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Some new results about the ubiquitous semigroup $\mathbb{H}$

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Abstract

The semigroup $\mathbb{H}$ is defined as $\bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$, where it has the algebraic structure (and topology) inherited from the right topological semigroup $(\beta\mathbb{N},+)$. Topological and algebraic copies of $\mathbb{H}$ are found in $(\beta S,\cdot)$ for any discrete semigroup which has some sequence with distinct finite products. And any compact Hausdorff right topological semigroup which has a countable dense set contained in its topological center is an image of $\mathbb{H}$ under a continuous homomorphism. (Thus the term "ubiquitous" in the title.) Much is already known about the structure of $\mathbb{H}$. In this paper we present several new results. Included are the following facts. (1) For any $n\in\mathbb{N}$, $\mathbb{H}$ is the union of n pairwise disjoint clopen copies of itself, each of which is a right ideal of $\mathbb{H}$ and $\mathbb{H}$ is the union of n pairwise disjoint clopen

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copies of itself, each of which is a left ideal of $\mathbb{H}$. (2) $\mathbb{H}$ contains $\mathfrak{c}$ pairwise disjoint clopen copies of itself, each of which is a right ideal of $\mathbb{H}$ and $\mathbb{H}$ contains $\mathfrak{c}$ pairwise disjoint clopen copies of itself, each of which is a left ideal of $\mathbb{H}$. (3) If S is a countable dense subgroup of $(\mathbb{R},+)$ and S_d is S with the discrete topology, then the set of ultrafilters in βS_d that converge to 0 (in the usual topology on S) is a copy of $\mathbb{H}$. (4) If S is the direct sum of countably many countable partial semigroups each of which has an identity and at least two elements, then the set of ultrafilters in βS_d that converge to the identity in the product topology on S is a copy of $\mathbb{H}$.

1 Introduction

Given a discrete space S, we take the Stone-Čech compactification βS of S to be the set of ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$, let $\overline{A} = \{p \in \beta S : A \in p\}$. Then $\{\overline{A} : A \subseteq S\}$ is a base for the open sets and a base for the closed sets of βS. If $(S,\cdot)$ is a discrete semigroup, the operation extends to βS so that $(\beta S,\cdot)$ becomes a right topological semigroup with S contained in its topological center. That is, for any $p \in \beta S$, the function $q \mapsto q \cdot p$ from βS to itself is continuous and for any $x \in S$, the function $q \mapsto x \cdot q$ from βS to itself is continuous. It follows that, for any $p, q \in \beta S$, $p \cdot q = \lim\limits_{s\to p}\lim\limits_{t\to q} st$, where s and t denote elements of S.

Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p\cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$ where $x^{-1}A = \{y \in S : x\cdot y \in A\}$. If the operation on S is written additively, we write that $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$ where $-x + A = \{y \in S : x + y \in A\}$. Any compact Hausdorff right topological semigroup has an idempotent. See Part I of [4] for much more information about the structure of βS.

We take $\mathbb{N}$ to be the set of positive integers and let $\omega = \mathbb{N} \cup \{0\}$. Given a set X, we let $\mathcal{P}_f(X) = \{F \subseteq X : F$ is finite and nonempty$\}$.

We define $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$. Then by [4, Lemma 6.8], $\mathbb{H}$ is a compact subsemigroup of $(\beta\mathbb{N},+)$ which contains all of the idempotents of $\beta\mathbb{N}$.

Let $(S,\cdot)$ be a discrete semigroup and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in S such that whenever F and H are distinct members of $\mathcal{P}_f(\mathbb{N})$, $\prod_{t\in F} x_t \neq \prod_{t\in H} x_t$, where the products are computed in increasing order of indices. For $m \in \mathbb{N}$, let $FP(\langle x_n\rangle_{n=m}^{\infty}) = \{\prod_{t\in F} x_t : F \in \mathcal{P}_f(\mathbb{N}$ and $\min F \geq m\}$. It was shown in [2, Theorem 5.6] that $\bigcap_{m=1}^{\infty} \overline{FP(\langle x_n\rangle_{n=m}^{\infty})}$ is topologically isomorphic to $\mathbb{H}$. By "topologically isomorphic" we mean there is a function $\varphi : \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n\rangle_{n=m}^{\infty})} \to \mathbb{H}$ which is both an isomorphism and a homeomorphism. In this case, that function is particularly simple. Define $f : FP(\langle x_n\rangle_{n=1}^{\infty}) \to \mathbb{N}$ by $f(\prod_{t\in F} x_t) = \sum_{t\in F} 2^{t-1}$, let $\widetilde{f} : \beta S \to \beta\mathbb{N}$ be the continuous extension of f, and let φ be the restriction of $\widetilde{f}$ to $\bigcap_{m=1}^{\infty} \overline{FP(\langle x_n\rangle_{n=m}^{\infty})}$.

In [7, Theorem 1], John Pym showed that one does not need a full semigroup structure to produce copies of $\mathbb{H}$. In fact, the only algebraic information needed to produce a copy of $\mathbb{H}$, is the rule for multiplying by an identity. (We shall have more to say about Pym's construction in Section 4.) In [6, Theorem 3.1], Talin Papazyan showed that if S is a discrete right cancellative and weakly left cancellative semigroup, then every G_δ subset of $S^* = \beta S \setminus S$ which contains an idempotent contains a copy of $\mathbb{H}$. (A semigroup S is *weakly left cancellative* provided that for all $a, b \in S$, $\{x \in S : ax = b\}$ is finite.)

The fact cited in the abstract that any compact Hausdorff right topological semigroup which has a countable dense set contained in its topological center is an image of $\mathbb{H}$ under a continuous homomorphism is established in [4, Theorem 6.4].

It is also known that $\mathbb{H}$ has a substantial algebraic structure. For example by [4, Corollary 7.36] $\mathbb{H}$ contains copies of the free group on $2^{\mathfrak{c}}$ generators.

Recall that if X is a topological space, $x \in X$, and p is an ultrafilter on X, that is $p \in \beta X_d$ where X_d is the set X with the discrete topology, one says that p *converges to* x if and only if p contains the neighborhood filter of x. If S is a semigroup (not necessarily commutative) with identity which is also a topological space, we let $O(S)$ be the set of ultrafilters on S that converge to the identity. In the event that $(S, \cdot)$ is a left topological semigroup with identity e, one has that $O(S)$ is a compact subsemigroup of $(\beta S_d, \cdot)$. (To see this, let $p, q \in O(S)$ and let A be an open neighborhood of e. Given $x \in A$, one may pick a neighborhood B_x of e such that $x \cdot B_x \subseteq A$ so $A \subseteq \{x \in S : x^{-1}A \in q\}$ so $A \in p \cdot q$.)

Let S be a dense subsemigroup of $(\mathbb{R}, +)$, $\big((-\infty, 0]. +\big)$, or $\big([0, \infty), +\big)$. In [3] several facts about $O(S)$ were obtained. We let $O^+(S) = \{p \in O(S) : (0, 1) \cap S \in p\}$ and let $O^-(S) = \{p \in O(S) : (-1, 0) \cap S \in p\}$. If $S \subseteq (0, \infty)$, then of course $O^-(S) = \emptyset$, but in any event, $O^+(S)$ and $O^-(S)$ have no isolated points. Note also that, given $p \in \beta S_d$, $p \in O^+(S)$ if and only if for every $\epsilon > 0$, $(0, \epsilon) \cap S \in p$ and $p \in O^-(S)$ if and only if for every $\epsilon > 0$, $(-\epsilon, 0) \cap S \in p$. If $0 \notin S$, then $O(S) = O^-(S) \cup O^+(S)$; if $0 \in S$, then $O(S) = O^-(S) \cup \{0\} \cup O^+(S)$.

Let $B(S)$ be the set of *bounded* ultrafilters on S. That is,

$$B(S) = \{p \in \beta S_d : (\exists n \in \mathbb{N})([-n,n] \cap S \in p)\}\,.$$

In [3, Theorem 2.6(b)] it was shown that $O(\mathbb{R})$ has all of the algebraic structure of $B(\mathbb{R})$ not already revealed by the algebra of $\mathbb{R}$. That is, $\mathbb{R} \times O(\mathbb{R})$ is isomorphic to $B(\mathbb{R})$. At the conclusion of this introduction we will show that a similar, though weaker, conclusion applies to any dense subgroup S of $(\mathbb{R},+)$ and $O^+(S)$ and $O^-(S)$ are isomorphic and homeomorphic. Consequently, there is additional interest in the algebraic structure of $O^+(S)$.

There are also applications of the algebraic structure of $O(S)$ to Ramsey Theory. See for example [1] and [8].

Some of our results have elementary proofs in the sense that we can explicitly describe the functions demonstrating that $O^+(S)$ or $O(S) \setminus \{0\}$ is topologically isomorphic to $\mathbb{H}$. We present these results in Section 2.

Our remaining results are based on [4, Theorem 7.24] and a modification thereof. In Section 3 we obtain statements (1), (2), and (3) stated in the abstract. In Section 4 we obtain a very general result about direct sums of sets with much less algebraic structure than a semigroup (the *ids* of [7]) and derive consequences of that result.

In Section 5 we present the proof of our modification of [4, Theorem 7.24].

As promised above, we show now that if S is a dense subgroup of $(\mathbb{R},+)$, then the ultrafilters converging to 0 contain much of the algebraic structure of βS_d.

Definition 1.1. Let S be a dense subsemigroup of $(\mathbb{R},+)$.

(a) Let $\iota : S \to [-\infty,\infty]$ be the inclusion function and let $\alpha : \beta S_d \to [-\infty,\infty]$ be its continuous extension.

(b) $B'(S) = \{p \in \beta S_d : \alpha(p) \in S\}$.

Notice that, if $S \neq \mathbb{R}$, then $B'(S) \neq B(S)$ because, given $x \in \mathbb{R} \setminus S$, $\mathcal{A} = \{S \cap (x-\epsilon, x+\epsilon) : \epsilon > 0\}$ is a set of subsets of S with the finite intersection property. If $p \in \beta S_d$ and $\mathcal{A} \subseteq p$, then $\alpha(p) = x$ so $p \in B(S) \setminus B'(S)$.

Theorem 1.2. *Let S be a dense subgroup of $(\mathbb{R},+)$. Define $\varphi : S_d \times O(S) \to B'(S)$ by $\varphi(x,p) = x + p$. Then φ is a continuous isomorphism onto $B'(S)$.*

Proof. Let $T = [-\infty,\infty]$ with the usual topology and for $x, y \in T$, define

$$x * y = \begin{cases} x + y & \text{if } x, y \in \mathbb{R}\,, \\ y & \text{if } y = \infty \text{ or } y = -\infty\,, \\ x & \text{if } y \in \mathbb{R} \text{ and either } x = \infty \text{ or } x = -\infty\,. \end{cases}$$

It is routine to establish that $(T,*)$ is a compact right topological semigroup and the topological center $\Lambda(T)=\mathbb{R}$. Note that $O(S)=\alpha^{-1}[\{0\}]$.

To see that φ is a homomorphism, let $(x,p),(y,q)\in S_d\times O(S)$. By [4, Theorem 4.23], $p+y=y+p$ so $\varphi\big((x,p)+(y,q)\big)=\varphi(x+y,p+q)=x+y+p+q=x+p+y+q=\varphi(x,p)+\varphi(y,q)$.

To see that φ is injective, assume that $(x,p),(y,q)\in S_d\times O(S)$ and $\varphi(x,p)=\varphi(y,q)$. Then $x+p=y+q$. By [4, Corollary 4.22] α is a homomorphism so $\alpha(x+p)=\alpha(x)+\alpha(p)=x+0=x$ and $\alpha(y+q)=\alpha(y)+\alpha(q)=y+0=y$ so $x=y$. Then by [4, Lemma 8.1] $p=q$.

Notice that we have also established that $\alpha[S_d\times O(S)]\subseteq B'(S)$.

To see that φ is continuous, let $(x,p)\in S_d\times O(S)$ and let U be a neighborhood of $\varphi(x,p)$ in $B'(S)$. Pick $A\in x+p$ such that $\overline{A}\cap B'(S)\subseteq U$. Since $A\in x+p$, $-x+A\in p$ so $\{x\}\times(\overline{-x+A}\cap O(S))$ is a neighborhood of (x,p) in $S_d\times O(S)$. If $q\in\overline{-x+A}\cap O(S)$, then $A\in x+q$ and $\alpha(x+q)=\alpha(x)+\alpha(q)=x+0\in S$. Therefore $\varphi[\{x\}\times(\overline{-x+A}\cap O(S))]\subseteq\overline{A}\cap B'(S)$.

To see that φ is surjective, let $p\in B'(S)$. Let $x=\alpha(p)$. Then $\alpha(-x+p)=\alpha(-x)+\alpha(p)=-x+x=0$, so $(x,-x+p)\in S_d\times O(S)$ and $\varphi(x,-x+p)=p$. □

Note that, even if $S=\mathbb{R}$, in which case $B'(S)=B(S)$, the function φ is not a homeomorphism. To see this, we claim that $\varphi[\{0\}\times O(S)]=O(S)$ is not open in $B'(S)$. To see this, pick $p\in O(S)\setminus\{0\}$ and suppose we have a neighborhood U of $p=0+p$ in $B'(S)$ such that $U\subseteq O(S)$. Pick $A\in p$ such that $\overline{A}\cap B'(S)\subseteq U$. Since $p\in\beta S_d$, $A\subseteq S$. Since p is not isolated, pick $x\in A\setminus\{0\}$. Then $x\in\overline{A}\cap B'(S)\subseteq O(S)$, while $\alpha(x)=x\neq 0$.

We remind the reader that any continuous injective function with a compact domain onto a Hausdorff space is a homeomorphism.

Theorem 1.3. *Let S be a dense subgroup of $(\mathbb{R},+)$. Then $O^+(S)$ and $O^-(S)$ are topologically isomorphic.*

Proof. Let $\psi:S\to S$ be defined by $\psi(x)=-x$ and let $\widetilde{\psi}:\beta S_d\to\beta S_d$ be the continuous extension of ψ which is injective because ψ is injective. By [4, Corollary 4.22], $\widetilde{\psi}$ is a homomorphism, hence an isomorphism and a homeomorphism. It is routine to verify that $\widetilde{\psi}[O^+(S)]=O^-(S)$. □

We remark that, if S is any countable regular topological space without isolated points, which is first countable, then the set of ultrafilters in βS_d which converge to any given point of S, is homeomorphic to $\mathbb{H}$. We leave the easy proof as an exercise for the reader.

We close the introduction with the remark that it is well known, and reasonably easy to see, that any compact non-empty G_δ subset of $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$ is either homeomorphic to $\mathbb{N}^*$ or to $\mathbb{H}$. Depending on set theoretic assumptions, $\mathbb{H}$ may or may not be homeomorphic to $\mathbb{N}^*$. (Under CH, it is a consequence of [5, Corollary 1.2.4] that $\mathbb{H}$ and $\mathbb{N}^*$ are homeomorphic, while it is shown in [9] that it is consistent that $\mathbb{H}$ and $\mathbb{N}^*$ are not homeomorphic.) It is a consequence of [4, Exercises 6.1.1 and 6.1.3] that $\mathbb{N}^*$ and $\mathbb{H}$ are not topologically isomorphic.

2 Elementary results

The results of this section utilize representations of numbers to various bases. We summarize some basic facts about these expansions now, leaving verification of the assertions to the reader.

First, of course, every $x\in\mathbb{N}$ has a unique expansion in the form $x=\sum_{t\in F}2^t$ for some $F\in\mathcal{P}_f(\omega)$ and we define $\mathrm{supp}_2(x)=F$. Given $n\in\mathbb{N}$, $x\in 2^n\mathbb{N}$ if and only if $\min\mathrm{supp}_2(x)\geq n$.

Every $x\in\mathbb{N}$ has a unique expression in the form $\sum_{t\in F}c_x(t)t!$ where $F\in\mathcal{P}_f(\mathbb{N})$ and $c_x\in\times_{t\in F}\{1,2,\ldots,t\}$ and we define $\mathrm{supp}_f(x)=F$. Given x and n in $\mathbb{N}$, $x\in n!\mathbb{N}$ if and only if $\min\mathrm{supp}_f(x)\geq n$.

Every $x\in\mathbb{Z}\setminus\{0\}$ has a unique expansion in the form $x=\sum_{t\in F}(-2)^t$ for some $F\in\mathcal{P}_f(\omega)$ and we define $\mathrm{supp}_{-2}(x)=F$. We also define $\mathrm{supp}_{-2}(0)=\emptyset$. Given $n\in\mathbb{N}$, $x\in 2^n\mathbb{Z}\setminus\{0\}$ if and only if $x\neq 0$ and $\min\mathrm{supp}_{-2}(x)\geq n$. Also, $x\in\mathbb{N}$ if and only if $x\neq 0$ and $\max\mathrm{supp}_{-2}(x)$ is even.

Let $\mathbb{D}=\{\frac{n}{2^k}:n\in\mathbb{Z}\text{ and }k\in\omega\}$, the set of *dyadic rationals*. Every $x\in\mathbb{D}\cap(0,1)$ has a unique expansion in the form $x=\sum_{t\in F}\frac{1}{2^t}$ where $F\in\mathcal{P}_f(\mathbb{N})$ and we define $\mathrm{supp}_{1/2}(x)=F$. Further every such expansion is in $\mathbb{D}\cap(0,1)$. We note that for $n\in\mathbb{N}$, $x\in\mathbb{D}\cap(0,\frac{1}{2^n})$ if and only if $\min\mathrm{supp}_{1/2}(x)>n$.

Every $x\in\mathbb{D}\cap\big((-\frac{2}{3},0)\cup(0,\frac{1}{3})\big)$ has a unique expression of the form $x=\sum_{t\in F}\frac{1}{(-2)^t}$ where $F\in\mathcal{P}_f(\mathbb{N})$ and we define $\mathrm{supp}_{-1/2}(x)=F$. Further, every such expression is in $\mathbb{D}\cap\big((-\frac{2}{3},0)\cup(0,\frac{1}{3})\big)$.

Every $x\in\mathbb{Q}\cap(0,1)$ has a unique expression in the form $\sum_{t\in F}\frac{a_x(t)}{t!}$ where $F\in\mathcal{P}_f(\mathbb{N})$ with $\min F>1$ and $a_x\in\times_{t\in F}\{1,2,\ldots,t-1\}$ and we define $\mathrm{supp}_f(x)=F$. Further, every such expression is in $\mathbb{Q}\cap(0,1)$.

Let $c=\sum_{t=1}^\infty\frac{2t}{(2t+1)!}$ and let $d=\sum_{t=1}^\infty\frac{2t-1}{(2t)!}$. Every $x\in(\mathbb{Q}\setminus\{0\})\cap(-c,d)$ has a unique expression in the form $\sum_{t\in F}(-1)^t\frac{b_x(t)}{t!}$ where $F\in\mathcal{P}_f(\mathbb{N})$ with $\min F>1$ and $b_x\in\times_{t\in F}\{1,2,\ldots,t-1\}$ and we define $\mathrm{supp}_{-f}(x)=F$.

Further, every such expression is in $(\mathbb{Q}\setminus\{0\})\cap(c,d)$.

One property that all of these expansions have in common is that if x and y have disjoint supports, then the support of $x+y$ is the union of the supports of x and y.

The fact in the following theorem that $\bigcap_{n=1}^{\infty}\overline{2^n\mathbb{Z}\setminus\{0\}}$ is a copy of $\mathbb{H}$ is [4, Exercise 7.2.1], except that we erroneously failed to exclude 0 in the statement of the exercise. But the intent of the exercise was to use [4, Theorem 7.24] which in turn cited [4, Lemma 7.4] whose proof is quite complicated. Here we present a simple description of the function which is simultaneously an isomorphism and a homeomorphism.

Theorem 2.1. *Define* $\varphi:\mathbb{Z}\setminus\{0\}\to\mathbb{N}$ *by* $\varphi(x)=\sum_{t\in\operatorname{supp}_{-2}(x)}2^t$ *and let* $\widetilde{\varphi}:\beta\mathbb{Z}\to\beta\mathbb{N}$ *be the continuous extension of* φ*. Then the restriction of* $\widetilde{\varphi}$ *to* $\bigcap_{n=1}^{\infty}\overline{2^n\mathbb{Z}\setminus\{0\}}$ *is an isomorphism and a homeomorphism onto* $\mathbb{H}$*.*

Proof. Since φ is injective, so is $\widetilde{\varphi}$. Given $n\in\mathbb{N}$, $2^n\mathbb{Z}\setminus\{0\}=\{x\in\mathbb{Z}\setminus\{0\}:\min\operatorname{supp}_{-2}(x)\geq n\}$ and $2^n\mathbb{N}=\{x\in\mathbb{N}:\min\operatorname{supp}_2(x)\geq n\}$. Therefore, $\widetilde{\varphi}[\bigcap_{n=1}^{\infty}\overline{2^n\mathbb{Z}\setminus\{0\}}]=\bigcap_{n=1}^{\infty}\overline{2^n\mathbb{N}}=\mathbb{H}$.

Thus it suffices to show that the restriction of $\widetilde{\varphi}$ to $\bigcap_{n=1}^{\infty}\overline{2^n\mathbb{Z}\setminus\{0\}}$ is a homomorphism. To do this, we use [4, Theorem 4.21]. Let $n\in\mathbb{N}$ and let $x\in 2^n\mathbb{Z}\setminus\{0\}$. Let $m=\max\operatorname{supp}_{-2}(x)+1$ and let $y\in 2^m\mathbb{Z}\setminus\{0\}$. Then

$$\operatorname{supp}_{-2}(x+y)=\operatorname{supp}_{-2}(x)\cup\ \operatorname{supp}_{-2}(y)$$

so $\varphi(x+y)=\varphi(x)+\varphi(y)$ as required. □

In the following corollary we get quite explicit descriptions of disjoint copies of $\mathbb{H}$ that are each left ideals of $\mathbb{H}$. Corollary 3.6 has a much stronger result (but with nonelementary proof).

Corollary 2.2. *Let* $E=\{x\in\mathbb{N}:\max\operatorname{supp}_2(x)$ *is even*$\}$*. Let* $\mathbb{H}_1=\mathbb{H}\cap\overline{E}$ *and let* $\mathbb{H}_2=\mathbb{H}\cap\overline{\mathbb{N}\setminus E}$*. Then* $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *partition* $\mathbb{H}$ *into clopen sets, each of which is topologically isomorphic to* $\mathbb{H}$ *and each of which is a left ideal of* $\mathbb{H}$*.*

Proof. Let $T=\bigcap_{n=1}^{\infty}\overline{2^n\mathbb{Z}\setminus\{0\}}$ and let φ be the function of Theorem 2.1. Now $T\cap\mathbb{N}^*=\mathbb{H}$ so $\widetilde{\varphi}[T\cap\mathbb{N}^*]$ is topologically isomorphic to $\mathbb{H}$. If $\psi:\mathbb{Z}\to\mathbb{Z}$ is defined by $\psi(x)=-x$, then ψ is a homomorphism so by [4, Corollary 4.22], its continuous extension $\widetilde{\psi}:\beta\mathbb{Z}\to\beta\mathbb{Z}$ is an isomorphism and a homeomorphism. Since $T\cap(-\mathbb{N}^*)=\widetilde{\psi}[T\cap\mathbb{N}^*]$, we have that $T\cap(-\mathbb{N}^*)$ is also topologically isomorphic to $\mathbb{H}$ and so also $\widetilde{\varphi}[T\cap(-\mathbb{N}^*)]$ is topologically isomorphic to $\mathbb{H}$. Using the fact that $\mathbb{N} = \{x \in \mathbb{Z} \setminus \{0\} : \max \operatorname{supp}_{-2}(x)$ is even$\}$, we see that $\widetilde{\varphi}[T \cap \mathbb{N}^*] = \mathbb{H}_1$ and $\widetilde{\varphi}[T \cap (-\mathbb{N}^*)] = \mathbb{H}_2$.

There are at least two ways to show that $\mathbb{H}_1$ and $\mathbb{H}_2$ are left ideals of $\mathbb{H}$. One is to use the fact that $\mathbb{N}^*$ and $-\mathbb{N}^*$ are both left ideals of $\beta\mathbb{Z}$. We will use a more direct route by letting $p \in \mathbb{H}_1$ and $q \in \mathbb{H}$ and showing that $q + p \in \mathbb{H}_1$. (The proof for $\mathbb{H}_2$ is essentially the same.) We have that $q + p \in \mathbb{H}$ so we only need to show that $E \in q + p$. We will in fact show that for all $x \in \mathbb{N}$, $-x + E \in p$. So let $x \in \mathbb{N}$ and let $m = \max \operatorname{supp}_2(x) + 1$. Then $2^m\mathbb{N} \cap E \subseteq -x + E$. $\square$

In the next two theorems we get a simple description of an isomorphism and a homeomorphism from $O^+(\mathbb{D})$ to $\mathbb{H}$ and from $O(\mathbb{D}) \setminus \{0\}$ to $\mathbb{H}$.

Theorem 2.3. *$O^+(\mathbb{D})$ and $O^-(\mathbb{D})$ are topologically isomorphic to $\mathbb{H}$.*

Proof. By Theorem 1.3 it suffices to show that $O^+(\mathbb{D})$ is a copy of $\mathbb{H}$. Define $\varphi : \mathbb{D} \cap (0,1) \to \mathbb{N}$ by $\varphi(x) = \sum_{t \in \operatorname{supp}_{1/2}(x)} 2^{t-1}$. Define φ at will for $x \in \mathbb{D} \setminus (0,1)$. Let $\widetilde{\varphi} : \beta\mathbb{D}_d \to \beta\mathbb{N}$ be the continuous extension of φ.

We claim that the restriction of $\widetilde{\varphi}$ to $O^+(\mathbb{D})$ is an isomorphism. It is injective because φ is injective on $\mathbb{D} \cap (0,1)$. To see that it is a homomorphism, we use [4, Theorem 4.21]. Let $x \in \mathbb{D} \cap (0,1)$ and let $m = \max \operatorname{supp}_{1/2}(x)$. Let $y \in \mathbb{D} \cap (0, \frac{1}{2^m})$. Then $\min \operatorname{supp}_{1/2}(y) > m$ so $\varphi(x+y) = \varphi(x) + \varphi(y)$.

Also, given $n \in \mathbb{N}$, $\varphi[\mathbb{D} \cap (0, \frac{1}{2^n})] = 2^n\mathbb{N}$ so $\widetilde{\varphi}[O^+(\mathbb{D})] = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}} = \mathbb{H}$. $\square$

Theorem 2.4. *$O(\mathbb{D}) \setminus \{0\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. Define $\varphi : \mathbb{N} \to \mathbb{D} \cap (-\frac{2}{3}, \frac{1}{3})$ by $\varphi(x) = \sum_{t \in \operatorname{supp}_2(x)} \frac{1}{(-2)^{t+1}}$ and let $\widetilde{\varphi} : \beta\mathbb{N} \to \beta\mathbb{D}_d$ be its continuous extension.

Let $c_0 = \frac{2}{3}$. For $n \in \omega$ let $d_{2n} = d_{2n+1} = \frac{1}{3 \cdot 4^n}$ and if $n > 0$, let $c_{2n-1} = c_{2n} = \frac{2}{3 \cdot 4^n}$. Then for each $n \in \omega$, $\varphi[2^n\mathbb{N}] = (\mathbb{D} \setminus \{0\}) \cap (-c_n, d_n)$. (With apologies, we leave the tedious verification of this assertion to the reader.)

As a consequence, we have that $\widetilde{\varphi}[\mathbb{H}] = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{D}_d}\big((-c_n, 0) \cup (0, d_n)\big) = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{D}_d}(-c_n, 0) \cup \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{D}_d}(0, d_n) = O^-(\mathbb{D}) \cup O^+(\mathbb{D}) = O(\mathbb{D}) \setminus \{0\}$.

As before one easily establishes that $\widetilde{\varphi}$ is an isomorphism and a homeomorphism. $\square$

We will obtain in Corollary 3.6 a much stronger result than the following.

Corollary 2.5. *There exist $\mathbb{H}_1$ and $\mathbb{H}_2$ that partition $\mathbb{H}$ into clopen sets, each of which is topologically isomorphic to $\mathbb{H}$ and each of which is a right ideal of $\mathbb{H}$.*

Proof. By Theorems 2.3 and 2.4 $O^-(\mathbb{D})$, $O^+(\mathbb{D})$, and $O(\mathbb{D}) \setminus \{0\}$ are all topologically isomorphic to $\mathbb{H}$. By [3, Lemma 2.5(d)] $O^-(\mathbb{D})$ and $O^+(\mathbb{D})$ are right ideals of $O(\mathbb{D}) \setminus \{0\}$. $\square$

We conclude this section by showing that Theorems 2.3 and 2.4 hold if $\mathbb{D}$ is replaced by $\mathbb{Q}$. The proofs are not quite as constructive because we produce explicit functions that are isomorphisms and homeomorphisms onto $\bigcap_{n=1}^{\infty} \overline{n\mathbb{N}}$ and then rely on the previously established fact that $\bigcap_{n=1}^{\infty} \overline{n\mathbb{N}}$ is a copy of $\mathbb{H}$.

Theorem 2.6. *$O^+(\mathbb{Q})$ and $O^-(\mathbb{Q})$ are topologically isomorphic to $\mathbb{H}$.*

Proof. It suffices to establish that $O^+(\mathbb{Q})$ is a copy of $\mathbb{H}$. Define $\varphi : \mathbb{Q} \cap (0,1) \to \mathbb{N}$ by $\varphi(x) = \sum_{t \in \mathrm{supp}_f(x)} a_x(t)(t-1)!$ defining φ at will on $\mathbb{Q} \setminus (0,1)$. Let $\widetilde{\varphi} : \beta\mathbb{Q}_d \to \beta\mathbb{N}$ be its continuous extension. As in previous proofs, one shows that $\widetilde{\varphi}[O^+(\mathbb{Q})] = \bigcap_{n=1}^{\infty} \overline{n!\mathbb{N}} = \bigcap_{n=1}^{\infty} \overline{n\mathbb{N}}$ and the restriction of $\widetilde{\varphi}$ to $O^+(\mathbb{Q})$ is an isomorphism. One then invokes [4, Corollary 7.26] which says that $\bigcap_{n=1}^{\infty} \overline{n\mathbb{N}}$ is topologically isomorphic to $\mathbb{H}$. $\square$

Theorem 2.7. *$O(\mathbb{Q}) \setminus \{0\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. Define $\varphi : \mathbb{N} \to \mathbb{Q} \setminus \{0\}$ by $\varphi(x) = \sum_{t \in \mathrm{supp}_f(x)} (-1)^{t+1} \frac{c_x(t)}{(t+1)!}$ and let $\widetilde{\varphi} : \beta\mathbb{N} \to \beta\mathbb{Q}_d$ be its continuous extension. Using $\widetilde{\varphi}$ one shows in a now familiar fashion that $O(\mathbb{Q}) \setminus \{0\}$ is topologically isomorophic to $\bigcap_{n=1}^{\infty} \overline{n\mathbb{N}}$. $\square$

3 Copies of $\mathbb{H}$ in $\mathbb{H}$

In this section we obtain results which are applications of theorems whose main ideas are due to Yevhen Zelenyuk in his proof that $\beta\mathbb{N}$ does not contain any non-trivial finite groups. A topology $\mathcal{T}$ on a group G is *left invariant* provided that for every $U \in \mathcal{T}$ and every $a \in G$, $aU \in \mathcal{T}$. A space is *zero-dimensional* provided it has a base of clopen sets.

Theorem 3.1. *Let G be a group with a left invariant zero-dimensional Hausdorff topology, and let X be a countable subspace of G which contains the identity e of G and has no isolated points. Suppose also that, for each $a \in X$, $aX \cap X$ is a neighborhood of a in X and that there is a neighborhood $V(a)$ of e in X, with $V(e) = X$, for which $aV(a) \subseteq X$. If the filter of neigborhoods of e in X has a countable base, then $\bigcap\{c\ell_{\beta X_d} W : W$ is a neighborhood of e in $X\} \setminus \{e\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. [4, Theorem 7.24]. $\square$

Corollary 3.2. *Let G be a countable metrizable topological group, with identity e, which has no isolated points. Then $O(G) \setminus \{e\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. To see that G is zero-dimensional, let d denote a metric for G. For every $a \in G$ and every $r > 0$, there exists $r' \in (0, r)$ for which $\{x \in G : d(a, x) = r'\} = \emptyset$. This implies that $\{x \in G : d(a, x) < r'\}$ is clopen. Let $X = G$ and for $a \in G$, let $V(a) = G$. Then apply Theorem 3.1. □

In Section 5 we will derive a generalization of Theorem 3.1 which replaces the assumption that G is a group with the assumption that G is a cancellative semigroup with identity. The proof of the generalization is quite complicated, but still significantly simpler than the proof of [4, Lemma 7.4] on which the proof of [4, Theorem 7.24] (and thus of Theorem 3.1) relies.

The reader who is primarily interested in the results about copies of $\mathbb{H}$ and has been through the proof of [4, Theorem 7.24], or is willing to take Theorem 3.1 on faith, is advised to ignore Section 5. On the other hand, if she has not been through the proof of [4, Theorem 7.24] and wants to understand the proofs of Theorems 3.4 and 3.5, she is advised to read the proof of Theorem 5.3.

Definition 3.3. Let E be an infinite subset of $\mathbb{N}$. Then $Y_E = \mathbb{H} \cap c\ell_{\beta\mathbb{N}}\{x \in \mathbb{N} : \max \operatorname{supp}_2(x) \in E\}$ and $Z_E = \mathbb{H} \cap c\ell_{\beta\mathbb{N}}\{x \in \mathbb{N} : \min \operatorname{supp}_2(x) \in E\}$.

It has been known for some time that for any infinite $E \subseteq \mathbb{N}$, Y_E is a left ideal of $\mathbb{H}$ and Z_E is a right ideal of $\mathbb{H}$. The new information in the following two theorems is that they are copies of $\mathbb{H}$.

Theorem 3.4. *Let E be an infinite subset of $\mathbb{N}$. Then Y_E is a clopen subset of $\mathbb{H}$ which is a left ideal of $\mathbb{H}$ and topologically isomorphic to $\mathbb{H}$.*

Proof. Let $X = \{0\} \cup \{x \in \mathbb{N} : \max \operatorname{supp}_2(x) \in E\}$. Then $Y_E = \mathbb{H} \cap \overline{X}$. Since E is infinite, $Y_E \neq \emptyset$, and Y_E is trivially clopen in $\mathbb{H}$. To verify that Y_E is a left ideal of $\mathbb{H}$, let $p \in Y_E$ and let $q \in \mathbb{H}$. We claim that for all $x \in \mathbb{N}$, $-x + X \in p$ so that $X \in q + p$. (Then, since $q + p \in \mathbb{H}$ we have $q + p \in Y_E$.) So let $x \in \mathbb{N}$ and let $n = 1 + \max \operatorname{supp}_2(x)$. Then $X \cap 2^n\mathbb{N} \in p$ and $X \cap 2^n\mathbb{N} \subseteq -x + X$.

We show that Y_E is topologically isomorphic to $\mathbb{H}$ using Theorem 3.1 with $G = \mathbb{Z}$. Let $\mathcal{B} = \{x + 2^n\mathbb{Z} : x \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$. It is routine to verify that $\mathcal{B}$ is a base for a Hausdorff zero-dimensional left invariant topology on $\mathbb{Z}$.

To see that X has no isolated points in the relative topology, let $a \in X$ and $n \in \mathbb{N}$. Pick $t \in E$ such that $t > n$ and $t > \max \operatorname{supp}_2(a)$ if $a \neq 0$. Then $a + 2^t \in (a + 2^n\mathbb{Z}) \cap X$.

We now let $a \in X$ and show that $(a + X) \cap X$ is a neighborhood of a in X. If $a = 0$ this is trivial. So assume that $a \neq 0$ and pick $n \in \mathbb{N}$ such that $n > \max \operatorname{supp}_2(a)$. We claim that $(a + 2^n\mathbb{Z}) \cap X \subseteq (a + X) \cap X$ so let $y \in (a + 2^n\mathbb{Z}) \cap X$. Pick $w \in \mathbb{Z}$ such that $y = a + 2^n w$. Since $2^n > a$, $w \geq 0$. If $w = 0$, then $y = a + 0 \in (a + X)$, so assume $w > 0$. Then $\max \operatorname{supp}_2(2^n w) = \max \operatorname{supp}_2(y) \in E$ so $2^n w \in X$.

Let $V(0) = X$. For $a \in X \setminus \{0\}$, pick $n \in \mathbb{N}$ with $n > \max \operatorname{supp}_2(a)$ and let $V(a) = 2^n\mathbb{Z} \cap X$. Then $a + V(a) \subseteq X$.

For $n \in \mathbb{N}$, let $W_n = X \cap 2^n\mathbb{Z} = (X \cap 2^n\mathbb{N}) \cup \{0\}$. Then $\langle W_n \rangle_{n=1}^{\infty}$ is a neighborhood base for 0 in X.

We have established that the hypotheses of Theorem 3.1 hold. Therefore $\bigcap_{n=1}^{\infty} \overline{W_n \setminus \{0\}} = \bigcap \{c\ell_{\beta X_d} W : W \text{ is a neighborhood of } 0 \text{ in } X\} \setminus \{0\}$ is topologically isomorphic to $\mathbb{H}$, and $\bigcap_{n=1}^{\infty} \overline{W_n \setminus \{0\}} = Y_E$. $\square$

One is tempted to modify slightly the proof of Theorem 3.4 to prove Theorem 3.5. However, if $\mathbb{N} \setminus E$ is infinite and $X = \{0\} \cup \{x \in \mathbb{N} : \min \operatorname{supp}_2(x) \in E\}$, it is not true that for each $a \in X$, $(a + X) \cap X$ is a neighborhood of a in X for the topology on $\mathbb{Z}$ generated by $\{x + 2^n\mathbb{Z} : x \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$. For example, let $t \in E$ and let $a = 2^t$. Suppose we have $n \in \mathbb{N}$ such that $(a + 2^n\mathbb{Z}) \cap X \subseteq (a + X) \cap X$. Pick $j \in \mathbb{N} \setminus E$ such that $j > \max\{t, n\}$. Then $a + 2^j \in \big((a + 2^n\mathbb{Z}) \cap X\big) \setminus (a + X)$.

Theorem 3.5. *Let E be an infinite subset of $\mathbb{N}$. Then Z_E is a clopen subset of $\mathbb{H}$ which is a right ideal of $\mathbb{H}$ and is topologically isomorphic to $\mathbb{H}$.*

Proof. Let $Y = \{x \in \mathbb{N} : \min \operatorname{supp}_2(x) \in E\}$. Then $Z_E = \mathbb{H} \cap \overline{Y}$. Since E is infinite, $Z_E \neq \emptyset$, and Z_E is trivially clopen in $\mathbb{H}$. To verify that Z_E is a right ideal of $\mathbb{H}$, let $p \in Z_E$ and let $q \in \mathbb{H}$. We claim that $Y \setminus \{0\} \subseteq \{x \in \mathbb{N} : -x + Y \in q\}$ so that $Y \in p + q$. (Then, since $p + q \in \mathbb{H}$ we have $p + q \in Z_E$.) So let $x \in Y \setminus \{0\}$ and let $n = 1 + \max \operatorname{supp}_2(x)$. Then $2^n\mathbb{N} \in q$ and $2^n\mathbb{N} \subseteq -x + Y$.

We show that Z_E is is topologically isomorphic to $\mathbb{H}$ using Theorem 3.1 with $G = \bigoplus_{n=0}^{\infty} \mathbb{Z}_2$. For $\vec{x} \in G$, let $\operatorname{supp}(\vec{x}) = \{n \in \omega : x_n \neq 0\}$. For each $n \in \mathbb{N}$, let

$$T_n = \{\vec{0}\} \cup \{\vec{x} \in G \setminus \{\vec{0}\} : \min \operatorname{supp}(\vec{x}) \in E \text{ and } \min \operatorname{supp}(\vec{x}) \geq n\}\,.$$

Let $\mathcal{B} = \{\vec{a} + T_n : \vec{a} \in G\,,\, n \in \mathbb{N}$, and if $\vec{a} \neq \vec{0}$, then $n > \max \operatorname{supp}(\vec{a})\}$. It is routine to verify that $\mathcal{B}$ is a base for a left invariant Hausdorff topology on G.

We show now that this topology is zero-dimensional by verifying that each member of $\mathcal{B}$ is closed. To this end let $\vec{a} + T_n \in \mathcal{B}$. Assume first that $\vec{a} = \vec{0}$. Let $\vec{x} \in G \setminus T_n$. Then $\vec{x} \neq \vec{0}$. Let $m = \min \operatorname{supp}(\vec{x})$ and let $k = 1 + \max \operatorname{supp}(\vec{x})$. Then $m \notin E$ or $m < n$. We claim that $(\vec{x} + T_k) \cap T_n = \emptyset$. Let $\vec{z} \in T_k \setminus \{\vec{0}\}$. Then $m = \min \operatorname{supp}(\vec{x} + \vec{z})$ so $\vec{x} + \vec{z} \notin T_n$.

Now assume that $\vec{a} \neq \vec{0}$ and $n > \max \operatorname{supp}(\vec{a})$. Let $\vec{x} \in G \setminus (\vec{a} + T_n)$. Then $\vec{x} \neq \vec{a}$. Assume first that $\max \operatorname{supp}(\vec{x}) < n$. Then there is some $i \in \{0, 1, \ldots, n-1\}$ such that $x_i \neq a_i$ so $(\vec{x} + T_n) \cap (\vec{a} + T_n) = \emptyset$. Thus we may assume that $k = 1 + \max \operatorname{supp}(\vec{x}) \geq n$. We claim that $(\vec{x} + T_k) \cap (\vec{a} + T_n) = \emptyset$, so suppose instead that we have $\vec{u} \in T_k$ and $\vec{v} \in T_n$ such that $\vec{x} + \vec{u} = \vec{a} + \vec{v}$. Then $\operatorname{supp}(\vec{x} + \vec{u}) = \operatorname{supp}(\vec{x}) \cup \operatorname{supp}(\vec{u})$ and $\operatorname{supp}(\vec{a} + \vec{v}) = \operatorname{supp}(\vec{a}) \cup \operatorname{supp}(\vec{v})$. Since $\max \operatorname{supp}(\vec{x}) \geq n > \max \operatorname{supp}(\vec{a})$ we have that $\operatorname{supp}(\vec{a})$ is an intial segment of $\operatorname{supp}(\vec{x})$ and $\operatorname{supp}(\vec{u})$ is a final segment of $\operatorname{supp}(\vec{v})$, so $\operatorname{supp}(\vec{v} - \vec{u}) = \operatorname{supp}(\vec{v}) \setminus \operatorname{supp}(\vec{u})$ and thus $\min \operatorname{supp}(\vec{v} - \vec{u}) = \min \operatorname{supp}(\vec{v}) \in E$ and $\min \operatorname{supp}(\vec{v}) \geq n$ so $\vec{v} - \vec{u} \in T_n$. Therefore $\vec{x} = \vec{a} + (\vec{v} - \vec{u}) \in \vec{a} + T_n$, a contradiction.

Let $X = \{\vec{0}\} \cup \{\vec{x} \in G \setminus \{\vec{0}\} : \min \operatorname{supp}(\vec{x}) \in E\}$. Trivially X has no isolated points and $(\vec{0} + X) \cap X$ is a neighborhood of $\vec{0}$ in X. Let $V(\vec{0}) = X$. Let $\vec{a} \in X \setminus \{\vec{0}\}$ and let $n = 1 + \max \operatorname{supp}(\vec{a})$. Then $(\vec{a} + T_n) \cap X \subseteq (\vec{a} + X) \cap X$ so $(\vec{a} + X) \cap X$ is a neighborhood of $\vec{a}$ in X. Let $V(\vec{a}) = T_n$. Then $\vec{a} + V(\vec{a}) \subseteq X$.

We have established that the hypotheses of Theorem 3.1 hold. Since $\{T_n : n \in \mathbb{N}\}$ is a neighborhood base for $\vec{0}$ in X, $\bigcap_{n=1}^{\infty} c\ell_{\beta X_d} T_n \setminus \{\vec{0}\}$ is a topological and algebraic copy of $\mathbb{H}$.

Define $\psi : \mathbb{N} \to G$ by, for $x \in \mathbb{N}$, $\psi(x) = \vec{y}$ where $\operatorname{supp}(\vec{y}) = \operatorname{supp}_2(x)$ and let $\widetilde{\psi} : \beta\mathbb{N} \to \beta G_d$ be its continuous extension. For each $n \in \mathbb{N}$, $\psi[2^n\mathbb{N} \cap Y] = T_n \setminus \{\vec{0}\}$, so the restriction of $\widetilde{\psi}$ to Z_E is an isomorphism and a homeomorphism onto $\bigcap_{n=1}^{\infty} c\ell_{\beta X_d} T_n \setminus \{\vec{0}\}$. $\square$

Corollary 3.6. *Let $n \in \mathbb{N}$.*

(1) There is a partition of $\mathbb{H}$ into n clopen subsets, each of which is topologically isomorphic to $\mathbb{H}$ and each of which is a left ideal of $\mathbb{H}$.

(2) There is a partition of $\mathbb{H}$ into n clopen subsets, each of which is a topological and algebraic copy of $\mathbb{H}$ and each of which is a right ideal of $\mathbb{H}$.

Proof. Partition $\mathbb{N}$ into n infinite sets $E(1), E(2), \ldots, E(n)$. Then $Y_{E(1)}$, $Y_{E(2)}, \ldots, Y_{E(n)}$ are as required for conclusion (1) and $Z_{E(1)}, Z_{E(2)}, \ldots, Z_{E(n)}$ are as required for conclusion (2). $\square$

Recall that a collection $\mathcal{A}$ of sets is *almost disjoint* if and only if the intersection of any two members of $\mathcal{A}$ is finite. Recall also that there exist a collection of $\mathfrak{c}$ almost disjoint subsets of $\mathbb{N}$ where $\mathfrak{c} = |\mathbb{R}|$. (An easy way of seeing this is to pick a sequence of rationals converging to α for every $\alpha \in \mathbb{R}$.) Then by Zorn's Lemma, one may pick a maximal almost disjoint family of cardinality $\mathfrak{c}$.

Corollary 3.7. *Let $\langle E(\sigma)\rangle_{\sigma<\mathfrak{c}}$ enumerate a maximal almost disjoint family of infinite subsets of* $\mathbb{N}$.

(1) For each $\sigma < \mathfrak{c}$, $Y_{E(\sigma)}$ is a clopen topological and algebraic copy of $\mathbb{H}$ which is a left ideal of $\mathbb{H}$, if $\sigma < \tau < \mathfrak{c}$, then $Y_{E(\sigma)} \cap Y_{E(\tau)} = \emptyset$, and $\bigcup_{\sigma<\mathfrak{c}} Y_{E(\sigma)}$ is dense in $\mathbb{H}$.

(2) For each $\sigma < \mathfrak{c}$, $Z_{E(\sigma)}$ is a clopen topological and algebraic copy of $\mathbb{H}$ which is a right ideal of $\mathbb{H}$, if $\sigma < \tau < \mathfrak{c}$, then $Z_{E(\sigma)} \cap Z_{E(\tau)} = \emptyset$, and $\bigcup_{\sigma<\mathfrak{c}} Z_{E(\sigma)}$ is dense in $\mathbb{H}$.

Proof. (1) If $\sigma < \tau < \mathfrak{c}$, pick $n \in \mathbb{N}$ such that $E(\sigma) \cap E(\tau) \subseteq \{1, 2, \ldots, n\}$. Then $2^{n+1}\mathbb{N} \cap \{x \in \mathbb{N} : \max \operatorname{supp}_2(x) \in E(\sigma)\} \cap \{x \in \mathbb{N} : \max \operatorname{supp}_2(x) \in E(\tau)\} = \emptyset$ so $Y_{E(\sigma)} \cap Y_{E(\tau)} = \emptyset$.

To see that $\bigcup_{\sigma<\mathfrak{c}} Y_{E(\sigma)}$ is dense in $\mathbb{H}$, suppose we have $A \subseteq \mathbb{N}$ such that $\emptyset \neq \overline{A} \cap \mathbb{H}$ and $(\overline{A} \cap \mathbb{H}) \cap \bigcup_{\sigma<\mathfrak{c}} Y_{E(\sigma)} = \emptyset$. Choose $x_1 \in A$. Inductively, having chosen $\langle x_t\rangle_{t=1}^n$, choose $x_{n+1} \in A$ such that $\min \operatorname{supp}_2(x_{n+1}) > \max \operatorname{supp}_2(x_n)$. For each $n \in \mathbb{N}$ let $y_n = \max \operatorname{supp}_2(x_n)$. By the maximality of $\{E(\sigma) : \sigma < \mathfrak{c}\}$ pick $\sigma < \mathfrak{c}$ such that $E(\sigma) \cap \{y_n : n \in \mathbb{N}\}$ is infinite. Let $D = \{n \in \mathbb{N} : y_n \in E(\sigma)\}$. Pick $p \in \mathbb{N}^*$ such that $\{x_n : n \in D\} \in p$. Then $p \in \overline{A} \cap \mathbb{H}$. Let $X = \{x \in \mathbb{N} : \max \operatorname{supp}_2(x) \in E(\sigma)\}$. Then $\{x_n : n \in D\} \subseteq X$ so $p \in \overline{X} \cap \mathbb{H} = Y_{E(\sigma)}$, a contradiction.

The proof of (2) is essentially the same, letting $y_n = \min \operatorname{supp}_2(x_n)$. $\square$

We saw in Theorems 2.3, 2.4, 2.6, and 2.7 by elementary arguments that if $S = \mathbb{D}$ or $S = \mathbb{Q}$, then $O^+(S)$, $O^-(S)$, and $O(S) \setminus \{0\}$ are copies of $\mathbb{H}$. We see now in the next two theorems, as a consequence of Theorem 3.1, that if S is any countable dense subgroup of $(\mathbb{R}, +)$, the same conclusions hold.

Theorem 3.8. *Let S be a countable dense subsemigroup of $\big([0,\infty), +\big)$ with the property that $b - a \in S$ whenever $a, b \in S$ and $a \leq b$. Then $O^+(S)$ is topologically isomorphic to* $\mathbb{H}$.

Proof. We apply Theorem 3.1 with $G = \mathbb{R}$ and $X = S$. We give G the zero dimensional first countable topology for which the sets of the form $[a, b)$, where $a, b \in \mathbb{R}$ and $a < b$, are a base. It is easy to check that the hypotheses of Theorem 3.1 are satisfied. □

One may also derive Theorem 3.8 using only Corollary 5.4 by letting $T_n = [0, \frac{1}{n}) \cap S$ and $\mathcal{B} = \{a + T_n : a \in S \text{ and } n \in \mathbb{N}\}$.

When we add the assumption that S is dense in $\mathbb{R}$, we need to add the assumption that $b - a \in S$ when $b < a$. But once we have done that, we in fact have a subgroup.

Theorem 3.9. *Let S be a countable dense subgroup of $(\mathbb{R}, +)$. Then $O(S) \setminus \{0\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. This is immediate from Corollary 3.2. □

4 Direct sums

Many of the results of this section depend on the notions of *id* and *oid* introduced by John Pym in [7]. Before introducing these objects, we recall the notions of *partial semigroup* and *adequate* partial semigroup.

Definition 4.1. (a) A *partial multiplication* on a set S is a function $*$ taking a nonempty subset D of $S \times S$ to S. If $(x, y) \in D$, we write $x * y$ for $*(x, y)$ and say that $x * y$ is *defined*.

(b) A *partial semigroup* is a pair $(S, *)$ where $*$ is a partial multiplication on S such that $(x * y) * z = x * (y * z)$ for all x, y, and $z \in S$ in the sense that if either side is defined, so is the other and they are equal.

(c) Let $(S, *)$ be a partial semigroup and let $F \in \mathcal{P}_f(S)$. Then $\sigma(F) = \{y \in S : x * y \text{ is defined for all } x \in F\}$.

(d) A partial semigroup $(S, *)$ is *adequate* if and only if for every $F \in \mathcal{P}_f(S)$, $\sigma(F) \neq \emptyset$.

Definition 4.2. (a) An *id* is a partial semigroup $(S, \cdot)$ with an identity 1. That is, for each $x \in S$, $1 \cdot x = x \cdot 1 = x$.

(b) If for each i in some index set I, S_i is an id, then $\bigoplus_{i \in I} S_i = \{\vec{x} \in \times_{i \in I} S_i : \{i \in I : x_i \neq 1\} \text{ is finite}\}$ is an *oid*.

(c) For $\vec{x} \in \bigoplus_{i \in I} S_i$, $\mathrm{supp}(\vec{x}\,) = \{i \in I : x_i \neq 1\}$.

(d) If $S = \bigoplus_{i\in I} S_i$ is an oid and $\vec{x}, \vec{y} \in S$, then the oid operation $\vec{x} \cdot \vec{y}$ is defined if and only if $\mathrm{supp}(\vec{x}\,) \cap \mathrm{supp}(\vec{y}\,) = \emptyset$ in which case $(\vec{x} \cdot \vec{y})_i = x_i \cdot y_i$ for each $i \in I$.

If each S_i is a semigroup, Definition 4.2(d) conflicts with the usual definition of the operation on $\bigoplus_{i\in I} S_i$ which is defined for all $\vec{x}$ and $\vec{y}$, but the values agree wherever they are defined for both operations.

By [4, Lemma 6.14.1], if I is infinite and for each $i \in I$, S_i is an id with at least two elements, then the oid $\bigoplus_{i\in I} S_i$ is an adequate partial semigroup.

Lemma 4.3. *Assume that I is infinite and for each $i \in I$, $(S_i, \cdot)$ is an id with at least two elements. Let $\delta S = \bigcap_{F\in\mathcal{P}_f(S)} c\ell_{\beta S_d}\sigma(F)$. For $p, q \in \delta S$ and $A \subseteq S$, agree that $A \in p \cdot q$ if and only if $\{\vec{x} \in S : \{\vec{y} \in S : \mathrm{supp}(\vec{x}\,) \cap \mathrm{supp}(\vec{y}\,) = \emptyset$ and $\vec{x} \cdot \vec{y} \in A\} \in q\} \in p$. Then $(\delta S, \cdot)$ is a semigroup.*

Proof. This is an immediate consequence of [4, Theorem 4.22.2]. □

In the case that $|S_n| = 2$ for each $n \in \mathbb{N}$, the following result was established by Pym in [7].

Theorem 4.4. *For each $n \in \mathbb{N}$ let S_n be a countable id with at least two members and identity 1_n, and let $S = \bigoplus_{n=1}^{\infty} S_n$. Let $\vec{1} = \langle 1_n \rangle_{n\in\mathbb{N}} \in S$. For $n \in \mathbb{N}$, let $U_n = \{\vec{1}\,\} \cup \{\vec{x} \in S \setminus \{\vec{1}\,\} : \min \mathrm{supp}(\vec{x}\,) > n\}$. Then $\delta S = \bigcap_{n=1}^{\infty} c\ell_{\beta S_d} U_n$ and $\delta S \setminus \{\vec{1}\,\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. To see that $\delta S = \bigcap_{n=1}^{\infty} c\ell_{\beta S_d} U_n$, assume first that $p \in \delta S$ and let $n \in \mathbb{N}$. For $i \in \{1, 2, \ldots, n\}$ pick $a_i \in S_i \setminus \{1_i\}$ and define $\vec{x} \in S$ by $x_i = a_i$ if $i \leq n$ and $x_i = 1_i$ otherwise. Then $\sigma(\{\vec{x}\,\}) = U_n$ so $U_n \in p$. Now assume that $p \in \bigcap_{n=1}^{\infty} c\ell_{\beta S_d} U_n$ and let $F \in \mathcal{P}_f(S)$. Let $n = \max \bigcup\{\mathrm{supp}(\vec{x}\,) : \vec{x} \in F\}$. Then $U_n \subseteq \sigma(F)$ so $\sigma(F) \in p$.

For $n \in \mathbb{N}$, if S_n is infinite, let $G_n = \mathbb{Z}$; if $|S_n| = k \in \mathbb{N} \setminus \{1\}$, let $G_n = \mathbb{Z}_k$. Let $G = \bigoplus_{n=1}^{\infty} G_n$ with the topology inherited from $\times_{n=1}^{\infty} G_n$ with the product topology where each G_n is discrete. Given $\vec{x} \in G$, let $\mathrm{supp}(\vec{x}\,) = \{n \in \mathbb{N} : x_n \neq 0\}$. For $n \in \mathbb{N}$, let $T_n = \{\vec{0}\,\} \cup \{\vec{x} \in G \setminus \{\vec{0}\,\} : \min \mathrm{supp}(\vec{x}\,) > n\}$. It is well known, and easy to see, that G is a Hausdorff zero-dimensional topological group with no isolated points. And $\{T_n : n \in \mathbb{N}\}$ is a neighborhood base for $\vec{0}$ in G. Thus by Corollary 3.2, $O(G) \setminus \{\vec{0}\,\}$ is topologically isomorphic to $\mathbb{H}$.

For each $n \in \mathbb{N}$, pick a bijection $\varphi_n : S_n \to G_n$ with $\varphi_n(1) = 0$, and define $\psi : S \to G$ by, for $n \in \mathbb{N}$ and $\vec{x} \in S$, $\psi(\vec{x}\,)_n = \varphi_n(x_n)$, noting that ψ is a bijection. Let $\widetilde{\psi} : \beta S_d \to \beta G_d$ be the continuous extension of ψ and note that $\widetilde{\psi}$ is also a bijection.

If $\vec{x},\vec{y}\in S$ and $\operatorname{supp}(\vec{x}\,)\cap\operatorname{supp}(\vec{y}\,)=\emptyset$, then $\psi(\vec{x}\cdot\vec{y}\,)=\psi(\vec{x}\,)+\psi(\vec{y}\,)$. That is, ψ is a surjective partial semigroup homomorphism from S to G so by [4, Theorem 4.22.3], the restriction of $\widetilde{\psi}$ to δS is a homomorphism into $\delta G=\beta G_d$.

Given $n\in\mathbb{N}$, $\psi[U_n]=T_n$ so $\widetilde{\psi}[\delta S\setminus\{\vec{1}\,\}]=\widetilde{\psi}[(\bigcap_{n=1}^\infty c\ell_{\beta S_d}[U_n])\setminus\{\vec{1}\,\}]=(\bigcap_{n=1}^\infty c\ell_{\beta G_d}T_n)\setminus\{\vec{0}\,\}]=O(G)\setminus\{\vec{0}\,\}$. □

Corollary 4.5. *For $n\in\mathbb{N}$, let S_n be a countable id with identity 1_n and at least two members and let $S=\bigoplus_{n=1}^\infty S_n$. Let $\vec{1}=\langle 1_n\rangle_{n\in\mathbb{N}}\in S$. Give S the topology it inherits from $\times_{n=1}^\infty S_n$ where each S_n is discrete. For $n\in\mathbb{N}$, let $U_n=\{\vec{1}\,\}\cup\{\vec{x}\in S\setminus\{\vec{1}\,\}:\min\operatorname{supp}(\vec{x}\,)>n\}$. Then $O(S)=\bigcap_{n=1}^\infty c\ell_{\beta S_d}U_n$ and $O(S)\setminus\{\vec{1}\,\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. This follows immediately from Theorem 4.4 and the fact that the sets U_n are a base for the neighborhoods of $\vec{1}$ in S. □

Corollary 4.6. *Enumerate the primes as $\langle p_n\rangle_{n=1}^\infty$. For $n\in\mathbb{N}$, let $T_n=\{x\in\mathbb{N}:$ for all $i\in\{1,2,\ldots,n\}$, p_i does not divide $x\}$. Then $(\bigcap_{n=1}^\infty(c\ell_{\beta\mathbb{N}}T_n)\setminus\{1\},\cdot)$ is topologically isomorphic to $\mathbb{H}$.*

Proof. $(\mathbb{N},\cdot)$ is isomorphic to $(\bigoplus_{n=1}^\infty\omega,+)$ so this follows from Corollary 4.5 with each $S_n=\omega$. □

We write $\mathbb{Q}^+=\mathbb{Q}\cap(0,\infty)$. Given $x\in\mathbb{Q}^+$ and a prime p we say that p does not occur in x provided that when x is written in lowest terms, p does not divide either the numerator or denominator of x.

Corollary 4.7. *Enumerate the primes as $\langle p_n\rangle_{n=1}^\infty$. For $n\in\mathbb{N}$, let $M_n=\{x\in\mathbb{Q}^+:$ for all $i\in\{1,2,\ldots,n\}$, p_i does not occur in $x\}$. Then $\big((\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{Q}_d^+}M_n)\setminus\{1\},\cdot)\big)$ is topologically isomorphic to $\mathbb{H}$.*

Proof. $(\mathbb{Q}^+,\cdot)$ is isomorphic to $(\bigoplus_{n=1}^\infty\mathbb{Z},+)$ so this follows from Corollary 4.5 with each $S_n=\mathbb{Z}$. □

This last corollary raises the question of whether, letting

$$R_n=\{x\in\mathbb{Q}\setminus\{0\}:\text{ for all } i\in\{1,2,\ldots,n\}\,,\ p_i\text{ does not occur in }x\}\,,$$

one has $\big((\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{Q}_d}R_n)\setminus\{1\},\cdot\big)$ is topologically isomorphic to $\mathbb{H}$. That is easily seen to be false because $-1\in(\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{Q}_d}R_n)\setminus\{1\}$ and $\mathbb{H}$ has no isolated points. Naturally, we ask whether $\big((\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{Q}_d}R_n)\setminus\{1,-1\},\cdot\big)$ is topologically isomorphic to $\mathbb{H}$. We see in fact it is not even isomorphic to any subset of $\beta\mathbb{N}$.

Theorem 4.8. *Enumerate the primes as* $\langle p_n\rangle_{n=1}^\infty$. *For* $n\in\mathbb{N}$, *let*

$$R_n=\{x\in\mathbb{Q}\setminus\{0\}:\ \text{for all } i\in\{1,2,\ldots,n\}\,,\ p_i \text{ does not occur in } x\}\,.$$

Then $\big((\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{Q}_d}R_n)\setminus\{1,-1\},\cdot\big)$ *is not isomorphic to any subset of* $\beta\mathbb{N}$.

Proof. Let $X=(\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{Q}_d}R_n)\setminus\{1,-1\}$. For $p\in\beta\mathbb{Q}_d$ let $-p=(-1)\cdot p$. By [4, Theorem 4.23], $\mathbb{Q}$ is contained in the center of $(\beta\mathbb{Q}_d,\cdot)$ so for $p,q\in\beta\mathbb{Q}_d$, $(-p)\cdot(-q)=p\cdot q$ and $(-p)\cdot q=p\cdot(-q)=-(p\cdot q)$. For $n\in\mathbb{N}$, $(-1)\cdot R_n=R_n$ so if $p\in X$, then $-p\in X$. By [4, Theorem 4.20], $(X,\cdot)$ is a compact right topological semigroup, so pick an idempotent $p\in X$. Then $\{p,-p\}$ is a two element subgroup of X. By Zelenyuk's Theorem [4, Theorem 7.17], $(\mathbb{N}^*,+)$ has no notrivial finite groups, so neither does $(\beta\mathbb{N},+)$. □

We say that a commutative semigroup S with identity 0 *has a base* if and only if there exist sequences $\langle b_n\rangle_{n=1}^\infty$ in S and $\langle k_n\rangle_{n=1}^\infty$ in $\mathbb{N}\setminus\{1\}$ such that each $x\in S\setminus\{0\}$ has a unique expression in the form $\sum_{n\in F}m_x(n)b_n$, where $F\in\mathcal{P}_f(\mathbb{N})$ and $m_x\in\times_{n\in F}\{1,2,\ldots,k_n-1\}$. We let $\operatorname{supp}(x)=F$. We have seen many examples of semigroups that have a base.

Theorem 4.9. *Let* S *be a semigroup with a base and let* $\langle b_n\rangle_{n=1}^\infty$ *and* $\langle k_n\rangle_{n=1}^\infty$ *be as in the definition. For* $n\in\mathbb{N}$, *let* $T_n=\{0\}\cup\{x\in S\setminus\{0\}:\min\operatorname{supp}(x)\geq n\}$. *Then* $(\bigcap_{n=1}^\infty c\ell_{\beta S_d}T_n)\setminus\{0\}$ *is topologically isomorphic to* $\mathbb{H}$.

Proof. Let $G=\bigoplus_{n=1}^\infty\mathbb{Z}_{k_n}$ and let G have the topology inherited from $\times_{n=1}^\infty\mathbb{Z}_{k_n}$ with the product topology. As in the proof of Theorem 4.4, we have that $O(G)\setminus\{\vec{0}\}$ is topologically isomorphic to $\mathbb{H}$.

For $n\in\mathbb{N}$, define $\vec{e}_n\in G$ by $\vec{e}_n(n)=1$ and $\vec{e}_n(j)=0$ if $j\neq n$. Define $\varphi:S\to G$ by $\varphi(0)=\vec{0}$ and if $x\in S\setminus\{0\}$, $\varphi(x)=\sum_{n\in\operatorname{supp}(x)}m_x(n)\vec{e}_n$. Let $\widetilde{\varphi}:\beta S_d\to\beta G_d$ be the continuous extension of φ. Then $\widetilde{\varphi}[\bigcap_{n=1}^\infty c\ell_{\beta S_d}T_n]=O(G)$ and by [4, Theorem 4.21] the restriction of $\widetilde{\varphi}$ to $\bigcap_{n=1}^\infty c\ell_{\beta S_d}T_n$ is a homomorphism, and since φ is bijective, this restriction is an isomorphism and a homeomorphism. □

As a consequence of Theorem 4.9, for any $k\in\mathbb{N}\setminus\{1\}$, using base $k+1$, one has that $\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{N}}k^n\mathbb{N}$ is topologically isomorphic to $\mathbb{H}$; using base $-(k+1)$, one has that $(\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{Z}}k^n)\setminus\{0\}$ is topologically isomorphic to $\mathbb{H}$. Using the factorial base, one can also conclude that $\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{N}}n!\mathbb{N}$ is topologically isomorphic to $\mathbb{H}$, a fact that we knew allready because $\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{N}}n!\mathbb{N}=\bigcap_{n=1}^\infty c\ell_{\beta\mathbb{N}}n\mathbb{N}$.

If for each $n\in\mathbb{N}$, S_n is a countable dense subsemigroup of $([0,\infty),+)$ or of $(\mathbb{R},+)$, then by Corollary 4.5, if each S_n is given the discrete topology and

$S = \bigoplus_{n=1}^{\infty} S_n$ is given the product topology, then $O(S)\setminus\{\vec{0}\}$ is topologically isomorphic to $\mathbb{H}$. We see now that with additional assumptions the same conclusion follows when S_n has the relative topology inherited from $\mathbb{R}$

Theorem 4.10. *For each $i \in \mathbb{N}$, let S_i be a countable dense subsemigroup of $([0,\infty),+)$ such that whenever $a \leq b$ in S_i, one has $b - a \in S_i$ and let S_i have the relative topology from $\mathbb{R}$ with its usual topology. Let $S = \bigoplus_{i=1}^{\infty} S_i$ with the topology inherited from $\times_{i=1}^{\infty} S_i$ with the product topology. Then $O(S) \setminus \{\vec{0}\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. As in the proof of Theorem 3.8, we give $\mathbb{R}$ the zero dimensional first countable topology which has the sets of the form $[a, b)$, where $a, b \in \mathbb{R}$ and $a \leq b$, as a base. For each $n \in \mathbb{N}$, let $\mathbb{R}_n$ denote a copy of $\mathbb{R}$ with this topology, and let T_n denote S_n with the relative topology that it has as a subspace of $\mathbb{R}_n$. It is easy to check that the hypotheses of Theorem 3.1 are satisfied with $G = \bigoplus_{n=1}^{\infty} \mathbb{R}_n$ and $X = \bigoplus_{n=1}^{\infty} T_n$. So $O(T)$ is a copy of $\mathbb{H}$. Our claim now follows from the fact that the neighborhoods of 0 in S and in $\bigoplus_{n=1}^{\infty} T_n$ coincide. □

Alternatively, one may prove Theorem 4.10 using Corollary 5.4 instead of Theorem 3.1 by letting $T_n = \{\vec{x} \in S : (\forall i \in \{1, 2, \ldots, n\})(x_i \in [0, \frac{1}{n}) \cap S_i)\}$.

Theorem 4.11. *For each $i \in \mathbb{N}$, let S_i be a countable dense subgroup of $(\mathbb{R}, +)$ and let S_i have the relative topology from $\mathbb{R}$ with its usual topology. Let $S = \bigoplus_{i=1}^{\infty} S_i$ with the topology inherited from $\times_{i=1}^{\infty} S_i$ with the product topology. Then $O(S) \setminus \{\vec{0}\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. This is an immediate consequence of Corollary 3.2. □

5 Generalization of Theorem 3.1

In this section we derive Theorem 5.3 which is a generalization of Theorem 3.1. It replaces the assumption that G is a group by the assumption that G is a cancellative semigroup with identity.

The proof of Theorem 5.3 is a consequence of Theorem 5.2 which is adapted from the proof of [4, Lemma 7.4] whose main ideas are due to Yevhen Zelenyuk. The proof of Theorem 5.2 is simpler than the proof of [4, Lemma 7.4] because it proves less. (Here we only include the conclusions needed to prove Theorem 5.3; in [4], additional conclusions were needed for other applications of Lemma 7.4.)

We incorporate now some of the notation from [4, Section 7.1].

Definition 5.1. (a) F will denote the free semigroup on the letters 0 and 1 with identity $\emptyset$.

(b) If $m \in \omega$ and $i \in \{0, 1, 2, \ldots, m\}$, s_i^m will denote the element of F consisting of i 0's followed by $m - i$ 1's. We also write $u_m = s_m^m$ (so $u_0 = \emptyset$).

(c) If $s \in F$, $l(s)$ will denote the length of s and $\mathrm{supp}_F(s) = \{i \in \{1, 2, \ldots, l(s)\} : s_i = 1\}$ where s_i is the i^{th} letter of s.

(d) If $s, t \in F$, we shall write $s << t$ if $\max \mathrm{supp}_F(s) + 1 < \min \mathrm{supp}_F(t)$.

(e) If $s, t \in F$, we define $s + t$ to be the element of F for which $l(s + t) = \max\{l(s), l(t)\}$ and $(s + t)_i = 1$ if and only if $s_i = 1$ or $t_i = 1$.

(f) Given any $t \in F$, t has a unique representation in the form $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_k}^{m_k}$ where $0 \leq i_0 < m_0 < i_1 < m_1 < \ldots < i_k \leq m_k$ (except that, if $k = 0$, the requirement is $0 \leq i_0 \leq m_0$). We shall call this the *canonical representation* of t. When we write $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_k}^{m_k}$ we shall assume that this is the canonical representation.

(g) Given $t \in F$, if $t = s_i^m$ for some $i, m \in \omega$, then $t' = \emptyset$ and $t^* = t$. Otherwise, if $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_{k+1}}^{m_{k+1}}$, then $t' = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_k}^{m_k}$ and $t^* = s_{i_{k+1}}^{m_{k+1}}$.

For example $011001110001 = s_1^3 + s_5^8 + s_{11}^{12}$ and $101110 = s_0^1 + s_2^5 + s_6^6$.

Theorem 5.2. *Assume that G is a cancellative semigroup with identity e and that X is a countably infinite subset of G with $e \in X$. Well order X in order type ω. Assume that G has a Hausdorff topology and there is a decreasing sequence $\langle T_n \rangle_{n=1}^{\infty}$ of subsets of G such that $\{T_n : n \in \mathbb{N}\}$ is a neighborhood base for e consisting of sets clopen in G and for each $a \in X$ there exists $m \in \mathbb{N}$ such that $\{aT_n : n \geq m\}$ is a neighborhood base for a consisting of sets clopen in G. Assume that X has no isolated points, for each $a \in X$, $aX \cap X$ is a neighborhood of a in X, and for each $a \in X$, we have a neighborhood $V(a)$ of e in X such that $aV(a) \subseteq X$ and $V(e) = X$. Assume that $\langle W_n \rangle_{n=1}^{\infty}$ is a sequence of neighborhoods of e in X.*

We can define $x(t) \in X$ and $X(t) \subseteq X$ for each $t \in F$ such that $x(\emptyset) = e$, $X(\emptyset) = X$ and for each $k \in \omega$ we can define $a_k \in X$ and $t_k \in F$ such that

(i) $a_k = \min X \setminus \{x(t) : t \in F \text{ and } l(t) \leq k\}$;

(ii) $a_k \in X(t_k{}^\frown 1)$;

(iii) $a_k = x(t_k{}^\frown 1)$; *and*

(iv) $l(t_k) = k$.

Further, for each $t \in F$,

(1) $X(t)$ *is clopen in* X*;*

(2) $x(t) \in X(t)$*;*

(3) $X(t) = X(t^\frown 0) \cup X(t^\frown 1)$ *and* $X(t^\frown 0) \cap X(t^\frown 1) = \emptyset$*;*

(4) $x(t^\frown 0) = x(t)$*;*

(5) $x(t) = x(t')x(t^*)$*;*

(6) $X(t) = x(t')X(t^*)$*; and*

(7) $X(t^*) \subseteq V\big(x(t')\big)$.

Further,

(8) for $s, t \in F$, $x(s) = x(t)$ *if and only if* $s = t$, $s = t^\frown 0 \ldots 0$ *or* $t = s^\frown 0 \ldots 0$*;*

(9) for each $n \in \mathbb{N} \setminus \{1\}$, $X(u_n) \subseteq W_{n-1}$*;*

(10) if $t, r \in F$ *and* $t << r$*, then* $x(t + r) = x(t)x(r)$*; and*

(11) for every $n \in \omega$, $x[u_n F] = X(u_n)$ *and in particular* $x[F] = X$.

Proof. Let $\mathcal{U} = \{T_n : n \in \mathbb{N}\}$. The proof proceeds by induction on $l(t)$ establishing hypotheses (i) through (iv) and (1) through (9).

When the induction is complete, conclusions (10) and (11) will be established.

We let $x(\emptyset) = e$ and $X(\emptyset) = X$. Let $a_0 = \min(X \setminus \{e\})$, let $t_0 = \emptyset$, pick $U_0 \in \mathcal{U}$ such that $a_0 \notin U_0$. Let $X(0) = U_0 \cap X$, $X(1) = X \setminus U_0$, $x(0) = e$, and $x(1) = a_0$.

Now let $n \in \mathbb{N}$, and assume we have chosen $\langle a_k \rangle_{k=0}^{n-1}$, $\langle t_k \rangle_{k=0}^{n-1}$, and $x(t)$ and $X(t)$ for all $t \in F$ with $l(t) \leq n$ satisfying (i), (ii), (iii), (iv), and (1)-(9), where as an induction hypothesis (8) includes the assertion that $l(s) \leq n$ and $l(t) \leq n$ and (3) and (4) include the assertion that $l(t) < n$.

We first verify the hypotheses for $n = 1$ and $l(t) \leq 1$. One can simply check hypotheses (i) through (iv) and (1) through (4). For (5), (6), and (7), note that $\emptyset' = \emptyset^* = \emptyset$, $0' = 1' = \emptyset$, $0^* = 0$, and $1^* = 1$. For hypothesis (8), if $l(s) \leq 1$, $l(t) \leq 1$, $t \neq s$ and $x(t) = x(s)$, then $\{t, s\} = \{\emptyset, 0\}$. Hypothesis (9) is vacuous. Thus the hypotheses are satisfied for $n = 1$.

By hypothesis (3), $\{X(t) : l(t) = n\}$ is a partition of X. Let

$$a_n = \min(X \setminus \{x(t) : l(t) \leq n\})$$

and pick the unique t_n with $l(t) = n$ such that $a_n \in X(t_n)$. Then hypotheses (i) and (iv) are satisfied. (We have to wait until $X(t_n{}^\frown 1)$ and $x(t_n{}^\frown 1)$ have been defined to verify (ii) and (iii).)

By hypothesis (6), $X(t_n) = x(t'_n)X(t^*_n)$ so pick $c_n \in X(t^*_n)$ such that $a_n = x(t'_n)c_n$.

For $s \in \{s^n_i : i \in \{0, 1, \ldots, n\}\}$, choose $b_s \in X(s)$ such that $b_s \neq x(s)$. (Recall that X has no isolated points so $X(s)$ is not finite.) If $s = t^*_n$, let $b_s = c_n$. (By hypothesis (5), $x(t_n) = x(t'_n)x(t^*_n) = x(t'_n)x(s)$ and $a_n = x(t'_n)c_n$. Since $a_n \neq x(t_n)$, $c_n \neq x(s)$.) Define $x(s^\frown 0) = x(s)$ and $x(s^\frown 1) = b_s$.

Now assume that $t \in F$, $l(t) = n$, and $t \notin \{s^n_i : i \in \{0, 1, \ldots, n\}\}$. Then $t^* \in \{s^n_i : i \in \{0, 1, \ldots, n\}\}$ so $x(t^{*\frown} 1)$ has been defined. We define $x(t^\frown 0) = x(t)$ and $x(t^\frown 1) = x(t')x(t^{*\frown} 1)$. Note that if $s \in \{s^n_i : i \in \{0, 1, \ldots, n\}\}$, then $s' = \emptyset$ and $s^* = s$ so $x(s^\frown 1) = ex(s^{*\frown} 1) = x(s')x(s^{*\frown} 1)$.

We choose $U_n \in \mathcal{U}$ such that

(a) $U_n \cap X \subseteq W_n$,

(b) $U_n \cap X \subseteq V\big(x(v)\big)$ for all $v \in F$ with $l(v) \leq n$,

(c) $x(v)U_n \cap X \subseteq x(v)X$ and $x(v)U_n \cap X$ is clopen in X for all $v \in F$ with $l(v) \leq n$,

(d) $a_n \notin x(t_n)U_n$,

(e) if $t \in F$ and $l(t) = n$, then $x(t)U_n \cap X \subseteq X(t)$, and

(f) if $t \in F$ and $l(t) = n$, then $x(t^\frown 1) \notin x(t)U_n$.

It is enough to show that for each of the items, a member of $\mathcal{U}$ can be found satisfying that item. It is trivially possible to satisfy (a) and (b). For (c), let $v \in F$ with $l(v) \leq n$. Pick $m \in \mathbb{N}$ such that $\{x(v)T_n : n \geq m\}$ is a neighborhood base for $x(v)$ in G consisting of sets clopen in G. Also $x(v)X \cap X$ is a neighborhood of $x(v)$ in X. So we may pick $k \geq m$ such that $x(v)T_k \cap X \subseteq x(v)X \cap X$. For (d) we use the fact that the topology is Hausdorff and $a_n \notin \{x(t) : l(t) \leq n\}$. For (e), given $t \in F$ with $l(t) = n$ we have that $x(t) \in X(t)$ which is clopen in X and for sufficiently large k, $x(t)T_k$ is clopen in G. To see that we can satisfy (f), assume $t \in F$ with

$l(t) = n$. By hypothesis (5) $x(t) = x(t')x(t^*)$, $x(t^*{}^\frown 1) = b_{t^*} \neq x(t^*)$, and $x(t^\frown 1) = x(t')x(t^*{}^\frown 1)$, so $x(t^\frown 1) \neq x(t)$.

Let $X(t^\frown 0) = x(t)U_n \cap X$ and $X(t^\frown 1) = X(t) \setminus X(t^\frown 0)$. This completes the inductive definitions for n.

We claim that for any $v \in F$ with $l(v) \leq n$, $x(v)U_n \cap X = x(v)(U_n \cap X)$, so let such v be given. By (c), $x(v)U_n \cap X \subseteq x(v)X$ so

$$x(v)U_n \cap X \subseteq x(v)U_n \cap x(v)X = x(v)(U_n \cap X)\,.$$

Also by (b), $U_n \cap X \subseteq V\big(x(v)\big)$ and by the hypotheses of the theorem, $x(v)V\big(x(v)\big) \subseteq X$, so $x(v)(U_n \cap X) \subseteq x(v)V\big(x(v)\big) \subseteq X$ so $x(v)(U_n \cap X) \subseteq x(v)U_n \cap X$.

We need to verify that hypotheses (ii), (iii), and (1) through (9) are satisfied. For (ii), using (d) we have that $a_n \notin x(t_n)U_n$ so $a_n \notin X(t_n{}^\frown 0)$. Also, $a_n \in X(t_n)$, so $a_n \in X(t_n) \setminus X(t_n{}^\frown 0) = X(t_n{}^\frown 1)$.

For (iii) we have $a_n = x(t'_n)c_n = x(t'_n)b_{t^*_n} = x(t'_n)x(t^*_n{}^\frown 1)$. If $t_n \notin \big\{s^n_i : i \in \{0, 1, \ldots, n\}\big\}$, then we defined $x(t_n{}^\frown 1) = x(t'_n)x(t^*_n{}^\frown 1)$ so $a_n = x(t_n{}^\frown 1)$. If $t_n \in \big\{s^n_i : i \in \{0, 1, \ldots, n\}\big\}$, then $t'_n = \emptyset$ and $t^*_n = t_n$ so $x(t_n{}^\frown 1) = ex(t^*_n{}^\frown 1) = x(t'_n)x(t^*_n{}^\frown 1)$ and thus $a_n = x(t_n{}^\frown 1)$.

To verify hypotheses (1) through (7), let $t \in F$ with $l(t) = n$. Since $x(t)U_n \cap X$ is clopen in X, hypothesis (1) holds immediately for $t^\frown 0$ and $t^\frown 1$. For hypothesis (2), $x(t^\frown 0) = x(t) \in x(t)U_n \cap X = X(t^\frown 0)$. Also, $x(t^*{}^\frown 1) = b_{t^*} \in X(t^*)$ so $x(t^\frown 1) = x(t')x(t^*{}^\frown 1) \in x(t')X(t^*) = X(t)$ by hypothesis (6) and $x(t^\frown 1) \notin x(t)U_n$ by (f) so $x(t^\frown 1) \in X(t) \setminus x(t)U_n \subseteq X(t) \setminus X(t^\frown 0) = X(t^\frown 1)$.

Since $x(t)U_n \cap X \subseteq X(t)$, (3) holds immediately as does (4).

Now let $v = t^\frown 1$ and $w = t^\frown 0$. We show that hypotheses (5) through (7) hold for v and w. There are four possibilities for t. That is $t = s^n_n$, $t = s^n_i$ for some $i < n$, $t = s^{m_0}_{i_0} + s^{m_1}_{i_1} + \ldots + s^{m_k}_{m_k}$, or $t = s^{m_0}_{i_0} + s^{m_1}_{i_1} + \ldots + s^{m_k}_{i_k}$ for some i_k with $m_{k-1} < i_k < m_k$. By separately considering these four cases, we determine that

- $v' = t'$,
- $v^* = t^*{}^\frown 1$,
- $t = w'$ or $t = w'$ followed by one or more 0's (so $x(t) = x(w')$),
- $x(t) = x(w')$, and
- $w^* = u_{n+1} = s^{n+1}_{n+1}$.

Now $x(v) = x(t')x(t^{*\frown}1) = x(v')x(v^*)$ and $x(w) = x(t) = x(t)e = x(w')x(w^*)$ so (5) holds.

$X(u_{n+1}) = X(u_n{}^\frown 0) = x(u_n)U_n \cap X = U_n \cap X$ so $X(w) = x(t)U_n \cap X = x(t)(U_n \cap X) = x(w')X(u_{n+1}) = x(w')X(w^*)$ so (6) holds for w. To verify (6) for v, we use the fact that $l(t^*) = n$.

$$\begin{aligned} X(v) &= X(t) \setminus X(t^\frown 0) \\ &= X(t) \setminus (x(t)U_n \cap X) \\ &= x(t')X(t^*) \setminus \big(x(t')x(t^*)(U_n \cap X)\big) \\ &= x(t')\big(X(t^*) \setminus x(t^*)(U_n \cap X)\big) \\ &= x(t')\big(X(t^*) \setminus (x(t^*)U_n \cap X)\big) \\ &= x(t')X(t^{*\frown}1) \\ &= x(v')X(v^*). \end{aligned}$$

To verify (7), $X(v^*) = X(t^{*\frown}1) \subseteq X(t^*) \subseteq V\big(x(t')\big) = V\big(x(v')\big)$ and $X(w^*) = X(u_{n+1}) = x(u_n)U_n \cap X = U_n \cap X \subseteq V\big(x(t)\big) = V\big(x(w')\big)$.

Hypothesis (8) follows immediately from the fact that for any t, $x(t^\frown 0) \in X(t^\frown 0)$, $x(t^\frown 1) \in X(t^\frown 1)$, and $X(t^\frown 0) \cap X(t^\frown 1) = \emptyset$.

For hypothesis (9), we have $X(u_{n+1}) = X(u_n{}^\frown 0) = x(u_n)U_n \cap X = U_n \cap X \subseteq W_n$.

This completes the inductive construction. We still need to verify that conclusions (10) and (11) hold.

If $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_k}^{m_k}$, one easily shows by induction on k using (5) that $x(t) = x(s_{i_0}^{m_0})x(s_{i_1}^{m_1}) \cdots x(s_{i_k}^{m_k})$. If $t << r$, $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_k}^{m_k}$, and $r = s_{j_0}^{n_0} + s_{j_1}^{n_1} + \ldots + s_{j_l}^{n_l}$, then the canonical representation of $t + r$ is $s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_k}^{m_k} + s_{j_0}^{n_0} + s_{j_1}^{n_1} + \ldots + s_{j_l}^{n_l}$ unless $i_k = m_k$ in which case it is $s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_{k-1}}^{m_{k-1}} + s_{j_0}^{n_0} + s_{j_1}^{n_1} + \ldots + s_{j_l}^{n_l}$. Since $x(s_{m_k}^{m_k}) = e$ one has in either case that $x(t + r) = x(t)x(r)$ so (10) holds.

To verify (11), we note first that each $a \in X$ occurs eventually as a value of x. For otherwise, let $a = \min(X \setminus x[F])$. There are only finitely many precessors so $a = a_n$ for some n and then by (iii), $a_n = x(t_n{}^\frown 1)$.

Now for $n \in \omega$ and $t \in F$, $x(u_n t) \in X(u_n t) \subseteq X(u_n)$ so $x[u_n F] \subseteq X(u_n)$. Now suppose that $a \in X(u_n) \setminus x[u_n F]$. We have seen that $a = x(v)$ for some v so $v \in F \setminus u_n F$. Since $e = x(u_n) \in x[u_n F]$, $a \neq e$ so $v \notin \{u_m : m < \omega\}$ so $\mathrm{supp}_F(v) \neq \emptyset$. Let $k = \min \mathrm{supp}_F(v)$. Then $k \leq n$ since otherwise $v \in u_n F$. Then $x(v) \in X(s_{k-1}^k)$ and $X(u_n) \subseteq X(u_k)$. Since $X(s_{k-1}^k) \cap X(u_k) = \emptyset$, we have $a = x(v) \notin X(u_n)$, a contradiction. □

Theorem 5.3. *Assume that G is a cancellative semigroup with identity e and that X is a countably infinite subset of G with $e \in X$. Assume that G has a Hausdorff topology and there is a decreasing sequence $\langle T_n \rangle_{n=1}^\infty$ of subsets* *of G such that for each $a \in X$ there exists $m \in \mathbb{N}$ such that $\{aT_n : n \geq m\}$ is a neighborhood base for a consisting of sets clopen in G. Assume that X has no isolated points, for each $a \in X$, $aX \cap X$ is a neighborhood of a in X, and for each $a \in X$, we have a neighborhood $V(a)$ of e in X such that $aV(a) \subseteq X$ and $V(e) = X$. Assume that $\langle W_n \rangle_{n=1}^{\infty}$ is a neighborhood base for e in X and let $Y = (\bigcap_{n=1}^{\infty} c\ell_{\beta X_d} W_n) \setminus \{e\}$. Then Y is is topologically isomorphic to $\mathbb{H}$.*

Proof. Pick $x(t)$ and $X(t)$ for $t \in F$ satisfying conclusions (1) through (11) of Theorem 5.2. Given $n \in \mathbb{N}$, $X(u_n)$ is a neighborhood of $x(u_n) = e$ in X so there is $m \in \mathbb{N}$ such that $W_m \subseteq X(u_n)$. By conclusion (9), for each n, $X(u_{n+1}) \subseteq W_n$. Therefore $V = \bigcap_{n=1}^{\infty} \overline{X(u_n) \setminus \{e\}}$.

Define $\psi : X \to \omega$ by $\psi(e) = 0$ and for $t \in F \setminus \{e\}$,

$$\psi\big(x(t)\big) = \textstyle\sum_{i \in \mathrm{supp}_F(t)} 2^{i-1}\,.$$

By conclusion (8) we have that ψ is well defined and injective. it is trivially surjective. Let $\widetilde{\psi} : \beta X_d \to \beta\omega$ be the continuous extension of ψ.

Given $n \in \mathbb{N}$, $X(u_n) \setminus \{e\} = x[u_n F] \setminus \{e\}$ so $X(u_n) \setminus \{e\} = \{x(t) : t \in F\,,\ \mathrm{supp}_F(t) \neq \emptyset$, and $\min \mathrm{supp}_F(t) > n\}$. Consequently, $\psi[X(u_n) \setminus \{e\}] = 2^n\mathbb{N}$ so that $\widetilde{\psi}[Y] = \mathbb{H}$.

We show that Y is a semigroup using [4, Theorem 4.20] and that $\widetilde{\psi}$ is a homomorphism using [4, Theorem 4.21]. Since $\widetilde{\psi}$ is injective, this will suffice.

So let $n \in \mathbb{N}$ and let $v \in X(u_n) \setminus \{e\}$. Pick $t \in F$ such that $\mathrm{supp}_F(t) \neq \emptyset$, $\min \mathrm{supp}_F(t) > n$, and $x(t) = v$. Let $m = \max \mathrm{supp}_F(t) + 1$ and let $w \in X(u_m) \setminus \{e\}$. To complete the proof we will show that $vw \in X(u_n) \setminus \{e\}$ and that $\psi(vw) = \psi(v) + \psi(w)$. Pick $r \in F$ such that $\mathrm{supp}_F(r) \neq \emptyset$, $\min \mathrm{supp}_F(r) > m$, and $x(r) = w$. Then $vw = x(t)x(r) = x(t+r)$ by conclusion (10) and $\mathrm{supp}_F(t+r) = \mathrm{supp}_F(t) \cup \mathrm{supp}_F(r)$. Thus $x(t+r) \in X(u_n) \setminus \{e\}$ and $\psi(vw) = \sum_{i \in \mathrm{supp}_F(t+r)} 2^{i-1} = \sum_{i \in \mathrm{supp}_F(t)} 2^{i-1} + \sum_{i \in \mathrm{supp}_F(r)} 2^{i-1} = \psi(v) + \psi(w)$. □

Most of our results involve semigroups written additively, and some follow from the following corollary.

Corollary 5.4. *Let $(S, +)$ be a countable cancellative semigroup with identity 0. Assume that there exist a decreasing sequence $\langle T_n \rangle_{n=1}^{\infty}$ of subsets of S and $m : S \to \mathbb{N}$ such that $m(0) = 1$, $\{a + T_n : a \in S$ and $n \geq m(a)\}$ is a base for a Hausdorff topology on S consisting of clopen sets, and S has no isolated points. Assume also that for each $a \in S$, $\{a + T_n : n \geq m(a)\}$ is a neighborhood base for a. Then $O(S) = (\bigcap_{n=1}^{\infty} c\ell_{\beta S_d} T_n)$ and $O(S) \setminus \{0\}$ is topologically isomorphic to $\mathbb{H}$.*

Proof. It is immediate that $O(S) = (\bigcap_{n=1}^{\infty} c\ell_{\beta S_d} T_n)$. We apply Theorem 5.3 with $G = S = X$. Given $a \in X$, $a + T_{m(a)} \subseteq (a + X) \cap X$ so $(a + X) \cap X$ is a neighborhood of a in X. For each $a \in X$, let $V(a) = X$. $\square$

The proof of Theorem 3.4 using Theorem 5.3 is verbatim the same as its proof in Section 3. The proof of Theorem 3.5 using Theorem 5.3 is slightly shorter than its proof in Section 3, since one may let $G = \omega$ and dispense with the last paragraph of the proof.

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