This paper was published in *Topology Proceedings* **23** (1998), 155-190. To the best of my knowledge, this is the final version as it was submitted to the publisher.–NH

Topologies on S Determined by Idempotents in βS

by

Neil Hindman[1]

Igor Protasov[2]

and

Dona Strauss[2]

Abstract. Given a discrete semigroup S, its Stone-Čech compactification βS has a rich algebraic structure. In particular, it has idempotents – usually many idempotents. Each of these idempotents in turn determines a topology on S in at least two different ways making S a left topological semigroup. (These are then topologies on S completely determined by the algebraic structure of S.) We investigate these topologies and the relationship between them, paying special attention to the existence of separate or joint continuity.

1. Introduction.

Given a discrete semigroup $(S,\cdot)$, the operation extends to the Stone-Čech compactification βS making $(\beta S,\cdot)$ a compact right topological semigroup (i.e., for each $p\in\beta S$, the function $\rho_p:\beta S\to\beta S$ defined by $\rho_p(q)=q\cdot p$, is continuous) with S contained in its topological center (i.e., the set of points p such that the function $\lambda_p:\beta S\to\beta S$ defined by $\lambda_p(q)=p\cdot q$, is continuous). We take the points of βS to be the ultrafilters on S and identify the principal ultrafilters with the points of S.

The topology of βS is defined by choosing the sets of the form $\overline{A}=\{p\in\beta S: A\in p\}$ as a basis for the open sets, where A denotes a subset of S. With this topology, $\overline{A}$ is clopen in βS and $\overline{A}=c\ell_{\beta S}(A)$.

Given $p,q\in\beta S$ and $A\subseteq S$, $A\in p\cdot q$ if and only if $\{x\in S: x^{-1}A\in q\}\in p$, where $x^{-1}A=\{y\in S: x\cdot y\in A\}$. We could also define $p\cdot q$ topologically, by $p\cdot q=\lim\limits_{s\to p}\lim\limits_{t\to q} s\cdot t$, where s and t denote members of S. See [8] for an elementary introduction to the semigroup $(\beta S,\cdot)$.

[1] This author gratefully acknowledges support received from the National Science Foundation (USA) via grant DMS 9424421.

[2] These authors would like to thank the London Mathematical Society for funding a visit by the second author to the University of Hull.

If S is commutative, we may use $+$ for the semigroup operation of S and for that of βS.

As with any compact Hausdorff right topological semigroup, βS has idempotents. Modest cancellation assumptions guarantee that $S^* = \beta S \backslash S$ is a subsemigroup of βS and hence has idempotents. (See [8, Theorem 4.28].) Usually, in fact there are many idempotents. For example, the semigroups $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$ both have $2^{\mathfrak{c}}$ idempotents.

There is a natural relation between topologies on S and filters on S, which can also be expressed as a relation between topologies on S and compact subsets of βS. For example, if S is a group, every left invariant topology on S is uniquely determined by the filter of neighborhoods of the identity. It is also uniquely determined by the compact subsemigroup of βS consisting of the ultrafilters on S which converge to the identity.

Topologies defined on S by using the algebra of βS are interesting because of their topological properties and because they provide a tool for analyzing the algebra of βS. On the one hand, they give rise to interesting examples in general topology. For example, if S is a countably infinite group, strongly right maximal idempotents in S^* define invariant regular topologies on S which are maximal subject to having no isolated points. (See [16] or [8, Section 9.2].) If S is a countably infinite Boolean group, strongly summable idempotents in S^* (whose existence follows from Martin's Axiom) define extremally disconnected non-discrete topologies on S which make S a topological group. (See [7], for example.)

On the other hand, topologies arising in this way have been an important tool in investigating the algebra of βS. For example, Zelenuk's Theorem states that S^* can contain no non-trivial finite groups if S is a countable torsion-free group. (See [18] or [8, Theorem 7.17].) The proof of Zelenuk's Theorem depends in an essential way on a topology defined by a finite subgroup of S^*.

In this paper, we study topologies on an arbitrary semigroup determined by idempotents in βS. Any idempotent in βS determines two natural topologies on S making S a left topological semigroup. If S is an infinite abelian group, these topologies both provide simple examples in which S is a semitopological semigroup where the operation is usually not jointly continuous. In fact, in one of these topologies, it cannot be demonstrated in ZFC that the operation is ever jointly continuous. (See [13, Theorem 7.3].)

In Section 2 we introduce a topology defined by an arbitrary filter on S or, equivalently, by an arbitrary compact subset of βS. These are topologies in which multiplication on the left is continuous and open. We introduce an operation of multiplication on filters which is useful in deciding when the filter of neighborhoods of each point of S is given by a left translation of the filter defining the topology. In Section 3 we introduce another topology determined by an ultrafilter, and investigate both topologies under the additional assumption that the ultrafilter is an idempotent. In Section 4 we investigate when the two topologies are in fact the same. In Section 5 we consider the questions of separate and joint continuity of the operation with respect to these topologies.

Topologies on a semigroup S defined by ultrafilters were first considered by T. Papazyan (see [12]). Relations between filters on a semigroup S and topologies on S were studied in [13], and there is some overlap between our paper and results in this paper. For example, Theorem 2.22 below occurs as Theorem 3.7 in [13]. We have included a proof of this theorem, however, for the sake of completeness, and because our terminology differs from that of [13]. We should mention that some of the results in our paper are known and can be found in the literature in the special case in which S is a group.

If S and T are topological spaces and $f : S \to T$, then for any ultrafilter p on S there is an ultrafilter $\overline{f}(p)$ on T defined by $\overline{f}(p) = \{B \subseteq T : f^{-1}[B] \in p\}$. We shall use the following obvious criterion for continuity: f is continuous at the point $s \in S$ if and only if $\overline{f}(p)$ converges to $f(s)$ whenever p is an ultrafilter on S which converges to s.

Another simple fact which we shall use without any further reference is that a left cancelable element of S is also a left cancelable element of βS [8, Lemma 8.1].

We conclude the introduction with a word about our notation. We shall use the overline notation, $\overline{U}$, strictly for the closure of the set U in βS. Closures with respect to other topologies will be indicated by $c\ell U$ (or $c\ell_{\mathcal{T}} U$ in case of ambiguity). We shall have occasion to use βS when S is a semigroup, such as the unit circle, which would usually be assumed to have a non-discrete topology. We shall always assume that βS denotes the Stone-Čech compactification of the semigroup S, with S having the discrete topology.

2. Topologies Determined by Filters on S.

We study in this section a topology defined in terms of an arbitrary filter on S and investigate the relationship with topologies making S a left topological semigroup (i.e., for each $x \in S$, $l_x = \lambda_x|S : S \to S$ is continuous) and satisfying the additional property that each l_x is an open map. (Notice that, if S is a group, since $l_x^{-1} = l_{x^{-1}}$, the "open" conclusion follows from the fact that S is left topological.)

2.1 Definition. Let S be a semigroup, let $\mathcal{F}$ be a filter on S, and let C be a subset of βS.

(a) $\mathcal{T}_{\mathcal{F}} = \{V \subseteq S : \text{for all } x \in V,\, x^{-1}V \in \mathcal{F}\}$.

(b) $\mathcal{T}_C = \{V \subseteq S : VC \subseteq \overline{V}\}$.

We first observe that the definitions are essentially the same. In particular, if $p \in S^* = \beta S \backslash S$, so that p is a filter on S and $\{p\}$ is a subset of βS, then $\mathcal{T}_p = \mathcal{T}_{\{p\}}$. We also note that we may just as well presume that C is compact. We omit the routine proof of the following lemma.

2.2 Lemma. *Let $\emptyset \neq C \subseteq \beta S$ and let $\mathcal{F} = \bigcap C$. Then $\mathcal{T}_{\mathcal{F}} = \mathcal{T}_C = \mathcal{T}_{\overline{C}}$.*

As a consequence of Lemma 2.2, we are justified in stating results about $\mathcal{T}_{\mathcal{F}}$ for an arbitrary filter $\mathcal{F}$ or about $\mathcal{T}_C$ for an arbitrary compact subset of βS, whichever is more appropriate in the context. The only difference is that topologies of the form $\mathcal{T}_C$ always include the discrete topology, because $\mathcal{T}_C$ is the discrete topology on S if $C = \emptyset$. The discrete topology on S is also a topology of the form $\mathcal{T}_{\mathcal{F}}$ in the case in which S has a right identity e (with $\mathcal{F} = \{A \subseteq S : e \in A\}$). However, it can happen that the discrete topology is not defined by a filter. For example, there is no filter $\mathcal{F}$ on $\mathbb{N}$ for which $\mathcal{T}_{\mathcal{F}}$ is the discrete topology on $\mathbb{N}$.

2.3 Definition. Let S be a semigroup and let C be a compact subset of βS. For each $x \in S$, we define a filter ϕ_x on S by

$$\phi_x = \{V \subseteq S : x \in V \text{ and } xC \subseteq \overline{V}\}\,.$$

Notice that if $C \neq \emptyset$ and $\mathcal{F} = \bigcap C$, then $\phi_x = \{V \subseteq S : x \in V \text{ and } x^{-1}V \in \mathcal{F}\}$.

We shall investigate whether, for each $x \in S$, ϕ_x is the filter of neighborhoods of x for the topology $\mathcal{T}_C$. The answer to this question lies in a generalization of the product of two ultrafilters and will be given in Theorem 2.16.

2.4 Theorem. *Let S be a semigroup and let C be a compact subset of βS. Then $\mathcal{T}_C$ is a topology on S and for each $x \in S$, l_x is both continuous and open with respect to $\mathcal{T}_C$. Further, for each $x \in S$, each $\mathcal{T}_C$-neighborhood of x is in ϕ_x.*

Proof. In the standard way, we can define a topology $\mathcal{T}$ on S by putting $\mathcal{T} = \{V \subseteq S : \text{for all } x \in V,\, V \in \phi_x\}$. Now $V \in \phi_x$ for all $x \in V$ if and only if $VC \subseteq \overline{V}$ and so $\mathcal{T} = \mathcal{T}_C$.

It is obvious that l_x is open with respect to $\mathcal{T}_C$ for each $x \in S$. To see that l_x is continuous, let $V \in \mathcal{T}_C$ and let $q \in C$. If $y \in l_x^{-1}[V]$, then $xy \in V$ and so $V \in xyq$ and $l_x^{-1}[V] \in yq$. Thus $l_x^{-1}[V] \in \mathcal{T}_C$. $\square$

2.5 Definition. Let S be a semigroup and let $\mathcal{F}$ and $\mathcal{G}$ be filters on S. Let $x \in S$.

(a) $\mathcal{F} \cdot \mathcal{G} = \{A \subseteq S : \{x \in S : x^{-1}A \in \mathcal{G}\} \in \mathcal{F}\}$.

(b) $\overline{\mathcal{F}} = \{p \in \beta S : \mathcal{F} \subseteq p\}$.

(c) $x\mathcal{F}^{\uparrow} = \{V \subseteq S : x^{-1}V \in \mathcal{F}\}$.

Observe, as is well known, if $\mathcal{F}$ is a filter on S, then $\overline{\mathcal{F}}$ is a closed subset of βS and $\mathcal{F} = \bigcap \overline{\mathcal{F}}$. Also, if C is a nonempty subset of βS and $\mathcal{F} = \bigcap C$, then $\overline{\mathcal{F}} = \overline{C}$.

It is not hard to prove that the operation $\cdot$ on filters is associative. We omit the routine verification of the following lemma.

2.6 Lemma. *Let S be a semigroup and let $\mathcal{F}$ and $\mathcal{G}$ be filters on S. Then $\mathcal{F} \cdot \mathcal{G}$ is a filter on S and $\overline{\mathcal{F}} \cdot \overline{\mathcal{G}} \subseteq \overline{\mathcal{F} \cdot \mathcal{G}}$.*

By [1, Lemma 5.15] one need not have $\overline{\mathcal{F}} \cdot \overline{\mathcal{G}} = \overline{\mathcal{F} \cdot \mathcal{G}}$.

Let S be a semigroup and let $\mathcal{F}$ be a filter on S. Let $C = \overline{\mathcal{F}}$. If $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$, then Lemma 2.6 implies that $CC \subseteq C$; i.e. that C is a subsemigroup of βS. Note that $CC = C$ implies that $\mathcal{F} \cdot \mathcal{F} \subseteq \mathcal{F}$.

2.7 Definition. Let S be a semigroup, let $\mathcal{F}$ be a filter on S, and let $U \subseteq S$. Then $U^{\star}(\mathcal{F}) = \{x \in U : x^{-1}U \in \mathcal{F}\}$.

Observe that, if $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$ and $U \in \mathcal{F}$, then $U^{\star}(\mathcal{F}) \in \mathcal{F}$.

2.8 Lemma. *Let S be a semigroup, let $\mathcal{F}$ be a filter on S, and let $U \subseteq S$. Then $\text{int}_{\mathcal{T}_{\mathcal{F}}} U \subseteq U^{\star}(\mathcal{F})$. If $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$, then $\text{int}_{\mathcal{T}_{\mathcal{F}}} U = U^{\star}(\mathcal{F})$.*

Proof. Write $U^{\star}$ for $U^{\star}(\mathcal{F})$. If $x \in \text{int}\, U$, then $x^{-1}(\text{int}\, U) \in \mathcal{F}$ and $x^{-1}(\text{int}\, U) \subseteq x^{-1}U$ and thus $x \in U^{\star}$.

Now assume that $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$. To see that $U^{\star} \subseteq \text{int}\, U$, we show that $U^{\star} \in \mathcal{T}_{\mathcal{F}}$. So let $x \in U^{\star}$. Then $x^{-1}U \in \mathcal{F}$ and so, since $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$, $(x^{-1}U)^{\star} \in \mathcal{F}$. A routine computation shows that $(x^{-1}U)^{\star} = x^{-1}U^{\star}$. □

2.9 Lemma. *Let S be a semigroup and let $\mathcal{F}$ and $\mathcal{G}$ be filters on S. Then $\mathcal{F} \cdot \mathcal{G} = \{U \subseteq S : V\overline{\mathcal{G}} \subseteq \overline{U}$for some $V \in \mathcal{F}\}$.*

Proof. Let $U \in \mathcal{F} \cdot \mathcal{G}$ and let $V = \{x \in S : x^{-1}U \in \mathcal{G}\}$. Then $V \in \mathcal{F}$ and $V\overline{\mathcal{G}} \subseteq \overline{U}$.

For the reverse inclusion, let $U \subseteq S$ and let $V \in \mathcal{F}$ with $V\overline{\mathcal{G}} \subseteq \overline{U}$. Then $V \subseteq \{x \in S : x^{-1}U \in \mathcal{G}\}$ and so $U \in \mathcal{F} \cdot \mathcal{G}$. □

The following definition is in [13].

2.10 Definition. Let S be a discrete semigroup and let C be a compact subset of βS. We say that C is *uniform* if, for every subset U of S such that $C \subseteq \overline{U}$, there exists a subset V of S such that $C \subseteq \overline{V}$ and $VC \subseteq \overline{U}$.

Notice that any uniform nonempty subset of βS is in fact a subsemigroup of βS.

2.11 Lemma. *Let S be a discrete semigroup, let C be a nonempty compact subset of βS and let $\mathcal{F} = \bigcap C$. Then C is uniform if and only if $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$.*

Proof. This follows easily from Lemma 2.9. □

2.12 Lemma. *Let S be a discrete semigroup. Then every finite subsemigroup of βS is uniform.*

Proof. Let $U \subseteq S$ such that $C \subseteq \overline{U}$ and let $V = \{x \in S : xC \subseteq \overline{U}\}$. It suffices to show that $C \subseteq \overline{V}$, so suppose instead that we have some $p \in C \backslash \overline{V}$. For each $q \in C$, let $D_q = \{x \in S : xq \notin \overline{U}\}$. Then $S \backslash V = \bigcup_{q \in C} D_q$ so, because C is finite, pick $q \in C$ such that $D_q \in p$. But then $pq \in C \backslash \overline{U}$, a contradiction. □

2.13 Lemma. *Let S be a semigroup and let $p \in S^*$. Let $C = \{q \in \beta S : qp = p\}$. Then C is uniform. Furthermore, if $W \in p$ and $W^\diamond = \{s \in S : s^{-1}W \in p\}$, then $W^\diamond \in \mathcal{T}_C$. If $C \neq \emptyset$, the sets of the form $W^\diamond$, with $W \in p$, are a base for the filter $\bigcap C$.*

Proof. Let $W \in p$. We first observe that $W^\diamond C \subseteq \overline{W^\diamond}$, i.e., $W^\diamond \in \mathcal{T}_C$. Indeed, let $s \in W^\diamond$ and let $q \in C$. Then $sqp = sp \in \overline{W}$ so that $\{t \in S : t^{-1}(s^{-1}W) \in p\} \in q$. That is, $(s^{-1}W)^\diamond = s^{-1}W^\diamond \in q$ as required.

We next observe that $C \subseteq \overline{W^\diamond}$. If $q \in C$, $W \in qp$ and so $\{s \in S : s^{-1}W \in p\} \in q$. I.e. $W^\diamond \in q$.

To see that C is uniform, let $U \subseteq S$ be such that $C \subseteq \overline{U}$. It suffices to show that there exists $W \in p$ such that $W^\diamond \subseteq U$. Suppose instead that $W^\diamond \backslash U \neq \emptyset$ for each $W \in p$. Then $\{W^\diamond \backslash U : W \in p\}$ has the finite intersection property. So pick $r \in \beta S$ such that $\{W^\diamond \backslash U : W \in p\} \subseteq r$. Then $r \in C \backslash \overline{U}$, a contradiction. □

The statement of the following lemma describes a method frequently used to define compact subsemigroups C of βS.

2.14 Lemma. *Let S be a semigroup and let $\mathcal{A} \subseteq \mathcal{P}(S)$ have the finite intersection property. Suppose that, for each $A \in \mathcal{A}$ and each $x \in A$, there exists $B \in \mathcal{A}$ such that $xB \subseteq A$. Then, if $C = \bigcap_{A \in \mathcal{A}} \overline{A}$, C is uniform.*

Proof. Since $C \backslash \overline{U} = \emptyset$, there is some finite $\mathcal{H} \subseteq \mathcal{A}$ such that $\bigcap \mathcal{H} \subseteq U$. Let $V = \bigcap \mathcal{H}$. We claim that $VC \subseteq \overline{U}$, in fact that $VC \subseteq \overline{V}$. To see this, let $x \in V$ and $q \in C$. To see that $xq \in \overline{V}$, let $A \in \mathcal{H}$. Then $x \in A$ so pick $B \in \mathcal{A}$ such that $xB \subseteq A$. Then $q \in B$ so $xq \in \overline{A}$ as required. □

We now see that the converse of Lemma 2.14 also holds.

2.15 Lemma. *Let S be a semigroup and let C be a nonempty compact subset of βS. If C is uniform, there is a family $\mathcal{A}$ of subsets of S such that $C = \bigcap_{A \in \mathcal{A}} \overline{A}$ and, for every $A \in \mathcal{A}$ and every $x \in A$, there exists $B \in \mathcal{A}$ satisfying $xB \subseteq A$.*

Proof. Let $\mathcal{F} = \bigcap C$ and write $U^\star$ for $U^\star(\mathcal{F})$. Let $\mathcal{A} = \{U^\star : U \in \mathcal{F}\}$. Since $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$ by Lemma 2.11, we have $\mathcal{A} \subseteq \mathcal{F}$ and so, since also each $U^\star \subseteq U$, $C = \bigcap_{A \in \mathcal{A}} \overline{A}$.

To complete the proof, we need to show that for each $U \in \mathcal{F}$ and each $x \in U^\star$, there exists $V \in \mathcal{F}$ such that $xV^\star \subseteq U^\star$. As we noted in the proof of Lemma 2.8, if $V = x^{-1}U$, then $V^\star = x^{-1}U^\star$. □

2.16 Theorem. *Let S be a semigroup and let $\mathcal{F}$ be a filter on S. If $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$, then, for every $x \in S$, ϕ_x is the filter of neighborhoods of x with respect to $\mathcal{T}_\mathcal{F}$.*

Proof. This follows immediately from Theorem 2.4 and Lemma 2.8. □

2.17 Corollary. *Let S be a semigroup and let C be a uniform subset of S. Then for each $x \in S$, ϕ_x is the filter of neighborhoods of x with respect to $\mathcal{T}_C$.*

Proof. If $C = \emptyset$, this is trivial, and if $C \neq \emptyset$, it follows from Theorem 2.16 and Lemma 2.11. □

To summarize, we have now established:

2.18 Corollary. *Let S be a semigroup and let $\mathcal{F}$ be a filter on S. If $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$, then $\mathcal{T}_\mathcal{F}$ is a topology on S making S a left topological semigroup such that for each $x \in S$, l_x is an open map and ϕ_x is the filter of neighborhoods of x.*

Proof. Theorems 2.4 and 2.16. □

2.19 Corollary. *Let S be a semigroup and let C be a finite subsemigroup of βS. Then, for each $x \in S$, ϕ_x is the filter of neighborhoods of x for the topology $\mathcal{T}_C$.*

Proof. Corollary 2.18 and Lemma 2.12. □

2.20 Corollary. *Let S be a semigroup and let $C \subseteq \beta S$ be defined as in Lemma 2.14. Then, for each $x \in S$, ϕ_x is the filter of neighborhoods of x for the topology $\mathcal{T}_C$.*

Proof. Corollary 2.18 and Lemma 2.14. □

We have a partial converse to Corollary 2.18.

2.21 Theorem. *Let S be a semigroup and let $\mathcal{F}$ be a filter on S. Suppose that S^* is a left ideal in βS and that $\overline{\mathcal{F}} \subseteq S^*$. Let $x \in S$. If ϕ_x is the filter of $\mathcal{T}_\mathcal{F}$-neighborhoods of x, then $x\mathcal{F}^\uparrow \subseteq (x\mathcal{F}^\uparrow) \cdot \mathcal{F}$. If, in addition, x is left cancelable, then $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$.*

Proof. Let $U \in x\mathcal{F}$. Then $\{x\} \cup U \in \phi_x$ and so $\{x\} \cup U$ is a $\mathcal{T}_\mathcal{F}$-neighborhood of x. Pick $V \in \phi_x \cap \mathcal{T}_\mathcal{F}$ for which $V \subseteq \{x\} \cup U$. Thus $V\overline{\mathcal{F}} \subseteq \overline{V} \subseteq \{x\} \cup \overline{U}$. Now $xW \subseteq V$ for some $W \in \mathcal{F}$. Since $x \notin xW\overline{\mathcal{F}}$, we have $xW\overline{\mathcal{F}} \subseteq \overline{U}$ and so $U \in (x\mathcal{F}^\uparrow) \cdot \mathcal{F}$ (by Lemma 2.9). Thus $x\mathcal{F}^\uparrow \subseteq (x\mathcal{F}^\uparrow) \cdot \mathcal{F}$. If x is left cancelable, this implies that $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$. $\square$

If S has an identity, we have a strong converse to Corollary 2.18.

2.22 Theorem. *Let S be a semigroup with identity e and let $\mathcal{T}$ be a topology on S making S a left topological semigroup such that, for each $x \in S$, l_x is an open map. Let $\mathcal{F}$ be the neighborhood filter of e with respect to $\mathcal{T}$. Then $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$ and $\mathcal{T} = \mathcal{T}_\mathcal{F}$.*

Proof. To see that $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$, let $U \in \mathcal{F}$. Pick $V \in \mathcal{T}$ such that $e \in V \subseteq U$. To see that $U \in \mathcal{F} \cdot \mathcal{F}$, we show that $V \subseteq \{x \in S : x^{-1}U \in \mathcal{F}\}$. So let $x \in V$. Then $x = l_x(e) \in V$ so pick $W \in \mathcal{F}$ such that $l_x[W] \subseteq V$. Then $W \subseteq x^{-1}V \subseteq x^{-1}U$ and so $x^{-1}U \in \mathcal{F}$.

To see that $\mathcal{T} \subseteq \mathcal{T}_\mathcal{F}$, let $U \in \mathcal{T}$ and let $x \in U$. Then $x = l_x(e)$ so pick $V \in \mathcal{F}$ such that $l_x[V] \subseteq U$. Then $V \subseteq x^{-1}U$ and thus $U \in \mathcal{T}_\mathcal{F}$.

Now let $U \in \mathcal{T}_\mathcal{F}$. To see that $U \in \mathcal{T}$, we let $x \in U$ and show that there exists $W \in \mathcal{T}$ such that $x \in W \subseteq U$. Now $x \in U$ so $x^{-1}U \in \mathcal{F}$. Pick $V \in \mathcal{T}$ such that $e \in V \subseteq x^{-1}U$. Then $x = l_x(e) \in l_x[V] \subseteq l_x[x^{-1}U] \subseteq U$. Since l_x is open, $l_x[V] \in \mathcal{T}$. $\square$

Notice that if S is a semigroup with identity e, then Corollary 2.18 and Theorem 2.22 set up a nearly one to one correspondence between filters $\mathcal{F}$ on S such that $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$ and topologies on S with the property that for each $x \in S$, l_x is continuous and open and ϕ_x is the filter of $\mathcal{T}_\mathcal{F}$-neighborhoods of x. The exception is, if $\mathcal{F}$ is a filter on S, $e \notin \bigcap \mathcal{F}$, and $\mathcal{G} = \big\{\{e\} \cup U : U \in \mathcal{F}\big\}$, then $\mathcal{F}$ and $\mathcal{G}$ may generate the same topology. Also, if $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$, then $\mathcal{G} \subseteq \mathcal{G} \cdot \mathcal{G}$. (This may be verified directly, or proved by invoking Corollary 2.18 and Theorem 2.22.)

2.23 Corollary. *Let S be a semigroup with identity e and let $\mathcal{T}$ be a topology on S making S a left topological semigroup such that, for each $x \in S$, l_x is an open map. Let $C = \{p \in \beta S : p$ converges to e with respect to $\mathcal{T}\}$. Then C is a compact subsemigroup of βS and $\mathcal{T} = \mathcal{T}_C$.*

Proof. Let $\mathcal{F} = \bigcap C$. Then $\mathcal{F}$ is the neighborhood filter of e, so by Theorem 2.22, $\mathcal{F} \subseteq \mathcal{F} \cdot \mathcal{F}$. By Lemma 2.2, $\mathcal{T}_C = \mathcal{T}_\mathcal{F}$, and by Lemma 2.6, $C \cdot C = \overline{\mathcal{F}} \cdot \overline{\mathcal{F}} \subseteq \overline{\mathcal{F}} = C$. $\square$

In view of Corollary 2.23, it is natural to ask whether the analogue of Corollary 2.18 holds for $\mathcal{T}_C$ whenever C is a compact subsemigroup of βS. That is, one would want to conclude that *if C is a compact subsemigroup of βS, then $\mathcal{T}_C$ is a topology on S making S a left topological semigroup such that for each $x \in S$, l_x is an open map and ϕ_x is the filter of neighborhoods of x.*

By applying Theorem 2.4, one sees that all conclusions follow immediately, except for the assertion that ϕ_x is the filter of neighborhoods of x. In the following theorem we show that this conclusion need not hold, even if S is an abelian group.

2.24 Theorem. *There exist a compact subsemigroup C of $(\beta\mathbb{Z}, +)$ and a set $W \subseteq \mathbb{Z}$ such that $W \in \phi_0$ but W is not a neighborhood of 0 with respect to $\mathcal{T}_C$.*

Proof. Let $\langle E_m \rangle_{m=1}^{\infty}$ be a sequence of pairwise disjoint infinite subsets of $2\mathbb{N}$ with $\min E_m > m$ for each m and let $D = \{2^m + 2^k : m \in \mathbb{N} \text{ and } k \in E_m\}$. Let $C = \mathbb{N}^* \backslash \overline{D}$.

To see that C is a subsemigroup of $(\beta\mathbb{Z}, +)$, it suffices to let $p, q \in \mathbb{N}^*$ and show that $D \notin p + q$. Suppose instead that we have $p, q \in \mathbb{N}^*$ with $D \in p + q$. Let $B = \{x \in \mathbb{N} : -x + D \in q\}$. For each $x \in \mathbb{N}$, define $F(x) \subseteq \{0\} \cup \mathbb{N}$ by $x = \Sigma_{t \in F(x)}\, 2^t$. Since $p \in \mathbb{N}^*$, B is infinite so pick $x \neq z \in B$ and let $l = \max\big(F(x) \cup F(z)\big)$. Pick $y \in (-x + D) \cap (-z + D)$ such that $y > 2^{l+1}$.

Let $j = \max F(y)$. Now $F(x + y) = \{m, k\}$ for some $m \in \mathbb{N}$ and some $k \in E_m$ and $F(z + y) = \{r, t\}$ for some $r \in \mathbb{N}$ and some $t \in E_r$. Since $x < 2^{l+1}$, $z < 2^{l+1}$, and $y > 2^{l+1}$, we have $k = \max F(x + y) \in \{j, j + 1\}$ and $t = \max F(z + y) \in \{j, j + 1\}$. Since k and t are both even, this says $k = t$ and thus $E_r \cap E_m \neq \emptyset$ and so $r = m$. But then $x + y = z + y$ while $x \neq z$, a contradiction. Thus C is a semigroup as claimed.

Notice that if $V \in \mathcal{T}_C$ and $0 \in V$, then there is some $m \in \mathbb{N}$ such that $\mathbb{N} \backslash D \subseteq V \cup \{1, 2, \ldots, m\}$. (For otherwise we can pick some $p \in C$ such that $V \notin p$ while $\{0\} \cup C \subseteq \overline{V}$.)

Let $W = \mathbb{Z} \backslash D$. Then trivially $W \in \phi_0$. Suppose that W is a neighborhood of 0 and pick $V \in \mathcal{T}_C$ such that $0 \in V \subseteq W$. Pick $m \in \mathbb{N}$ such that $\mathbb{N} \backslash D \subseteq V \cup \{1, 2, \ldots, m\}$. Then $2^m \in V$. Since V is open and l_{2^m} is continuous (by Theorem 2.4) pick $U \in \mathcal{T}_C$ such that $0 \in U$ and $l_{2^m}[U] \subseteq V$. Pick $k \in \mathbb{N}$ such that $\mathbb{N} \backslash D \subseteq U \cup \{1, 2, \ldots, k\}$. Pick $y \in E_m$ such that $y > k$. Then $2^y \in U$ and so $2^m + 2^y \in V \subseteq W$, a contradiction. □

The semigroup C in Theorem 2.24 is not uniform by Lemma 2.11 and Theorem 2.16. We now show that there is an interesting class of semigroups which are not uniform.

2.25 Definition. Let S be a discrete semigroup and let $p \in \beta S$. Then C_p denotes the smallest compact subsemigroup of βS which contains p.

The following is from [5].

2.26 Lemma. *Let T be a compact right topological semigroup and let $p \in T$. The smallest compact subsemigroup of T with p as a member is the smallest compact subset C of T such that $p \in C$ and $pC \subseteq C$.*

Proof. Let $x \in C$ and let $D = \{y \in C : yx \in C\}$. Then D is compact, $p \in D$, and $pD \subseteq D$ and thus $D = C$. So C is a semigroup. $\square$

Notice that by [8, Theorem 8.10], for any cancellative semigroup S, there is a large set of right cancelable elements of βS.

2.27 Theorem. *Let G be a discrete countable group and let $p \in G^*$ be a right cancelable element of βG. Then the subsemigroup C_p of βG is not uniform.*

Proof. We first consider the case in which $G = \mathbb{Z}$. We shall show that if $p \in \mathbb{Z}^* \cap \overline{\{2^n : n \in \mathbb{N}\}}$, then C_p is not uniform.

For each $n \in \mathbb{N}$, we define $\mathrm{supp}(n) \subseteq \omega$ by $n = \sum_{i \in \mathrm{supp}(n)} 2^i$. We define mappings f and c from $\mathbb{N}$ to ω by $f(n) = \min\big(\mathrm{supp}(n)\big)$ and $c(n) = |\mathrm{supp}(n)|$, and use $\overline{f}$ and $\overline{c}$ to denote their continuous extensions mapping $\beta\mathbb{N}$ to $\beta\omega$. Denote also by $\sigma_p(n)$ the sum of p with itself n times. (We cannot use np for this sum because np is the product of n with p in the semigroup $(\beta\mathbb{N}, \cdot)$, which is something else entirely.)

Let $U = \{n \in \mathbb{N} : f(n) > c(n)\}$ and let $V = \{n \in \mathbb{N} : f(n) + 1 < c(n)\}$. We shall show that $C_p \subseteq \overline{U \cup V}$.

It is easy to show by induction that $\overline{c}\big(\sigma_p(n)\big) = n$ for every $n \in \mathbb{N}$, and hence that $\sigma_p(n) \in \overline{U}$. So $\overline{\{\sigma_p(n) : n \in \mathbb{N}\}} \subseteq \overline{U}$. We claim that, for any $q \in C_p$ such that $\overline{c}(q) \in \mathbb{N}^*$, we have $p + q \in \overline{V}$. To see this, it suffices to show that $\{2^n : n \in \mathbb{N}\} \subseteq \{x \in \mathbb{N} : -x + V \in q\}$. So let $n \in \mathbb{N}$ be given. Now $\overline{\mathbb{N}2^{n+1}}$ is a subsemigroup of $\beta\mathbb{Z}$ and $p \in \overline{\mathbb{N}2^{n+1}}$ so that $C_p \subseteq \overline{\mathbb{N}2^{n+1}}$. Thus $\mathbb{N}2^{n+1} \cap \{x \in \mathbb{N} : c(x) > n + 1\} \in q$ and $\mathbb{N}2^{n+1} \cap \{x \in \mathbb{N} : c(x) > n + 1\} \subseteq -2^n + V$.

Let $D = \overline{\{\sigma_p(n) : n \in \mathbb{N}\}} \cup \overline{\{p + q : q \in C_p \text{ and } \overline{c}(q) \in \mathbb{N}^*\}}$. We claim that $p + D \subseteq D$ so that, by Lemma 2.26, $D = C_p$ and consequently $C_p \subseteq \overline{U \cup V}$. To see this, let $r \in D$. Then $r \in C_p$ and $\overline{c}(r) \in \mathbb{N}^*$ so that $p + r \in \{p + q : q \in C_p \text{ and } \overline{c}(q) \in \mathbb{N}^*\}$.

Let $W \subseteq \mathbb{Z}$ satisfy $C_p \subseteq \overline{W}$. We claim that $W + C \not\subseteq \overline{U \cup V}$. To see this, we note that, for any $n \in \mathbb{N}$, $\overline{f}\big(2^n + \sigma_p(n)\big) = n$ and $\overline{c}\big(2^n + \sigma_p(n)\big) = n + 1$ and so $2^n + \sigma_p(n) \notin \overline{U \cup V}$. Since $p \in \overline{\{2^n : n \in \mathbb{N}\}}$, $2^n \in W$ for some $n \in \mathbb{N}$ and thus $WC \not\subseteq \overline{U \cup V}$. So C_p is not uniform.

We now turn to the general case. Let G be a countable group and let $p \in G^*$ be right cancelable in βG. By [8,Theorem 8.61], there is a set $T \in p$, a compact subsemigroup T_∞ of βG satisfying $p \in T_\infty \subseteq \overline{T}$ and an injective map $\phi : T \to \mathbb{N}$ for which $\widetilde{\phi}(p) \in \overline{\{2^n : n \in \mathbb{N}\}}$ (where $\widetilde{\phi} : \overline{T} \to \beta\mathbb{N}$ denotes the continuous extension of ϕ). We observe that, since ϕ is injective, $\widetilde{\phi}$ is injective as well. It was also shown in the proof of [8,Theorem 8.61] that, for each $t \in T$, there exists a set U_t satisfying $T_\infty \subseteq \overline{U_t} \subseteq \overline{T}$, for which $\phi(tu) = \phi(t) + \phi(u)$ whenever $u \in U_t$. Thus, for every $t \in T$ and $x \in T_\infty$, we have $\widetilde{\phi}(tx) = \phi(t) + \widetilde{\phi}(x)$, as can be seen by letting u converge to x in the equation $\phi(tu) = \phi(t) + \phi(u)$. This implies that $\widetilde{\phi}|T_\infty$ is a homomorphism.

Let $q = \widetilde{\phi}(p)$ and let C_q denote the smallest compact subsemigroup of $\beta\mathbb{N}$ which contains q. Since $\widetilde{\phi}[C_p]$ is a compact subsemigroup of $\beta\mathbb{N}$ which contains q, $\widetilde{\phi}[C_p] \supseteq C_q$. Similarly, $\widetilde{\phi}^{-1}[C_q] \supseteq C_p$. So $\widetilde{\phi}[C_p] = C_q$. We have seen that C_q is not a uniform subset of $\beta\mathbb{Z}$, and it follows easily that C_p cannot be a uniform subset of βG. □

We are naturally concerned with when the topology is Hausdorff.

2.28 Theorem. *Let S be a semigroup, let C be a compact subset of βS.*
(a) If $\mathcal{T}_C$ is Hausdorff, then $aC \cap bC = \emptyset$ whenever $a \neq b$ in S.
(b) If C is uniform and if $aC \cap bC = \emptyset$ whenever $a \neq b$ in S, then $\mathcal{T}_C$ is Hausdorff.

Proof. (a). This is immediate from the definition of $\mathcal{T}_C$.

(b). Let $a \neq b$ and notice that $a \notin bC$ and $b \notin aC$. (For suppose that $a = bq$ where $q \in C$. Then $aq = bqq$ and $qq \in C$.) Thus $\{a\} \cup aC$ and $\{b\} \cup bC$ are disjoint compact subsets of βS so pick disjoint subsets V and W of S such that $\{a\} \cup aC \subseteq \overline{V}$ and $\{b\} \cup bC \subseteq \overline{W}$. By Corollary 2.17, V and W are neighborhoods of a and b respectively. □

We conclude this section with some technical results that will be useful later.

2.29 Theorem. *Let S be a semigroup and let C be a compact subset of βS. Suppose that, for every $x \in S$, ϕ_x is the filter of $\mathcal{T}_C$-neighborhoods of x. If $\mathcal{T}_C$ is regular, then for every $a \in S$ and every $x \in \beta S$ for which $ax \neq a$, $axC \cap aC \neq \emptyset$ implies that $ax \in aC$.*

Proof. Suppose that $axy \in aC$ for some $y \in C$, but that $ax \notin aC$. Then there is a subset U of S such that $aC \subseteq \overline{U}$ and $ax \notin \overline{U}$. Since $\{a\} \cup U$ is a neighborhood of a in the regular topology $\mathcal{T}_C$, we can pick a neighborhood V of a in $\mathcal{T}_C$ for which $c\ell_{\mathcal{T}_C} V \subseteq \{a\} \cup U$.

Now $axy \in aC \subseteq \overline{V}$ and thus $\{b \in S : aby \in \overline{V}\} \in x$. Also, $ax \notin \overline{\{a\} \cup U}$ and so $\{b \in S : ab \notin \{a\} \cup U\} \in x$. Pick $b \in S$ such that $ab \notin \{a\} \cup U$ and $aby \in \overline{V}$. Then $ab \notin c\ell_{\mathcal{T}_C} V$. So pick a neighborhood W of ab such that $W \cap V = \emptyset$. Since W is a neighborhood of ab in $\mathcal{T}_C$ and $y \in C$, $aby \in \overline{W}$, a contradiction. □

2.30 Corollary. *Let S be a semigroup and let C be a compact subset of βS. Suppose that, for every $x \in S$, ϕ_x is the filter of $\mathcal{T}_C$−neighborhoods of x. If $\mathcal{T}_C$ is regular and S has a left cancelable element, then for every $x \in S^*$, $xC \cap C \neq \emptyset$ implies that $x \in C$.*

Proof. Let a denote a left cancelable element of S. For every infinite subset B of S, aB is infinite. So $ax \in S^*$ and hence $ax \neq a$. Now $xC \cap C \neq \emptyset$ implies that $axC \cap aC \neq \emptyset$. Since a is left cancelable in βS ([8, Lemma 8.1]), our conclusion follows from Theorem 2.29. □

3. Two Topologies Determined by Idempotents in βS.

We have already defined $\mathcal{T}_p = \mathcal{T}_{\{p\}}$ for $p \in \beta S$, this topology being defined for arbitrary filters on S or subsets of βS. We introduce now another topology $\mathcal{V}_p$ which is defined only for $p \in S^*$ and compare this topology with $\mathcal{T}_p$.

3.1 Definition. Let S be a semigroup and let $p \in S^*$. Define $r_p : S \to \beta S$ by $r_p(s) = s \cdot p$. (So $r_p = (\rho_p)|S$.) $\mathcal{V}_p = \{r_p^{-1}[U] : U$ is open in $\beta S\}$.

Observe that trivially $\mathcal{V}_p$ is a topology on S.

In the following two lemmas, we describe convergence in the topologies $\mathcal{T}_p$ and $\mathcal{V}_p$.

3.2 Lemma. *Let S be a semigroup and let p be an idempotent in S^*. Let $a \in S$ and let $\langle a_\iota \rangle_{\iota \in I}$ be a net in $S \backslash \{a\}$.*

(i) $\langle a_\iota \rangle_{\iota \in I}$ converges to a in the topology $\mathcal{T}_p$ if and only if $\langle a_\iota \rangle_{\iota \in I}$ converges to ap in βS.

(ii) $\langle a_\iota \rangle_{\iota \in I}$ converges to a in the topology $\mathcal{V}_p$ if and only if $\langle a_\iota p \rangle_{\iota \in I}$ converges to ap in βS.

Proof. Statement (i) follows immediately from Theorem 2.16, and (ii) follows immediately from the definition of $\mathcal{V}_p$. □

3.3 Lemma. *Let S be a semigroup and let p be an idempotent in S^*. Let $q \in \beta S$ and let $a \in S$.*

(i) q converges to a with respect to the topology $\mathcal{T}_p$ if and only if $q = a$ or $q = ap$.

(ii) q converges to a with respect to the topology $\mathcal{V}_p$ if and only if $qp = ap$.

Proof. Statement (i) follows from Corollary 2.19, and (ii) follows easily from the definition of $\mathcal{V}_p$. □

3.4 Theorem. *Let S be a semigroup and let p be an idempotent in S^*. Then $\mathcal{V}_p \subseteq \mathcal{T}_p$.*

Proof. This is immediate by Lemma 3.2 and the fact that, if $\langle a_\iota \rangle_{\iota \in I}$ converges to ap in βS, then $\langle a_\iota p \rangle_{\iota \in I}$ converges to $app = ap$ in βS. □

3.5 Theorem. *Let S be a semigroup and let p and q be idempotents in βS. Statement (ii) implies statement (iii) which implies statement (iv). If S has a left cancelable element, then (i) implies (ii) and (iii) and (iv) are equivalent.*

(i) $\mathcal{T}_p \subseteq \mathcal{T}_q$.

(ii) $p = q$ *or* q *is a left identity for* βS.

(iii) $qp = p$.

(iv) $\mathcal{V}_p \subseteq \mathcal{V}_q$.

Proof. That (ii) implies (iii) is trivial. To see that (iii) implies (iv), assume that $qp = p$. To see that $\mathcal{V}_p \subseteq \mathcal{V}_q$, let $a \in S$, let $r \in \beta S$, and assume that r converges to a with respect to $\mathcal{V}_q$. Then by Lemma 3.3, $rq = aq$ and so $rp = rqp = aqp = ap$. Thus r converges to a with respect to $\mathcal{V}_p$.

Now assume that S has a left cancelable element t.

To see that (i) implies (ii), assume that $\mathcal{T}_p \subseteq \mathcal{T}_q$. Now tq converges to t with respect to $\mathcal{T}_q$ by Lemma 3.3 and so tq converges to t with respect to $\mathcal{T}_p$. Thus by Lemma 3.3, either $tq = t$ or $tq = tp$. If $tq = tp$, then $q = p$, and if $tq = t$, then q is a left identity for βS.

To see that (iv) implies (iii), assume that $\mathcal{V}_p \subseteq \mathcal{V}_q$. Then by Lemma 3.3, tq converges to t with respect to $\mathcal{V}_q$ and thus with respect to $\mathcal{V}_p$. Therefore by Lemma 3.3, $tqp = tp$ and so $qp = p$. □

Recall that a semigroup S is *weakly left cancellative* if and only if for any $a, b \in S$, $\{x \in S : ax = b\}$ is finite. Similarly, S is *weakly right cancellative* if and only if for any $a, b \in S$, $\{x \in S : xa = b\}$ is finite.

3.6 Theorem. *Let S be an infinite discrete semigroup which is weakly left cancellative and right cancellative. If $|S| = \kappa$, there are 2^{2^κ} non-comparable topologies of the form $\mathcal{V}_p$ on S.*

Proof. By [8, Lemma 6.31], there is a κ-sequence $\langle t_\iota \rangle_{\iota < \kappa}$ in S which has distinct finite products. This means that expressions of the form $t_{\iota_1} t_{\iota_2} \cdots t_{\iota_n}$, where $\iota_1 < \iota_2 < \ldots < \iota_n$, are unique. Let T denote the set of all elements of this form. We define $f : T \to T$ by $f(t_{\iota_1} t_{\iota_2} \ldots t_{\iota_n}) = t_{\iota_1}$ if $\iota_1 < \iota_2 < \ldots < \iota_n$, and we use $\overline{f} : \overline{T} \to \overline{T}$ to denote the continuous extension of f.

For each $\iota < \kappa$, let $A_\iota = \{t \in T : f(t) > \iota\}$ and let $T_\infty = \bigcap_{\iota < \kappa} \overline{A_\iota}$. Let $t \in A_\iota$. If $t = t_{\iota_1} t_{\iota_2} \ldots t_{\iota_n}$, then $tA_{\iota_n} \subseteq A_\iota$. It follows that T_∞ is a semigroup [8, Theorem 4.20].

We claim that $\overline{f}(xy) = \overline{f}(x)$ for every $x, y \in T_\infty$, and that $\overline{f}(xy) = x$ if $x \in \overline{\{t_\iota : \iota < \kappa\}}$ and $y \in T_\infty$. To see this, let $t = t_{\iota_1} t_{\iota_2} \ldots t_{\iota_n}$, where $\iota_1 < \iota_2 < \ldots < \iota_n$. For every $u \in A_{\iota_n}$, we have $f(tu) = f(t) = t_{\iota_1}$. Our claim now follows by letting u converge to y and then letting t converge to x.

Now $|\overline{f}[T_\infty]| = 2^{2^\kappa}$, because $\overline{f}[T_\infty]$ is the set of uniform ultrafilters on $\{t_\iota : \iota < \kappa\}$ [8, Theorem 3.58]. For each $x \in \overline{f}[T_\infty]$, we can choose an idempotent p_x in the right ideal xT_∞ of T_∞ [8, Theorem 2.7]. If x and y are distinct elements of $\overline{f}[T_\infty]$, then $p_x p_y \neq p_y$, because $\overline{f}(p_x p_y) = x$ and $\overline{f}(p_y) = y$. We shall show that, as a consequence, $\mathcal{V}_{p_y} \not\subseteq \mathcal{V}_{p_x}$.

If $\mathcal{V}_{p_y} \subseteq \mathcal{V}_{p_x}$, then $t_0 p_x$ converges to t_0 with respect to $\mathcal{V}_{p_y}$, because $t_0 p_x$ converges to t_0 with respect to $\mathcal{V}_{p_x}$. So $t_0 p_x p_y = t_0 p_y$, by Lemma 3.3. Now there are disjoint subsets P and Q of A_0 such that $P \in p_x p_y$ and $Q \in p_y$. Since $t_0 P \cap t_0 Q = \emptyset$, we have contradicted the equation $t_0 p_x p_y = t_0 p_y$. □

Notice that the number of topologies of the form $\mathcal{V}_p$ can be very small, even in a left cancellative or a right cancellative semigroup. A semigroup S is said to be a right (left) zero semigroup if $ab = b$ ($ab = a$) for every $a, b \in S$. If S is a right zero semigroup, then there is exactly one topology on S of the form $\mathcal{V}_p$ – the trivial topology $\{\emptyset, S\}$. If S is a left zero semigroup, there is again exactly one topology on S of the form $\mathcal{V}_p$ – the discrete topology. In this case, the discrete topology is also the only topology on S of the form $\mathcal{T}_p$.

We are naturally interested in determining when the topologies $\mathcal{T}_p$ and $\mathcal{V}_p$ are Hausdorff. The answer turns out to be the same in both cases.

3.7 Theorem. *Let S be a semigroup and let p be an idempotent in S^*. The following statements are equivalent.*
(a) $\mathcal{V}_p$ is Hausdorff.
(b) $\mathcal{T}_p$ is Hausdorff.
(c) For all $x \neq y$ in S, $x \cdot p \neq y \cdot p$.

Proof. That (a) implies (b) follows from Theorem 3.4.

That (b) implies (c) follows from Theorem 2.28.

To see that (c) implies (a), pick disjoint open subsets U and V of βS such that $x \cdot p \in U$ and $y \cdot p \in V$. Then $r_p^{-1}[U]$ and $r_p^{-1}[V]$ are disjoint neighborhoods of x and y respectively with respect to $\mathcal{V}_p$. □

If S is weakly left cancellative, then $\mathcal{T}_p$ is a T_1 topology on S. However, $\mathcal{V}_p$ is not

T_0 unless it is Hausdorff. To see this, note that, for any $a, b \in S$, if $a \cdot p = b \cdot p$, then $b \in c\ell_{\mathcal{V}_p}(\{a\})$ and $a \in c\ell_{\mathcal{V}_p}(\{b\})$.

3.8 Theorem. *Let S be a semigroup and let p be an idempotent in S^*. If $\mathcal{V}_p$ is Hausdorff, then r_p defines a homeomorphism from $(S, \mathcal{V}_p)$ to the subspace Sp of βS.*

Proof. This is immediate from the definition of $\mathcal{V}_p$ and Theorem 3.7. □

In light of Theorem 3.7, one might believe that often $\mathcal{V}_p = \mathcal{T}_p$. We shall see in the next section that that is not true.

Two natural questions are raised by Theorem 3.7.

3.9 Question. *For which semigroups S will there exist an idempotent $p \in S^*$ and points $x \neq y$ in S such that $x \cdot p = y \cdot p$?*

3.10 Question. *For which semigroups S will there exist an idempotent $p \in S^*$ such that whenever $x \neq y$ in S one has $x \cdot p \neq y \cdot p$?*

We would argue that the second question is the more interesting because it guarantees the existence of Hausdorff topologies with the special properties of $\mathcal{T}_p$ and $\mathcal{V}_p$.

3.11 Theorem. *Let S be a semigroup such that S^* is a subsemigroup of βS and let $x, y \in S$. If $\{z \in S : x \cdot z = y \cdot z\}$ is infinite, then there is an idempotent $p \in S^*$ such that $x \cdot p = y \cdot p$.*

Proof. Since $\{z \in S : x \cdot z = y \cdot z\}$ is infinite, there exists $q \in S^*$ such that $x \cdot q = y \cdot q$. We claim that $\{q \in S^* : x \cdot q = y \cdot q\}$ is a right ideal of S^*. To see this, let $q \in S^*$ such that $x \cdot q = y \cdot q$ and let $r \in S^*$. Then by assumption $q \cdot r \in S^*$ and $x \cdot q \cdot r = y \cdot q \cdot r$. Thus, by [8, Theorem 2.7], there is an idempotent in $\{q \in S^* : x \cdot q = y \cdot q\}$. □

3.12 Theorem. *Let S be a left cancellative semigroup and let $x \neq y$ in S. Then there is an idempotent $p \in S^*$ such that $x \cdot p = y \cdot p$ if and only if $\{z \in S : x \cdot z = y \cdot z\}$ is infinite.*

Proof. The necessity is a consequence of [8, Lemma 8.5].

By [8, Corollary 4.29], S^* is a subsemigroup of βS and so the sufficiency is a consequence of Theorem 3.11. □

3.13 Corollary. *If S is a cancellative semigroup, then for every idempotent $p \in S^*$, $\mathcal{T}_p$ and $\mathcal{V}_p$ are Hausdorff.*

Proof. This is an immediate consequence of Theorems 3.7 and 3.12. $\square$

In the following Theorem, we see that $(S,\mathcal{V}_p)$ is extremally disconnected, and that $(S,\mathcal{T}_p)$ satisfies a condition stronger than being extremally disconnected. (We are not assuming that extremally disconnected spaces have to be Hausdorff.)

3.14 Theorem. *Let S be a semigroup and let p be an idempotent in S^*. $(S,\mathcal{V}_p)$ is extremally disconnected. For any two disjoint subsets A and B of S, $c\ell_{\mathcal{T}_p}A\cap c\ell_{\mathcal{T}_p}B\subseteq A\cup B$.*

Proof. First suppose that A and B are disjoint $\mathcal{V}_p$-open subsets of S. Then $A=r_p^{-1}[U]$ and $B=r_p^{-1}[V]$, where U and V are disjoint open subsets of βS. Since βS is extremally disconnected ([8, Theorem 3.18]), $c\ell_{\beta S}U\cap c\ell_{\beta S}V=\emptyset$. It follows that $c\ell_{\mathcal{V}_p}A\cap c\ell_{\mathcal{V}_p}B=\emptyset$, because this set is contained in $r_p^{-1}[c\ell_{\beta S}U]\cap r_p^{-1}[c\ell_{\beta S}V]$.

Now suppose that A and B are arbitrary disjoint subsets of S. Suppose that $a\in c\ell_{\mathcal{T}_p}A\cap c\ell_{\mathcal{T}_p}B$. If $a\notin A\cup B$, it follows from Lemma 3.2, that $ap\in\overline{A}\cap\overline{B}$. This is a contradiction, because $\overline{A}\cap\overline{B}=\emptyset$. $\square$

We show now that the topology $\mathcal{T}_p$ has an interesting maximal property. (Observe that a point $a\in S$ is isolated with respect to the topology $\mathcal{T}_p$ if and only if $ap=a$.)

3.15 Theorem. *Let S be a semigroup and let p be an idempotent in S^*. If $\mathcal{T}$ is a topology without isolated points and $\mathcal{T}_p\subseteq\mathcal{T}$, then $\mathcal{T}_p=\mathcal{T}$.*

Proof. Since $\mathcal{T}$ has no isolated points neither does $\mathcal{T}_p$. Let $a\in S$. We show that every $\mathcal{T}$-neighborhood of a is a $\mathcal{T}_p$-neighborhood of a.

Let $\mathcal{W}=\{V\subseteq S:\{a\}\cup V$ is a $\mathcal{T}_p$-neighborhood of $a\}$ and let $\mathcal{U}=\{V\subseteq S:\{a\}\cup V$ is a $\mathcal{T}$-neighborhood of $a\}$. Since a is not an isolated point, $\emptyset\notin\mathcal{W}$ and thus $\mathcal{W}$ is a filter. Consequently $\mathcal{W}=ap$. Then $\mathcal{U}$ is a filter containing the ultrafilter $\mathcal{W}$ and so $\mathcal{U}=\mathcal{W}$. $\square$

We now observe that there is a compact subsemigroup C of βS for which $\mathcal{T}_C$ coincides with $\mathcal{V}_p$ if p is an idempotent and if $S=\mathbb{N}$ or S is a group. (In that event, the hypotheses in (ii) below are satisfied.)

3.16 Theorem. *Let S be a semigroup and let p be an idempotent in S^*. Let $C=\{q\in\beta S:qp=p\}$.*

(i) $\mathcal{V}_p\subseteq\mathcal{T}_C$;

(ii) If S is cancellative and if $aS^*=S^*$ for every $a\in S$, then $\mathcal{V}_p=\mathcal{T}_C$.

Proof. (i). Suppose that the ultrafilter $q \in \beta S$ converges to $a \in S$ with respect to $\mathcal{T}_C$. By Lemma 2.13 and Theorem 2.16, $\{a\} \cup aU \in q$ whenever $U \subseteq S$ satisfies $C \subseteq \overline{U}$. Thus $q = a$ or $q \in aC$. So $qp = ap$ and therefore q converges to a in $\mathcal{V}_p$ (by Lemma 3.3).

(ii). Suppose that the ultrafilter $q \in \beta S$ converges to $a \in S$ with respect to $\mathcal{V}_p$. Then $qp = ap$. If $q \in S$, then by [8, Lemma 6.28] $q = a$ (and thus q converges to a with respect to the topology $\mathcal{T}_C$). So assume that $q \in S^*$. Then $q = ax$ for some $x \in S^*$. So $axp = ap$ and therefore $xp = p$ and thus $x \in C$. It follows that $q \in aC$ and that q converges to a with respect to $\mathcal{T}_C$. □

In our next theorem, we relate a property of a topological group to an algebraic property of the semigroup of ultrafilters converging to the identity.

3.17 Theorem. *Let (G, τ) be a Hausdorff topological group with identity e, and let C denote the set of ultrafilters on G which converge to e. If G can be embedded topologically and algebraically in a compact Hausdorff topological group, then C contains all the idempotents of βG.*

Proof. Suppose that G can be embedded topologically and algebraically in a compact Hausdorff topological group H and assume that in fact $G \subseteq H$. Let $p \in \beta G$ be idempotent. There is a unique element x in H with the property that, if V is a neighborhood of x in H, $V \cap G \in p$. Since every member of p contains three elements of the form a, b, ab, it follows that $x^2 = x$ and hence that $x = e$. So p converges to e and $p \in C$. □

3.18 Definition. Let S be a semigroup. A subset U of S is said to be *syndetic* if there is a finite subset F of S for which $S \subseteq \bigcup_{s \in F} s^{-1}U$.

Recall that any compact right topological semigroup T has a smallest two sided ideal $K(T)$. (See, for example, [8, Theorem 2.8].)

3.19 Theorem. *Let (G, τ) be a Hausdorff topological group with identity e, and let C denote the set of ultrafilters on G which converge to e. Then (G, τ) is totally bounded if and only if $C \cap \overline{K(\beta G)} \neq \emptyset$. In this case (G, τ) can be embedded topologically and algebraically in a compact Hausdorff topological group. So C contains all the idempotents of βG.*

Proof. Necessity. Suppose that $C \cap \overline{K(\beta G)} = \emptyset$. Since C is compact, pick $U \subseteq G$ such that $C \subseteq \overline{U}$ and $\overline{U} \cap K(\beta G) = \emptyset$. Then U is a neighborhood of the identity, so pick a finite subset F of G such that $G = \bigcup_{t \in F} t^{-1}U$. Pick $p \in K(\beta G)$ and $t \in F$ such that $p \in \overline{t^{-1}U}$. Then $tp \in \overline{U} \cap K(\beta G)$, a contradiction.

Sufficiency. Let U be a τ-neighborhood of e. We can choose a τ-neighborhood V of e such that $VV^{-1} \subseteq U$. Since $C \subseteq \overline{V}$, there exists $p \in \overline{V} \cap K(\beta G)$. Now $r_p^{-1}[\overline{V}] \subseteq VV^{-1} \subseteq U$. Since $r_p^{-1}[\overline{V}]$ is syndetic [8, Theorem 4.39], so is U. Thus (G, τ) is totally bounded. It follows that G has a completion which is a compact Hausdorff topological group (see, for example, [9], Theorem 32 and Exercise Q in chapter 6). By Theorem 3.17, C contains all the idempotents of βG. □

In the case in which S is a group, the following theorem is Theorem 4.5 in [16].

3.20 Theorem. *Let S be a discrete semigroup with a left identity e and let $p \in S^*$ be an idempotent. Then every $\mathcal{V}_p$-neighborhood of e in S is syndetic if and only if $p \in K(\beta S)$.*

Proof. The sets of the form $r_p^{-1}[\overline{A}]$, where $A \in p$, are a base for the $\mathcal{V}_p$-neighborhoods of e. These are all syndetic if and only if $p \in K(\beta S)$ [8, Theorem 4.39]. □

4. When the Topologies Coincide.

Notice that the topology $\mathcal{V}_p$ is trivially regular (where we are not assuming that regular spaces are Hausdorff). Consequently, if $\mathcal{V}_p = \mathcal{T}_p$ one necessarily has that $\mathcal{T}_p$ is regular, and this in fact characterizes when $\mathcal{V}_p = \mathcal{T}_p$ in the event that S is a group [8, Theorem 9.15]. Another characterization in that theorem is that p must be strongly right maximal in S^*. (An idempotent p is *strongly right maximal* in S^* if and only if the equation $p = q \cdot p$ has the unique solution $q = p$ in S^*. Similarly p is *strongly left maximal* in S^* if and only if the equation $p = p \cdot q$ has the unique solution $q = p$ in S^*. See [8] for information about strongly right maximal idempotents, including the fact that their existence in $\mathbb{N}^*$ can be established in ZFC.) We investigate here when these characterizations apply in an arbitrary semigroup.

4.1 Theorem. *Let S be a semigroup which has a left cancelable element. If p is an idempotent in S^* and $\mathcal{T}_p$ is regular, then p is strongly right maximal in S^*.*

Proof. This is immediate from Corollary 2.30. □

Notice that the following theorem in particular says that if p is strongly right maximal and S is a group, then $\mathcal{T}_p = \mathcal{V}_p$.

4.2 Theorem. *Let S be a semigroup with an identity and let p be a strongly right maximal idempotent in S^*. Suppose that the set of right cancelable elements of S is a member of p. If x is an invertible element of S, then every neighborhood of x with respect to $\mathcal{T}_p$ is also a neighborhood of x with respect to $\mathcal{V}_p$.*

Proof. Let $q \in \beta S$ converge to x with respect to $\mathcal{V}_p$. Then $qp = xp$ (by Lemma 3.3) and so $x^{-1}qp = p$.

We claim that $q = x$ or else $x^{-1}q \in S^*$. To see this, suppose that $x^{-1}q = a \in S\backslash\{e\}$. Then $ap = p$ and hence (by [8, Theorem 3.35]) $\{b \in S : ab = b = eb\} \in p$. We obtain a contradiction by choosing a right cancelable element b in this set.

If $q = x$, then q converges to x with respect to $\mathcal{T}_p$. Otherwise $x^{-1}q \in S^*$. Since p is strongly right maximal, this implies that $x^{-1}q = p$ and $q = xp$. Thus q again converges to x with respect to $\mathcal{T}_p$ by Lemma 3.3. □

5. Separate and Joint Continuity.

We are interested in determining in our more general setting when the operation in S is separately or jointly continuous with respect to $\mathcal{V}_p$ or $\mathcal{T}_p$.

Ellis' Theorem [4] says that a locally compact Hausdorff semitopological semigroup which is algebraically a group, is in fact a topological group. There is a standard example, namely $(\mathbb{R}, +)$ with the half open interval topology, showing that a group which is a topological semigroup, need not satisfy continuity of the inverse. Probably the simplest example of a group which has a Hausdorff topology making it a semitopological semigroup but not a topological semigroup is provided by $(\mathbb{Z}, +)$ with the topology $\mathcal{T}_p$ where p is any idempotent in $\mathbb{Z}^*$. (See [8, Exercise 9.2.7].) As a consequence of the Theorems 5.1 and 5.8, one sees that almost as simple an example is provided by $(\mathbb{Z}, +)$ with the topology $\mathcal{V}_p$.

5.1 Theorem. *Let S be any semigroup and let p be an idempotent in S^*. Then S is a left topological semigroup with respect to both of the topologies $\mathcal{T}_p$ and $\mathcal{V}_p$.*

Proof. Let $x \in S$ and let $l_x : S \to S$ be defined by $l_x(y) = xy$. (That is, $l_x = (\lambda_x)|S$.) That l_x is continuous with respect to $\mathcal{T}_p$ follows immediately from Theorem 2.4. To see that l_x is continuous with respect to $\mathcal{V}_p$, let $U \in \mathcal{V}_p$ and pick W open in βS such that $U = r_p^{-1}[W]$. Then $l_x^{-1}[U] = r_p^{-1}\big[\lambda_x^{-1}[W]\big]$ and so $l_x^{-1}[U] \in \mathcal{V}_p$. □

As a consequence of Theorem 5.1, if S is commutative, then it is a semitopological semigroup with respect to both $\mathcal{T}_p$ and $\mathcal{V}_p$. We now see that there is a close relation between S being commutative and being semitopological for either $\mathcal{T}_p$ or $\mathcal{V}_p$.

5.2 Theorem. *Let S be a discrete semigroup and let p be an idempotent in S^*.*

(i) S is right topological with respect to $\mathcal{T}_p$ if and only if $spt \in \{st, stp\}$ for every $s, t \in S$;

(ii) *S is right topological with respect to $\mathcal{V}_p$ if and only if, for every $s, t \in S$ and every $q \in \beta S$, $qp = sp$ implies that $qtp = stp$.*

Proof. We observe that S is right topological with respect to a given topology on S if and only if, for every $s, t \in S$, qt converges to st whenever q is an ultrafilter on S which converges to s. Thus our theorem follows easily from Lemma 3.3. □

5.3 Corollary. *Let S be a discrete semigroup which contains a left cancelable element and let p be an idempotent in S^*.*

(i) *S is right topological with respect to $\mathcal{T}_p$ if and only if $pt \in \{t, pt\}$ for every $t \in S$;*

(ii) *S is right topological with respect to $\mathcal{V}_p$ if and only if $ptp = tp$ for every $t \in S$.*

Proof. (i). The sufficiency follows immediately from Theorem 5.2 (i) and the necessity follows from Theorem 5.2 (i) by taking s to be left cancelable.

(ii). The necessity follows from Theorem 5.2 (ii) by taking s to be left cancelable and $q = sp$. For the sufficiency, let $s, t \in S$, let $q \in \beta S$, and assume that $qp = sp$. Then $qtp = qptp = sptp = stp$. □

5.4 Corollary. *Let S be a discrete semigroup and let p be an idempotent in S^*. If S is right topological with respect to $\mathcal{T}_p$, then S is also right topological with respect to $\mathcal{V}_p$.*

Proof. It follows from Theorem 5.2 that, for every $t \in S$, $S = \{b \in S : bpt = bt\} \cup \{b \in S : bpt = btp\}$. If $q \in \beta S$, we have (i) $\{b \in S : bpt = bt\} \in q$ and hence $qpt = qt$; or (ii) $\{b \in S : bpt = btp\} \in q$ and hence $qpt = qtp$. Suppose that $qp = sp$ for some $s \in S$. In case (i), $qtp = qptp = sptp$. Since $spt \in \{st, stp\}$, $qtp = stp$. In case (ii), we again have $qtp = qtpp = qptp = sptp$ so that, again because $spt \in \{st, stp\}$, $qtp = stp$. □

Notice that the following corollary tells us that, at least in a cancellative semigroup, a great deal of commutativity is needed for S to be right topological with respect to $\mathcal{T}_p$.

5.5 Corollary. *Let S be a semigroup and assume that either*

(a) *S is left cancellative and weakly right cancellative, or*

(b) *S is right cancellative and has a left cancelable element.*

Then there is an idempotent $p \in S^*$ such that S is right topological with respect to $\mathcal{T}_p$ if and only if for every finite subset F of S, $\{x \in S : sx = xs$ for all $s \in F\}$ is infinite.

Proof. Necessity. Pick an idempotent $p \in S^*$ such that S is right topological with respect to $\mathcal{T}_p$. It suffices to show that for each $s \in S$, $\{x \in S : sx = xs\} \in p$. (For then, given a finite subset F of S, $\{x \in S : sx = xs$ for all $s \in F\} \in p$.) To this end, let $s \in S$

be given. By Corollary 5.3, either $ps = s$ or $ps = sp$. Since S is (at least) weakly right cancellative, $ps \neq s$ and so $ps = sp$.

We claim that, if $f, g : S \to S$ and if g is injective, then $\overline{f}(p) = \overline{g}(p)$ implies that $\{x \in S : f(x) = g(x)\} \in p$. To see this, we observe that we can define $h : S \to S$ such that $h \circ g(t) = t$ for every $t \in S$ and $g \circ h(t) = t$ for every $t \in g[S]$. Since $\overline{h \circ f}(p) = \overline{h} \circ \overline{f}(p) = \overline{h} \circ \overline{g}(p) = \overline{h \circ g}(p) = p$, it follows from the deBruijn-Erdős Lemma (see [8, Theorem 3.35]) that $\{x \in S : hf(x) = x\} \in p$. Now $\{x \in S : f(x) \in g[S]\} \in p$. If $h \circ f(x) = x$ and if $f(x) \in g[S]$, then $f(x) = g(x)$.

If S is left cancellative, $l_s : S \to S$ is injective; if S is right cancellative, $r_s : S \to S$ is injective. In either case, the equation $ps = sp$ implies that $\{x \in S : xs = sx\} \in p$.

Sufficiency. For $s \in S$, let $C_s = \{x \in S : sx = xs\}$ and let $T = S^* \cap \bigcap_{s \in S} \overline{C_s}$. Since for every finite subset F of S, $\{x \in S : sx = xs \text{ for all } s \in F\}$ is infinite, $T \neq \emptyset$. Since S is either right or left cancellative, we have by [8, Corollary 4.29] that S^* is a subsemigroup of βS. For each $s \in S$, C_s is a subsemigroup of βS and so by [8, Corollary 4.18] $\overline{C_s}$ is a subsemigroup of βS. Thus T is a compact semigroup and so contains an idempotent p. Since for each $s \in S$, $sp = ps$ we have by Theorem 5.2 (i) that S is right topological with respect to $\mathcal{T}_p$. □

5.6 Theorem. *Let S be the free semigroup on the letters a and b and let p be an idempotent in S^*. Then S is not a right topological semigroup with respect to $\mathcal{V}_p$.*

Proof. Since S is cancellative, it suffices to show that $ptp \neq tp$ for some $t \in S$ (by Corollary 5.3).

Since $p \in S^*$, $\{a, b\} \notin p$ and so either $aS \in p$ or $bS \in p$. Assume without loss of generality that $aS \in p$. We note that aS is a right ideal of S and therefore that $\overline{aS}$ is a right ideal of βS by [8, Corollary 4.18]. So $pbp \in \overline{aS}$. However, $bp \in \overline{bS}$. Since $aS \cap bS = \emptyset$, $pbp \neq bp$. □

Notice that if S is the free semigroup on $\{a, b\}$ and p is an idempotent in S^*, then one can conclude that S is not right topological with respect to $\mathcal{T}_p$ either by invoking Theorem 5.5 or by invoking Corollary 5.4 and Theorem 5.6.

In view of the fact that one does not expect right continuity to hold for noncommutative semigroups, we now restrict our attention to commutative semigroups.

5.7 Definition. Let $(S, +)$ be a commutative semigroup and let $n \in \mathbb{N}$. The map $s \mapsto ns$ from S to itself has a continuous extension to a map from βS to itself. If $q \in \beta S$, nq will denote the image of q under this map.

It is easy to see that $n(p + q) = np + nq$ for every $p, q \in \beta S$.

We shall use $\mathbb{T}$ to denote the circle group $\mathbb{R}/\mathbb{Z}$. We shall use the number $x \in [0,1)$ to stand for the element $x + \mathbb{Z}$ of $\mathbb{T}$.

5.8 Theorem. *Let $(S,+)$ be a discrete semigroup which can be algebraically embedded in $\mathbb{T}$ and let p be an idempotent in S^*. Then with respect to the topology $\mathcal{V}_p$ on S the map $s \mapsto 2s$ is not continuous at any point of S.*

Proof. Suppose that this map is continuous at $a \in S$. Since $a + p$ converges to a with respect to $\mathcal{V}_p$, $2a + 2p$ converges to $2a$ and so $2a + 2p + p = 2a + p$ (by Lemma 3.3). This implies that $2p + p = p$, because S is necessarily cancellative.

We shall show that this equation cannot hold for any idempotent $p \in \mathbb{T}^*$. (We are assuming that $\beta\mathbb{T}$ denotes the Stone-Čech compactification of $\mathbb{T}$, with $\mathbb{T}$ having the discrete topology.)

We may suppose that $(0, \frac{1}{2}) \in p$, as we could replace p by $-p$ otherwise. For $i \in \{0,1\}$, we define $X_i \subseteq \mathbb{T}$ by $X_i = \bigcup_{n=1}^{\infty} [\frac{1}{2^{2n-i+1}}, \frac{1}{2^{2n-i}})$. We choose $i \in \{0,1\}$ such that $X_i \in p$. We note that p and $2p$, being idempotent, converge to 0 in the ordinary topology on $\mathbb{T}$. If $x \in X_i$ and $x < \frac{1}{4}$, then $2x \in X_j$ where $j \equiv i + 1 \pmod 2$. Since p converges to 0 in the ordinary topology, $\{y \in \mathbb{T} : 2x + y \in X_j\} \in p$. Allowing first y and then x to converge to p, shows that $2p + p \in \mathrm{cl}_{\beta\mathbb{T}}(X_j)$. Since $p \in \mathrm{cl}_{\beta\mathbb{T}}(X_i)$, we have contradicted the assumption that $2p + p = p$. $\square$

We now show that, in any abelian group, $+$ cannot be jointly continuous in $\mathcal{V}_p$, except in the case in which p has a Boolean subgroup as a member. The analogous result for an arbitrary group with the topology $\mathcal{T}_p$ was proved in [13]. It was shown in [13] that, for any group G and any idempotent $p \in G^*$, joint continuity of the group operation in $\mathcal{T}_p$ implies that p has a countable Boolean group as a member and that there is a P-point in ω^*. So joint continuity of the group operation in $\mathcal{T}_p$ cannot be established in ZFC.

It follows from Martin's axiom that, for any countable abelian group G, there are strongly summable idempotents in G^* (see [7]). If G is Boolean and if $p \in G^*$ is strongly summable, then the topologies $\mathcal{T}_p$ and $\mathcal{V}_p$ coincide and are topologies for which G is a topological group.

5.9 Theorem. *Let $(G,+)$ be an abelian group and let $B = \{x \in G : 2x = 0\}$. Let $p \in G^*$ be an idempotent such that $B \notin p$. Then the maps $x \mapsto 2x$ and $x \mapsto 3x$ cannot both be continuous in $\mathcal{V}_p$.*

Proof. Suppose that both these maps are continuous in $\mathcal{V}_p$. Since p converges to 0 in $\mathcal{V}_p$, so do $2p$ and $3p$. Thus $2p + p = 3p + p = p$ (by Lemma 3.3).

Now G can be embedded in the direct sum of a family of copies of $\mathbb{T}$. We shall show that there is no non-principal ultrafilter p on a direct sum of this kind which satisfies the equation $p+p=2p+p=3p+p=p$.

Let $H=\bigoplus_{\iota<\kappa}\mathbb{T}_\iota$, where each $\mathbb{T}_\iota$ is a copy of $\mathbb{T}$ and κ is a cardinal. We make the inductive assumption that κ is the smallest cardinal for which there exists a non-principal ultrafilter p on the direct sum of κ copies of $\mathbb{T}$ satisfying $2p+p=3p+p=p$. (We saw in the proof of Theorem 5.8 that this implies that $\kappa>1$.)

For each $\iota<\kappa$, we use $\pi_\iota : H\to\mathbb{T}_\iota$ to denote the projection map. We also use $\sigma_\iota : H\to\bigoplus_{\lambda\le\iota}\mathbb{T}_\lambda$ to denote the natural projection map and $\overline{\sigma_\iota} : \beta H\to\beta(\bigoplus_{\lambda\le\iota}\mathbb{T}_\lambda)$ to denote its continuous extension. Since $\overline{\sigma_\iota}$ is a homomorphism [8, Theorem 4.8], $\overline{\sigma_\iota}(p)$ is idempotent and $2\overline{\sigma_\iota}(p)+\overline{\sigma_\iota}(p)=3\overline{\sigma_\iota}(p)+\overline{\sigma_\iota}(p)=\overline{\sigma_\iota}(p)$ for each ι. Our inductive assumption implies that $\overline{\sigma_\iota}(p)$ is a principal ultrafilter, which is therefore the identity of $\bigoplus_{\lambda\le\iota}\mathbb{T}_\lambda$.

For each $x\in H\backslash\{0\}$, let $f(x)$ denote the first element of κ for which $\pi_{f(x)}(x)\neq 0$ and let $g(x)=\pi_{f(x)}(x)$. We shall show that the equation $2p+p=p$ implies that $\{x\in H\backslash\{0\} : g(x)=\frac{1}{2}\}\in p$.

To see this, for $i\in\{0,1\}$, let X_i be defined as in the proof of Theorem 5.8. If $\{x\in H : g(x)=\frac{1}{2}\}\notin p$, we may suppose that $\{x\in H\backslash\{0\} : g(x)\in(0,\frac{1}{2})\}\in p$ and choose $i\in\{0,1\}$ such that $C=\{x\in H\backslash\{0\} : g(x)\in X_i\}\in p$. Choose any $x\in C$. Then $g(2x)\in X_j\cup[\frac{1}{2},1)$, where $j\equiv i+1 \pmod 2$. Now, if $D_x=\{y\in H : \pi_\lambda(y)=0$ for all $\lambda\le f(x)\}$, our inductive assumption implies that $D_x\in p$. If $x\in C$ and $y\in D_x$, then $g(2x+y)=g(2x)$. Allowing first y and then x to converge to p, shows that $\{2x+y : x\in C$ and $y\in D_x\}\in 2p+p$. Thus $\{z\in H\backslash\{0\} : g(z)\in X_j\cup[\frac{1}{2},1)\}\in p$. This contradicts the assumption that $2p+p=p$ and establishes that $\{x\in H\backslash\{0\} : g(x)=\frac{1}{2}\}\in p$.

Now the equation $2p+p=p$ holds if $2p$ is substituted for p. It follows that $\{x\in H\backslash B : g(2x)=\frac{1}{2}\}\in p$. For each $x\in H\backslash B$, let $h(x)$ denote the first element of κ for which $\pi_{h(x)}(x)\notin\{0,\frac{1}{2}\}$. Then $\{x\in H\backslash B : \pi_{h(x)}(x)=\frac{1}{4}\}\in p$ or $\{x\in H\backslash B : \pi_{h(x)}(x)=\frac{3}{4}\}\in p$. However, each of these possibilities is easily seen to contradict the equation $3p+p=p$. □

5.10 Remark. *Suppose that $(G,+)$ is an abelian group and that $p\in G^*$ is idempotent. Let $B=\{x\in G : 2x=0\}$. Then B is clopen with respect to both of the topologies $\mathcal{T}_p$ and $\mathcal{V}_p$.*

Proof. If $B\in p$, then, for any $q\in\beta G$, $q+p\in\overline{B}$ if and only if $q\in\overline{B}$. Thus $B=r_p^{-1}[\overline{B}]$ and $G\backslash B=r_p^{-1}[\overline{G\backslash B}]$ so B is clopen with respect to $\mathcal{V}_p$ and thus with respect to $\mathcal{T}_p$.□

5.11 Theorem. *Let S be a cancellative semigroup and let $a, b \in S$. If the semigroup operation of S is jointly continuous at (a, b) with respect to $\mathcal{T}_p$, then $\{x \in S : xb = b\} \in p$ or $\{x \in S : xbx = b\} \in p$.*

Proof. Define $f : S \to S$ by $f(x) = axbx$. Let $\langle x_\iota \rangle_{\iota \in I}$ be a net in S which converges to p in βS. By Lemma 3.2, $\langle ax_\iota \rangle_{\iota \in I}$ and $\langle bx_\iota \rangle_{\iota \in I}$ converge to a and b respectively with respect to $\mathcal{T}_p$. So $\langle ax_\iota bx_\iota \rangle_{\iota \in I}$ converges to ab with respect to $\mathcal{T}_p$. This implies that $\langle ax_\iota bx_\iota \rangle_{\iota \in I}$ converges to ab or to abp in βS. So $\overline{f}(p) = ab$ or $\overline{f}(p) = abp$. We note that the map $s \mapsto abs$ from S to itself is injective. It follows from the argument used in the proof of Corollary 5.5 that $\{x \in S : axbx = ab\} \in p$ or $\{x \in S : axbx = abx\} \in p$. Our claim then follows from the assumption that S is cancellative. □

5.12 Corollary. *Let S be a commutative and cancellative semigroup and let p be an idempotent in S^*. If $\{x \in S : x^2$ is the identity of $S\} \notin p$, then the map $x \mapsto x^2$ is not continuous at any point of S with respect to the topology $\mathcal{T}_p$.*

Proof. Suppose that this map is continuous at the point $a \in S$. Then, exactly as in the proof of Theorem 5.11 with $a = b$, we can deduce that $\{x \in S : x^2a = a\} \in p$ or $\{x \in S : xa = a\} \in p$. Our claim then follows from the observation that, for any $x \in S$, the equation $xa = a$ can only hold if x is the unique identity of S. □

Statement (f) of the following theorem deals with strongly summable ultrafilters. An ultrafilter p on a commutative semigroup $(S, +)$ is *strongly summable* if and only if for every $A \in p$, there exists $B \subseteq S$ such that $FS(B) \subseteq A$ and $FS(B) \in p$, where $FS(B) = \{\Sigma F : F$ is a finite nonempty subset of $B\}$. Their existence follows from the continuum hypothesis (or Martin's Axiom) and is independent of ZFC. See [7] and [8, Chapter 12] for information about strongly summable ultrafilters.

5.13 Theorem. *Let $(S, +)$ be a commutative and cancellative semigroup with identity 0 and let p be an idempotent in S^*. Statements (a), (b), (c), and (d) are equivalent. If $\{x \in S : 2x = 0\} \in p$, then these statements imply statements (e) and (f) and are implied by statement (g).*

(a) For all $x, y \in S$, $+$ is continuous at (x, y) with respect to the topology $\mathcal{T}_p$.
(b) The operation $+$ is continuous at $(0, 0)$ with respect to the topology $\mathcal{T}_p$.
(c) There exist $x, y \in S$ such that $+$ is continuous at (x, y) with respect to the topology $\mathcal{T}_p$.
(d) For all $A \in p$, there exists $B \in p$ such that $B + B \subseteq A \cup \{0\}$.
(e) The operation $+$ is continuous at $(0, 0)$ with respect to the topology $\mathcal{V}_p$.

(f) *The ultrafilter p is both strongly right maximal and strongly left maximal in S^*.*

(g) *The ultrafilter p is strongly summable.*

Proof. That (a) implies (b) and (b) implies (c) is trivial. To see that (c) implies (d), pick $x, y \in S$ such that $+$ is continuous at (x, y) with respect to the topology $\mathcal{T}_p$. Let $A \in p$. By Theorem 2.16, $\{x + y\} \cup (x + y + A)$ is a neighborhood of $x + y$ so pick neighborhoods U and V of x and y respectively such that $U + V \subseteq \{x+y\} \cup (x+y+C)$. Pick C and D in p such that $\{x\} \cup (x + C) \subseteq U$ and $\{y\} \cup (y + D) \subseteq V$. Let $B = C \cap D$ and let $u, v \in B$. Then $x + y + u + v \in \{x + y\} \cup (x + y + A)$ so by cancellation $u + v \in \{0\} \cup A$.

To see that (d) implies (a), let $x, y \in S$ and let W be a neighborhood of $x + y$ with respect to $\mathcal{T}_p$. Pick $A \in p$ such that $\{x+y\} \cup (x+y+A) \subseteq W$ and pick $B \in p$ such that $B + B \subseteq A$. Let $C = B \cap A$. Then $\{x\} \cup (x + C)$ is a neighborhood of x, $\{y\} \cup (y + C)$ is a neighborhood of y, and $\big(\{x\} \cup (x + C)\big) + \big(\{y\} \cup (y + C)\big) \subseteq \{x + y\} \cup (x + y + A)$.

Now let $D = \{x \in S : 2x = 0\}$ and assume that $D \in p$.

To see that (d) implies (e), let U be an open neighborhood of 0 with respect to $\mathcal{V}_p$ and pick V open in βS such that $U = r_p^{-1}[V]$. Then $p = 0 + p \in V$ so pick $C \in p$ such that $\overline{C} \subseteq V$. Let $A = \{x \in S : -x + C \in p\}$. Then $A \in p$ and $0 \in A$ so pick $B \in p$ such that $B + B \subseteq A \cup \{0\} = A$. Let $W = r_p^{-1}[\overline{B}]$. Since $0 + p \in \overline{B}$, W is a neighborhood of 0 with respect to $\mathcal{V}_p$. We claim that $W + W \subseteq U$.

To this end, let $y, z \in W$. If $y + z = 0$, then $y + z \in U$, so assume that $y + z \neq 0$. Now $-y + B \in p$ and $-z + B \in p$ so pick $a \in (-y + B) \cap (-z + B) \cap D$. Then $y + a \in B$ and $z + a \in B$ and so $y + z = y + a + z + a \in B + B \subseteq A$. Then $-(y + z) + C \in p$. That is, $y + z \in r_p^{-1}[\overline{C}] \subseteq U$.

To see that (d) implies (f), let $q \in S^*$ such that $q \neq p$ and suppose that either $p = p + q$ or $p = q + p$. Pick $C \in q \backslash p$. We may presume that $0 \notin C$. Let $A = D \backslash C$ and pick $B \in p$ such that $B + B \subseteq A \cup \{0\}$. We may presume that $B \subseteq A$.

We show that in either case there exists $x \in B$ and $y \in C$ such that $x + y \in B$. Then $x + x + y \in B + B$ and $x \in B \subseteq A \subseteq D$ so that $x + x = 0$ and thus $y \in A \cup \{0\}$, a contradiction.

<u>Case 1</u>. $p = p + q$. Then $\{x \in S : -x + B \in q\} \in p$ so pick $x \in B$ such that $-x + B \in q$ and pick $y \in C \cap (-x + B)$.

<u>Case 2</u>. $p = q + p$. Then $\{y \in S : -y + B \in p\} \in q$ so pick $y \in C$ such that $-y + B \in q$ and pick $x \in B \cap (-y + B)$.

To see that (g) implies (d), let $A \in p$. Pick $X \subseteq S$ such that $FS(X) \subseteq A \cap D$ and

$FS(X) \in p$. Let $B = FS(X)$. To see that $B + B \subseteq A \cup \{0\}$, let $a, b \in B$ and pick finite nonempty subsets F and G of S such that $a = \Sigma F$ and $b = \Sigma G$. If $F = G$, then $a + b = 0$. Otherwise $a + b = \Sigma(F \Delta G) \in B \subseteq A$. □

Of course, if S is an abelian group, one has by standard arguments that statement (e) in Theorem 5.13 is equivalent to statements (a), (b), and (c) with $\mathcal{T}_p$ replaced by $\mathcal{V}_p$.

References

[1] J. Berglund and N. Hindman, *Filters and the weak almost periodic compactification of a discrete semigroup*, Trans. Amer. Math. Soc. **284** (1984), 1-38.

[2] A. Blass and N. Hindman, *On strongly summable and union ultrafilters*, Trans. Amer. Math. Soc. **304** (1987), 83-99.

[3] R. Ellis, *A note on the continuity of the inverse*, Proc. Amer. Math. Soc. **8** (1957), 372-373.

[4] R. Ellis, *Locally compact transformation groups*, Duke Math. J. **24** (1957), 119-125.

[5] A. El-Mabhouh, J. Pym and D. Strauss, *Subsemigroups of $\beta\mathbb{N}$*, Topology and its Applications, **60** (1994), 87-100.

[6] N. Hindman, *Strongly summable ultrafilters on $\mathbb{N}$ and small maximal subgroups of $\beta\mathbb{N}$*, Semigroup Forum **42** (1991), 63-75.

[7] N. Hindman, I. Protasov, and D. Strauss, *Strongly summable ultrafilters on abelian groups*, Matem. Studii **9**, (1998).

[8] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification – theory and applications*, de Gruyter, Berlin, 1998.

[9] J. Kelley, *General Topology*, van Nostrand, New York, 1955.

[10] V. Malykhin, *Extremally disconnected and similar groups*, Soviet Math. Dokl. **16** (1975), 21-25.

[11] P. Matet, *Some filters of partitions*, J. Symbolic Logic **53** (1988), 540-553.

[12] T. Papazyan, *Extremal topologies on a semigroup*, Topology and its Applications **39** (1991), 229-243.

[13] I. Protasov, *Filters and topologies on semigroups*, Matem. Studii **3** (1994), 15-28.

[14] I. Protasov, *Ultrafilters on abelian groups close to being Ramsey ultrafilters*, Matem. Studii **7** (1997), 133-138.

[15] I. Protasov, *Finite groups in βG*, Algebra and Discrete Math, to appear.

[16] I. Protasov, *Maximal topologies on groups*, Siberian Math. J., to appear.

[17] S. Shelah, *Proper forcing*, Springer-Verlag, Berlin, 1982.

[18] E. Zelenuk, *Finite groups in $\beta\mathbb{N}$ are trivial*, Ukranian National Academy of Sciences Institute of Mathematics, Preprint 96.3 (1996).

Neil Hindman
Department of Mathematics
Howard University
Washington, DC 20059
USA

I. Protasov
Department of Mathematics
Kiev State University
Kiev
Ukrania

Dona Strauss
Department of Pure Mathematics
University of Hull
Hull HU6 7RX
UK