. □

We close with two open questions about semigroups satisfying SFC.

Definition 4.2. Let $(S,\cdot)$ be a semigroup satisfying SFC. Then $LIM_0(S)=\{\eta\in l_\infty(S)^*$: there exists a Følner net $\langle F_\alpha\rangle_{\alpha\in D}$ in $\mathcal{P}_f(S)$ such that $\langle\mu_\alpha\rangle_{\alpha\in D}$ converges to η in the weak* topology$\}$, where for $f\in l_\infty(S)$,

$$\mu_\alpha(f)=\tfrac{1}{|F_\alpha|}\textstyle\sum_{t\in F_\alpha}f(t)\,.$$

Question 4.3. *Let $(S,\cdot)$ be a semigroup satisfying SFC. Is $LIM(S)$ the weak* closed convex hull of $LIM_0(S)$?*

By [7, Theorem 2.12] the answer to Question 4.3 is "yes" if S is left cancellative.

Definition 4.4. Let $(S,\cdot)$ be a semigroup satisfying SFC. Then $\Delta(S)=\{p\in\beta S:(\forall A\in p)(d(A)>0)\}$.

Question 4.5. *Let $(S,\cdot)$ be a semigroup satisfying SFC. Is $\Delta(S)$ a two sided ideal of βS?*

It is always true that $\Delta(S)$ is a left ideal of βS. By [8, Corollary 6.6] the answer to Question 4.5 is "yes" if either S is left cancellative or there exists $b \in \mathbb{N}$ such that for each $x \in S$, ρ_x is at most b-to-1. (These are the same two conditions that were known to imply that thick sets must have density 1 before this current paper.)

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