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On additive properties of sets defined by the Thue-Morse word

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Abstract

In this paper we study some additive properties of subsets of the set $\mathbb{N}$ of positive integers: A subset A of $\mathbb{N}$ is called k-*summable* (where $k \in \mathbb{N}$) if A contains $\left\{ \sum_{n\in F} x_n \,|\, \emptyset \neq F \subseteq \{1, 2, \ldots, k\}\right\}$ for some k-term sequence of natural numbers $\langle x_t\rangle_{t=1}^{k}$ satisfying uniqueness of finite sums. We say $A \subseteq \mathbb{N}$ is *finite FS-big* if A is k-summable for each positive integer k. We say is $A \subseteq \mathbb{N}$ is *infinite FS-big* if for each positive integer k, A contains $\{\sum_{n\in F} x_n \,|\, \emptyset \neq F \subseteq \mathbb{N} \text{ and } \#F \leq k\}$ for some infinite sequence of natural numbers $\langle x_t\rangle_{t=1}^{\infty}$ satisfying uniqueness of finite sums. We say $A \subseteq \mathbb{N}$ is an *IP-set* if A contains $\{\sum_{n\in F} x_n \,|\, \emptyset \neq F \subseteq \mathbb{N} \text{ and } \#F < \infty\}$ for some infinite sequence of natural numbers $\langle x_t\rangle_{t=1}^{\infty}$. By the Finite Sums Theorem [5], the collection of all IP-sets is *partition regular*, i.e., if A is an IP-set then for any finite partition of A, one cell of the partition is an IP-set. Here we prove that the collection of all finite FS-big sets is also partition regular. Let $\mathbb{T} = 011010011001011010010110011010\ldots$ denote the Thue-Morse word fixed by the morphism $0 \mapsto 01$ and $1 \mapsto 10$. For each factor u of $\mathbb{T}$ we consider the set $\mathbb{T}\big|_u \subseteq \mathbb{N}$ of all occurrences of u in $\mathbb{T}$. In this note we characterize the sets $\mathbb{T}\big|_u$ in terms of the additive properties defined above. Using the Thue-Morse word we show that the collection of all infinite FS-big sets is not partition regular.

Keywords: Partition regularity, additive combinatorics, IP-sets, Thue-Morse infinite word.
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1. Introduction

A fundamental result in Ramsey theory, originally due to Issai Schur [12], states that given a finite partition of the natural numbers $\mathbb{N}$, one cell of the partition contains two points x, y and their sum $x + y$. An extension of Schur's Theorem, which we will call the *finite Finite Sums Theorem* states that whenever $\mathbb{N}$ is finitely partitioned, there exist arbitrarily large sets of numbers all of whose sums belong to the same element of the partition. The finite Finite Sums Theorem is an easy consequence of Rado's Theorem [10]. Given a finite sequence $\langle x_t\rangle_{t=1}^{k}$ or an infinite sequence $\langle x_t\rangle_{t=1}^{\infty}$ in $\mathbb{N}$ we say that the sequence satisfies *uniqueness of finite sums* provided that whenever F and H are finite nonempty subsets of the domain of the the sequence and $\sum_{t\in F} x_t = \sum_{t\in H} x_t$, one must have $F = H$. For k a positive integer, we say that a subset A of $\mathbb{N}$

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is k-*summable*, if A contains all finite sums of distinct terms of some k-term sequence $\langle x_n \rangle_{n=1}^{k}$ of natural numbers satisfying uniqueness of finite sums. We say that $A \subseteq \mathbb{N}$ is k^∞-*summable* if there exists an infinite sequence $\langle x_n \rangle_{n=1}^{\infty}$ of natural numbers satisfying uniqueness of finite sums such that A contains all sums of at most k distinct terms of $\langle x_n \rangle_{n=1}^{\infty}$. As a consequence of the (infinite) Finite Sums Theorem, given any finite partition of $\mathbb{N}$, one element of the partition is k^∞-summable.

In this paper we consider three different families of subsets of $\mathbb{N}$ each characterized by an additive property which may be regarded as an extension of the finite Finite Sums Theorem: finite FS-big, infinite FS-big, and IP-sets. A subset A of $\mathbb{N}$ is called *finite FS-big* if it is k-summable for every positive integer k. A subset A of $\mathbb{N}$ is called *infinite FS-big* if it is k^∞-summable for every positive integer k. A subset A of $\mathbb{N}$ is called an *IP-set* if A contains all finite sums of distinct terms of some infinite sequence $\langle x_n \rangle_{n=1}^{\infty}$ of natural numbers.

A collection of sets $\mathcal{S}$ is said to be *partition regular* if for each $A \in \mathcal{S}$, whenever A is partitioned into finitely many sets, at least one set of the partition is in $\mathcal{S}$. By the Finite Sums Theorem, the collection of all IP-sets is partition regular. Other examples of partition regular families are sets having positive upper density, and sets having arbitrarily long arithmetic progressions. (This latter fact is an almost immediate consequence of van der Waerden's Theorem [15]. Assume $A \subseteq \mathbb{N}$ contains arbitrarily long arithmetic progressions. Let $k, r \in \mathbb{N}$, and let $A = \bigcup_{i=1}^{r} C_i$. By van der Waerden's Theorem pick n such that whenever $\{1, 2, \ldots, n\}$ is partitioned into r classes, one class contains a length k arithmetic progression. Pick a and d in $\mathbb{N}$ such that $\{a+d, a+2d, \ldots, a+nd\} \subseteq A$. For $i \in \{1, \ldots, r\}$ let $B_i = \{t \in \{1, 2, \ldots, n\} \mid a + td \in C_i\}$. Pick i, b, and c such that $\{b+c, b+2c, \ldots, b+kc\} \subseteq B_i$. Then $\{a+bd+cd, a+bd+2cd, \ldots, a+bd+kcd\} \subseteq C_i$.)

We shall show in Section 2 that the collection of all finite FS-big subsets of $\mathbb{N}$ is partition regular (see Theorem 2.3). In contrast, for any fixed value of k, the property of being k-summable or k^∞-summable is not partition regular. For example, the set $A = \{n \in \mathbb{N} \mid n \not\equiv 0 \bmod 3\}$ is clearly 2^∞-summable. On the other hand $A = A_1 \cup A_2$ where $A_1 = \{n \in \mathbb{N} \mid n \equiv 1 \mod 3\}$ and $A_2 = \{n \in \mathbb{N} \mid n \equiv 2 \mod 3\}$, and neither A_i is 2-summable. Nevertheless, for each fixed k we could consider the set

$$\mathcal{R}^\infty(k) = \{A \subseteq \mathbb{N} \mid \text{whenever } r \in \mathbb{N} \text{ and } A = \bigcup_{i=0}^{r} A_i,\ \exists\, 0 \le i \le r \text{ such that } A_i \text{ is } k^\infty\text{-summable}\}$$

Then each $\mathcal{R}^\infty(k)$ is non-empty. In fact every IP-set belongs to $\mathcal{R}^\infty(k)$. It is a difficult open question of Imre Leader's [3, Question 8.1] whether there is any member of $\mathcal{R}^\infty(2)$ which is not an IP-set. In general, the question of determining whether a given subset $A \subseteq \mathbb{N}$ is in $\mathcal{R}^\infty(k)$ or is an IP-set is typically quite difficult, even if for every A, either A or its complement belongs to $\mathcal{R}^\infty(k)$ or is an IP-set.

In this note we focus on sets A which are defined in terms of the binary expansions of its elements. In this respect it is natural to consider the *Thue-Morse infinite word* $\mathbb{T} = t_0 t_1 t_2 t_3 \ldots \in \{0,1\}^\omega$ where t_n is defined as the sum modulo 2 of the digits in the binary expansion of n. The Thue-Morse word is 2-automatic [2]: In fact t_n is computed by feeding the binary expansion of n in the deterministic finite automata depicted in Figure 1. Starting from the initial state labeled 0, we read the binary expansion of n starting from the most significant digit. Then t_n is the corresponding terminal state. For example, the binary representation of 13 is 1101 and the path 1101 starting at 0 terminates at vertex 1. Whence $t_{13} = 1$.

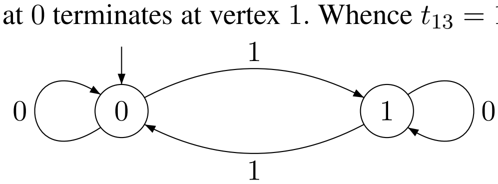

Figure 1: The Thue-Morse automaton

The origins of $\mathbb{T}$ go back to the beginning of the last century with the works of the Norwegian mathematician Axel Thue [13, 14]. Thue noted that every binary word of length four contains a square, that is two consecutive equal blocks XX. He then asked whether it was possible to find an infinite word on 3 distinct symbols which avoided squares. He also asked whether there exists an infinite binary word without cubes, that is with no three consecutive equal blocks. Thue showed that in each case the answer is positive and constructed this very special infinite word $\mathbb{T}$ to produce the desired words. In fact the word $\mathbb{T}$ contains no fractional power greater than 2, i.e., contains no word of the form XXX' where X' is a prefix of X. Thue's work originally appeared in an obscure Norwegian journal and for many years remained largely unknown and unappreciated.

A few years later in the 1920s, Marston Morse and Gustav Hedlund [8, 9] were pioneering a new branch of mathematics known as Symbolic Dynamics, inspired by the study of various classical dynamical systems dating back to Newton. The basic idea of symbolic dynamics consists in dividing up the set of possible states into a finite number of pieces. By discretizing both space and time, one could model a dynamical system (X, T) by a space consisting of infinite words of abstract symbols, each symbol corresponding to a state of the system, and a shift operator corresponding to the dynamics. Thus from this point of view, the orbits of motion are described as *symbolic trajectories* or *flows*. A periodic orbit would give rise to a periodic infinite word, while an aperiodic orbit would correspond to an aperiodic infinite word.

Curiously enough, these foundational works of Morse and Hedlund exhibited strong ties with the earlier work of Thue. This connection stems through the use of infinite words to describe infinite geodesic curves on a surface of negative curvature. And so, the word $\mathbb{T}$ originally defined by Thue to study combinatorial properties of words was rediscovered in 1921 by Morse [7] in connection with differential geometry. He proved that every surface of negative curvature having at least two normal segments, admits a continuum of recurrent aperiodic geodesics.

An alternative definition of the Thue-Morse word which will be useful to us is in terms of the morphism $\tau : \{0,1\} \to \{0,1\}^*$ given by $0 \mapsto 01$ and $1 \mapsto 10$. More precisely, iterating τ on the symbol 0 gives

$$0 \mapsto 01 \mapsto 0110 \mapsto 01101001 \mapsto 0110100110010110 \mapsto \cdots .$$

In general, $\tau^{n+1}(0) = \tau^n(0)\overline{\tau^n(0)}$ where $\overline{\tau^n(0)}$ is obtained from $\tau^n(0)$ by exchanging 0's and 1's. In particular, since $\tau^n(0)$ is a prefix of $\tau^{n+1}(0)$, the sequence $(\tau^n(0))_{n\geq 0}$ tends in the limit to the infinite word

$$\mathbb{T} = 0110100110010110100101100110100110010110\ldots$$

For more background and information on the Thue-Morse word we refer the reader to [1] or [2].

Let $\overline{\mathbb{T}}$ denote the word obtained from $\mathbb{T}$ by exchange of 0's and 1's, i.e., $\overline{\mathbb{T}}$ is the fixed point of the Thue-Morse morphism beginning in 1. We consider subsets of $\mathbb{N}$ defined by the Thue-Morse word via occurrences of its factors. More precisely, writing $\mathbb{T} = t_0t_1t_2\ldots$ with $t_i \in \{0,1\}$, for each factor u of $\mathbb{T}$ we set

$$\mathbb{T}\big|_u = \{n \in \mathbb{N} \,|\, t_nt_{n+1}\ldots t_{n+|u|-1} = u\}.$$

In other words, $\mathbb{T}\big|_u$ denotes the set of all occurrences of u in $\mathbb{T}$. The main result of this note is to obtain a full characterization of each of the sets $\mathbb{T}\big|_u$ in terms of the three additive properties defined above. We show that factors of the Thue-Morse word can be split into three classes: one corresponding to factors u for which $\mathbb{T}\big|_u$ is an IP-set; these factors are precisely all prefixes of $\mathbb{T}$. The second class consists of all factors u such that $\mathbb{T}\big|_u$ is infinite FS-big but not an IP-set; this corresponds to all prefixes of $\overline{\mathbb{T}}$. Finally, for all remaining factors u of $\mathbb{T}$, the set $\mathbb{T}\big|_u$ is not 3-summable, and in some cases not even 2-summable (see Theorem 3.1). We also show that the set $\mathbb{T}\big|_1$ may be partitioned into two cells neither of which is 2^∞-summable (see Lemma 3.3). Thus, the collection of all infinite FS-big sets is not partition regular (see Corollary 3.5). As pointed out to us by the referees of this paper, this latter point may be proved independently of the Thue-Morse word (either directly using the binary representation of digits without reference to $\mathbb{T}$, or via other digital representations of the integers). Our use of $\mathbb{T}$ is merely one of convenience as it provides a uniform framework on which to investigate the various additive properties defined above.

We conclude this introduction with some notation that we will be using. We denote the set of all k-summable subsets of $\mathbb{N}$ by Σ_k and the set of all finite FS-big subsets of $\mathbb{N}$ by Σ. Thus, $\Sigma = \bigcap_{k\geq 1} \Sigma_k$. We denote the set of all k^∞-summable sets by Σ_k^∞ and the set of all infinite FS-big sets by Σ^∞ so that $\Sigma^\infty = \bigcap_{k\geq 1} \Sigma_k^\infty$. It is immediate that $\Sigma_{k+1}^\infty \subseteq \Sigma_k^\infty$ and $\Sigma_k^\infty \subseteq \Sigma_k$.

We have seen that $\{n \in \mathbb{N} \,|\, n \not\equiv 0 \bmod 3\} \in \Sigma_2^\infty \setminus \Sigma_3$. More generally, for each $k > 2$ we have that $\{n \in \mathbb{N} \,|\, n \not\equiv 0 \bmod k\} \in \Sigma_{k-1}^\infty \setminus \Sigma_k$. This follows immediately from the following simple lemma which is likely a well known fact but the authors were unable to find it in the literature. We thus include a proof here for the sake of completeness.

Lemma 1.1. *Given any set S of k nonnegative integers, some subset L of S sums up to 0 modulo k, i.e., $\sum_{x\in L} x \equiv 0 \bmod k$ for some $L \subseteq S$.*

Proof. Equivalently, given a k-term sequence $\langle x_i\rangle_{i=1}^k$ in the cyclic group $\mathbb{Z}_k$ of order k, we will show that some subsequence sums up to 0. To see this, for each $0 \leq i \leq k$, define sets $C_i \subseteq \mathbb{Z}_k$ recursively as follows: $C_0 = \{0\}$ and for $i \geq 0$, set $C_{i+1} = C_i \bigcup (C_i + x_{i+1})$, in other words

$$C_{i+1} = \{0\} \bigcup \{\sum_{i\in F} x_i \,|\, F \subseteq \{1, 2\ \ldots, i+1\}\}.$$

We claim that $0 \in C_i + x_{i+1}$ for some $0 \leq i \leq k-1$, i.e., some subsequence of $\langle x_j\rangle_{j=1}^k$ sums to 0. In fact, for each $0 \leq i \leq k-1$, if $0 \notin C_i + x_{i+1}$ then $\#C_{i+1} > \#C_i$ (since $0 \in C_i$). Thus if for every $0 \leq i \leq k-1$ we had that $0 \notin C_i + x_{i+1}$, then $\#C_k \geq k + \#C_0 = k+1$, a contradiction (since $C_k \subseteq \mathbb{Z}_k$). $\square$

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2. Finite FS-big sets

In this section we prove that the collection of all finite FS-big sets is partition regular (see Theorem 2.3 below). Surprisingly the authors were unable to find a proof of this fact in the existing literature. Thus we take this opportunity to present two different derivations: Our first proof is a straightforward application of the so-called finite Finite Unions Theorem (see Theorem 2.2 below) and is by now quite routine to experts in Ramsey theory. Our second proof uses a clever argument suggested to us by Imre Leader which establishes Theorem 2.2 using only the finite Finite Sums Theorem (Theorem 2.4) and Ramsey's Theorem [11].

Throughout this section we will use the notation $\mathrm{Fin}(A)$ for the set of all non-empty finite subsets of A and let $FS(\langle x_t\rangle_{t\in A}) = \{\sum_{t\in F} x_t \,|\, F \in \mathrm{Fin}(A)\}$.

We first observe that we had some choices to make when we defined k-summable. That is, we could have defined A to be k-summable$_1$ if there is a sequence $\langle x_t\rangle_{t=1}^k$ such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq A$; we could have defined A to be k-summable$_2$ if there is an increasing sequence $\langle x_t\rangle_{t=1}^k$ such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq A$; and we could have defined A to be k-summable$_3$ if there is a sequence $\langle x_t\rangle_{t=1}^k$ satisfying uniqueness of finite sums such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq A$. (We actually chose k-summable$_3$ because it generalizes most naturally to arbitrary semigroups as in Theorem 2.3.) These notions are progressively strictly stronger. For example if $k > 1$, $\{1, 2, \ldots, k\}$ is k-summable$_1$ but not k-summable$_2$. And if $k > 1$, $\{1, 2, \ldots, \frac{k^2+k}{2}\}$ is k-summable$_2$ but not k-summable$_3$. However, for the notion of finite FS-big subsets of $\mathbb{N}$, it does not matter which choice was made for k-summable. The reason is that for each k there is some m such that if A is an m-summable$_1$ subset of $\mathbb{N}$, then A is k-summable$_2$. Similarly, for each k there is some m such that if A is an m-summable$_2$ subset of $\mathbb{N}$, then A is k-summable$_3$.

The main key for proving that finite FS-big sets are partition regular is the finite Finite Unions Theorem. The first proof that we will present, and much the simpler of the two, uses a standard compactness argument and the infinite Finite Unions Theorem.

Theorem 2.1 (Infinite Finite Unions Theorem)**.** *Let* $r \in \mathbb{N}$. *If* $\mathrm{Fin}(\mathbb{N}) = \bigcup_{i=1}^{r} \mathcal{F}_i$*, then there exist* $i \in \{1, 2, \ldots, r\}$ *and a sequence* $\langle F_t \rangle_{t=1}^{\infty}$ *in* $\mathrm{Fin}(\mathbb{N})$ *such that for each* $t \in \mathbb{N}$, $\max F_t < \min F_{t+1}$ *and for each* $H \in \mathrm{Fin}(\mathbb{N})$, $\bigcup_{t \in H} F_t \in \mathcal{F}_i$.

Proof. This is actually stated in [5]. A much easier proof is in [6, Corollary 5.17]. It is an immediate corollary of the (infinite) Finite Sums Theorem, because, given any sequence $\langle x_t \rangle_{t=1}^{\infty}$ in $\mathbb{N}$ one may choose a sequence $\langle F_n \rangle_{n=1}^{\infty}$ in $\mathrm{Fin}(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max F_n < \min F_{n+1}$ and for each n and l in $\mathbb{N}$, if $2^l \leq \sum_{t \in F_n} x_t$, then 2^{l+1} divides $\sum_{t \in F_{n+1}} x_t$. (That is, the maximum of the binary support of $\sum_{t \in F_n} x_t$ is less than the minimum of the binary support of $\sum_{t \in F_{n+1}} x_t$.) □

Theorem 2.2 (Finite Finite Unions Theorem)**.** *Let* $r, k \in \mathbb{N}$. *There is some* $m \in \mathbb{N}$ *such that whenever* $\mathrm{Fin}(\{1, 2, \ldots, m\}) = \bigcup_{i=1}^{r} \mathcal{F}_i$*, there exist* $i \in \{1, 2, \ldots, r\}$ *and a sequence* $\langle F_t \rangle_{t=1}^{k}$ *in* $\mathrm{Fin}(\{1, 2, \ldots, m\})$ *such that for each* $t \in \{1, 2, \ldots, k-1\}$*, if any,* $\max F_t < \min F_{t+1}$ *and for each* $H \in \mathrm{Fin}(\{1, 2, \ldots, k\})$, $\bigcup_{t \in H} F_t \in \mathcal{F}_i$.

Proof. Suppose not. For each $m \in \mathbb{N}$ pick a function $\psi_m : \mathrm{Fin}(\{1, 2, \ldots, m\}) \to \{1, 2, \ldots, r\}$ with the property that there do not exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle F_t \rangle_{t=1}^{k}$ in $\mathrm{Fin}(\{1, 2, \ldots, m\})$ such that for each $t \in \{1, 2, \ldots, k-1\}$, if any, $\max F_t < \min F_{t+1}$ and for each $H \in \mathrm{Fin}(\{1, 2, \ldots, k\})$, $\psi_m(\bigcup_{t \in H} F_t) = i$. Define $\sigma_m : \mathrm{Fin}(\mathbb{N}) \to \{1, 2, \ldots, r\}$ by $\sigma_m(F) = \psi_m(F)$ if $F \subseteq \{1, 2, \ldots, m\}$ and $\sigma_m(F) = 1$ otherwise.

Give $\{1, 2, \ldots, r\}$ the discrete topology and let $X = \times_{F \in \mathrm{Fin}(\mathbb{N})} \{1, 2, \ldots, r\}$ with the product topology. Then X is compact and $\langle \sigma_m \rangle_{m=1}^{\infty}$ is a sequence in X so pick a cluster point φ of $\langle \sigma_m \rangle_{m=1}^{\infty}$. Pick by Theorem 2.1, $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle F_t \rangle_{t=1}^{\infty}$ in $\mathrm{Fin}(\mathbb{N})$ such that for each $t \in \mathbb{N}$, $\max F_t < \min F_{t+1}$ and for each $H \in \mathrm{Fin}(\mathbb{N})$, $\varphi(\bigcup_{t \in H} F_t) = i$. Let

$$U = \left\{ \mu \in X \mid \mu(F_i) = \varphi(F_i) \text{ for all } i \in \{1, 2, \ldots, k\} \right\}.$$

Then U is a neighborhood of φ in X so pick $m > \max F_k$ such that $\sigma_m \in U$. Then for each $H \in \mathrm{Fin}(\{1, 2, \ldots, k\})$, $\psi_m(\bigcup_{t \in H} F_t) = \sigma_m(\bigcup_{t \in H} F_t) = \varphi(\bigcup_{t \in H} F_t) = i$, a contradiction. □

The definition of FS-big makes sense in an arbitrary semigroup $(S, +)$. (Even though we are writing the semigroup additively, we are not assuming commutativity, so we need to specify that the sums are taken in increasing order of indices.) The reader should be cautioned that an arbitrary semigroup might have no nontrivial sequences satisfying uniqueness of finite sums, in which case $\Sigma = \emptyset$. However, if S is cancellative, then by [6, Lemma 6.31], any infinite subset of S contains a sequence satisfying uniqueness of finite products.

Theorem 2.3. *Let* $(S, +)$ *be a semigroup. The collection* Σ *of all finite FS-big subsets of* S *is partition regular.*

Proof. Suppose $A \subseteq S$ is finite FS-big and $A = \bigcup_{i=1}^{r} B_i$ for some $r \in \mathbb{N}$. Let $k \in \mathbb{N}$. We shall show that there are some $i \in \{1, 2, \ldots, r\}$ and some $\langle x_t \rangle_{t=1}^{k}$ satisfying uniqueness of finite sums such that $FS(\langle x_t \rangle_{t=1}^{k}) \subseteq B_i$. By the pigeon hole principle, there is thus one i which contains such a set for arbitrarily large k, and thus for all k.

By Theorem 2.2 pick $m \in \mathbb{N}$ such that whenever $\mathrm{Fin}(\{1, 2, \ldots, m\}) = \bigcup_{i=1}^{r} \mathcal{F}_i$, then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle F_t \rangle_{t=1}^{k}$ in $\mathrm{Fin}(\{1, 2, \ldots, m\})$ such that for each $t \in \{1, 2, \ldots, k-1\}$, if any, $\max F_t < \min F_{t+1}$ and for each $H \in \mathrm{Fin}(\{1, 2, \ldots, k\})$, $\bigcup_{t \in H} F_t \in \mathcal{F}_i$. Since A is finite FS-big we may pick $\langle y_t \rangle_{t=1}^{m}$ satisfying uniqueness of finite sums with $FS(\langle y_t \rangle_{t=1}^{m}) \subseteq A$. For each $i \in \{1, 2, \ldots, r\}$

let $\mathcal{F}_i = \{H \in \mathrm{Fin}(\{1,2,\ldots,m\}) \mid \sum_{t\in H} y_t \in B_i\}$. Pick $i \in \{1,2,\ldots,r\}$ and a sequence $\langle F_t\rangle_{t=1}^k$ in $\mathrm{Fin}(\{1,2,\ldots,m\})$ such that for each $t \in \{1,2,\ldots,k-1\}$, if any, $\max F_t < \min F_{t+1}$ and for each $H \in \mathrm{Fin}(\{1,2,\ldots,k\})$, $\bigcup_{t\in H} F_t \in \mathcal{F}_i$. For $n \in \{1,2,\ldots,k\}$ let $x_n = \sum_{t\in F_n} y_t$. Then since $\max F_t < \min F_{t+1}$ when $t < k$, if $H \in \mathrm{Fin}(\{1,2,\ldots,k\})$ and $K = \bigcup_{n\in H} F_n$, then $\sum_{n\in H} x_n = \sum_{t\in K} y_t \in B_i$. Further it is an easy exercise to show that, since $\langle y_t\rangle_{t=1}^m$ satisfies uniqueness of finite sums, so does $\langle x_t\rangle_{t=1}^k$. □

Notice that, since we did the above proof for an arbitrary semigroup, it would not be good enough to have $F_t \cap F_l = \emptyset$ when $t \neq l$. For example, if $F_1 = \{1,3\}$, $F_2 = \{2\}$, $H = \{1,2\}$, and $K = \bigcup_{n\in H} F_n$, then $K = \{1,2,3\}$. Thus $\sum_{n\in H} x_n = y_1 + y_3 + y_2$ which need not equal $y_1 + y_2 + y_3 = \sum_{t\in K} y_t$.

As we remarked earlier, the finite Finite Sums Theorem, (sometimes called Folkman's Theorem), has been known, or at least easily knowable, since the proof of Rado's Theorem [10] was published in 1933.

Theorem 2.4 (Finite Finite Sums Theorem)**.** *Let $k, r \in \mathbb{N}$. There exists $m \in \mathbb{N}$ such that whenever $\{1,2,\ldots,m\} = \bigcup_{i=1}^r B_i$, there exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle x_t\rangle_{t=1}^k$ such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq B_i$.*

Proof. This is an easy consequence of Rado's Theorem. See [6, Exercise 15.3.1]. □

While it is immediate that Theorem 2.2 implies Theorem 2.4 (by means of the binary support of integers), it is by no means obvious that one can derive Theorem 2.2 from Theorem 2.4. We are grateful to Imre Leader for providing an argument which establishes Theorem 2.2 using only Theorem 2.4 and Ramsey's Theorem [11].

Lemma 2.5. *Let $k, r, s \in \mathbb{N}$ with $k \leq s$. There exists $m \in \mathbb{N}$ such that whenever A is a set with $\#A = m$, and $\mathrm{Fin}(A) = \bigcup_{i=1}^r \mathcal{F}_i$, there exist $\varphi : \{1,2,\ldots,k\} \to \{1,2,\ldots,r\}$ and $B \subseteq A$ with $\#B = s$ such that for all $C \in \mathrm{Fin}(B)$, if $t = \#C$ and $t \leq k$, then $C \in \mathcal{F}_{\varphi(t)}$.*

Proof. We proceed by induction on k (for all r and all $s \geq k$). For $k = 1$ the conclusion is an immediate consequence of the pigeon hole principle.

Now assume that the lemma holds for k. Let $r, s \in \mathbb{N}$ be given with $s \geq k+1$. By Ramsey's Theorem pick $n \in \mathbb{N}$ such that if $\#D = n$ and $\{C \subseteq D \mid \#C = k+1\} \subseteq \bigcup_{i=1}^r \mathcal{F}_i$, then there exist $i \in \{1,2,\ldots,r\}$ and $B \subseteq D$ such that $\#B = s$ and $\{C \subseteq B \mid \#C = k+1\} \subseteq \mathcal{F}_i$.

Pick m as guaranteed by the induction hypothesis for k, r, and n (with n replacing s) and let $\#A = m$. Assume that $\mathrm{Fin}(A) = \bigcup_{i=1}^r \mathcal{F}_i$. Pick $\varphi : \{1,2,\ldots,k\} \to \{1,2,\ldots,r\}$ and $D \subseteq A$ with $\#D = n$ such that for all $C \in \mathrm{Fin}(D)$, if $t = \#C$ and $t \leq k$, then $C \in \mathcal{F}_{\varphi(t)}$. Then $\{C \subseteq D \mid \#C = k+1\} \subseteq \bigcup_{i=1}^r \mathcal{F}_i$, so pick $\varphi(k+1) \in \{1,2,\ldots,r\}$ and $B \subseteq D$ such that $\#B = s$ and $\{C \subseteq B \mid \#C = k+1\} \subseteq \mathcal{F}_\varphi(k+1)$. □

Second proof of Theorem 2.2 Let $k, r \in \mathbb{N}$ and pick by Theorem 2.4 $s \in \mathbb{N}$ such that whenever $\{1,2,\ldots,s\} = \bigcup_{i=1}^r C_i$, there exist $x_1, x_2, \ldots, x_k$ and $i \in \{1,2,\ldots,r\}$ such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq C_i$. Let $k' = s$ and pick m as guaranteed by Lemma 2.5 for r, k', and s. Let $\mathrm{Fin}(\{1,2,\ldots,m\}) = \bigcup_{i=1}^r \mathcal{F}_i$. Pick $\varphi : \{1,2,\ldots,s\} \to \{1,2,\ldots,r\}$ and $B \subseteq \{1,2,\ldots,m\}$ with $\#B = s$ such that for all $C \in \mathrm{Fin}(B)$, if $t = \#C$ and $t \leq s$, then $C \in \mathcal{F}_{\varphi(t)}$. Pick $x_1, x_2, \ldots, x_k$ and $i \in \{1,2,\ldots,r\}$ such that $\varphi[FS(\langle x_t\rangle_{t=1}^k)] = \{i\}$. Pick $\langle F_t\rangle_{t=1}^k$ with $\max F_t < \min F_{t+1}$ for all $t < k$ such that $\#F_t = x_t$, which one can do since $\sum_{t=1}^k x_t \leq s$. □

3. Sets defined by the Thue-Morse word

In this section we define a class of subsets of $\mathbb{N}$ defined by the occurrences of factors in the Thue-Morse word.

Theorem 3.1. *Let u be a factor of the Thue-Morse word $\mathbb{T} = 011010011001011010\ldots$. Then*

1. *If u is a prefix of $\mathbb{T}$ then $\mathbb{T}\big|_u$ is an IP-set.*
2. *If u is a prefix of $\overline{\mathbb{T}}$ then $\mathbb{T}\big|_u$ is infinite FS-big but is not an IP-set.*
3. *If u is neither a prefix of $\mathbb{T}$ nor a prefix of $\overline{\mathbb{T}}$ then $\mathbb{T}\big|_u$ is not 3-summable. Moreover $\mathbb{T}\big|_u$ is 2-summable if and only u is a prefix of $\tau^n(010)$ or of $\tau^n(101)$ for some $n \geq 0$.*

Before we begin with the proof of Theorem 3.1 we introduce some useful notation: For each positive integer n we will denote the binary expansion of n by $[n]_2$, i.e., if $n = r_k 2^k + r_{k-1}2^{k-1} + \ldots + r_0 2^0$ with $r_k = 1$ and $r_i \in \{0,1\}$ we write $[n]_2 = r_k r_{k-1} \ldots r_0$. We define the *support* of n, denote $\mathrm{supp}(n)$ by $\mathrm{supp}(n) = \{i \in \{0,1,\ldots,k\} \mid r_i = 1\}$. For instance, $\mathrm{supp}(19) = \{0,1,4\}$. Thus

$$t_n = 0 \Leftrightarrow \#\mathrm{supp}(n) \text{ is even.}$$

Finally, for each length n we denote by $\mathrm{pref}_n\mathbb{T}$ the prefix of $\mathbb{T}$ of length n.

Proof of Theorem 3.1, part 1. It follows from the definition of the Thue-Morse word $\mathbb{T}$ that if $u = u_1 u_2 \ldots u_k \in \{0,1\}^k$ is a factor of $\mathbb{T}$, then $m \in \mathbb{T}\big|_u$ if and only if

$$\#\mathrm{supp}(m+j) \equiv u_{j+1} \bmod 2$$

for each $0 \leq j \leq |u|-1$. Thus, if $2^n > m + |u| - 1$ then $\#\mathrm{supp}(2^{n+1} + 2^n + m + j) = 2 + \#\mathrm{supp}(m+j)$ for each $0 \leq j \leq |u|-1$ from which it follows that $2^{n+1} + 2^n + m \in \mathbb{T}\big|_u$. Hence if $0 \in \mathbb{T}\big|_u$ (equivalently if u is a prefix of $\mathbb{T}$), then there is a sequence of positive integers of the form $2^n + 2^{n+1}$ whose finite sums are all in $\mathbb{T}\big|_u$. Thus, $\mathbb{T}\big|_u$ is an IP-set. This completes the proof of 1.

We will need the following lemma in the proof of 2.:

Lemma 3.2. *Let i, j, k and r be positive integers with r odd and $j \leq k-2$. If $[r]_2 = r_j r_{j-1} \ldots r_0$ then*

$$\#\mathrm{supp}\big(r2^i(2^k - 1)\big) = k.$$

Proof. Since $\#\mathrm{supp}\big(r2^i(2^k-1)\big) = \#\mathrm{supp}\big(r(2^k-1)\big)$ it suffices to show that $\#\mathrm{supp}\big(r(2^k-1)\big) = k$.

We have that $[r2^k]_2 = r_l r_{l-1} \ldots r_0 0^k$. Thus $[r2^k - 1]_2 = r_l r_{l-1} \ldots r_0 1^{k-1}$.

Since $l \leq k-2$ we have

$$\#\mathrm{supp}\big(r(2^k-1)\big) = \#\mathrm{supp}(r2^k - 1 - r + 1) = \#\mathrm{supp}(r) + k - 1 - \#\mathrm{supp}(r) + 1 = k$$

where the last $+1$ term comes from the fact that $r_0 = 1$ since r is odd. □

Proof of Theorem 3.1, part 2. We first note that those n's for which $\#\mathrm{supp}(n)$ is odd and $[n]_2$ ends in 0^l correspond to occurrences of $\mathrm{pref}_{2^l}\overline{\mathbb{T}}$.

Let u be a prefix of $\overline{\mathbb{T}}$ and k a positive integer. To prove that $\mathbb{T}_u \in \Sigma^\infty_{2k-1}$ consider the sequence $\langle x_n \rangle_{n=0}^\infty$ of numbers whose binary representation is given by

$$[x_n]_2 = 110^{2n+j}1^{2k-1}0^l \text{ where } j = \lceil \log_2(2k-1) \rceil \text{ and } l = \lceil \log_2 |u| \rceil .$$

Consider any $r \leq 2k-1$ distinct numbers x_{n_i} and consider their sum

$$\sum_{i=1}^r x_{n_i} = \sum_{i=1}^r (2^{2n_i+2k-1+l+j} + 2^{2n_i+2k+l+j}) + r(2^{2k-1}-1)2^l.$$

By Lemma 3.2 it follows that $\#\mathrm{supp}(r(2^{2k-1}-1)2^l) = 2k-1$ and hence that $\#\mathrm{supp}(\sum_{i=1}^r x_{n_i}) = 2k-1+2r$. As this is an odd number, and $[\sum_{i=1}^r x_{n_i}]_2$ ends in at least $l = \lceil \log_2 |u| \rceil$ many 0's, it follows that $\sum_{i=1}^r x_{n_i}$ is an occurrence of u.

Next we will prove that if u is a prefix of $\overline{\mathbb{T}}$ then $\mathbb{T}\big|_u$ is not an IP-set. We will make use of the following lemma:

Lemma 3.3. *There exists a partition of the set $\mathbb{T}\big|_1$ into two sets neither of which is in Σ_2^∞.*

Proof. Consider the partition $\mathbb{T}\big|_1 = A_0 \cup A_1$ defined as follows: Let A_0 be the set of all $n \in \mathbb{T}\big|_1$ such that the $\min\bigl(\mathrm{supp}(n)\bigr)$ is even, and let A_1 be the set of all $n \in \mathbb{T}\big|_1$ such that the $\min\bigl(\mathrm{supp}(n)\bigr)$ is odd. For instance, $25 = 2^4 + 2^3 + 2^0$, and hence the least nonzero digit is in position 0, so $25 \in A_0$. We will show that neither A_i is in Σ_2^∞. Fix $i \in \{0,1\}$ and suppose to the contrary that A_i is in Σ_2^∞, i.e., there is an infinite sequence $\langle x_n \rangle_{n=1}^\infty$ in A_i satisfying uniqueness of finite sums such that for every $n \neq m$ we have $x_n + x_m \in A_i$. Note first that for each $n > 1$ we have $\mathrm{supp}(x_n) \cap \mathrm{supp}(x_1) \neq \emptyset$. Otherwise $\#\mathrm{supp}(x_1 + x_n)$ would be even. Therefore, there exists a positive constant M such that $\min\bigl(\mathrm{supp}(x_n)\bigr) \leq M$ for each $n \in \mathbb{N}$. By the pigeon hole principle there exists a positive integer r and an infinite subsequence $x_{n_1}, x_{n_2}, \ldots$ of the sequence $\langle x_n \rangle_{n=1}^\infty$ such that $\min\bigl(\mathrm{supp}(x_{n_j})\bigr) = r$ for each $j \in \mathbb{N}$. Again by the pigeon hole principle there exists infinitely many of the x_{n_j} whose binary expansions also agree in position $r+1$. Thus there exists $n \neq m$ such that $\min\bigl(\mathrm{supp}(x_n)\bigr) = \min\bigl(\mathrm{supp}(x_m)\bigr) = r$ and such that $r + 1 \in \mathrm{supp}(x_n)$ if and only if $r + 1 \in \mathrm{supp}(x_m)$. It is readily verified that $\min\bigl(\mathrm{supp}(x_n + x_m)\bigr) = r + 1$. Hence $x_n + x_m \in A_{1-i}$. $\square$

Remark. It is not difficult to see that the sets A_0 and A_1 from the proof of Lemma 3.3 are both finite FS-big. So they provide examples of sets which are finite FS-big but not 2^∞-summable.

It follows from the above lemma that $\mathbb{T}\big|_1$ is not an IP-set. In fact, the property of being an IP-set is partition regular, so for any finite partition of $\mathbb{T}\big|_1$ one element of the partition must be an IP-set and in particular must be in Σ_2^∞. But this contradicts Lemma 3.3. Let u be a prefix of $\overline{\mathbb{T}}$. Since $\mathbb{T}\big|_u \subseteq \mathbb{T}\big|_1$ it follows that $\mathbb{T}\big|_u$ is not an IP-set.

Proof of Theorem 3.1, part 3. We will make use of the following lemma:

Lemma 3.4. *Let u be a factor of $\mathbb{T}$ which is neither a prefix of $\mathbb{T}$ nor a prefix of $\overline{\mathbb{T}}$. Then there exists a nonnegative integer k such that one of the two following properties holds:*

1. *For each $x \in \mathbb{T}\big|_u$, $[x]_2$ ends in 10^k.*
2. *For each $x \in \mathbb{T}\big|_u$ either $[x]_2$ ends in 110^k or in 10^{k+1} and both cases happen. Furthermore, u is a prefix of $\tau^n(aba)$ for some nonnegative integer n, with a, b distinct letters.*

Proof. Let u be a factor of $\mathbb{T}$ which is neither a prefix of $\mathbb{T}$ nor a prefix of $\overline{\mathbb{T}}$ and let $\{a, b\} = \{0, 1\}$. If x is an occurrence of aa, then the $\#\mathrm{supp}(x)$ and $\#\mathrm{supp}(x+1)$ have the same parity, and hence $[x]_2$ ends in 1 and the statement is verified for all factors beginning with aa.

We can then assume that u begins with ab and we will proceed by induction on $|u|$. Clearly the shortest such u is of the kind aba. Then for each $x \in \mathbb{T}_u$, the number of 1's in the binary expansion of x and of $x+2$ have the same parity. It follows that $[x]_2$ must end in 10 or 11 (and in fact it is easily verified that both are possible). Thus the result of the lemma is verified with $k = 0$.

Next suppose $|u| = N \geq 4$ and that the claim is true for all factors u of length smaller than N. If u begins in aba, then $\mathbb{T}\big|_u \subseteq \mathbb{T}\big|_{aba}$ and hence, as we have just seen if $x \in \mathbb{T}\big|_u$ we have that $[x]_2$ must end in either 11 or 10. Otherwise u must begin in either 0110 or 1001. In this case, let v denote the longest prefix of u which is either a prefix of $\mathbb{T}$ or of $\overline{\mathbb{T}}$. Then we can write $u = va\lambda$ where v begins in either 0110 or 1001, $a \in \{0,1\}$ and $v \in \{0,1\}^*$. Since both $v0$ and $v1$ are factors of $\mathbb{T}$ it follows that $v = \tau(v')$ for some v' strictly shorter than v such that v' begins in 01 or 10, and $v'a$ is a factor of $\mathbb{T}$ which is neither a prefix of $\mathbb{T}$ nor of $\overline{\mathbb{T}}$. By the induction hypothesis we deduce that there exists a k such that for all $x' \in \mathbb{T}\big|_{v'a}$ we have that $[x']_2$ ends in either 110^k or in 10^{k+1}. Moreover, since every occurrence of va in $\mathbb{T}$ (and hence of u) is the image of τ of an occurrence in $\mathbb{T}$ of $v'a$ it follows that if $x \in \mathbb{T}\big|_u$ then $x = 2x'$ for some $x' \in \mathbb{T}\big|_{v'a}$. Whence $[x]_2$ ends in either 110^{k+1} or in 10^{k+2}. We have thus proved that if u is neither a prefix of $\mathbb{T}$ nor of $\overline{\mathbb{T}}$ and u begins in ab, then there exists a k such that for any $x \in \mathbb{T}\big|_u$ either $[x]_2$ ends in 110^k or in 10^{k+1}. If only one of these cases occurs, then clearly property 1 holds and we are done. Assume then that both cases occur, we need to prove that u is a prefix of $\tau^n(aba)$ for some nonnegative integer n (i. e. that we are in case 2). It is not difficult to prove (given the definition of τ) that every factor of $\mathbb{T}$ of length at least 4 either appears only in odd positions or only in even positions. Since we are assuming that there exist $x, y \in \mathbb{T}\big|_u$ such that $[x]_2$ ends in 110^k and $[y]_2$ ends in 10^{k+1}, it must be $k > 0$ and u occurs only in even positions. Again from the definition of τ, it is easy to see that if $|u|$ is odd, then there exists a unique letter c such that every occurrence of u is followed by c. Hence there exists a unique $\alpha \in \{0, 1, \varepsilon\}$ such that $|u\alpha|$ is even and $\mathbb{T}\big|_u = \mathbb{T}\big|_{u\alpha}$. From the uniformity of τ, since $u\alpha$ is a factor of $\mathbb{T}$ of even length which appears only in even positions, there exists u' shorter than u such that $\tau(u') = u\alpha$ and $\mathbb{T}\big|_{u\alpha} = \{2x, x \in \mathbb{T}\big|_{u'}\}$. Hence, for each $x \in \mathbb{T}\big|_{u'}$ either $[x]_2$ ends in 110^{k-1} or in 10^k and both cases actually happen, thus, by induction hypothesis, u' is a prefix of $\tau^n(aba)$ for some n and u is a thus a prefix of $\tau^{n+1}(aba)$. □

We are now able to easily prove item 3. of our main theorem. First of all, let us observe that it is readily verified that $\{3, 15, 18\} \subseteq \mathbb{T}\big|_{010}$ and $\{35, 47, 82\} \subseteq \mathbb{T}\big|_{101}$ and hence $\{2^n \cdot 3, 2^n \cdot 15, 2^n \cdot 18\} \subseteq \mathbb{T}\big|_{\tau^n(010)}$ and $\{2^n \cdot 35, 2^n \cdot 47, 2^n \cdot 82\} \subseteq \mathbb{T}\big|_{\tau^n(101)}$ which proves that if $aba \in \{010, 101\}$, then $\mathbb{T}\big|_{\tau^n(aba)}$ are 2-summable for every nonnegative integer n. Clearly then, if u is a prefix of some $\tau^n(aba)$, $\mathbb{T}\big|_u$ is 2- summable as well.

Let u be a factor of $\mathbb{T}$. In case point 1 of the preceding lemma holds, we have that there exists a nonnegative integer k such that each $x \in \mathbb{T}\big|_u$ ends 10^k. But then for any $x, y \in \mathbb{T}\big|_u$, it follows that $[x + y]_2$ ends in 0^{k+1} and hence $x + y \notin \mathbb{T}\big|_u$.

Thanks to point 2 of the preceding lemma we have thus proved that $\mathbb{T}\big|_u$ is 2-summable if and only if u is a prefix of $\tau^n(aba)$ for some n and a, b distinct letters (considering that prefixes of $\mathbb{T}$ and of $\overline{\mathbb{T}}$ are as well prefixes of $\tau^n(aba)$). We are left to prove that if u is neither a prefix $\mathbb{T}$ nor a prefix of $\overline{\mathbb{T}}$, then $\mathbb{T}\big|_u$ is not 3-summable. Of course, the statement is trivial if $\mathbb{T}\big|_u$ is not 2-summable, so, as observed before, we can assume that point 2 of Lemma 3.4 holds, that is there exists k such that $[x]_2$ ends in 100^k or 110^k for each $x \in \mathbb{T}_u$ and both cases happen. Consider three points $x, y, z \in \mathbb{T}\big|_u$. If $[x]_2$ ends in 100^k, then $[x + y]_2$ ends in 000^k or 010^k; in either way it follows that $x + y \notin \mathbb{T}\big|_u$. On the other hand if $[x]_2$ and $[y]_2$ both end in 110^k, then $[x + y]_2$ ends in 100^k and hence as above it follows that $x + y + z \notin \mathbb{T}\big|_u$. It follows that $\mathbb{T}\big|_u$ is not 3-summable, and the statement is complete. □

Remark. We proved part 1 of Theorem 3.1 directly using the numeration system, though it actually follows from parts 2, 3, and the Finite Sums Theorem [5].
As a corollary of Theorem 3.1 2. and Lemma 3.3 we obtain:

Corollary 3.5. *Σ^∞ is not partition regular, i.e., there exists a set $A \subseteq \mathbb{N}$ which is infinite FS-big and a partition of $A = A_0 \cup A_1$ such that neither A_i is 2^∞-summable.*

One of the two referees suggested the following alternative proof of Corollary 3.5. For each positive integer n, let $\mathrm{supp}_3(n)$ denote the support of the ternary expansion of n. Let $D = FS(\langle 3^n \rangle_{n \in \mathbb{N}})$. We note that for any $m, n \in D$, if $m + n \in D$ then $\mathrm{supp}_3(m) \cap \mathrm{supp}_3(n) = \emptyset$. Let $(E_i)_{i=1}^\infty$ be a partition of $\mathbb{N}$ into infinite disjoint sets. For each $i \in \mathbb{N}$ set

$$D_i = \{n \in D : \mathrm{supp}_3(n) \subseteq E_i \text{ and } \#\mathrm{supp}_3(n) \leq i\}$$

and put $A = \bigcup_{i=1}^\infty D_i$. Then clearly $A \in \Sigma^\infty$. However, let B_0 and B_1 be a partition of $\mathbb{N}$ so that no two integers in $\mathbb{N}$ of the form x and $2x$ are both in B_0 or both in B_1. For $i \in \{0, 1\}$, put

$$A_i = \{n \in A : \#\mathrm{supp}_3(n) \in B_i\}.$$

It is clear we cannot have $x, y \in A_i$ satisfying $x + y \in A_i$ and $\#\mathrm{supp}_3(x) = \#\mathrm{supp}_3(y)$. Thus A_i is not 2^∞-summable.

We next derive two additional consequences of Theorem 3.1. For this purpose, we recall some terminology which will be needed. Let $\mathbb{A}$ be a finite non-empty set, and let $\mathbb{A}^\omega$ denote the set of all right infinite words $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \mathbb{A}$. We endow $\mathbb{A}^\omega$ with the topology generated by the metric

$$d(x, y) = \frac{1}{2^n} \text{ where } n = \min\{k : x_k \neq y_k\}$$

whenever $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ are two elements of $\mathbb{A}^\omega$. (This is also the product topology when $\mathbb{A}$ has the discrete topology.) Let $T : \mathbb{A}^\omega \to \mathbb{A}^\omega$ denote the *shift* transformation defined by $T : (x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}$. A point $x \in X$ is said to be *uniformly recurrent* in X if for every neighborhood V of x the set $\{n \,|\, T^n(x) \in V\}$ is syndetic, i.e., of bounded gap. Two points $x, y \in \mathbb{A}^\omega$ are said to be *proximal* if for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) < \epsilon$.

Let X be a closed and T-invariant subset of $\mathbb{A}^\omega$; the pair (X, T) is called a *subshift* of $\mathbb{A}^\omega$. A subshift (X, T) is said to be *minimal* whenever X and the empty set are the only T-invariant closed subsets of X. To each $x \in \mathbb{A}^\omega$ is associated the subshift $(\Omega(x), T)$ where $\Omega(x)$ is the shift orbit closure of x. A point $x \in \mathbb{A}^\omega$ is called *distal* if the only point in $\Omega(x)$ proximal to x is x itself. If $x \in \mathbb{A}^\omega$ is uniformly recurrent, then the associated subshift $(\Omega(x), T)$ is minimal. And, if $(\Omega(x), T)$ is minimal, then every point of $\Omega(x)$ is uniformly recurrent. (For the proofs of the last two assertions see for example [4, Theorems 1.17 and 1.15].) It is well known that the Thue-Morse word is uniformly recurrent. (See for example [8, p. 832].)

As an application of Theorem 3.1 we have the following corollary. In the proof of this corollary we use some facts from [6] about the algebraic structure of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$, the points of which are the ultrafilters on $\mathbb{N}$. Given an ultrafilter $p \in \beta\mathbb{N}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ in a compact Hausdorff space X, $p\text{-}\lim\limits_{n\in\mathbb{N}} x_n$ is the unique point $y \in X$ with the property that for every neighborhood U of y, $\{n \in \mathbb{N} | x_n \in U\} \in p$.

Corollary 3.6. *The Thue-Morse word $\mathbb{T}$ is distal. In particular, for each $n \geq 0$, exactly one of the sets $\{T^n(\mathbb{T})\big|_0, T^n(\mathbb{T})\big|_1\}$ is an IP-set.*

Proof. Suppose $x \in \Omega(\mathbb{T})$ is proximal to $\mathbb{T}$. Then, since $\mathbb{T}$ is uniformly recurrent, we have by [6, Theorem 19.26] that there exists a (minimal) idempotent ultrafilter $p \in \beta\mathbb{N}$ with $p\text{-}\lim\limits_{n\in\mathbb{N}} T^n(\mathbb{T}) = x$. Given a prefix u of x, $U = \{y \in \mathbb{A}^\omega | u \text{ is a prefix of } y\}$ is a neighborhood of x so $\{n \in \mathbb{N} | T^n(\mathbb{T}) \in U\} \in p$; that is $\mathbb{T}\big|_u \in p$. Therefore by [6, Theorem 5.12] $\mathbb{T}\big|_u$ is an IP-set. By Theorem 3.1 it follows that u is a prefix of $\mathbb{T}$ and hence $x = \mathbb{T}$ as required. Having established that $\mathbb{T}$ is distal, it follows that $T^n(\mathbb{T})$ is distal for each $n \geq 0$. Finally, let us fix $n \geq 0$, and let $a \in \{0, 1\}$ denote the initial symbol of $T^n(\mathbb{T})$. We claim that $T^n(\mathbb{T})\big|_a$ is an IP-set while $T^n(\mathbb{T})\big|_{\overline{a}}$ is not, where $\overline{a} := 1 - a$. Since $T^n(\mathbb{T})$ is uniformly recurrent, it follows from [6, Theorem 19.23] that there exists a idempotent ultrafilter $p \in \beta\mathbb{N}$ with $p\text{-}\lim\limits_{m\in\mathbb{N}} T^m(T^n(\mathbb{T})) = T^n(\mathbb{T})$. Then as above, $T^n(\mathbb{T})\big|_a \in p$ so by [6, Theorem 5.12] we have that $T^n(\mathbb{T})\big|_a$ is an IP-set. Now suppose on the other hand that $T^n(\mathbb{T})\big|_{\overline{a}}$ is also an IP-set. Then by [6, Theorem 5.12] there exists an idempotent $q \in \beta\mathbb{N}$ such that $T^n(\mathbb{T})\big|_{\overline{a}} \in q$. We claim that $q\text{-}\lim\limits_{m\in\mathbb{N}} T^m\big(T^n(\mathbb{T})\big)$ is proximal to $T^n(\mathbb{T})$ for which it suffices by [6, Lemma 19.22] to show that $q\text{-}\lim\limits_{r\in\mathbb{N}} T^r\big(q\text{-}\lim\limits_{m\in\mathbb{N}} T^m\big(T^n(\mathbb{T})\big)\big) = q\text{-}\lim\limits_{k\in\mathbb{N}} T^k\big(T^n(\mathbb{T})\big)$. To this end

$$
\begin{aligned}
q\text{-}\lim_{r\in\mathbb{N}} T^r\big(q\text{-}\lim_{m\in\mathbb{N}} T^m\big(T^n(\mathbb{T})\big)\big) &= q\text{-}\lim_{r\in\mathbb{N}} q\text{-}\lim_{m\in\mathbb{N}} T^{r+m}\big(T^n(\mathbb{T})\big) \text{ by [6, Theorem 3.49]}\\
&= (q+q)\text{-}\lim_{k\in\mathbb{N}} T^k\big(T^n(\mathbb{T})\big) \text{ by [6, Theorem 4.5]}\\
&= q\text{-}\lim_{k\in\mathbb{N}} T^k\big(T^n(\mathbb{T})\big)\,.
\end{aligned}
$$

Since $q\text{-}\lim_{m\in\mathbb{N}} T^m\big(T^n(\mathbb{T})\big)$ is proximal to $T^n(\mathbb{T})$ and $T^n(\mathbb{T})$ is distal, we have that $q\text{-}\lim_{m\in\mathbb{N}} T^m(T^n(\mathbb{T})) = T^n(\mathbb{T})$. Thus, $T^n(\mathbb{T})\big|_a \in q$ from which it follows that $\emptyset = T^n(\mathbb{T})\big|_a \cap T^n(\mathbb{T})\big|_{\overline{a}} \in q$, a contradiction. □

Corollary 3.7. *Let N be a positive integer and set $x = t_N t_{N-1} \dots t_0\mathbb{T} \in \Omega(\mathbb{T})$ where $\mathbb{T} = t_0t_1t_2\dots$. Consider the partition $\mathbb{N} = A_0 \cup A_1$ where $A_0 = x\big|_0$ and $A_1 = x\big|_1$. Then $A_i - n$ is an IP-set for each $i \in \{0,1\}$ and $0 \le n \le N$. On the other hand, for each $n > N$, exactly one of the sets $\{A_0 - n, A_1 - n\}$ is an IP-set.*

Proof. For $a \in \{0,1\}$ we put $\overline{a} = 1 - a$. We first note that since $a\mathbb{T} \in \Omega(\mathbb{T})$ for some $a \in \{0,1\}$, by iteratively applying the morphism $0 \mapsto 01$, $1 \mapsto 10$ we have that both $t_nt_{n-1}\dots t_0\mathbb{T} \in \Omega(\mathbb{T})$ and $\overline{t}_n\overline{t}_{n-1}\dots\overline{t}_0\mathbb{T} \in \Omega(\mathbb{T})$ for each $n \ge 0$. Fix a positive integer N and put $x = t_Nt_{N-1}\dots t_0\mathbb{T}$ and $y = \overline{t}_N\overline{t}_{N-1}\dots\overline{t}_0\mathbb{T}$. Then for each $0 \le n \le N$, we have that $T^n(x)$ and $T^n(y)$ are proximal and begin in distinct symbols. Whence applying [6, Theorem 19.26 & Theorem 5.12] we deduce that $A_0 - n = T^n(x)\big|_0$ and $A_1 - n = T^n(x)\big|_1$ are both IP-sets for $0 \le n \le N$. On the other hand, applying Corollary 3.6 we see that for each $n > N$, exactly one of the sets $\{A_0 - n, A_1 - n\}$ is an IP-set. □

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