Semigroup Forum
Springer-Verlag New York Inc.

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This paper was published in Semigroup Forum **59** (1999), 33-55. As published, there was a gap in the proof of Lemma 4.10, wherein it was implicitly assumed that if $x, y \in S$ and $x < y$, then $y - x \in S$. This version corrects that proof, as well as a few typos in Section 4. Also the hypotheses for Theorem 5.3 have been strengthened. Other than that, to the best of my knowledge, this is the final version as it was submitted to the publisher. -NH

The Semigroup of Ultrafilters Near 0

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Communicated by Karl H. Hofmann

Abstract. The set 0^+ of ultrafilters on $(0,1)$ that converge to 0 is a semigroup under the restriction of the usual operation $+$ on $\beta\mathbb{R}_d$, the Stone-Čech compactification of the discrete semigroup $(\mathbb{R}_d, +)$. It is also a sumbsemigroup of $\beta\big((0,1)_d, \cdot\big)$. The interaction of these operations has recently yielded some strong results in Ramsey Theory. Since $(0^+, \cdot)$ is an ideal of $\beta\big((0,1)_d, \cdot\big)$, much is known about the structure of $(0^+, \cdot)$. On the other hand, $(0^+, +)$ is far from being an ideal of $(\beta\mathbb{R}_d, +)$ so little about its algebraic structure follows from known results.

We characterize here the smallest ideal of $(0^+, +)$, its closure, and those sets "central" in $(0^+, +)$, that is, those sets which are members of minimal idempotents in $(0^+, +)$. We derive new combinatorial applications of those sets that are central in $(0^+, +)$.

1. Introduction.

Given a discrete semigroup $(S, \cdot)$, it is well known that one can extend the operation $\cdot$ to βS, the Stone-Čech compactification of S so that $(\beta S, \cdot)$ is a right topological semigroup (i.e. for each $p \in \beta S$, the function $\rho_p : \beta S \longrightarrow \beta S$, defined by $\rho_p(q) = q \cdot p$, is continuous) with S contained in the topological center (i.e. for each $x \in S$, the function $\lambda_x : \beta S \longrightarrow \beta S$, defined by $\lambda_x(p) = x \cdot p$, is continuous). Further, this operation has frequently proved to be useful in Ramsey Theory. See [10] for an elementary introduction to the semigroup $(\beta S, \cdot)$ and its combinatorial applications.

It is also well known that if S is not discrete, such an extension may not be possible. (See Section 2 of this paper where it is shown how bad the situation is for any dense subsemigroup of $([0,\infty], +)$.) Since known facts about compact right topological semigroups were utilized in the combinatorial applications, it

* This author acknowledges support received from the National Science Foundation (USA) via grant DMS 9424421.

seemed that the Stone-Čech compactification was not likely to be a useful tool in the study, say, of Ramsey Theory in the real interval $(0,1)$.

Surprisingly, however, it *has* turned out to be possible to use the algebraic structure of $\beta\mathbb{R}_d$ to obtain Ramsey Theoretic results that are stated in terms of the usual topology on $\mathbb{R}$. (Given a topological space X, the notation X_d represents the set X with the discrete topology.) For example, it was shown in [2], as a corollary to a much stronger result, that if $\mathcal{F}$ is a finite partition of $(0,1)$ and if either every member of $\mathcal{F}$ is Lebesgue measurable or every member of $\mathcal{F}$ is a Baire set (i.e. a member of the smallest σ-algebra containing the open sets and the nowhere dense sets), then there exist some $A \in \mathcal{F}$ and some sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $\Sigma_{n\in F}\, x_n \in A$ and $\Pi_{n\in F}\, x_n \in A$ whenever F is a finite nonempty subset of $\mathbb{N}$.

Specifically, consider the semigroup $\big((0,1),\cdot\big)$ and let

$$0^+ = \bigcap\nolimits_{\epsilon>0} c\ell_{\beta(0,1)_d}\ (0,\epsilon)\ .$$

Then 0^+ is a two sided ideal of $(\beta(0,1)_d,\cdot)$, so contains the smallest ideal of $(\beta(0,1)_d,\cdot)$. (See [3] for basic information about the smallest ideal of a compact right topological semigroup.) This simple algebraic fact then yielded the results mentioned above.

Many things are known about the smallest ideal of $(\beta S,\cdot)$ where $(S,\cdot)$ is any discrete semigroup, and these facts automatically apply to the smallest ideal of $(0^+,\cdot)$, since that is the same as the smallest ideal of $(\beta(0,1)_d,\cdot)$. For example [8, Corollary 4.6] the closure of the smallest ideal of $(0^+,\cdot)$ is itself an ideal of $(0^+,\cdot)$.

It turns out that 0^+ is also a subsemigroup of $(\beta\mathbb{R}_d,+)$. However, it is far from being an ideal of $(\beta\mathbb{R}_d,+)$, so a description of the smallest ideal of $(0^+,+)$ does not follow from known results about arbitrary discrete semigroups. In Section 3 we characterize the members of the smallest ideal of $(0^+,+)$ and its closure. We also describe those subsets of $\mathbb{R}$ that have idempotents in $(0^+,+)$ in their closure.

Especially important in the combinatorial applications have been the "central" sets, i.e. those sets with idempotents in the intersection of their closure with the smallest ideal. In Section 4 we describe sets that are "central near 0", and in Section 5 we derive new combinatorial results from their existence.

For most of our results, we don't actually need to work with all of $(\mathbb{R},+)$, so we derive these results for an arbitrary subsemigroup of $(\mathbb{R},+)$ which is dense in $(0,\infty)$, such as the positive rationals.

We take the points of βS (where S is discrete) to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. If $(S,+)$ is a discrete semigroup, $p,q \in \beta S$ and $A \subseteq S$, then $A \in p+q$ if and only if $\{x \in S : -x+A \in q\} \in p$, where $-x+A = \{y \in S : x+y \in A\}$. We again refer the reader to [10] for an elementary introduction, with the caution that $(\beta S,+)$ is taken there to be left rather than right topological.

Given a set A, we write $\mathcal{P}_f(A)$ for the set of finite nonempty subsets of A.

2. Stone-Čech Compactifications of Subsemigroups of $(\mathbb{R},+)$.

The plural "compactifications" in the section title reflects the fact that we will deal here with both βS and βS_d, where S is a dense subsemigroup of $\big([0,\infty),+\big)$. We show first why one cannot do algebra nicely on βS. Then we establish some elementary facts about the algebra of βS_d.

We have mentioned that we take the points of βS_d to be the ultrafilters on S. We will not be doing enough with βS to care what particular construction one takes. We merely assume that S is a dense subspace of the compact Hausdorff space βS and that given any continuous $f : S \longrightarrow X$, there is a continuous extension f^β of f taking βS to X.

As we have remarked, in the case of βS_d one gets an operation $+$ such that ρ_p is continuous for each $p \in \beta S_d$ and λ_x is continuous for each $x \in S$, where $\rho_p(q) = q + p$ and $\lambda_x(q) = x + q$.

We see below that if S is any dense subsemigroup of $\big([0,\infty),+\big)$, one cannot do this. The point of requiring that λ_x be continuous for each $x \in S$ is that it provides a connection between the operation of βS and that of S which is needed for the combinatorial applications. We will return to further discussion of this point after Theorem 2.1.

Note that in the following theorem we do not assume that the operation on βS is associative.

Theorem 2.1. *Let S be a dense subsemigroup of $\big([0,\infty),+\big)$ and denote also by $+$ an extension to βS such that λ_x is continuous for each $x \in S$. There is a point p of βS such that ρ_p is discontinuous at each point of S.*

Proof. Choose sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ in S such that $x_1 < 1$ and for each $n \in \mathbb{N}$, $x_{n+1} \leq \frac{1}{2}x_n$ and $y_{n+1} \geq y_n + 1$. let

$$A = \textstyle\bigcup_{m=1}^\infty \bigcup_{n=1}^m \big([y_m + x_{4n+1}, y_m + x_{4n}] \cap S\big)$$

and

$$B = \textstyle\bigcup_{m=1}^\infty \bigcup_{n=1}^m \big([y_m + x_{4n+3}, y_m + x_{4n+2}] \cap S\big).$$

Note that, since the listed endpoints are all in S, one has $z+A$ and $z+B$ are disjoint closed subsets of S for any $z \in S$, and consequently $c\ell_{\beta S}(z+A) \cap c\ell_{\beta S}(z+B) = \emptyset$. (See [7] for this and other basic facts about the Stone-Čech compactification.)

Now we claim that $c\ell_{\beta S}\ B \cap \bigcap_{n=1}^\infty c\ell_{\beta S}(A - x_{4n+1}) \neq \emptyset$, for which it suffices to note that given $k \in \mathbb{N}$,

$$[y_k + x_{4k+3}, y_k + x_{4k+2}] \cap S \subseteq B \cap \textstyle\bigcap_{n=1}^k (A - x_{4n+1}).$$

Pick $p \in c\ell_{\beta S}\ B \cap \bigcap_{n=1}^\infty c\ell_{\beta S}(A - x_{4n+1})$.

Now let $z \in S$. We show that ρ_p is not continuous at z. We claim first that $z + p \in c\ell_{\beta S}(z+B)$. To see this let U be an arbitrary neighborhood of $z+p$ and (by the continuity of λ_z) pick a neighborhood W of p such that $\lambda_z[W] \subseteq U$. Pick $w \in W \cap B$. Then $z + w \in U \cap (z+B)$. Since, as already observed, $c\ell_{\beta S}(z+A) \cap c\ell_{\beta S}(z+B) = \emptyset$, we have $z + p = \rho_p(z) \notin c\ell_{\beta S}(z+A)$.

Suppose ρ_p is continuous at $z \in S$ and pick a neighborhood V of z such that $\rho_p[V] \cap c\ell_{\beta S}(z+A) = \emptyset$. Pick $\epsilon > 0$ such that $S \cap (z, z+\epsilon) \subseteq V$, and pick $n \in \mathbb{N}$ such that $x_{4n+1} < \epsilon$. Then $z + x_{4n+1} \in V$ so $(z+x_{4n+1})+p \notin c\ell_{\beta S}(z+A)$ so, by the continuity of $\lambda_{z+x_{4n+1}}$, pick a neighborhood U of p such that $\big((z + x_{4n+1}) + U\big) \cap (z+A) = \emptyset$. (We include the parentheses on $z + x_{4n+1}$ because we are not assuming that $+$ is associative on βS.) Now $p \in c\ell_{\beta S}(A - x_{4n+1})$ so pick $y \in U \cap (A - x_{4n+1})$. Then $z + x_{4n+1} + y \in \big((z + x_{4n+1}) + U\big) \cap (z+A)$, a contradiction. ■

Let us suppose that we decide not to worry about the requirement that λ_x be continuous. If one can show that one can extend the operation $+$ to βS so that $(\beta S,+)$ is a compact right topological semigroup, then one still has the structure theorems of compact right topological semigroups to work with. In fact, if $S=[0,\infty)$, it follows from [1, Theorem 6] that this can be done.

The extension produced in [1, Theorem 6], however, has the property that if $p\in\beta S\backslash S$, then for all $q\in\beta S$, $q+p=p$, so the structure theorems become trivial. Now if $S=[0,\infty)$, then all points of $\beta S\backslash S$ "live at infinity". We will be dealing in this paper mostly with points "living at zero" on semigroups S dense in $(0,\infty)$.

We see in the following theorem that in this case *any* extension making $(\beta S,+)$ a right topological semigroup must have a trivial operation for all points living at 0.

Theorem 2.2. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ and assume that one has an extension of $+$ to βS so that βS is a right topological semigroup. If $p\in\bigcap_{n=1}^\infty c\ell_{\beta S}\big((0,1/n)\cap S\big)$ and $q\in\beta S$, then $q+p=q$.*

Proof. Since ρ_p is continuous for any $p\in\beta S$, it suffices to show that for all $p\in\bigcap_{n=1}^\infty c\ell_{\beta S}\big((0,1/n)\cap S\big)$ and all $x\in S$, $x+p=x$.

Consider the identity $\iota:S\longrightarrow S\subseteq[0,\infty]$, and let $\alpha:\beta S\longrightarrow[0,\infty]$ be the continuous extension of ι. Given $p\in\beta S$ and $x\in S$ with $p\neq x$, we claim that $\alpha(p)\neq x$. To see this, suppose instead that $\alpha(p)=x$ and consider a continuous $f:\beta S\longrightarrow[0,1]$ such that $f(p)=1$ and $f(x)=0$. Pick an $\epsilon>0$ such that $S\cap(x-\epsilon,x+\epsilon)\subseteq f^{-1}[0,1/3)$. Then $(x-\epsilon,x+\epsilon)$ is a neighborhood of $\alpha(p)$ so pick a neighborhood V of p, such that $\alpha[V]\subseteq(x-\epsilon,x+\epsilon)$. Picking $y\in V\cap f^{-1}(2/3,1]\cap S$, one has $y=\alpha(y)\in(x-\epsilon,x+\epsilon)$, a contradiction.

Next observe that if $x\in S$ and $p\in\beta S$, then $\alpha(p+x)=\alpha(p)+x$, where the addition on the right hand side is the usual addition in the semigroup $[0,\infty]$. To see this, note that $\alpha\circ\rho_x$ and $\rho_x\circ\alpha$ are continuous functions agreeing on the dense subset S of βS.

As a final preliminary, observe that if $\alpha(p)=0$ and $x\in S$, then $p+x=x$. Indeed, we have $\alpha(p+x)=\alpha(p)+x=x$, so $p+x=x$.

To complete the proof, let $p\in\bigcap_{n=1}^\infty c\ell_{\beta S}\big((0,1/n)\cap S\big)$, and note that $\alpha(p)=0$. To show that $q+p=q$ for all $q\in\beta S$, it suffices to show that ρ_p is equal to the identity on S. So let $x\in S$ be given and pick any $y\in S$. Now $p+y=y$ so $(x+p)+y=x+(p+y)=x+y$. Thus $x+y=\alpha(x+y)=\alpha\big((x+p)+y\big)=\alpha(x+p)+y$. Now right cancellation holds at all points of $[0,\infty]$ except ∞, so $x=\alpha(x+p)$ so $x=x+p$ as required. ∎

We do not know in general whether one can make βS into a right topological semigroup at all. As we remarked earlier, one can if $S=[0,\infty)$. We pause now to observe that one also can if $S=(0,\infty)$.

Theorem 2.3. *Let $S=(0,\infty)$. There is an extension of $+$ to βS such that $(\beta S,+)$ is a right topological semigroup.*

Proof. Given $x\in S$ define $r_x:S\longrightarrow S\subseteq\beta S$ by $r_x(y)=y+x$ and let ${r_x}^\beta$ be its continuous extension to βS. As in the proof of Theorem 2.2, let $\alpha:\beta S\longrightarrow[0,\infty]$ be the continuous extension of the identity function. For $p,q\in\beta S$ define

$$p+q=\begin{cases} q & \text{if } \alpha(q)=\infty\\ {r_q}^\beta(p) & \text{if } q\in S\\ p & \text{if } \alpha(q)=0\ .\end{cases}$$

Then trivially $+$ extends addition on S and ρ_p is continuous for each $q \in \beta S$.

To check associativity, let $p, q, t \in \beta S$. If $\alpha(t) = \infty$, then $p + (q + t) = p + t = t = (p + q) + t$. If $\alpha(t) = 0$, then $p + (q + t) = p + q = (p + q) + t$. So assume $t \in S$. If $\alpha(q) = \infty$, we observe that $\alpha(q + t) = \infty$ so that $(p + q) + t = q + t = p + (q + t)$. If $\alpha(q) = 0$, we saw in the proof of Theorem 2.2 that $q + t = t$. Thus $(p + q) + t = p + t = p + (q + t)$. Finally assume $q \in S$. Then ${r_t}^\beta \circ {r_q}^\beta$ and ${r_{q+t}}^\beta$ are continuous functions agreeing on S, hence on βS so ${r_t}^\beta \circ {r_q}^\beta(p) = {r_{q+t}}^\beta(p)$. That is, $(p + q) + t = p + (q + t)$. ■

Since we are interested in algebra near 0, Theorem 2.2 completely eliminates any hope of using βS. Also, we know that ultrafilters are naturals for Ramsey Theory type combinatorial applications as was again demonstrated in [2]. Consequently, we will concentrate on βS_d, the Stone-Čech compactification of the set S with the discrete topology.

One immediately sees a difference in where the elements of $\beta S_d \backslash S$ live. For example, we saw that if $S = (0, \infty)$, then all elements of $\beta S \backslash S$ either live at 0 or at ∞. On the other hand we see that there are many elements of βS_d residing at each point of $[0, \infty]$.

In the following definition, we supress the dependence of α and the sets x^+ and x^- on the choice of S.

Definition 2.4. Let S be a subsemigroup of $(\mathbb{R}, +)$.

(a) Let $\alpha : \beta S_d \longrightarrow [-\infty, \infty]$ be the continuous extension of the identity function.

(b) $B(S) = \big\{p \in \beta S_d : \alpha(s) \notin \{-\infty, \infty\}\big\}$.

(c) Given $x \in \mathbb{R}$,

$$x^+ = \{p \in B(S) : \alpha(p) = x \text{ and } (x, \infty) \cap S \in p\} \text{ and}$$
$$x^- = \{p \in B(S) : \alpha(p) = x \text{ and } (-\infty, x) \cap S \in p\}.$$

(d) $U = \bigcup_{x \in \mathbb{R}} x^+$ and $D = \bigcup_{x \in \mathbb{R}} x^-$.

The set $B(S)$ is the set of "bounded" ultrafilters on S. That is, an ultrafilter $p \in \beta S_d$ is in $B(S)$ if and only if there is some $n \in \mathbb{N}$ with $[-n, n] \cap S \in p$. We collect some routine information about the notions defined above.

Lemma 2.5. *Let S be a semigroup of $(\mathbb{R}, +)$.*

(a) Let $x \in \mathbb{R}$ and let $p \in \beta S_d$. Then $p \in x^+$ if and only if for every $\epsilon > 0$, $(x, x + \epsilon) \cap S \in p$. Also, $p \in x^-$ if and only if for every $\epsilon > 0$, $(x - \epsilon, x) \cap S \in p$.

(b) Let $x \in \mathbb{R}$. Then $x^+ \neq \emptyset$ if and only if $x \in c\ell_{\mathbb{R}}\big((x, \infty) \cap S\big)$ and $x^- \neq \emptyset$ if and only if $x \in c\ell_{\mathbb{R}}\big((-\infty, x) \cap S\big)$.

(c) Let $p, q \in B(S)$, let $x = \alpha(p)$, and let $y = \alpha(q)$. If $p \in x^+$, then $p + q \in (x + y)^+$. If $p \in x^-$, then $p + q \in (x + y)^-$.

(d) $B(S) \backslash S = U \cup D$. If U and D are nonempty, they are disjoint right ideals of $(B(S), +)$. In particular, $B(S)$ is not commutative.

(e) If $0^+ \neq \emptyset$, then 0^+ is a compact subsemigroup of $(B(S), +)$.

(f) If $x \in S$ and $p \in \beta S_d$, then $x + p = p + x$

Proof. The proofs of (a) and (b) are routine exercises and (d) and (e) follow from (c). We establish (c) and (f).

To verify (c) assume first that $p \in x^+$. To see that $p + q \in (x + y)^+$, let $\epsilon > 0$ be given and let $A = (x + y, x + y + \epsilon) \cap S$. To see that $A \in p + q$, we show that $(x, x + \epsilon) \cap S \subseteq \{z \in S : -z + A \in q\}$. So let $z \in (x, x + \epsilon) \cap S$. Let $\delta = \min\{z - x, x + \epsilon - z\}$. Since $\alpha(q) = y$, we have $(y - \delta, y + \delta) \cap S \in q$

and $(y-\delta,y+\delta)\cap S\subseteq -z+A$ so $-z+A\in q$ as required. The proof that $p+q\in(x+y)^-$ if $p\in x^-$ is nearly identical.

To see that (f) holds, note that λ_x and ρ_x are continuous functions agreeing on S, hence on βS_d. ■

Why do we restrict our attention to 0^+? On the one hand, it is a subsemigroup of $(\beta\mathbb{R}_d,+)$ and for any other $x\in\mathbb{R}$, x^+ and x^- are not semigroups. On the other hand we see in the following theorem that 0^+ holds all of the algebraic structure of $B(\mathbb{R})$ not already revealed by $\mathbb{R}$. (We restrict our attention to $\mathbb{R}$ here because we do not know what happens with other subsemigroups of $\mathbb{R}$, not even in the case of $\mathbb{Q}$.)

Theorem 2.6. *Let $S=\mathbb{R}$.*

(a) 0^+ and 0^- are isomorphic.

(b) The function $\varphi:\mathbb{R}_d\times(\{0\}\cup 0^+\cup 0^-)\longrightarrow B(\mathbb{R})$ defined by $\varphi(x,p)=x+p$ is a continuous isomorphism onto $B(\mathbb{R})$.

Proof. (a). Define $\tau:0^+\longrightarrow 0^-$ by $\tau(p)=-p$, where $-p=\{-A:A\in p\}$. It is routine to verify that τ takes 0^+ one-to-one onto 0^-. Let $p,q\in 0^+$. To see that $\tau(p+q)=\tau(p)+\tau(q)$ it suffices, since $\tau(p+q)$ and $\tau(p)+\tau(q)$ are both ultrafilters, to show that $\tau(p+q)\subseteq\tau(p)+\tau(q)$. So let $A\in\tau(p+q)$. Then $-A\in p+q$ so $B=\{x\in\mathbb{R}:-x+-A\in q\}\in p$ and hence $-B\in\tau(p)$. Then $-B\subseteq\{x\in\mathbb{R}:-x+A\in\tau(q)\}$ so $A\in\tau(p)+\tau(q)$ as required.

(b) To see that φ is a homomorphism, let (x,p) and (y,q) be in $\mathbb{R}_d\times(\{0\}\cup 0^+\cup 0^-)$. Since $p+y=y+p$, we have $\varphi(x,p)+\varphi(y,q)=\varphi(x+y,p+q)$.

To see that φ is one-to-one, assume we have $\varphi(x,p)=\varphi(y,q)$. By Lemma 2.5(c) we have $x=\alpha(x+p)=\alpha(y+q)=y$. Then $x+p=x+q$ so $p=-x+x+p=-x+x+q=q$.

To see that φ is onto $B(\mathbb{R})$, let $q\in B(\mathbb{R})$ and let $x=\alpha(q)$. Let $p=-x+q$. Then $q=x+p=\varphi(x,p)$.

To see that φ is continuous, let $(x,p)\in\mathbb{R}_d\times(\{0\}\cup 0^+\cup 0^-)$ and let $A\in x+p$. Then $-x+A\in p$ so $\{x\}\times\big(c\ell_{\beta S_d}(-x+A)\big)$ is a neighborhood of (x,p) contained in $\varphi^{-1}[c\ell\beta S_d A]$. ■

At first glance it might seem that if say $S=\mathbb{Q}$ one ought to have $\varphi(x,p)=x+p$ again as an isomorphism from $\mathbb{R}_d\times(\{0\}\cup 0^+\cup 0^-)$ to $B(S)$. However, if $x\notin S$ and $p\in\{0\}\cup 0^+\cup 0^-$, $x+p$ is not defined (in βS_d). It is true that $\varphi(x,p)=x+p$ does define a one-to-one homomorphism from $S_d\times(\{0\}\cup 0^+\cup 0^-)$ to $B(S)$, but it is not onto because $\alpha[B(S)]=\mathbb{R}$.

3. Idempotents and the Smallest Ideal of 0^+.

For reasons discussed at the end of Section 2, as well as the fact that 0^+ has yielded useful combinatorial results, we are interested in studying the algebra of 0^+ (defined in terms of some semigroup S of $(\mathbb{R},+)$). Since $\{x\in\mathbb{R}:x\le 0\}$ does not contribute to 0^+ at all, we will assume that $S\subseteq(0,\infty)$. Since we want $0^+\neq\emptyset$ we need $0\in c\ell_R S$. Once we have that, we know that S is dense in $(0,\infty)$, so that will be our standing assumption about S.

As we remarked in the introduction, 0^+ has an interesting and useful multiplicative structure. But much is known of this structure because 0^+ is a two sided ideal of $(\beta(0,1)_d,\cdot)$, so results about the algebraic structure of the Stone-Čech compactification of an arbitrary discrete semigroup apply.

On the other hand, 0^+ is far from being an ideal of $B(S)$. In fact, as the referee kindly pointed out, 0^+ is a prime subsemigroup of βS, because it is the inverse image of zero under the homomorphism $\beta S \longrightarrow ([0,\infty],+)$. Consequently, no general results apply to $(0^+,+)$ beyond those that apply to any compact right topological semigroup.

Among the consequences of the fact that $(0^+,+)$ is a compact right topological semigroup is the fact that it must contain idempotents [4, Corollary 2.10]. If S is an arbitrary discrete semigroup, it is a result of Galvin's (see [9, Theorem 2.5]) that a set $A \subseteq S$ is a member of some idempotent in βS if and only if there is some sequence $\langle x_n\rangle_{n=1}^\infty$ in S with $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq A$, where $FS(\langle x_n\rangle_{n=1}^\infty) = \{\Sigma_{n\in F}\, x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$. We have a similar characterization of members of idempotents in 0^+.

Theorem 3.1. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ and let $A \subseteq S$. There exists $p = p+p$ in 0^+ with $A \in p$ if and only if there is some sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that $\Sigma_{n=1}^\infty\, x_n$ converges and $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq A$.*

Proof. Necessity. Let $A_1 = A$ and let $B_1 = \{x \in S : -x + A_1 \in p\}$. Then $B_1 \in p$ so pick $x_1 \in B_1 \cap A_1$ and let $A_2 = A_1 \cap (-x_1 + A_1) \cap (0,1/2)$. Inductively, given $A_n \in p$, let $B_n = \{x \in S : -x + A_n \in p\}$ and pick $x_n \in A_n \cap B_n$. Let $A_{n+1} = A_n \cap (-x_n + A_n) \cap (0, \frac{1}{2^n})$. Then one easily sees that $\langle x_n\rangle_{n=1}^\infty$ is as required. (To see for example that $x_2 + x_4 + x_5 \in A$, one has $x_5 \in A_5 \subseteq -x_4 + A_4$ so $x_4 + x_5 \in A_4 \subseteq A_3 \subseteq -x_2 + A_2$ so $x_2 + x_4 + x_5 \subseteq A_2 \subseteq A_1 = A$.)

Sufficiency. Let $T = \bigcap_{m=1}^\infty c\ell_{\beta S_d} FS(\langle x_n\rangle_{n=m}^\infty)$. It suffices to show that T is a subsemigroup of 0^+, for then as a compact right topological semigroup, T must contain an idempotent.

Observe that since $\Sigma_{n=1}^\infty\, x_n$ converges, for each $\epsilon > 0$ there is some $m \in \mathbb{N}$ with $FS(\langle x_n\rangle_{n=m}^\infty) \subseteq (0,\epsilon) \cap S$. Consequently $T \subseteq 0^+$. To see that T is a subsemigroup of 0^+, let $p, q \in T$ and let $m \in \mathbb{N}$ be given. To see that $FS(\langle x_n\rangle_{n=m}^\infty) \in p + q$, we show that $FS(\langle x_n\rangle_{n=m}^\infty) \subseteq \{x \in S : -x + FS(\langle x_n\rangle_{n=m}^\infty) \in q\}$. Given $y \in FS(\langle x_n\rangle_{n=m}^\infty)$, pick F with $y = \Sigma_{n\in F}\, x_n$ and $\min F \geq m$. Let $r = \max F + 1$. Then $FS(\langle x_n\rangle_{n=r}^\infty) \subseteq -y + FS(\langle x_n\rangle_{n=m}^\infty)$ so $-y + FS(\langle x_n\rangle_{n=m}^\infty) \in q$. ■

As a compact right topological semigroup, 0^+ has a smallest two sided ideal [3, Theorem 1.3.11]. We now turn our attention to characterizing the smallest ideal of 0^+ and its closure. Deducing the parallels with known theory becomes progressively less straight forward as we proceed. If $(S,+)$ is a discrete semigroup we know from [8, Corollary 3.6] that any $p \in \beta S$ is in the smallest ideal of βS if and only if , for each $A \in p$, $\{x \in S : -x + A \in p\}$ is "syndetic". (A subset B of S is *syndetic* if and only if there is a finite nonempty subset F of S such that $S \subseteq \bigcup_{t\in F} -t + B$. The terminology comes from topological dynamics.)

Definition 3.2. Let S be a dense subsemigroup of $\big((0,\infty),+\big)$.

(a) K is the smallest ideal of $(0^+,+)$.

(b) A subset B of S is *syndetic near 0* if and only if for every $\epsilon > 0$ there exist some $F \in \mathcal{P}_f\big((0,\epsilon) \cap S\big)$ and some $\delta > 0$ such that $S \cap (0,\delta) \subseteq \bigcup_{t\in F}(-t + B)$.

Theorem 3.3. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ and let $p \in 0^+$. The following statements are equivalent.*

(a) $p \in K$.

(b) For all $A \in p$, $\{x \in S : -x + A \in p\}$ is syndetic near 0.

(c) For all $r \in 0^+$, $p \in 0^+ + r + p$.

Proof. (a) implies (b). Let $A \in p$, let $B = \{x \in S : -x+A \in p\}$, and suppose that B is not syndetic near 0. Pick $\epsilon > 0$ such that for all $F \in \mathcal{P}_f\big((0,\epsilon)\cap S\big)$ and all $\delta > 0$, $\big(S \cap (0,\delta)\big)\setminus\bigcup_{t\in F}(-t+B) \neq \emptyset$.

Let $\mathcal{G} = \{\big(S\cap(0,\delta)\big)\setminus\bigcup_{t\in F}(-t+B) : F \in \mathcal{P}_f\big((0,\epsilon)\cap S\big)$ and $\delta > 0\}$. Then $\mathcal{G}$ has the finite intersection property so pick $r \in \beta S_d$ with $\mathcal{G} \subseteq r$. Since $\{S \cap (0,\delta) : \delta > 0\} \subseteq r$ we have $r \in 0^+$.

Pick a minimal left ideal L of 0^+ with $L \subseteq 0^+ + r$, by [3, Proposition 1.2.4]. Since K is the union of all of the minimal right ideals of 0^+ [3, Theorem 1.3.11], pick a minimal right ideal R of 0^+ with $p \in R$. Then $L \cap R$ is a group [3, Theorem 1.3.11] so let q be the identity of $L \cap R$. Then $R = q + 0^+$, so $p \in q + 0^+$ so $p = q + p$ so $B \in q$. Also $q \in 0^+ + r$ so pick $w \in 0^+$ such that $q = w+r$. Then $(0,\epsilon)\cap S \in w$ and $\{t \in S : -t+B \in r\} \in w$ so pick $t \in (0,\epsilon)\cap S$ such that $-t+B \in r$. But $\big(S\cap(0,1)\big)\setminus(-t+B) \in \mathcal{G} \subseteq r$, a contradiction.

(b) implies (c). Let $r \in 0^+$. For each $A \in p$, let $B(A) = \{x \in S : -x + A \in p\}$ and let $C(A) = \{t \in S : -t + B(A) \in r\}$. Observe that for any $A_1, A_2 \in p$, one has $B(A_1 \cap A_2) = B(A_1) \cap B(A_2)$ and $C(A_1 \cap A_2) = C(A_1) \cap C(A_2)$.

We claim that for every $A \in p$ and every $\epsilon > 0$, $C(A) \cap (0,\epsilon) \neq \emptyset$. To see this, let $A \in p$ and $\epsilon > 0$ be given and pick $F \in \mathcal{P}_f\big((0,\epsilon)\cap S\big)$ and $\delta > 0$ such that $(0,\delta)\cap S \subseteq \bigcup_{t\in F}\big(-t+B(A)\big)$. Since $(0,\delta)\cap S \in r$ we have $\bigcup_{t\in F}\big(-t+B(A)\big) \in r$ and hence there is some $t \in F$ with $-t+B(A) \in r$. Then $t \in C(A) \cap (0,\epsilon)$.

Thus $\{(0,\epsilon)\cap C(A) : \epsilon > 0$ and $A \in p\}$ has the finite intersection property so pick $q \in \beta S_d$ with $\{(0,\epsilon)\cap C(A) : \epsilon > 0$ and $A \in p\} \subseteq q$. Then $q \in 0^+$. We claim that $p = q + r + p$ for which it suffices (since both are ultrafilters) to show that $p \subseteq q + r + p$. Let $A \in p$ be given. Then $\{t \in S : -t + B(A) \in r\} = C(A) \in q$ so $B(A) \in q + r$, so $A \in q + r + p$ as required.

(c) implies (a). Pick $r \in K$. ■

If $(S,+)$ is any discrete semigroup, we know from [8, Theorem 4.5] that any $p \in \beta S$ is in the closure of the smallest ideal of βS if and only if each $A \in p$ is "piecewise syndetic". (A subset A of S is *piecewise syndetic* if and only if there is some $F \in \mathcal{P}_f(S)$ such that for any $G \in \mathcal{P}_f(S)$ there is some $x \in S$ with $G + x \subseteq \bigcup_{t\in F}(-t+A)$.)

Definition 3.4. Let S be a dense subsemigroup of $\big((0,\infty),+\big)$. A subsets A of S is *piecewise syndetic near 0* if and only if there exist sequences $\langle F_n\rangle_{n=1}^\infty$ and $\langle \delta_n\rangle_{n=1}^\infty$ such that

(1) for each $n \in \mathbb{N}$, $F_n \in \mathcal{P}_f\big((0,1/n)\cap S\big)$ and $\delta_n \in (0,1/n)$ and

(2) for all $G \in \mathcal{P}_f(S)$ and all $\mu > 0$ there is some $x \in (0,\mu)\cap S$ such that for all $n \in \mathbb{N}$, $\big(G \cap (0,\delta_n)\big) + x \subseteq \bigcup_{t\in F_n}(-t+A)$.

Theorem 3.5. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ and let $A \subseteq S$. Then $K \cap c\ell_{\beta S_d}A \neq \emptyset$ if and only if A is piecewise syndetic near 0.*

Proof. Necessity. Pick $p \in K \cap c\ell_{\beta S_d}A$ and let $B = \{x \in S : -x + A \in p\}$. By Theorem 3.3, B is piecewise syndetic near 0. Inductively for $n \in \mathbb{N}$ pick $F_n \in \mathcal{P}_f\big((0,1/n)\cap S\big)$ and $\delta_n \in (0,1/n)$ (with $\delta_n \leq \delta_{n-1}$ if $n > 1$) such that $S \cap (0,\delta_n) \subseteq \bigcup_{t\in F_n}(-t+B)$.

Let $G \in \mathcal{P}_f(S)$ be given. If $G \cap (0,\delta_1) = \emptyset$, the conclusion is trivial, so assume $G \cap (0,\delta_1) \neq \emptyset$ and let $H = G \cap (0,\delta_1)$. For each $y \in H$, let

$m(y) = \max\{n \in \mathbb{N} : y < \delta_n\}$. For each $y \in H$ and each $n \in \{1, 2, \ldots, m(y)\}$, we have $y \in \bigcup_{t \in F_n}(-t + B)$ so pick $t(y, n) \in F_n$ such that $y \in -t(y, n) + B$. Then given $y \in H$ and $n \in \{1, 2, \ldots, m(y)\}$, one has $-(t(y, n) + y) + A \in p$.

Now let $\mu > 0$ be given. Then $(0, \mu) \in p$ so pick

$$x \in (0, \mu) \cap \bigcap_{y \in H} \bigcap_{n=1}^{m(y)} (-(t(y, n) + y) + A) \ .$$

Then given $n \in \mathbb{N}$ and $y \in G \cap (0, \delta_n)$, one has $y \in H$ and $n \leq m(y)$ so $t(y, n) + y + x \in A$ so

$$y + x \in -t(y, n) + A \subseteq \bigcup_{t \in F_n}(-t + A) \ .$$

Sufficiency. Pick $\langle F_n \rangle_{n=1}^{\infty}$ and $\langle \delta_n \rangle_{n=1}^{\infty}$ satisfying (1) and (2) of Definition 3.4. Given $G \in \mathcal{P}_f(S)$ and $\mu > 0$, let

$$C(G, \mu) = \{x \in (0, \mu) \cap S : \text{for all } n \in \mathbb{N}, \big(G \cap (0, \delta_n)\big) + x \subseteq \bigcup_{t \in F_n}(-t + A)\} \ .$$

By assumption each $C(G, \mu) \neq \emptyset$. Further, given G_1 and G_2 in $\mathcal{P}_f(S)$ and $\mu_1, \mu_2 > 0$, one has $C(G_1 \cup G_2, \min\{\mu_1, \mu_2\}) \subseteq C(G_1, \mu_1) \cap C(G_2, \mu_2)$ so $\{C(G, \mu) : G \in \mathcal{P}_f(S) \text{ and } \mu > 0\}$ has the finite intersection property so pick $p \in \beta S_d$ with $\{C(G, \mu) : G \in \mathcal{P}_f(S) \text{ and } \mu > 0\} \subseteq p$. Note that since each $C(G, \mu) \subseteq (0, \mu)$, one has $p \in 0^+$.

Now we claim that for each $n \in \mathbb{N}$, $0^+ + p \subseteq c\ell_{\beta S_d}\big(\bigcup_{t \in F_n}(-t + A)\big)$, so let $n \in \mathbb{N}$ and let $q \in 0^+$. To show that $\bigcup_{t \in F_n}(-t + A) \in q + p$, we show that

$$(0, \delta_n) \cap S \subseteq \{y \in S : -y + \bigcup_{t \in F_n}(-t + A) \in p\} \ .$$

So let $y \in (0, \delta_n) \cap S$. Then $C(\{y\}, \delta_n) \in p$ and $C(\{y\}, \delta_n) \subseteq -y + \bigcup_{t \in F_n}(-t + A)$.

Now pick $r \in (0^+ + p) \cap K$ (since $0^+ + p$ is a left ideal of 0^+). Given $n \in \mathbb{N}$, $\bigcup_{t \in F_n}(-t + A) \in r$ so pick $t_n \in F_n$ such that $-t_n + A \in r$. Now each $t_n \in F_n \subseteq (0, 1/n)$ so $\lim\limits_{n \to \infty} t_n = 0$ so pick $q \in 0^+ \cap c\ell_{\beta S_d}\{t_n : n \in \mathbb{N}\}$. Then $q + r \in K$ and $\{t_n : n \in \mathbb{N}\} \subseteq \{t \in S : -t + A \in r\}$ so $A \in q + r$. ■

Since $(0^+, +)$ is a compact right topological semigroup, the closure of any right ideal is again a right ideal. (This is well known and a very easy exercise.) Consequently $c\ell_{0^+} K = c\ell_{\beta S_d} K$ is a right ideal of 0^+. On the other hand, if S is any discrete semigroup, we know from [8, Corollary 4.6] that the closure of the smallest ideal of βS is a two sided ideal of βS. We do not know whether $c\ell_{0^+} K$ is a left ideal of 0^+, but would conjecture that it is not.

4. Sets Central Near 0.

In this section we characterize those sets that are members of minimal idempotents in $(0^+, +)$. From [12, Theorem 2.1] we have a relatively complicated characterization of those subsets $\mathcal{A}$ of $\mathcal{P}(S)$ which extend to a member of the smallest ideal of βS. This characterization was called "collectionwise piecewise syndetic" in [13] and used there to characterize central sets, that is members of idempotents in the smallest ideal of βS.

Definition 4.1. Let S be a dense subsemigroup of $\big((0, \infty), +\big)$.

(a) A set $A \subseteq S$ is *central near 0* if and only if there is some idempotent $p \in K$ with $A \in p$.

(b) A family $\mathcal{A} \subseteq \mathcal{P}(S)$ is *collectionwise piecewise syndetic near 0* if and only if there exist functions

$$F : \mathcal{P}_f(\mathcal{A}) \longrightarrow \times_{n=1}^{\infty} \mathcal{P}_f\big((0,1/n) \cap S\big)$$

and

$$\delta : \mathcal{P}_f(\mathcal{A}) \longrightarrow \times_{n=1}^{\infty} (0,1/n)$$

such that for every $\mu > 0$, every $G \in \mathcal{P}_f(S)$, and every $\mathcal{H} \in \mathcal{P}_f(\mathcal{A})$, there is some $t \in (0,\mu) \cap S$ such that for every $n \in \mathbb{N}$ and every $\mathcal{F} \in \mathcal{P}_f(\mathcal{H})$,

$$\big(G \cap (0, \delta(\mathcal{F})_n)\big) + t \subseteq \bigcup_{x \in F(\mathcal{F})_n} (-x + \bigcap \mathcal{F}) \ .$$

Theorem 4.2. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ and let $\mathcal{A} \subseteq \mathcal{P}(S)$. There exists $p \in K$ such that $\mathcal{A} \subseteq p$ if and only if $\mathcal{A}$ is collectionwise piecewise syndetic near 0.*

Proof. Necessity. For each $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, let $B(\mathcal{F}) = \{x \in S : -x + \bigcap \mathcal{F} \in p\}$. Then by Theorem 3.3, $B(\mathcal{F})$ is syndetic near 0, so for each $n \in \mathbb{N}$, pick $F(\mathcal{F})_n \in \mathcal{P}_f\big((0,1/n) \cap S\big)$ and $\delta(\mathcal{F})_n \in (0,1/n)$ such that

$$S \cap (0, \delta(\mathcal{F})_n) \subseteq \bigcup_{x \in F(\mathcal{F})_n} \big(-x + B(\mathcal{F})\big) \ .$$

We may presume that $\delta(\mathcal{F})_n < 1/n$.

We have thus defined

$$F : \mathcal{P}_f(\mathcal{A}) \longrightarrow \times_{n=1}^{\infty} \mathcal{P}_f\big((0,1/n) \cap S\big)$$

and

$$\delta : \mathcal{P}_f(\mathcal{A}) \longrightarrow \times_{n=1}^{\infty} (0,1/n) \ .$$

To see that these functions are as required, let $\mu > 0$, $G \in \mathcal{P}_f(S)$, and $\mathcal{H} \in \mathcal{P}_f(\mathcal{A})$ be given.

Pick $m \in \mathbb{N}$ such that $1/m < \min G$. For each $(y, n, \mathcal{F})$ such that $n \in \{1, 2, \ldots, m\}$, $\mathcal{F} \in \mathcal{P}_f(\mathcal{H})$, and $y \in (0, \delta(\mathcal{F})_n) \cap G$, pick $x(y, n, \mathcal{F}) \in F(\mathcal{F})_n$ such that $x(y, n, \mathcal{F}) + y \in B(\mathcal{F})$, that is $-(x(y, n, \mathcal{F}) + y) + \bigcap \mathcal{F} \in p$. Let

$$\mathcal{B} = \{-(x(y,n,\mathcal{F}) + y) + \bigcap \mathcal{F} : \ n \in \{1,2,\ldots,m\}, \mathcal{F} \in \mathcal{P}_f(\mathcal{H}), \text{ and } y \in (0, \delta(\mathcal{F})_n) \cap G\} \ .$$

If $\mathcal{B} = \emptyset$, the conclusion is trivial, so we may assume $\mathcal{B} \neq \emptyset$ and hence $\mathcal{B} \in \mathcal{P}_f(p)$. Pick $t \in \bigcap \mathcal{B} \cap (0,\mu)$. Let $n \in \mathbb{N}$ and $\mathcal{F} \in \mathcal{P}_f(\mathcal{H})$ be given. If $G \cap (0, \delta(\mathcal{F})_n) = \emptyset$, the conclusion holds, so assume $G \cap (0, \delta(\mathcal{F})_n) \neq \emptyset$ and let $y \in G \cap (0, \delta(\mathcal{F})_n)$. Then $y < \delta(\mathcal{F})_n < 1/n$ and $y \in G$ so $n < m$. Thus $t \in -(x(y, n, \mathcal{F}) + y) + \bigcap \mathcal{F}$ so

$$y + t \in -x(y, n, \mathcal{F}) + \bigcap \mathcal{F} \subseteq \bigcup_{x \in F(\mathcal{F})_n} (-x + \bigcap \mathcal{F}) \ .$$

Sufficiency. Pick functions F and δ as guaranteed by the assumption that $\mathcal{A}$ is collectionwise piecewise syndetic near 0. Given $\mu > 0$, $G \in \mathcal{P}_f(S)$, and $\mathcal{H} \in \mathcal{P}_f(\mathcal{A})$, pick $t(\mathcal{H}, G, \mu) \in (0,\mu) \cap S$ such that for every $n \in \mathbb{N}$ and every $\mathcal{F} \in \mathcal{P}_f(\mathcal{H})$,

$$\big(G \cap (0, \delta(\mathcal{F})_n)\big) + t(\mathcal{H}, G, \mu) \subseteq \bigcup_{x \in F(\mathcal{F})_n} (-x + \bigcap \mathcal{F}) \ .$$

For each $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$ and every $y \in S$, let $D(\mathcal{F}, y) =$

$$\{t(\mathcal{H}, G, \mu) : \mathcal{H} \in \mathcal{P}_f(\mathcal{A}),\ G \in \mathcal{P}_f(S),\ y \in G,\ \mathcal{F} \subseteq \mathcal{H},\ \text{and } \mu > 0\}\ .$$

Then $\{D(\mathcal{F}, y) : \mathcal{F} \in \mathcal{P}_f(\mathcal{A}) \text{ and } y \in S\} \cup \{(0, \mu) : \mu > 0\}$ has the finite intersection property. Indeed, given $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$, $y_1, y_2, \ldots, y_n$, and $\mu_1, \mu_2, \ldots, \mu_n$, let

$$\mathcal{H} = \textstyle\bigcup_{i=1}^n \mathcal{F}_i, G = \{y_1, y_2, \ldots, y_n\}, \text{ and } \mu = \min\{\mu_1, \mu_2, \ldots, \mu_n\}\ .$$

Then $t(\mathcal{H}, G, \mu) \in \bigcap_{i=1}^n \big(D(\mathcal{F}_i, y_i) \cap (0, \mu_i)\big)$. So pick $u \in 0^+$ such that $\{D(\mathcal{F}, y) : \mathcal{F} \in \mathcal{P}_f(\mathcal{A}) \text{ and } y \in S\} \subseteq u$.

Now we claim that for each $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$ and each $n \in \mathbb{N}$,

$$0^+ + u \subseteq \textstyle\bigcup_{x \in F(\mathcal{F})_n} c\ell_{\beta S_d}(-x + \bigcap \mathcal{F})\ .$$

So let $q \in 0^+$ and let $A = \bigcup_{x \in F(\mathcal{F})_n}(-x + \bigcap \mathcal{F})$. We claim that $(0, \delta(\mathcal{F})_n) \cap S \subseteq \{y \in S : -y + A \in u\}$, so that, since $(0, \delta(\mathcal{F})_n) \cap S \in q$, we have $A \in q + u$. Let $y \in (0, \delta(\mathcal{F})_n) \cap S$. It suffices to show that $D(\mathcal{F}, y) \subseteq -y + A$. So let $\mathcal{H} \in \mathcal{P}_f(\mathcal{A})$ with $\mathcal{F} \subseteq \mathcal{H}$, let $G \in \mathcal{P}_f(S)$ with $y \in G$, and let $\mu > 0$ be given. Then $y \in G \cap (0, \delta(\mathcal{F})_n)$ so $y + t(\mathcal{H}, G, \mu) \in A$ as required.

Pick a minimal left ideal L of 0^+ with $L \subseteq 0^+ + u$. Then

$$L \subseteq \textstyle\bigcap_{\mathcal{F} \in \mathcal{P}_f(\mathcal{A})} \bigcap_{n=1}^\infty \bigcup_{x \in F(\mathcal{F})_n} c\ell_{\beta S_d}(-x + \bigcap \mathcal{F})\ .$$

Pick $r \in L$. For each $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$ and each $n \in \mathbb{N}$, pick $x(\mathcal{F}, n) \in F(\mathcal{F})_n$ such that $-x(\mathcal{F}, n) + \bigcap \mathcal{F} \in r$. For each $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, let

$$\mathcal{E}(\mathcal{F}) = \{x(\mathcal{H}, n) : \mathcal{H} \in \mathcal{P}_f(\mathcal{A}),\ \mathcal{F} \subseteq \mathcal{H},\ \text{and } n \in \mathbb{N}\}\ .$$

We claim that $\{\mathcal{E}(\mathcal{F}) : \mathcal{F} \in \mathcal{P}_f(\mathcal{A})\} \cup \{(0, \delta) : \delta > 0\}$ has the finite intersection property. Indeed, given $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ and $\delta_1, \delta_2, \ldots, \delta_m$, pick $n \in \mathbb{N}$ such that $1/n < \min\{\delta_1, \delta_2, \ldots, \delta_m\}$ and let $\mathcal{H} = \bigcup_{i=1}^m \mathcal{F}_i$. Then $x(\mathcal{H}, n) \in \bigcap_{i=1}^m \mathcal{E}(\mathcal{F}_i) \cap \bigcap_{i=1}^m (0, \delta_i)$. So pick $w \in 0^+$ such that $\{\mathcal{E}(\mathcal{F}) : \mathcal{F} \in \mathcal{P}_f(\mathcal{A})\} \subseteq w$. Let $p = w + r$. Then $p \in L \subseteq K$. To see that $\mathcal{A} \subseteq p$, let $A \in \mathcal{A}$. We show that $\mathcal{E}(\{A\}) \subseteq \{x \in S : -x + A \in r\}$. Let $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$ with $A \in \mathcal{F}$ and let $n \in \mathbb{N}$. Then $-x(\mathcal{F}, n) + \bigcap \mathcal{F} \in r$ and $-x(\mathcal{F}, n) + \bigcap \mathcal{F} \subseteq -x(\mathcal{F}, n) + A$. ■

We formalize the notion of "tree" below. We write $\omega = \{0, 1, 2, 3, \ldots\}$, the first infinite ordinal and recall that each ordinal is the set of its predecessors. (So $3 = \{0, 1, 2\}$ and $0 = \emptyset$ and, if f is the function

$$\{(0, 3), (1, 5), (2, 9), (3, 7), (4, 5)\}\ ,$$

then $f_{|3} = \{(0, 3), (1, 5), (2, 9)\}$.)

Definition 4.3. T is a *tree in* A if and only if T is a set of functions and for each $f \in T$, $\text{domain}(f) \in \omega$ and $\text{range}(f) \subseteq A$ and if $\text{domain}(f) = n > 0$, then $f_{|n-1} \in T$. T is a *tree* if and only if for some A, T is a tree in A.

Definition 4.4. (a) Let f be a function with $\text{domain}(f) = n \in \omega$ and let x be given. Then $f^\frown x = f \cup \{(n, x)\}$.

(b) Given a tree T and $f \in T$, $B_f = B_f(T) = \{x : f^\frown x \in T\}$.

(c) Let $(S, +)$ be a semigroup and let $A \subseteq S$. Then T is a $*$-*tree* in A if and only if T is a tree in A and for all $f \in T$ and all $x \in B_f$, $B_{f^\frown x} \subseteq -x + B_f$.

(d) Let $(S, +)$ be a semigroup and let $A \subseteq S$. Then T is a *FS-tree in* A if and only if T is a tree in A and for all $f \in T$, $B_f = \{\Sigma_{t \in F}\, g(t) : g \in T,$ $f \subsetneq g$, and $\emptyset \neq F \subseteq \text{dom}(g) \backslash \text{dom}(f)\}$.

Lemma 4.5. *Let $(S,+)$ be a semigroup and let p be an idempotent in βS_d. If $A\in p$, then there is a FS-tree T in A such that for each $f\in T$, $B_f\in p$.*

Proof. This is [13, Lemma 3.6]. ■

Lemma 4.6. *Any FS-tree is a $*$-tree.*

Proof. Let T be a FS-tree. Then given $f\in T$ and $x\in B_f$, we claim that $B_{f^\frown x}\subseteq -x+B_f$. To this end let $y\in B_{f^\frown x}$ and pick $g\in T$ with $f^\frown x\subsetneq g$ and pick $F\subseteq \mathrm{dom}(g)\backslash\mathrm{dom}(f^\frown x)$ such that $y\in\Sigma_{t\in F}\; g(t)$. Let $n=\mathrm{dom}(f)$ and let $G=F\cup\{n\}$. Then $x+y=\Sigma_{t\in G}\; g(t)$ and $G\subseteq\mathrm{dom}(g)\backslash\mathrm{dom}(f)$, so $x+y\in B_f$ as required. ■

When we say that $\langle C_F\rangle_{F\in I}$ is a "downward directed family" we mean that I is a directed set and whenever $F,G\in I$ with $F\leq G$, one has $C_G\subseteq C_F$.

Theorem 4.7. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ and let $A\subseteq S$. Statements (a), (b), (c), and (d) are equivalent and are implied by statement (e). If S is countable, all five statements are equivalent.*

(a) A is central near 0.

(b) There is a FS-tree T in A such that $\{B_f:f\in T\}$ is collectionwise piecewise syndetic near 0.

*(c) There is a *-tree T in A such that $\{B_f:f\in T\}$ is collectionwise piecewise syndetic near 0.*

(d) There is a downward directed family $\langle C_F\rangle_{F\in I}$ of subsets of A such that

(i) for all $F\in I$ and all $x\in C_F$, there is some $G\in I$ with $C_G\subseteq -x+C_F$ and

(ii) $\{C_F:F\in I\}$ is collectionwise piecewise syndetic near 0.

(e) There is a decreasing sequence $\langle C_n\rangle_{n=1}^\infty$ of subsets of A such that

(i) for all $n\in\mathbb{N}$ and all $x\in C_n$, there is some $m\in\mathbb{N}$ with $C_m\subseteq -x+C_n$ and

(ii) $\{C_n:n\in\mathbb{N}\}$ is collectionwise piecewise syndetic near 0.

Proof. (a) implies (b). By Lemma 4.5 pick a FS-tree T in A such that for each $f\in T$, $B_f\in p$. By Theorem 4.2, $\{B_f:f\in T\}$ is collectionwise piecewise syndetic near 0.

That (b) implies (c) follows from Lemma 4.6.

(c) implies (d). Let T be given as guaranteed by (c). Let $I=\mathcal{P}_f(T)$ and for $F\in I$, let $C_F=\bigcap_{f\in F}\; B_f$. Since $\{B_f:f\in T\}$ is collectionwise piecewise syndetic near 0, so is $\{C_F:F\in I\}$. Given $F\in I$ and $x\in C_F$, let $G=\{f^\frown x:f\in F\}$. For each $f\in F$ we have $B_{f^\frown x}\subseteq -x+B_f$ by the definition of $*$-tree so

$$C_G=\bigcap_{f\in F}\; B_{f^\frown x}\subseteq\bigcap_{f\in F}(-x+B_f)=-x+C_F\;.$$

(d) implies (a). Let $M=\bigcap_{F\in I}\; c\ell_{\beta S_d}\; C_F$. We claim that M is a subsemigroup of βS_d. To this end, let $p,q\in M$ and let $F\in I$. To see that $C_F\in p+q$, we show that $C_F\subseteq\{x\in S:-x+C_F\in q\}$. Let $x\in C_F$ and pick $G\in I$ such that $C_G\subseteq -x+C_F$. Then $C_G\in q$ so $-x+C_F\in q$.

By Theorem 4.2 we have $M\cap K\neq\emptyset$. Since K is the union of all minimal left ideals of 0^+ (see [3, Theorem 1.3.11], pick a minimal left ideal L of K with $M\cap L\neq\emptyset$. Then $M\cap L$ is a compact semigroup so by [4, Corollary 2.10] there is some $p=p+p$ in $M\cap L$. Since each $C_F\subseteq A$, we have $p\in K\cap c\ell_{\beta S_d}\;A$.

That (e) implies (d) is trivial.

Now assume that S is countable. We show that (c) implies (e), so let T be as guaranteed by (c). Since T is countable, enumerate T as $\langle f_n\rangle_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, let $C_n = \bigcap_{k=1}^{n} B_{f_k}$. Then $\{C_n : n \in \mathbb{N}\}$ is collectionwise piecewise syndetic near 0. Let $n \in \mathbb{N}$ be given and let $x \in C_n$. Then for each k, $B_{f_k \frown x} \subseteq -x + B_{f_k}$. Pick $m \in \mathbb{N}$ such that $\big\{f_k \frown x : k \in \{1,2,\ldots,n\}\big\} \subseteq \big\{f_k : k \in \{1,2,\ldots,m\}\big\}$. Then $C_m \subseteq -x + C_n$. ■

The following consequence of Theorem 4.7 will be useful later when we investigate combined additive and multiplicative structures. The requirement that $y < 1$ is not essential, but makes the proof simpler since under that assumption, if $x < 1/n$, then $yx < 1/n$.

Lemma 4.8. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ and let $y \in S \cap (0,1)$ such that for all $x \in S$, $x/y \in S$ and $yx \in S$. If $A \subseteq S$ and $y^{-1}A$ is central near 0, then A is central near 0.*

Proof. Note that, while we standardly define $y^{-1}A$ to be $\{x \in S : yx \in A\}$, under the current hypotheses we in fact have that $y^{-1}A = \{x/y : x \in A\}$. Pick by Theorem 4.7 a downward directed family $\langle C_F\rangle_{F\in I}$ of subsets of $y^{-1}A$ such that

(i) for all $F \in I$ and all $x \in C_F$, there is some $G \in I$ with $C_G \subseteq -x + C_F$ and

(ii) $\{C_F : F \in I\}$ is collectionwise piecewise syndetic near 0.

For each $F \in I$, let $D_F = yC_F$. Then $\langle D_F\rangle_{F\in I}$ is a downward directed family of subsets of yA and trivially for all $F \in I$ and all $x \in D_F$, there is some $G \in I$ with $D_G \subseteq -x + D_F$. Thus it suffices to show that $\mathcal{B} = \{D_F : F \in I\}$ is collectionwise piecewise syndetic near 0.

Let $\mathcal{A} = \{C_F : F \in I\}$ and pick

$$F : \mathcal{P}_f(\mathcal{A}) \longrightarrow \times_{n=1}^{\infty} \mathcal{P}_f\big((0,1/n) \cap S\big)$$

and

$$\delta : \mathcal{P}_f(\mathcal{A}) \longrightarrow \times_{n=1}^{\infty} (0,1/n)$$

as guaranteed by Definition 4.1. Given $\mathcal{F} \in \mathcal{P}_f(\mathcal{B})$, choose $\tau(\mathcal{F}) \in \mathcal{P}_f(I)$ such that $\mathcal{F} = \{D_F : F \in \tau(\mathcal{F})\}$. Define $\sigma : \mathcal{P}_f(\mathcal{B}) \longrightarrow \mathcal{P}_f(\mathcal{A})$ by $\sigma(\mathcal{F}) = \{C_F : F \in \tau(\mathcal{F})\}$ and note that for any $\mathcal{F} \in \mathcal{P}_f(\mathcal{B})$, $y \cdot \sigma(\mathcal{F}) = \mathcal{F}$. Define

$$F' : \mathcal{P}_f(\mathcal{B}) \longrightarrow \times_{n=1}^{\infty} \mathcal{P}_f\big((0,1/n) \cap S\big)$$

and

$$\delta' : \mathcal{P}_f(\mathcal{B}) \longrightarrow \times_{n=1}^{\infty} (0,1/n)$$

by $F' = y \cdot (F \circ \sigma)$ and $\delta' = y \cdot (\delta \circ \sigma)$.

To see that F' and δ' are as required by Definition 4.1, let $\mu > 0$, $G \in \mathcal{P}_f(S)$, and $\mathcal{H} \in \mathcal{P}_f(\mathcal{B})$ be given. Then $y^{-1}G \in \mathcal{P}_f(S)$ and $\sigma(\mathcal{H}) \in \mathcal{P}_f(\mathcal{A})$ so pick $t \in (0,\mu) \cap S$ such that for all $n \in \mathbb{N}$ and all $\mathcal{F} \in \mathcal{P}_f\big(\sigma(\mathcal{H})\big)$,

$$y^{-1}G \cap (0, \delta(\mathcal{F})_n) + t \subseteq \bigcup_{x \in F(\mathcal{F})_n} (-x + \bigcap \mathcal{F}) \,.$$

Then $yt \in (0,\mu) \cap S$. Let $n \in \mathbb{N}$ and $\mathcal{F} \in \mathcal{P}_f(\mathcal{H})$ be given. Then $\sigma(\mathcal{F}) \in \mathcal{P}_f\big(\sigma(\mathcal{H})\big)$ so

$$y^{-1}G \cap (0, \delta\big(\sigma(\mathcal{F})\big)_n) + t \subseteq \bigcup_{x \in F(\sigma(\mathcal{F}))_n} \big(-x + \bigcap \sigma(\mathcal{F})\big) \,.$$

Thus

$$G \cap (0, y \cdot \delta\big(\sigma(\mathcal{F})\big)_n)) + yt \subseteq \bigcup_{x \in y \cdot F(\sigma(\mathcal{F}))_n} \big(-x + \bigcap y \cdot \sigma(\mathcal{F})\big) \,.$$

That is

$$G \cap (0, \delta'(\mathcal{F})_n) + yt \subseteq \bigcup_{x \in F'(\mathcal{F})_n} (-x + \bigcap \mathcal{F}) \,,$$

as required. ■

The main reason central sets in a commutative semigroup $(S,+)$ are of combinatorial interest is that they satisfy the Central Sets Theorem, a generalization of [5, Proposition 8.21], and as a consequence have rich combinatorial structure. (See [14] for example.) We now show that sets that are central near 0 satisfy a version of the Central Sets Theorem (Theorem 4.11). As a consequence, we obtain new Ramsey Theoretic conclusions about arbitrary finite partitions of S, where S is any dense subsemigroup of $\big((0,\infty),+\big)$.

Definition 4.9. Let S be a dense subsemigroup of $\big((0,\infty),+\big)$.

(a) $\Phi=\{f: f:\mathbb{N}\longrightarrow\mathbb{N}$ and for all $n\in\mathbb{N}$, $f(n)\leq n\}$.

(b) $\mathcal{Y}=\{\langle\langle y_{i,t}\rangle_{t=1}^\infty\rangle_{i=1}^\infty$: for each $i\in\mathbb{N}$, $\langle y_{i,t}\rangle_{t=1}^\infty$ is a sequence in $S\cup -S\cup\{0\}$ and $\Sigma_{t=1}^\infty|y_{i,t}|$ converges$\}$.

(c) Given $Y=\langle\langle y_{i,t}\rangle_{t=1}^\infty\rangle_{i=1}^\infty$ in $\mathcal{Y}$ and $A\subseteq S$, A is a *J_Y-set near 0* if and only if for all $n\in\mathbb{N}$ there exist $a\in(0,1/n)\cap S$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that $\min H\geq n$ and for each $i\in\{1,2,\ldots,n\}$, $a+\Sigma_{t\in H}\,y_{i,t}\in A$.

(d) Given $Y\in\mathcal{Y}$, $J_Y=\{p\in 0^+ :$ for all $A\in p$, A is a J_Y-set near $0\}$.

(e) $J=\bigcap_{Y\in\mathcal{Y}} J_Y$.

The following result is similar to [13, Lemma 2.5] and is based on an argument from [6]. It tells us that $J\neq\emptyset$.

Lemma 4.10. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$. Let $Y\in\mathcal{Y}$. Then $K\subseteq J_Y$.*

Proof. Let $Y=\langle\langle y_{i,t}\rangle_{t=1}^\infty\rangle_{i=1}^\infty$ and let $p\in K$. To see that $p\in J_Y$, let $A\in p$. To see that A is a J_Y-set near 0, let $n\in\mathbb{N}$ be given.

For $k\in\mathbb{N}$ let

$$\begin{aligned} I_k = \ & \{(a+\Sigma_{t\in H}\,y_{1,t}, a+\Sigma_{t\in H}\,y_{2,t},\ldots,a+\Sigma_{t\in H}\,y_{n,t}) : \\ & a\in(0,1/n)\cap S, H\in\mathcal{P}_f(\mathbb{N}), \min H\geq k, \text{ and} \\ & \text{for all } i\in\{1,2,\ldots,n\}, a+\Sigma_{t\in H}\,y_{i,t}\in(0,1/k)\cap S\} \end{aligned}$$

and let

$$E_k = I_k\cup\{(a,a,\ldots,a): a\in(0,1/k)\cap S\}\,.$$

Let $W=\times_{i=1}^n 0^+$ and let $Z=\times_{i=1}^n \beta S_d$. Let $E=\bigcap_{k=1}^\infty c\ell_Z E_k$ and let $I=\bigcap_{k=1}^\infty c\ell_Z I_k$. We claim that $\emptyset\neq I\subseteq E$. To see this, it suffices to let $k\in\mathbb{N}$ and show that $I_k\neq\emptyset$. We may assume that $k\geq n$. Pick $t\geq k$ such that for all $i\in\{1,2,\ldots,n\}$, $|y_{i,t}|<\frac{1}{2nk}$ and let $a=\sum_{i=1}^n|y_{i,t}|$. Then $a\in S\cap(0,\frac{1}{2k})\subseteq S\cap(0,1/n)$ and given $j\in\{1,2,\ldots,n\}$, $a+y_{j,t}\in S\cap(0,1/k)$.

We claim that E is a subsemigroup of W and I is an ideal of E. First to see that $E\subseteq W$, note that for each k, $E_k\subseteq\times_{i=1}^n\big((0,1/k)\cap S\big)$ so $c\ell_{\beta S_d}E_k\subseteq\times_{i=1}^n c\ell_Z\big((0,1/k)\cap S\big)$ so $E\subseteq W$

Now let $\vec{q},\vec{r}\in E$. We show that $\vec{q}+\vec{r}\in E$ and if either $\vec{q}\in I$ or $\vec{r}\in I$, then $\vec{q}+\vec{r}\in I$. So let $k\in\mathbb{N}$ be given and let U be an open neighborhood of $\vec{q}+\vec{r}$. We show that $U\cap E_k\neq\emptyset$ and if either $\vec{q}\in I$ or $\vec{r}\in I$, then $U\cap I_k\neq\emptyset$. Pick a neighborhood V of $\vec{q}$ such that $V+r\subseteq U$ and pick $\vec{x}\in V\cap E_{2k}$, with $\vec{x}\in I_{2k}$ if $\vec{q}\in I$. If $\vec{x}\notin I_{2k}$, pick $a\in(0,\frac{1}{2k})\cap S$ such that $\vec{x}=(a,a,\ldots,a)$ and let $H=\emptyset$. If $\vec{x}\in I_{2k}$, pick $a\in(0,\frac{1}{2k})\cap S$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that $\min H>2k$, $\vec{x}=(a+\Sigma_{t\in H}\,y_{1,t},a+\Sigma_{t\in H}\,y_{2,t},\ldots,a+\Sigma_{t\in H}\,y_{n,t})$, and each $a+\Sigma_{t\in H}\,y_{i,t}\in(0,\frac{1}{2k})$. If $H=\emptyset$, let $m=2k$. If $H\neq\emptyset$, let $m=\max H+1$.

Now $\vec{x}+\vec{r}\in U$ so pick a neighborhood R of $\vec{r}$ with $\vec{x}+R\subseteq U$. Pick $\vec{w}\in R\cap E_m$ with $\vec{w}\in I_m$ if $\vec{r}\in I$. If $\vec{w}\notin I_m$, pick $b\in(0,1/m)\cap S$ such that

$\vec{w} = (b, b, \ldots, b)$ and let $G = \emptyset$. If $\vec{w} \in I_m$, pick $b \in (0, 1/m) \cap S$ and $G \in \mathcal{P}_f(\mathbb{N})$ such that $\min G \geq m$, $\vec{w} = (b + \Sigma_{t\in G}\; y_{1,t}, b + \Sigma_{t\in G}\; y_{2,t}, \ldots, b + \Sigma_{t\in G}\; y_{n,t})$, and each $b + \Sigma_{t\in g}\; y_{i,t} \in (0, 1/m)$. If $G \cup H = \emptyset$, we have $\vec{x} + \vec{w} = (a+b, a+b, \ldots, a+b) \in E_k$. If $G \cup H \neq \emptyset$, then

$$\vec{x} + \vec{w} = (a+b+\Sigma_{t\in G\cup H}\; y_{1,t}, a+b+\Sigma_{t\in G\cup H}\; y_{2,t}, \ldots, a+b+\Sigma_{t\in G\cup H}\; y_{n,t}) \in I_k\ .$$

Now let $\vec{p} = (p, p, \ldots, p)$. We claim that $\vec{p} \in E$. Indeed, let $B \in p$, so that $\times_{i=1}^{n} c\ell_{\beta S_d}\; B$ is a basic neighborhood of $\vec{p}$. Let $k \in \mathbb{N}$ be given. Then $(0, 1/k) \cap S \in p$ so pick $a \in B \cap (0, 1/k)$. Then $(a, a, \ldots, a) \in E_k \cap \times_{i=1}^{n} c\ell_{\beta S_d}\; B$.

Let M be the smallest ideal of W. By [13, Lemma 2.1], $M = \times_{i=1}^{n} K$ so $\vec{p} \in M$ and hence $M \cap E \neq \emptyset$. Thus by [3, Corollary 1.2.15] the smallest ideal of E is $E \cap M$ so $\vec{p}$ is in the smallest ideal of E so $\vec{p} \in I$. Thus $I_n \cap \times_{i=1}^{n} c\ell_{\beta S_d}\; A \neq \emptyset$. ∎

The following is our promised version of the Central Sets Theorem.

Theorem 4.11. *Let S be a dense subsemigroup of $\big((0,\infty), +\big)$ and let A be central near 0 in S. Let $Y = \langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty} \in \mathcal{Y}$. Then there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in S and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(a) for each $n \in \mathbb{N}$, $a_n < 1/n$ and $\max H_n < \min H_{n+1}$ and

(b) for each $f \in \Phi$, $FS(\langle a_n + \Sigma_{t\in H_n}\; y_{f(n),t}\rangle_{n=1}^{\infty}) \subseteq A$.

Proof. Pick an idempotent $p \in K$ with $A \in p$. By Lemma 4.10 we have $p \in J_Y$. Let $A_1 = A$ and let

$$B_1 = A_1 \cap \{x \in S \cap (0,1) : -x + A_1 \in p\}\ .$$

Then $B_1 \in p$ so, since $p \in J_Y$, pick $a_1 \in S \cap (0,1)$ and $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $a_1 + \Sigma_{t\in H_1}\; y_{1,t} \in B_1$. Let

$$A_2 = A_1 \cap (-(a_1 + \Sigma_{t\in H_1}\; y_{1,t}) + A_1)\ .$$

Inductively, given $A_n \in p$, let

$$B_n = A_n \cap \{x \in S \cap (0, 1/n) : -x + A_n \in p\}\ .$$

Let $m = \max(H_{n-1} \cup \{n\}) + 1$. Pick $a_n \in S \cap (0, 1/m)$ and $H_n \in \mathcal{P}_f(\mathbb{N})$ with $\min H_n \geq m$ such that for all $i \in \{1, 2, \ldots, m\}$, $a_n + \Sigma_{t\in H_n}\; y_{i,t} \in B_n$. Let

$$A_{n+1} = A_n \cap \textstyle\bigcap_{k=1}^{n}(-(a_n + \Sigma_{t\in H_n}\; y_{k,t}) + A_n)\ .$$

Let $f \in \Phi$ be given. We show by induction on $|F|$ that for $F \in \mathcal{P}_f(\mathbb{N})$, $\Sigma_{n\in F}(a_n + \Sigma_{t\in H_n}\; y_{f(n),t}) \in A_k$ where $k = \min F$. If $|F| = 1$, we have $a_k + \Sigma_{t\in H_k}\; y_{f(k),t} \in B_k \subseteq A_k$. So assume $|F| > 1$, let $G = F\backslash\{k\}$ and let $\ell = \min G$. Then

$$\Sigma_{n\in G}(a_n + \Sigma_{t\in H_n}\; y_{f(n),t}) \in A_\ell \subseteq A_{k+1} \subseteq -(a_k + \Sigma_{t\in H_k}\; y_{f(k),t}) + A_k$$

so $\Sigma_{n\in F}(a_n + \Sigma_{t\in H_n}\; y_{f(n),t}) \in A_k$ as required. ∎

5. Combinatorial Applications.

There are some immediate simple applications of Theorem 4.11. As an example we have the following.

Corollary 5.1. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ and let $\langle x_n\rangle_{n=1}^\infty$ be any sequence in S such that $\lim\limits_{n\to\infty} x_n = 0$. Assume $r\in\mathbb{N}$ and $S=\bigcup_{i=1}^r A_i$. then there is some $i\in\{1,2,\ldots,r\}$ such that for every $\delta>0$ and every $\ell\in\mathbb{N}$ there is an arithmetic progression $\{a,a+d,\ldots,a+\ell d\}\subseteq A_i\cap(0,\delta)$ with increment $d\in FS(\langle x_n\rangle_{n=1}^\infty)$.*

Proof. Pick $j\in\{1,2,\ldots,r\}$ such that A_j is central near 0. By thinning the sequence $\langle x_n\rangle_{n=1}^\infty$, we may presume that $\Sigma_{n=1}^\infty x_n$ converges. For each $i,t\in\mathbb{N}$, let $y_{i,t}=i\cdot x_t$ and let $Y=\langle\langle y_{i,t}\rangle_{t=1}^\infty\rangle_{i=1}^\infty$. Since $Y\in\mathcal{Y}$, Theorem 4.11 applies, so pick sequences $\langle a_n\rangle_{n=1}^\infty$ in S and $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ as guaranteed.

Let $\delta>0$ and $\ell\in\mathbb{N}$ be given and pick $n\in\mathbb{N}$ such that $n\geq\ell$ and $1/n<\delta$. Let $a=a_n$ and let $d=\Sigma_{t\in H_n} x_t$. ■

Many of the classical results of Ramsey Theory are naturally stated as instances of the following problem: Let $u,v\in\mathbb{N}$ and let A be a $u\times v$ matrix with rational entries. Is it true that whenever $\mathbb{N}=\bigcup_{i=1}^r B_i$ there will exist $i\in\{1,2,\ldots,r\}$ and $\vec{x}\in\mathbb{N}^v$ with all entries of $A\vec{x}$ in B_i?

For example van der Waerden's Theorem gives an affirmative answer to this question for all matrices of the form

$$A=\begin{pmatrix}1&0\\1&1\\1&2\\\vdots&\vdots\\1&\ell\end{pmatrix}.$$

See [11] for a more thorough discussion of this point.

Definition 5.2. Let $u,v\in\mathbb{N}$ and let A be a $u\times v$ matrix with integer entries. Then

(a) A satisfies the *first entries condition* if and only if each row of A is not $\vec{0}$ and whenever $i,j\in\{1,2,\ldots,u\}$ and $t=\min\{k\in\{1,2,\ldots,v\}:a_{i,k}\neq 0\}=\min\{k\in\{1,2,\ldots,v\}:a_{j,k}\neq 0\}$, one has $a_{i,t}=a_{j,t}>0$.

(b) A number c is a *first entry* of A if and only if for some $i\in\{1,2,\ldots,u\}$ and some $t\in\{1,2,\ldots,v\}$, $t=\min\{k\in\{1,2,\ldots,v\}:a_{i,k}\neq 0\}$ and $c=a_{i,t}$.

It is proved in [11, Theorem 3.1] that an affirmative answer to the question above can be given for A if and only if there exist $m\in\mathbb{N}$ and a $u\times m$ matrix B which satisfies the first entries condition such that for each $\vec{y}\in\mathbb{N}^m$, there exists $\vec{x}\in\mathbb{N}^v$ such that $A\vec{x}=B\vec{y}$. (In [11] the matrices are allowed to have entries from $\mathbb{Q}$. The two versions are easily seen to be equivalent.)

Theorem 5.3. *Let G be a dense subgroup of $(\mathbb{R},+)$ and let $S=G\cap(0,\infty)$. Let $u,v\in\mathbb{N}$, let A be a $u\times v$ matrix with entries from $\mathbb{Z}$ which satisfies the first entries condition, and let $B\subseteq S$ be central near 0. Assume that for every first entry c of A, $cS\cap B$ is central near 0. Then there exist sequences $\langle x_{1,t}\rangle_{t=1}^\infty,\langle x_{2,t}\rangle_{t=1}^\infty,\ldots,\langle x_{v,t}\rangle_{t=1}^\infty$ in S such that for each $i\in\{1,2,\ldots,v\}$, $\sum_{t=1}^\infty x_{i,t}$ converges and for each $F\in\mathcal{P}_f(\mathbb{N})$, $A\vec{x_F}\in B^u$, where*

$$x_F=\begin{pmatrix}\Sigma_{t\in F}\ x_{1,t}\\\Sigma_{t\in F}\ x_{2,t}\\\vdots\\\Sigma_{t\in F}\ x_{v,t}\end{pmatrix}.$$

Proof. We proceed by induction on v. If $v = 1$, then by deleting repeated rows we have that $A = (c)$ for some $c \in \mathbb{N}$. Now $cS \cap B$ is central near 0, so in particular there is some idempotent $p \in 0^+$ such that $cS \cap B \in p$. By Theorem 3.1, there is some sequence $\langle y_n \rangle_{n=1}^\infty$ in S such that $\lim\limits_{n\to\infty} y_n = 0$ and $FS(\langle y_n \rangle_{n=1}^\infty) \subseteq cS \cap B$. By thinning the sequence, we may assume that $\sum_{n=1}^\infty y_n$ converges. For each $n \in \mathbb{N}$, let $x_{1,n} = y_n/c$. Since each $y_n \in cS$ we have that each $x_{1,n} \in S$.

Now let $v \in \mathbb{N}$ be given and assume the result is true for v. Let A be a $u \times (v+1)$ matrix satisfying the first entries condition with the property that for each first entry c of A, $cS \cap B$ is central near 0. By adding additional rows if necessary, we may assume that we have some $\ell \in \{1, 2, \ldots, u-1\}$ and some $c \in \mathbb{N}$ such that for each $j \in \{1, 2, \ldots, u\}$,

$$a_{j,1} = \begin{cases} 0 & \text{if } j \le \ell \\ c & \text{if } j > \ell\ . \end{cases}$$

Let D be the $\ell \times v$ matrix defined by $d_{j,i} = a_{j,i+1}$ and note that D satisfies the first entries condition and all of the first entries of D are first entries of A. Choose sequences $\langle x_{1,t} \rangle_{t=1}^\infty, \langle x_{2,t} \rangle_{t=1}^\infty, \ldots, \langle x_{v,t} \rangle_{t=1}^\infty$ in S as guaranteed by the induction hypothesis for D and B. For each $j \in \{\ell+1, \ell+2, \ldots, u\}$ and each $t \in \mathbb{N}$, let $y_{j,t} = \Sigma_{i=2}^{v+1} a_{j,i} \cdot x_{i-1,t}$. By simultaneously thinning the original sequences we may assume that for each $j \in \{\ell+1, \ell+2, \ldots, u\}$ and each $k \in \mathbb{N}$, $\Sigma_{t=k}^\infty |y_{j,t}| < 1/k$. For $j \in \mathbb{N} \backslash \{\ell+1, \ell+2, \ldots, u\}$ and $t \in \mathbb{N}$, let $y_{i,t} = y_{u,t}$. Let $Y = \langle \langle y_{i,t} \rangle_{t=1}^\infty \rangle_{i=1}^\infty$ and note that $Y \in \mathcal{Y}$.

By Theorem 4.11 choose sequences $\langle b_n \rangle_{n=1}^\infty$ in S and $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that

(a) for each $n \in \mathbb{N}$, $b_n < 1/n$ and $\max H_n < \min H_{n+1}$ and
(b) for each $f \in \Phi$, $FS(\langle b_n + \Sigma_{t \in H_n}\, y_{f(n),t} \rangle_{n=1}^\infty) \subseteq B \cap cS$.

By discarding the first u terms (so that $f \in \Phi$ takes on values up to u) we may presume that for each $j \in \{\ell+1, \ell+2, \ldots, u\}$ and each $F \in \mathcal{P}_f(\mathbb{N})$ one has $\Sigma_{n \in F}(b_n + \Sigma_{t \in H_n}\, y_{j,t} \in B \cap cS$. For each $n \in \mathbb{N}$, let $z_{1,n} = b_n/c$ and for each $n \in \mathbb{N}$ and each $i \in \{2, 3, \ldots, v+1\}$, let $z_{i,n} = \Sigma_{t \in H_n}\, x_{i-1,t}$. We claim that the sequences $\langle z_{1,t} \rangle_{t=1}^\infty, \langle z_{2,t} \rangle_{t=1}^\infty, \ldots, \langle z_{v+1,t} \rangle_{t=1}^\infty$ are as required.

Certainly for each $i \in \{1, 2, \ldots, v\}$, $\sum_{t=1}^\infty z_{i,t}$ converges. Let $j \in \{1, 2, \ldots, u\}$ and $F \in \mathcal{P}_f(\mathbb{N})$ be given. We show that

$$\Sigma_{i=1}^{v+1}\, a_{j,i} \cdot \Sigma_{n \in F}\, z_{i,n} \in B\ .$$

Let $G = \bigcup_{n \in F}\, H_n$. First assume $j \in \{1, 2, \ldots, \ell\}$. Then

$$\begin{aligned} \Sigma_{i=1}^{v+1}\, a_{j,i} \cdot \Sigma_{n \in F}\, z_{i,n} &= \Sigma_{i=2}^{v+1}\, a_{j,i} \cdot \Sigma_{n \in F} \Sigma_{t \in H_n}\, x_{i-1,t} \\ &= \Sigma_{i=1}^{v}\, d_{j,i} \cdot \Sigma_{t \in G}\, x_{i,t} \in B \end{aligned}$$

by the induction hypothesis.

Now assume $j \in \{\ell+1, \ell+2, \ldots, u\}$. Then

$$\begin{aligned} &\Sigma_{i=1}^{v+1}\, a_{j,i} \cdot \Sigma_{n \in F}\, z_{i,n} \\ &\qquad = c \cdot \Sigma_{n \in F}\, z_{1,n} + \Sigma_{i=2}^{v+1}\, a_{j,i} \cdot \Sigma_{n \in F} \Sigma_{t \in H_n}\, x_{i-1,t} \\ &\qquad = \Sigma_{n \in F}(b_n + \Sigma_{t \in H_n} \Sigma_{i=2}^{v+1}\, a_{j,i} \cdot x_{i-1,t}) \\ &\qquad = \Sigma_{n \in F}(b_n + \Sigma_{t \in H_n}\, y_{j,t})\ . \end{aligned}$$

■

We do not know whether the requirement that for every first entry c of A, $cS \cap B$ is central near 0 in Theorem 5.3 is needed. In the most natural examples of S, it holds. In case $S = \mathbb{R} \cap (0, \infty)$ or $S = \mathbb{Q} \cap (0, \infty)$, it is trivial since for every $c \in \mathbb{N}$, $cS = S$. We see now that it also holds in case that $S = \mathbb{D} \cap (0, \infty)$, where $\mathbb{D}$ is the set of dyadic rationals.

Theorem 5.4. *Let $S = \mathbb{D} \cap (0,\infty)$, let $c \in \mathbb{N}$, and let B be central near 0 in S. Then $B \cap cS$ is central near 0.*

Proof. Pick an idempotent $p \in K$ such that $B \in p$. We claim that $B \cap cS \in p$. Suppose instead that $B \backslash cS \in p$. Pick by Theorem 3.1 some sequence $\langle x_n \rangle_{n=1}^\infty$ with $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq B \backslash cS$. Let $m = (c-1)^2 + 1$ and pick some $k \in \mathbb{N}$ such that $\{2^k x_1, 2^k x_2, \ldots, 2^k x_m\} \subseteq \mathbb{N}$. If any $n \in \{1,2,\ldots,m\}$ has $c | 2^k x_n$, then for some $a \in \mathbb{N}$, $x_n = c \cdot \frac{a}{2^k} \in cS$, a contradiction. Thus for each $n \in \{1,2,\ldots,m\}$, there is some $i \in \{1,2,\ldots,c-1\}$ such that $x_n \equiv i (\mathrm{mod} c)$. Pick by the pigeon hole principle some $F \subseteq \{1,2,\ldots,m\}$ and some $i \in \{1,2,\ldots,c-1\}$ such that $|F| = c$ and for each $n \in F$, $n \equiv i (\mathrm{mod} c)$. Then $c | \Sigma_{n \in F}\, x_n$ so, as above, $\Sigma_{n \in F}\, x_n \in cS$, a contradiction. ∎

We are now ready to derive some combined additive and multiplicative results for some (additive) semigroups S for which S is also a semigroup under multiplication.

Note that the conclusion "$A\vec{x_F} \in B^u$" in Theorem 5.3 can be rewritten as "for each $i \in \{1,2,\ldots,u\}$, $\Sigma_{j=1}^v\, a_{i,j} \cdot \Sigma_{t \in F}\, x_{j,t} \in B$". Because we don't have the convenient matrix notation available for the multiplicative result, we state the conclusion of the following theorem in the above form.

Theorem 5.5. *Let S be a subsemigroup of $\big((0,1),\cdot\big)$. Let $u,v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Z}$ which satisfies the first entries condition, and let $B \subseteq S$ be central in $(S,\cdot)$. Assume that for every first entry c of A, $B \cap \{x^c : x \in S\}$ is central in $(S,\cdot)$. Then there exist sequences $\langle x_{1,t} \rangle_{t=1}^\infty, \langle x_{2,t} \rangle_{t=1}^\infty, \ldots, \langle x_{v,t} \rangle_{t=1}^\infty$ in S such that for each $F \in \mathcal{P}_f(\mathbb{N})$ and for each $i \in \{1,2,\ldots,u\}$, $\Pi_{j=1}^v (\Pi_{t \in F}\, x_{j,t})^{a_{i,j}} \in B$.*

Proof. This is the translation to multiplicative notation of what was proved in [14, Lemma 2.4]. ∎

We now show that in many circumstances we will be able to utilize Theorems 5.3 and 5.5 simultaneously.

In the following theorem, we ignore the fact that an ultrafilter on $S \cap (0,1)$ is not quite the same thing as an ultrafilter on S with $(0,1)$ as a member and so pretend that $0^+ \subseteq \beta\big(S \cap (0,1)\big)_d$.

Theorem 5.6. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ such that $S \cap (0,1)$ is a subsemigroup of $\big((0,1),\cdot\big)$ and assume that for each $y \in S \cap (0,1)$ and each $x \in S$, $x/y \in S$ and $yx \in S$. Let $r \in \mathbb{N}$ and let $S \cap (0,1) = \bigcup_{i=1}^r B_i$. Then there is some $i \in \{1,2,\ldots,r\}$ such that B_i is central near 0 and B_i is central in $(S \cap (0,1),\cdot)$.*

Proof. Let $M = \{p \in 0^+ :$ for all $A \in p$, A is central near $0\}$. Note that if $p + p = p \in K$, then $p \in M$ so $M \neq \emptyset$. We claim that M is a left ideal of $\beta\big(S \cap (0,1)\big)_d$. To see this, let $p \in M$, let $q \in \beta\big(S \cap (0,1)\big)_d$, and let $A \in q \cdot p$. Then $\{y \in S \cap (0,1) : y^{-1}A \in p\} \in q$, so pick $y \in S \cap (0,1)$ such that $y^{-1}A \in p$. Then $y^{-1}A$ is central near 0, so by Lemma 4.8, A is central near 0.

Since M is a left ideal of $\beta\big(S \cap (0,1)\big)_d$, pick a minimal left ideal L of $\beta\big(S \cap (0,1)\big)_d$ with $L \subseteq M$ and pick $p = p \cdot p \in L$ and note that p is in the smallest ideal of $\big(\beta\big(S \cap (0,1)\big)_d,\cdot\big)$ so all of its members are central in $(S \cap (0,1),\cdot)$. Pick $i \in \{1,2,\ldots,r\}$ such that $B_i \in p$. Then B_i is central in $(S \cap (0,1),\cdot)$ and, since $p \in M$, B_i is central near 0. ∎

We see now that as a consequence of Theorems 5.3, 5.5, and 5.6, we obtain combined additive and multiplicative results of varying strength, depending on the choice of S. We illustrate the results with the semigroups $\mathbb{R}^+ = \mathbb{R} \cap (0, \infty)$, $\mathbb{Q}^+ = \mathbb{Q} \cap (0, \infty)$, and $\mathbb{D}^+ = \mathbb{D} \cap (0, \infty)$. (Recall that $\mathbb{D}$ is the set of dyadic rationals.)

Note that there is no loss of power in the next result by applying the same matrix both additively and multiplicatively. Indeed, if A and B are matrices satisfying the first entries condition, the matrix

$$\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$$

also satisfies the first entries condition (where $\mathbf{O}$ is a matrix of the appropriate size with all zero entries).

Theorem 5.7. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Z}$ which satisfies the first entries condition, let $r \in \mathbb{N}$, and let $\mathbb{R} \cap (0,1) = \bigcup_{k=1}^{r} B_k$. There exists $k \in \{1, 2, \ldots, r\}$ and there exist sequences*

$$\langle x_{1,t} \rangle_{t=1}^{\infty}, \langle x_{2,t} \rangle_{t=1}^{\infty}, \ldots, \langle x_{v,t} \rangle_{t=1}^{\infty} \text{ and } \langle y_{1,t} \rangle_{t=1}^{\infty}, \langle y_{2,t} \rangle_{t=1}^{\infty}, \ldots, \langle y_{v,t} \rangle_{t=1}^{\infty}$$

in $\mathbb{R} \cap (0,1)$ such that for each $j \in \{1, 2, \ldots, v\}$, $\lim_{t \to \infty} x_{j,t} = 0$ and for each $F \in \mathcal{P}_f(\mathbb{N})$ and each $i \in \{1, 2, \ldots, u\}$, $\Sigma_{j=1}^{v} a_{i,j} \cdot \Sigma_{t \in F} x_{j,t} \in B_k$ and $\Pi_{j=1}^{v} (\Pi_{t \in F} y_{j,t})^{a_{i,j}} \in B_k$.

Proof. For each $y \in \mathbb{R} \cap (0,1)$ and each $x \in \mathbb{R}$, $x/y \in \mathbb{R}$ and $yx \in \mathbb{R}$ so, by Theorem 5.6, pick $k \in \{1, 2, \ldots, r\}$ such that B_k is central near 0 and is also central in $(\mathbb{R} \cap (0,1), \cdot)$.

Pick sequences $\langle x_{1,t} \rangle_{t=1}^{\infty}, \langle x_{2,t} \rangle_{t=1}^{\infty}, \ldots, \langle x_{v,t} \rangle_{t=1}^{\infty}$ as guaranteed by Theorem 5.3.

Given $c \in \mathbb{N}$, $\{x^c : x \in \mathbb{R} \cap (0,1)\} = \mathbb{R} \cap (0,1)$, so pick sequences $\langle y_{1,t} \rangle_{t=1}^{\infty}, \langle y_{2,t} \rangle_{t=1}^{\infty}, \ldots, \langle y_{v,t} \rangle_{t=1}^{\infty}$ in $\mathbb{R} \cap (0,1)$ as guaranteed by Theorem 5.5. ∎

We see now that Theorem 5.7 fails badly in $\mathbb{Q}$.

Theorem 5.8. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

let $B_1 = \{x^3 : x \in \mathbb{Q} \cap (0,1)\}$ and let $B_2 = \big(\mathbb{Q} \cap (0,1)\big) \setminus B_1$. There do not exist $k \in \{1,2\}$ and $x_1, x_2, x_3, y_1, y_2, y_3$ in $\mathbb{Q} \cap (0,1)$ such that for each $i \in \{1,2,3\}$, $\Sigma_{j=1}^{3} a_{i,j} \cdot x_j \in B_k$ and $\Pi_{j=1}^{3} {y_j}^{a_{i,j}} \in B_k$.

Proof. Suppose one has such $k \in \{1,2\}$ and $x_1, x_2, x_3, y_1, y_2, y_3$ in $\mathbb{Q} \cap (0,1)$. Since ${y_3}^3 \in B_k$, we have $k = 1$. But then we have $\{x_1, x_2, x_1 + x_2\} \subseteq B_1$. Multiplying by the product of the denominators, one obtains a solution in positive integers to the equation $a^3 + b^3 = c^3$, contradicting an instance of Fermat's Last Theorem which has long been known to be valid. ∎

We do get a weaker version of Theorem 5.7 to hold in $\mathbb{Q}$. In this weaker version, restricting A and C to be the same size is merely a notational convenience; either can be expanded by adding rows and columns to fit without disturbing the first entries condition.

Theorem 5.9. *Let $u, v \in \mathbb{N}$, let A and C be $u \times v$ matrices with entries from $\mathbb{Z}$ which satisfy the first entries condition, let $r \in \mathbb{N}$, and let $\mathbb{Q} \cap (0,1) = \bigcup_{k=1}^{r} B_k$. Assume that 1 is the only first entry of C. There exists $k \in \{1, 2, \ldots, r\}$ and there exist sequences $\langle x_{1,t}\rangle_{t=1}^{\infty}, \langle x_{2,t}\rangle_{t=1}^{\infty}, \ldots, \langle x_{v,t}\rangle_{t=1}^{\infty}$ and $\langle y_{1,t}\rangle_{t=1}^{\infty}, \langle y_{2,t}\rangle_{t=1}^{\infty}, \ldots, \langle y_{v,t}\rangle_{t=1}^{\infty}$ in $\mathbb{Q} \cap (0,1)$ such that for each $j \in \{1, 2, \ldots, v\}$, $\lim_{t\to\infty} x_{j,t} = 0$ and for each $F \in \mathcal{P}_f(\mathbb{N})$ and each $i \in \{1, 2, \ldots, u\}$, $\Sigma_{j=1}^{v} a_{i,j} \cdot \Sigma_{t\in F} x_{j,t} \in B_k$ and $\Pi_{j=1}^{v}(\Pi_{t\in F} y_{j,t})^{c_{i,j}} \in B_k$.*

Proof. For each $y \in \mathbb{Q} \cap (0,1)$ and each $x \in \mathbb{Q}$, $x/y \in \mathbb{Q}$ and $yx \in \mathbb{Q}$ so, by Theorem 5.6, pick $k \in \{1, 2, \ldots, r\}$ such that B_k is central near 0 and is also central in $(\mathbb{Q} \cap (0,1), \cdot)$.

Pick sequences $\langle x_{1,t}\rangle_{t=1}^{\infty}, \langle x_{2,t}\rangle_{t=1}^{\infty}, \ldots, \langle x_{v,t}\rangle_{t=1}^{\infty}$ as guaranteed by Theorem 5.3.

Since 1 is the only first entry of C, we may pick sequences

$$\langle y_{1,t}\rangle_{t=1}^{\infty}, \langle y_{2,t}\rangle_{t=1}^{\infty}, \ldots, \langle y_{v,t}\rangle_{t=1}^{\infty}$$

in $\mathbb{Q} \cap (0,1)$ as guaranteed by Theorem 5.5. ■

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