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# ALGEBRA IN THE STONE-ČECH COMPACTIFICATION–AN UPDATE

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Abstract. The first edition of the book *Algebra in the Stone-Čech compactification* was published in 1998 and the second edition in 2012. Since that time there have been many new results published about the algebraic structure of the Stone-Čech compactification $\beta S$ of the discrete semigroup $S$ and the combinatorial applications of that structure, mostly in the area of Ramsey Theory. We present here, with proofs so far as possible, what we believe to be some of the most significant of these new results.

## Part 1. Introduction

There has been a substantial amount of research on the algebraic structure of the Stone-Čech compactification of a discrete semigroup or its combinatorial applications since the publication of [72]. In this paper we present a few of what we feel are the most significant and striking of these results.

We shall assume that the reader is familiar with the basic structure of $\beta S$ as presented in [72, Part I]. We will provide detailed proofs of the results we present. The only result that we use and do not prove is the density Hales-Jewett Theorem, Theorem 2.1.

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In Part 2 of this paper we present some new Ramsey theoretic applications.

Early in the applications of the algebraic structure of $\beta S$ to Ramsey Theory came some results about the combined additive and multiplicative structure of $\mathbb{N}$. Specifically, it was shown in [57] that if $\mathbb{N}$ is finitely colored there exist sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^\infty)\cup FP(\langle y_n\rangle_{n=1}^\infty)$ is monochromatic, where $FS(\langle x_n\rangle_{n=1}^\infty)=\{\sum_{t\in F}x_t : F\in\mathcal{P}_f(\mathbb{N})\}$ and $FP(\langle y_n\rangle_{n=1}^\infty)=\{\prod_{t\in F}y_t : F\in\mathcal{P}_f(\mathbb{N})\}$ and $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of $X$. Shortly thereafter it was shown that there is a 2-coloring of $\mathbb{N}$ for which there is no sequence $\langle x_n\rangle_{n=1}^\infty$ with $FS(\langle x_n\rangle_{n=1}^\infty)\cup FP(\langle x_n\rangle_{n=1}^\infty)$ monochromatic.

Since at least 1985 the first author of the current paper has maintained that it is a fact that if $m,r\in\mathbb{N}$ and $\mathbb{N}$ is $r$-colored, there exists $\langle x_n\rangle_{n=1}^m$ such that $FS(\langle x_n\rangle_{n=1}^m)\cup FP(\langle x_n\rangle_{n=1}^m)$ is monochromatic. Note that he has not claimed that he could prove that fact. And the only instance that has been proved is $m=r=2$. That remains the situation today, but dramatic progress has been made recently, beginning with the result [108] of Joel Moriera that whenever $r\in\mathbb{N}$ and $\mathbb{N}$ is $r$-colored, there exist a color class $C$ and infinitely many $y$ such that $\{x\in\mathbb{N}:\{x,xy,x+y\}\subseteq C\}$ is infinite – in fact that set is piecewise syndetic. We present that result in Section 1.

Noticeably missing from the above result is $y$ itself. In Section 2 we present the result [25] of Matt Bowen and Marcin Sabok that whenever $r\in\mathbb{N}$ and $\mathbb{Q}$ is $r$-colored, there exist a color class $C$ and infinitely many $y$ such that $\{x\in\mathbb{N}:\{x,y,xy,x+y\}\subseteq C\}$ is infinite. That is, the claim above is valid for $m=2$ and all $r$, provided one replaces the requirement that $x$ and $y$ come from $\mathbb{N}$ by the requirement that they come from $\mathbb{Q}$.

In [117] Alessandro Sisto proved that whenever $\mathbb{N}\setminus\{1\}$ is 2-colored, there exist infinitely many monochromatic exponential triples, that is sets of the form $\{a,b,b^a\}$. In [114] Julian Sahasrabudhe extended this result to any finite coloring of $\mathbb{N}\setminus\{1\}$. In Section 3 we present the very simple proof [44] of Sahasrabudhe's result by Mauro Di Nasso and Mariaclara Ragosta as well as a new infinitary extension.

In Section 4 we present a new result of Vitaly Bergelson, John Johnson, and Joel Moreira about configurations of polynomials from $\mathbb{Z}^j$ to $\mathbb{Z}$ with zero constant terms for $j\in\mathbb{N}$.

In Part 3 we present some new results about the algebraic structure of $\beta S$.

In a handwritten manuscript written in 1978, Eric K. van Douwen asked whether there exist topological and algebraic copies of $\beta\mathbb{N}$ in $\mathbb{N}^*$. That question was answered in the negative in [122], where it was shown that if $\varphi:\beta\mathbb{N}\to\mathbb{N}^*$ is a continuous homorphism, then $\varphi[\beta\mathbb{N}]$ is finite.

The question then immediately arose as to whether the image could be nontrivial. That question remained open for 29 years. We present the strong affirmative answer by Yevhen Zelenyuk [132] in Section 5.

In Section 6 we present results from [80] showing that if $S$ is a countably infinite cancellative semigroup, then several simply defined algebraic subsets are not at all simple topologically. Specifically under assumptions a bit weaker than cancellativity, the set of idempotents, $K(\beta S)$, $p+\beta S$ for any $p \in S^*$, and $S^*S^*$ are not Borel.

Given idempotents $p$ and $q$ in $(\beta S,+)$, $p \leq_R q$ if and only if $p = q+p$, $p \leq_L q$ if and only if $p = p+q$, and $p \leq q$ if and only if $p = q+p = p+q$. We write $p <_R q$ provided $p \leq_R q$ and it is not true that $q \leq_R p$.

In [95, Theorem 5.4] it was shown that there exists a sequence $\langle p_n \rangle_{n=1}^{\infty}$ of idempotents in $\beta\mathbb{N}$ such that $p_n <_R p_{n+1}$ for each $n \in \mathbb{N}$. (It was also shown in [95] that for each countable ordinal $\lambda$, there is a sequence $\langle p_\sigma \rangle_{\sigma<\lambda}$ of idempotents in $\beta\mathbb{N}$ such that $p_\sigma > p_\tau$ whenever $\sigma < \tau < \lambda$.) In Section 7 we will present the result from [79] that there are increasing $<_R$ chains of idempotents in $\beta\mathbb{N}$ of length $\omega_1$.

One of the oldest questions about the algebra of the Stone-Čech compactification was whether every point of $\beta\mathbb{Z}\setminus\mathbb{Z} = \mathbb{Z}^*$ is a member of some maximal orbit closure of the shift function. This question was asked to Mary Ellen Rudin by some now anonymous analysts in the late 1970's or early 1980's before it was widely known that $\beta\mathbb{Z}$ *had* an algebraic structure. The *shift function* $\sigma : \mathbb{Z} \to \mathbb{Z}$ is defined by $\sigma(n) = n+1$. Letting $\widetilde{\sigma} : \beta\mathbb{Z} \to \beta\mathbb{Z}$ be its continuous extension, one has for $p \in \mathbb{Z}^*$ that the *orbit closure* of $p$ is $c\ell\{\widetilde{\sigma}^n(p) : n \in \mathbb{Z}\} = \beta\mathbb{Z} + p$. So the question was whether every point of $\mathbb{Z}^*$ is a member of a maximal principal left ideal of $\beta\mathbb{Z}$. This question was finally answered in the affirmative recently by Yevhen Zelenyuk who showed [133] that there does not exist a strictly increasing sequence of principal left ideals of $\beta\mathbb{Z}$. We present this result in Section 8. Notice that as an immediate consequence, there does not exist a sequence of idempotents $\langle p_n \rangle_{n=1}^{\infty}$ such that $p_n <_L p_{n+1}$ for each $n$.

## Part 2. Sums, Products, Exponents, and Polynomials

### 1. $x$, $xy$, $x+y$ IN $\mathbb{N}$

In this section we present Moreira's proof [108] that if $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, then there exist $i \in \{1,2,\ldots,r\}$ and infinitely many $y$ such that $\{x \in \mathbb{N} : \{x, xy, x+y\} \subseteq C_i\}$ is piecewise syndetic in $(\mathbb{N},+)$. We also derive the result of Bergelson and Moreira [16, Theorem 4.1] that a similar result holds in any infinite field.

**Lemma 1.1.** *Let $(S,+)$ be an infinite semigroup, let $L$ be a minimal left ideal of $(\beta S,+)$, and let $A$ be a subset of $S$ such that $\overline{A}\cap L\neq\emptyset$. There exists $E$, a syndetic subset of $S$, such that for all $F\in\mathcal{P}_f(E)$ there exists $X\subseteq S$ such that $\overline{X}\cap L\neq\emptyset$ and $F+X\subseteq A$.*

*Proof.* Pick $q\in\overline{A}\cap L$ and let $E=\{x\in S:-x+A\in q\}$. By [72, Theorem 4.39], $E$ is syndetic in $S$. Let $F\in\mathcal{P}_f(E)$ and let $X=\bigcap_{f\in F}(-f+A)$. Then $F+X\subseteq A$ and since $X\in q$, $\overline{X}\cap L\neq\emptyset$. □

**Definition 1.2.** A *semiring* is a triple $(S,+,\cdot)$ such that $(S,+)$ is a commutative semigroup, $(S,\cdot)$ is a semigroup, and for all $a,b,c\in S$, $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$.

The following result is due to John H. Johnson, Jr. in a personal communication. In the case $S=\mathbb{N}$, it provides a simplified proof of a special case of [58, Corollary 3.8] which was in turn a simplification of a special case of [11, Theorem C].

**Theorem 1.3.** *Let $(S,+,\cdot)$ be an infinite semiring, let $L$ be a minimal left ideal of $(\beta S,+)$, let $A$ be a subset of $S$ such that $\overline{A}\cap L\neq\emptyset$, let $v$ be an idempotent in $(\beta S,+)$, and let $M\in\mathcal{P}_f(S)$. Then*

$$\{n\in S:\overline{A}\cap L\cap\textstyle\bigcap_{m\in M}(\overline{-mn+A})\neq\emptyset\}\in v\,.$$

*In particular, If $A$ is piecewise syndetic in $(S,+)$ and $M\in\mathcal{P}_f(S)$, then*

$$\{n\in S:A\cap\textstyle\bigcap_{m\in M}(-mn+A)\text{ is piecewise syndetic in }(S,+)\}$$

*is an $IP^*$-set in $(S,+)$.*

*Proof.* Let $C=\{n\in S:\overline{A}\cap L\cap\bigcap_{m\in M}(\overline{-mn+A})\neq\emptyset\}$. To show that $C\in v$ it suffices to show that for every $B\in v$, $C\cap B\neq\emptyset$, so let $B\in v$. Since $v$ is an idempotent, pick a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $S$ such that $FS(\langle x_n\rangle_{n=1}^{\infty})\subseteq B$.

We claim that

$(*)$ if $n\in S$ and there exists $X\subseteq A$ such that $\overline{X}\cap L\neq\emptyset$ and $\{mn:m\in M\}+X\subseteq A$, then $n\in C$.

To establish $(*)$, let $n\in S$ and assume we have $X\subseteq A$ such that $\overline{X}\cap L\neq\emptyset$ and $\{mn:m\in M\}+X\subseteq A$. Pick $r\in\overline{X}\cap L$. Since $X\subseteq A$, we have that $r\in\overline{A}\cap L$. To see that $n\in C$ we show that for $m\in M$, $(-mn+A)\in r$. Given $m\in M$, we have $mn+X\subseteq A$ so $X\subseteq(-mn+A)$ so $(mn+A)\in r$.

Pick by Lemma 1.1 a syndetic set $E\subseteq S$ such that for all $F\in\mathcal{P}_f(E)$ there exists $X\subseteq S$ such that $\overline{X}\cap L\neq\emptyset$ and $F+X\subseteq A$.

For $m\in M$, define $f_m\in{}^{\mathbb{N}}S$ by $f_m(t)=mx_t$. By [72, Theorem 14.8.3] $E$ is a J-set so pick by [61, Theorem 4.1] some $a\in E$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that for $m \in M$, $a + \sum_{t\in H} f_m(t) \in E$. Let $F = \{a\} \cup \big\{a + \sum_{t\in H} f_m(t) : m \in M\big\}$. Pick $X \subseteq \mathbb{N}$ such that $\overline{X} \cap L \neq \emptyset$ and $F + X \subseteq A$.

We claim that $\sum_{t\in H} x_t \in C$, so that $B \cap C \neq \emptyset$ as required. We have that $a + X \subseteq A$ and $\{m \sum_{t\in H} x_t : m \in M\} + (a + X) \subseteq A$ so by $(*)$ it suffices to show that $\overline{a + X} \cap L \neq \emptyset$. By the continuity of $\lambda_a$, $\overline{a + X} = a + \overline{X}$. Pick $r \in \overline{X} \cap L$. Then $a + r \in L$ and $a + r \in a + \overline{X} = \overline{a + X}$. □

**Lemma 1.4.** *Let $(S, +, \cdot)$ be an infinite semiring. For all $x \in S$ and all $p, q \in \beta S$, $x(p + q) = xp + xq$ and $(p + q)x = px + qx$.*

*Proof.* For $p \in \beta S$, let $l_p$, $r_p$, $\lambda_p$, and $\rho_p$ be functions from $\beta S$ to $\beta S$ defined by, for $q \in \beta S$, $l_p(q) = pq$, $r_p(q) = qp$, $\lambda_p(q) = p + q$, and $\rho_p(q) = q + p$. Recall that for each $p \in \beta S$, $r_p$ and $\rho_p$ are continuous and for each $x \in S$, $l_x$ and $\lambda_x$ are continuous.

Let $x \in S$ and let $p, q \in \beta S$. To see that $x(p + q) = xp + xq$, it suffices that $l_x \circ \rho_q$ and $\rho_{xq} \circ l_x$ agree on $S$, so let $y \in S$. We need to show that $x(y + q) = xy + xq$ which is true because $l_x \circ \lambda_y$ and $\lambda_{xy} \circ l_x$ agree on $S$.

To see that $(p + q)x = px + qx$ it suffices that $r_x \circ \rho_q$ and $\rho_{qx} \circ r_x$ agree on $S$, so let $y \in S$. We need to show that $(y + q)x = yx + qx$ which is true because $r_x \circ \lambda_y$ and $\lambda_{yx} \circ r_x$ agree on $S$. □

In the proofs of Lemmas 1.5, 1.6, and 1.7 we use the fact that, by [72, Theorem 1.67], a point $x \in \beta S$ is in $K(\beta S)$ if and only if for each $q \in \beta S$ there exists $u \in \beta S$ such that $x = u + q + x$.

**Lemma 1.5.** *Let $A \subseteq \mathbb{N}$ be piecewise syndetic in $(\mathbb{N}, +)$ and let $y \in \mathbb{N}$. Then $Ay$ is piecewise syndetic in $(\mathbb{N}, +)$.*

*Proof.* Pick $x \in \overline{A} \cap K(\beta\mathbb{N})$. Pick an idempotent $q \in K(\beta\mathbb{N})$. By [72, Lemma 5.19.2], $\frac{1}{y} \cdot q \in \beta\mathbb{N}$, where $\frac{1}{y} \cdot q$ is the product in $(\beta\mathbb{Q}_d, \cdot)$. Pick $u \in \beta\mathbb{N}$ such that $x = u + \frac{1}{y} \cdot q + x$. By Lemma 1.4, $y$ distributes over $\beta\mathbb{N}$ and it is easy to verify that $y \cdot \frac{1}{y} \cdot q = q$ so $xy = uy + q + xy \in K(\beta\mathbb{N}) \cap \overline{Ay}$. □

**Lemma 1.6.** *Let $(S, +, \cdot)$ be a field, let $y \in S \setminus \{0\}$, and let $A \subseteq S$ be piecewise syndetic in $(S, +)$. Then $Ay$ is piecewise syndetic in $(S, +)$.*

*Proof.* Pick $x \in \overline{A} \cap K(\beta S, +)$ and pick an idempotent $q$ in $K(\beta\mathbb{N}, +)$. Then $qy^{-1} \in \beta S$. Pick $u \in \beta S$ such that $x = u + qy^{-1} + x$. By Lemma 1.4, $y$ distributes over $\beta S$ so $xy = uy + qy^{-1}y + xy = uy + q + xy \in K(\beta S) \cap Ay$. □

**Lemma 1.7.** *Let $y \in \mathbb{N}$ and let $A$ be a piecewise syndetic subset of $\mathbb{N}$ such that $A \subseteq \mathbb{N}y$. Then $A/y$ is piecewise syndetic.*

*Proof.* Pick $x \in \overline{A} \cap K(\beta\mathbb{N})$. Then $x \in \overline{y\mathbb{N}} = y\beta\mathbb{N}$ so pick $z \in \beta\mathbb{N}$ such that $x = yz$. Pick $q \in K(\beta\mathbb{N})$. Then $yq \in \beta\mathbb{N}$ so pick $u \in \beta\mathbb{N}$ such that $yz = u + yq + yz$. Then $u \in \overline{y\mathbb{N}}$ so $u = yw$ for some $w \in \beta\mathbb{N}$. Then $yz = y(w + q + z)$ by [72, Lemma 13.1] so by [72, Lemma 8.1], $z = w + q + z \in K(\beta\mathbb{N}) \cap \overline{A/y}$. $\square$

**Definition 1.8.** Let $(S, \cdot)$ be a semigroup, let $m \in \mathbb{N}$, and let $\langle y_t \rangle_{t=1}^m$ be a sequence in $S$. The sequence satisfies *uniqueness of finite products* if and only if, whenever $H, K \in \mathcal{P}_f(\{1, 2, \ldots, m\})$ and $H \neq K$, then $\prod_{t \in H} y_t \neq \prod_{t \in K} y_t$. If $\langle y_t \rangle_{t=1}^\infty$ is an infinite sequence in $S$, then the sequence satisfies *uniqueness of finite products* if and only if, whenever $H, K \in \mathcal{P}_f(\mathbb{N})$ and $H \neq K$, then $\prod_{t \in H} y_t \neq \prod_{t \in K} y_t$.

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**Lemma 1.9.** *Let $(S, \cdot)$ be a group with identiy $1$, let $m \in \mathbb{N}$, let $\langle y_t \rangle_{t=1}^m$ be a sequence with $FP(\langle y_t \rangle_{t=1}^m) \subseteq S \setminus \{1\}$ satifying uniqlueness of finite products, and let $A$ be an infinite subset of $S$. There exists $y_{m+1} \in A$ such that $FP(\langle y_t \rangle_{t=1}^{m+1}) \subseteq S \setminus \{1\}$ and $\langle y_t \rangle_{t=1}^{m+1}$ satifies uniqueness of finite products.*

*Proof.* Let $B = FP(\langle y_t \rangle_{t=1}^m)$. Pick

$$y_{m+1} \in A \setminus \left(\{1\} \cup B \cup \{b^{-1} : b \in B\} \cup \{b^{-1}c : b, c \in B\}\right).$$

Then $y_{m+1}$ is as required. $\square$

**Theorem 1.10.** *Let $S$ be $\mathbb{N}$ or an infinite field, let $r \in \mathbb{N}$, and let $S = \bigcup_{i=1}^r C_i$. There exist $i \in \{1, 2, \ldots, r\}$ an injective sequence $\langle z_n \rangle_{n=1}^\infty$ in $S$, and a sequence $\langle E_n \rangle_{n=1}^\infty$ of piecewise syndetic subsets of $(S, +)$ such that for each $n \in \mathbb{N}$, $E_n \subseteq S z_n$ and if $w \in E_n$ and $x = w z_n^{-1}$, then $\{x, x z_n, x + z_n\} \subseteq C_i$.*

*Proof.* All references in this proof to piecewise syndetic sets refer to sets piecewise syndetic in $(S, +)$. Choose $t_0 \in \{1, 2, \ldots, r\}$ such that $C_{t_0}$ is piecewise syndetic and let $B_0 = C_{t_0}$. By Lemma 1.3 with $M = \{1\}$, pick $y_1 \in S \setminus \{0, 1\}$ such that $B_0 \cap (B_0 - y_1)$ is piecewise syndetic and let $D_1 = B_0 \cap (B_0 - y_1)$. By Lemma 1.5 or 1.6, $y_1 D_1$ is piecewise syndetic. Since $y_1 D_1 = \bigcup_{i=1}^r (y_1 D_1 \cap C_i)$, pick $t_1 \in \{1, 2, \ldots, r\}$ such that $y_1 D_1 \cap C_{t_1}$ is piecewise syndetic and let $B_1 = (y_1 D_1 \cap C_{t_1})$.

Let $k \in \mathbb{N}$ and assume we have chosen $\langle y_j \rangle_{j=1}^k$, $\langle B_j \rangle_{j=0}^k$, $\langle t_j \rangle_{j=0}^k$, and $\langle D_j \rangle_{j=1}^k$ satisfying the following induction hypotheses.

(1) For $j \in \{1, 2, \ldots, k\}$, $y_j \in S$ and
  (a) if $S = \mathbb{N}$ and $j > 1$, $y_j > y_{j-1}$;
  (b) if $S$ is a field, then $FP(\langle y_t \rangle_{t=1}^k) \subseteq S \setminus \{0, 1\}$ and $FP(\langle y_t \rangle_{t=1}^k)$ satifies uniqueness of finite products.
(2) For $j \in \{1, 2, \ldots, k\}$, $D_j$ is a piecewise syndetic subset of $S$.
(3) For $j \in \{0, 1, \ldots, k\}$, $t_j \in \{1, 2, \ldots, r\}$.

(4) For $j \in \{0, 1, \ldots, k\}$, $B_j$ is a piecewise syndetic subset of $S$.
(5) For $j \in \{0, 1, \ldots, k\}$, $B_j \subseteq C_{t_j}$.
(6) For $j \in \{1, 2, \ldots, k\}$, $B_j \subseteq y_j D_j$.
(7) For $j < m$ in $\{0, 1, \ldots, k\}$, $B_m \subseteq y_m y_{m-1} \cdots y_{j+1} B_j$.
(8) For $m \in \{1, 2, \ldots, k\}$, $D_m \subseteq B_{m-1} \cap (B_{m-1} - y_m)$ and, if $m > 1$, then $D_m \subseteq \bigcap_{j=1}^{m-1} (B_{m-1} - (y_{m-1} y_{m-2} \cdots y_j)^2 y_m)$.

All hypotheses hold for $k = 1$.

For $j \in \{1, 2, \ldots, k\}$, let $u_j = y_k y_{k-1} \cdots y_j$ and let $M = \{1, u_1^2, u_2^2, \ldots, u_k^2\}$. By Lemma 1.3,

$$A = \{y \in S : B_k \cap (B_k - y) \cap \textstyle\bigcap_{j=1}^{k} (B_k - u_j^2 y) \text{ is piecewise syndetic}\}$$

is an $IP^*$-set in $(S, +)$. If $S = \mathbb{N}$, pick $y_{k+1} \in A$ with $y_{k+1} > y_k$. If $S$ is a field, then by Lemma 1.9 applied to the group $(S \setminus \{0\}, \cdot)$ pick $y_{k+1} \in A$ such that $FP(\langle y_t \rangle_{t=1}^{k+1}) \subseteq S \setminus \{0, 1\}$ and $FP(\langle y_t \rangle_{t=1}^{k+1})$ satifies uniqueness of finite products. Let $D_{k+1} = B_k \cap (B_k - y_{k+1}) \cap \bigcap_{j=1}^{k} (B_k - u_j^2 y_{k+1})$. Note that hypotheses (1), (2), and (8) hold at $k+1$.

By Lemma 1.5 or 1.6, $y_{k+1} D_{k+1}$ is piecewise syndetic and

$$y_{k+1} D_{k+1} = \textstyle\bigcup_{i=1}^{r} (y_{k+1} D_{k+1} \cap C_i)$$

so pick $t_{k+1} \in \{1, 2, \ldots, r\}$ such that $y_{k+1} D_{k+1} \cap C_{t_{k+1}}$ is piecewise syndetic and let $B_{k+1} = y_{k+1} D_{k+1} \cap C_{t_{k+1}}$. Note that hypotheses (3), (4), (5), and (6) hold for $k+1$.

We need to verify hypothesis (7) so let $j < m$ in $\{0, 1, \ldots, k+1\}$ be given. If $m \leq k$, then (7) holds by assumption so assume that $m = k+1$. We have $B_{k+1} \subseteq y_{k+1} D_{k+1} \subseteq y_{k+1} B_k$. If $j = k$, we are done, so assume that $j < k$ in which case by (7) at $k$ we have $B_k \subseteq y_k y_{k-1} \cdots y_{j+1} B_j$ so $B_{k+1} \subseteq y_{k+1} y_k \cdots y_{j+1} B_j$ as required.

The construction is complete. Pick $i \in \{1, 2, \ldots, r\}$ such that $\{k \in \mathbb{N} : t_k = i\}$ is infinite and let $G = \{k \in \mathbb{N} : t_k = i\}$. We then choose a sequence $\langle k(n) \rangle_{n=0}^{\infty}$ in $G$ so that, letting $z_n = y_{k(n)} y_{k(n)-1} \cdots y_{k(n-1)+1}$ for $n \in \mathbb{N}$, we have $\langle z_n \rangle_{n=1}^{\infty}$ is an injective sequence. (This is either because $\langle y_n \rangle_{n=1}^{\infty}$ is increasing in $\mathbb{N}$ or satisfies uniqueness of finite products in the field $S$.)

For $n \in \mathbb{N}$, let $E_n = B_{k(n)}$. Then each $E_n$ is piecewise syndetic. Also,

$$E_n = B_{k(n)} \subseteq y_{k(n)} y_{k(n)-1} \cdots y_{k(n-1)+1} B_{k(n-1)} = z_n B_{k(n-1)} \subseteq z_n S\,.$$

Let $w \in E_n$ and let $x = w z_n^{-1}$. We need to show that $\{x, x z_n, x + z_n\} \subseteq C_i$. Now $x z_n = w \in E_n = B_{k(n)} \subseteq C_{t_{k(n)}} = C_i$. Also $x z_n \in E_n \subseteq z_n B_{k(n-1)}$ so $x \in B_{k(n-1)} \subseteq C_{t_{k(n-1)}} = C_i$. It remains to show that

$x+z_n\in C_i$. Now

$$\begin{aligned}
&z_n(x+z_n)\\
&= w+z_n^2 \in B_{k(n)}+z_n^2 \subseteq y_{k(n)}D_{k(n)}+z_n^2\\
&\subseteq y_{k(n)}(B_{k(n)-1}-y_{k(n)}y_{k(n)-1}^2y_{k(n)-2}^2\cdots y_{k(n-1)+1}^2)+z_n^2\\
&\subseteq y_{k(n)}(y_{k(n)-1}y_{k(n)-2}\cdots y_{k(n-1)+1}B_{k(n-1)}\\
&\quad - y_{k(n)}y_{k(n)-1}^2y_{k(n)-2}^2\cdots y_{k(n-1)+1}^2)+z_n^2\\
&= y_{k(n)}y_{k(n)-1}y_{k(n)-2}\cdots y_{k(n-1)+1}B_{k(n-1)}\\
&\quad - y_{k(n)}^2y_{k(n)-1}^2y_{k(n)-2}^2\cdots y_{k(n-1)+1}^2+z_n^2\\
&= z_nB_{k(n-1)}\,.
\end{aligned}$$

So $x+z_n\in B_{k(n-1)}\subseteq C_{t_{k(n-1)}}=C_i$. $\square$

**Corollary 1.11.** *Let $S$ be $\mathbb{N}$ or an infinite field, let $r\in\mathbb{N}$, and let $S=\bigcup_{i=1}^r C_i$. There exist $i\in\{1,2,\ldots,r\}$ and infinitely many $y$ such that $\{x\in\mathbb{N}:\{x,xy,x+y\}\subseteq C_i\}$ is piecewise syndetic.*

*Proof.* Pick $i$, $\langle z_n\rangle_{n=1}^\infty$, and $\langle E_n\rangle_{n=1}^\infty$ as guaranteed by Theorem 1.10. Given $n\in\mathbb{N}$, if $y=z_n$, then $E_ny^{-1}\subseteq\{x\in\mathbb{N}:\{x,xy,x+y\}\subseteq C_i\}$ and by Lemma 1.7 or 1.6, $E_ny^{-1}$ is piecewise syndetic. $\square$

## 2. $x$, $y$, $x+y$ AND $xy$ IN $\mathbb{Q}$

In this section we present the proof by Bowen and Sabok [25] that if $r\in\mathbb{N}$ and $\mathbb{Q}=\bigcup_{i=1}^r C_i$, there exist $i\in\{1,2,\ldots,r\}$ and infinitely many $y$ such that $\{x\in\mathbb{Q}\setminus\{0\}:\{x,y,x+y,xy\}\subseteq C_i\}$ is infinite.

Throughout this section we let $S=\mathbb{Q}\setminus\{0\}$ and for $n\in\mathbb{N}$, we will let $[n]=\{1,2,\ldots,n\}$. We denote the characteristic function of a set $A$ by $\chi_A$.

We will use the density Hales-Jewett Theorem, which we will not prove. See [72, Section 14.2] for the terminology surrounding the Hales-Jewett Theorem.

**Theorem 2.1** (Density Hales-Jewett)**.** *Let $n\in\mathbb{N}$ and $\eta\in(0,1)$. There exists $r\in\mathbb{N}$ such that whenever $C\subseteq[n]^r$ and $|C|\geq\eta n^r$, there is a length $r$ variable word $w$ over the alphabet $[n]$ such that $\{w(t):t\in[n]\}\subseteq C$.*

*Proof.* This is due to Furstenberg and Katznelson in [53]. For a simplified elementary proof see [113] which is an anonymous collaborative effort. $\square$

The next two lemmas are consequences of [7, Theorems 3.2 and 7.5] respectively.

**Lemma 2.2.** *Let $F\in\mathcal{P}_f(\mathbb{Q})$, let $\mathcal{F}:F\to\mathbb{N}$, let $0<\eta<\delta<1$, let $\lambda$ be a left invariant mean on $(\mathbb{Q},+)$, let $A\subseteq\mathbb{Q}$ such that $\lambda(\chi_A)\geq\delta$, and let $R=\{t\in\mathbb{Q}:\sum_{x\in F\cap(A-t)}\mathcal{F}(x)\geq\eta\sum_{x\in F}\mathcal{F}(x)\}$. Then $\lambda(\chi_R)\geq\dfrac{\delta-\eta}{1-\eta}$.*

*Proof.* Define $g : \mathbb{Q} \to [0,1]$ by $g(t) = \dfrac{\sum_{x\in F\cap(A-t)} \mathcal{F}(x)}{\sum_{x\in F} \mathcal{F}(x)}$. Then for $t \in \mathbb{Q}$,

$$\begin{aligned} g(t) &= \frac{1}{\sum_{x\in F} \mathcal{F}(x)} \textstyle\sum_{x\in F} \mathcal{F}(x) \cdot \chi_{(A-t)}(x) \\ &= \frac{1}{\sum_{x\in F} \mathcal{F}(x)} \textstyle\sum_{x\in F} \mathcal{F}(x) \cdot \chi_{(-x+A)}(t)\,, \end{aligned}$$

so

$$\begin{aligned} \lambda(g) &= \frac{1}{\sum_{x\in F} \mathcal{F}(x)} \cdot \big( \textstyle\sum_{x\in F} \mathcal{F}(x) \cdot \lambda(\chi_{(-x+A)})\big) \\ &= \frac{1}{\sum_{x\in F} \mathcal{F}(x)} \cdot \big( \textstyle\sum_{x\in F} \mathcal{F}(x) \cdot \lambda(\chi_A)\big) \end{aligned}$$

since $\lambda$ is invariant. Therefore $\lambda(g) = \dfrac{\lambda(\chi_A)}{\sum_{x\in F} \mathcal{F}(x)} \cdot \sum_{x\in F} \mathcal{F}(x) = \lambda(\chi_A)$.

Since $\lambda$ is additive, $\lambda(\chi_A) = \lambda(g) \le \lambda(g\chi_R) + \lambda(g\chi_{\mathbb{Q}\setminus R})$. Since $g\chi_R \le \chi_R$, $\lambda(g\chi_R) \le \lambda(\chi_R)$. For $t \in \mathbb{Q} \setminus R$, $\sum_{x\in F\cap(A-t)} \mathcal{F}(x) < \eta \sum_{x\in F} \mathcal{F}(x)$ so $g(t) = \dfrac{\sum_{x\in F\cap(A-t)} \mathcal{F}(x)}{\sum_{x\in F} \mathcal{F}(x)} < \eta$ and $\lambda(\chi_{\mathbb{Q}\setminus R}) = 1 - \lambda(\chi_R)$ so $\lambda(\chi_A) \le \lambda(\chi_R) + \eta\big(1 - \lambda(\chi_R)\big)$. Therefore $\lambda(\chi_A) - \eta \le \lambda(\chi_R)\cdot(1-\eta)$ so $\lambda(\chi_R) \ge \dfrac{\delta-\eta}{1-\eta}$. □

**Lemma 2.3.** *Let $n \in \mathbb{N}$ and $0 < \delta < 1$. Let $\lambda$ be an invariant mean on $(\mathbb{Q},+)$ and for $A \subseteq \mathbb{Q}$, let $d(A) = \lambda(\chi_A)$. There exist $r \in \mathbb{N}$ and $\beta > 0$ such that for any $A \subseteq S$ with $d(A) > \delta$ and any $q_1, q_2, \ldots, q_n \in \mathbb{Q}$,*

$$\{x \in S : d\big(\textstyle\bigcap_{i=1}^n (A - q_i x)\big) \ge \beta\} \text{ is } IP_r^*\,.$$

*Proof.* Pick $\eta$ such that $0 < \eta < \delta$. Pick by Theorem 2.1, $r \in \mathbb{N}$ such that whenever $C \subseteq [n]^r$ and $|C| \ge \eta n^r$, there is a length $r$ variable word $w$ over the alphabet $[n]$ such that $\{w(t) : t \in [n]\} \subseteq C$. Let $A \subseteq S$ with $d(A) > \delta$ and let $q_1, q_2, \ldots, q_n \in \mathbb{Q}$.

Let $\beta = \dfrac{\delta-\eta}{(1-\eta)(n+1)^r}$. Let $s_1, s_2, \ldots, s_r \in \mathbb{Q}$. We need to show that there exists $x \in FS(\langle s_i\rangle_{i=1}^r)$ such that $d\big(\bigcap_{i=1}^n (A - q_i x)\big) \ge \beta$.

Define $\psi : [n]^r \to \mathbb{Q}$ by, for $w = l_1 l_2 \cdots l_r \in [n]^r$, $\psi(w) = \sum_{i=1}^r q_{l_i} s_i$. Let $F = \{\psi(w) : w \in [n]^r\}$ and define $\mathcal{F} : F \to \mathbb{N}$ by

$$\mathcal{F}(x) = |\{w \in [n]^r : \psi(w) = x\}|\,.$$

Let $R = \{t \in \mathbb{Q} : \sum_{x\in F\cap(A-t)} \mathcal{F}(x) \ge \eta \sum_{x\in F} \mathcal{F}(x)\}$. Notice that $\sum_{x\in F} \mathcal{F}(x) = n^r$ so $R = \{t \in \mathbb{Q} : \sum_{x\in F\cap(A-t)} \mathcal{F}(x) \ge \eta n^r\}$. By Lemma 2.2, $d(R) \ge \dfrac{\delta-\eta}{1-\eta}$.

Now

$$\begin{aligned}\textstyle\sum_{x\in F\cap(A-t)}\mathcal{F}(x) &= \textstyle\sum_{x\in F\cap(A-t)}\left|\{w\in[n]^r : \psi(w)=x\}\right|\\ &= \left|\{w\in[n]^r:\psi(w)\in A-t\}\right|\\ &= \left|\{w\in[n]^r : t+\psi(w)\in A\}\right|,\end{aligned}$$

so $R=\{t\in\mathbb{Q}:|\{w\in[n]^r:t+\psi(w)\in A\}|\geq\eta n^r\}$.

For a length $r$ variable word $w$ over $[n]$, let

$$B_w=\{t\in R:\{t+\psi\big(w(k)\big):k\in[n]\}\subseteq A\}\,.$$

We claim that $R\subseteq\bigcup\{B_w : w$ is a length $r$ variable word over $[n]\}$. To see this, let $t\in R$ and let $C=\{w\in[n]^r : t+\psi(w)\in A\}$. Then $|C|\geq\eta n^r$ so by the choice of $r$, there is a length $r$ variable word $w$ such that $\{w(k):k\in[n]\}\subseteq C$. That is, $t\in B_w$.

Now we claim that there is a length $r$ variable word $w$ over $[n]$ such that $d(B_w)\geq\beta$. There are $(n+1)^r-n^r<(n+1)^r$ variable words over $[n]$. If for each variable word $w$ one had $d(B_w)<\beta$, then we would have $d(R)<\beta\cdot(n+1)^r=\dfrac{\delta-\eta}{1-\eta}$, a contradiction. So pick a length $r$ variable word $w=l_1l_2\cdots l_r$ over $[n]$ such that $d(B_w)\geq\beta$.

Let $\alpha=\{i\in[r]:l_i=v\}$, where $v$ is the variable. For $k\in[n]$, $\psi\big(w(k)\big)=\sum_{i\in[r]\setminus\alpha}q_{l_i}s_i+\sum_{i\in\alpha}q_ks_i$. Let $u=\sum_{i\in[r]\setminus\alpha}q_{l_i}s_i$ and let $x=\sum_{i\in\alpha}s_i$. Then $\psi\big(w(k)\big)=u+q_kx$ so for each $t\in B_w$, $t+u+q_kx\in A$ so $B_w+u\subseteq\bigcap_{k=1}^n(A-q_kx)$ and $d(B_w+u)=d(B_w)\geq\beta$ so $d(\bigcap_{k=1}^n(A-q_kx)\geq\beta$. □

Notice that one may change the conclusion of Lemma 2.3 to

$$\{x\in S:d\big(A\cap\textstyle\bigcap_{i=1}^n(A-q_ix)\big)\geq\beta\}\text{ is }IP_r^*\,,$$

by replacing $n$ by $n+1$ and letting $q_{n+1}=0$.

**Lemma 2.4.** *Let $k,r,N\in\mathbb{N}$ and let $T_1$, $T_2,\ldots$, $T_k$ be subsets of $S$ that are thick in $(S,\cdot)$. Then there exist $\big\langle\langle S_{l,i}\rangle_{l=1}^k\big\rangle_{i=1}^{N-1}$ such that*

(1) *for $l\in\{1,2,\ldots,k\}$ and $i\in\{1,2,\ldots,N-1\}$*
   (a) *$S_{l,i}$ is a finite $IP_r$ set in $(\mathbb{Q},+)$ and*
   (b) *$S_{l,i}\subseteq T_l$; and*

(2) *for $1\leq i\leq j<N$ and $l_i,l_{i+1},\ldots,l_j\in\{1,2,\ldots,k\}$, $S_{l_i,i}\cdot S_{l_{i+1},i+1}\cdots S_{l_j,j}\subseteq T_{l_i}$.*

*Proof.* Note that if $F$ is $IP_r$ in $(\mathbb{Q},+)$ and $t\in S$, then $Ft$ is $IP_r$ in $(\mathbb{Q},+)$. Consequently, if $V$ is thick in $(S,\cdot)$, then $V$ contains an $IP_r$ set.

We claim that if $V$ is thick in $(S,\cdot)$, $F\in\mathcal{P}_f(S)$, and $R=\{t\in S:Ft\subseteq V\}$, then $R$ is thick in $(S,\cdot)$. To see this, let $G\in\mathcal{P}_f(S)$ be given. Let $H=FG$. Pick $a\in S$ such that $Ha\subseteq T$. Then $FGa\subseteq T$ so $Ga\subseteq R$.

Now we construct $\big\langle\langle S_{l,i}\rangle_{l=1}^k\big\rangle_{i=1}^{N-1}$ by downward induction on $i$. To begin, for $l\in\{1,2,\ldots,k\}$ pick a finite $IP_r$ set $S_{l,N-1}\subseteq T_l$.

Now let $m\in\{2,3,\ldots,N-1\}$ and assume we have $\big\langle\langle S_{l,i}\rangle_{l=1}^k\big\rangle_{i=m}^{N-1}$ satisfying (1) and (2). Let

$$R=\{1\}\cup\bigcup_{j=m}^{N-1}\{S_{l_m,m}\cdot S_{l_{m+1},m+1}\cdots S_{l_j,j}:\\ l_m,l_{m+1},\ldots,l_j\in\{1,2,\ldots,k\}\}\,.$$

For $l\in\{1,2,\ldots,k\}$ pick a finite $IP_r$ set $S_{l,m-1}\subseteq\{x\in S:xR\subseteq T_l\}$. Since $1\in R$, each $S_{l,m-1}\subseteq T_l$. To verify (2), let $m-1\le j<N$ and $l_{m-1},l_m,\ldots,l_j\in\{1,2,\ldots,k\}$ be given. If $j=m-1$ there is nothing to show, so assume $j\ge m$. Then $S_{l_m,m}\cdot S_{l_{m+1},m+1}\cdots S_{l_j,j}\subseteq R$ so $S_{l_{m-1},m-1}\cdot S_{l_m,m}\cdots S_{l_j,j}\subseteq T_{l_{m-1}}$. $\square$

**Lemma 2.5.** *Let $n\in\mathbb{N}$ and let $S=\bigcup_{i=1}^n C_i$. There exist $k\in\mathbb{N}$, subsets $Y_1,Y_2,\ldots,Y_k$ of $\{1,2,\ldots,n\}$, and $F\in\mathcal{P}_f(S)$ such that*

(i) *for all $l\in\{1,2,\ldots,k\}$, $\bigcup_{m\in Y_l}C_m$ is thick in $(S,\cdot)$ and*
(ii) *$(\forall x\in S)(\exists l\in\{1,2,\ldots,k\})(\forall m\in Y_l)(\exists f\in F)(fx\in C_m)$.*

*Proof.* For $Y\subseteq\{1,2,\ldots,n\}$, let $C_Y=\bigcup_{m\in Y}C_m$. Let
$\mathcal{T}=\{Y\subseteq\{1,2,\ldots,n\}:C_Y$ is thick in $(S,\cdot)\}$ and let
$\mathcal{S}=\{Y\subseteq\{1,2,\ldots,n\}:C_Y$ is syndetic in $(S,\cdot)\}$.

Note that $\mathcal{T}\neq\emptyset$ and $\mathcal{S}\neq\emptyset$ since $S$ is both thick and syndetic in $(S,\cdot)$. For $Y\in\mathcal{S}$, pick $F_Y\in\mathcal{P}_f(S)$ such that $S=\bigcup_{t\in F_Y}t^{-1}C_Y$ and let $F=\bigcup_{Y\in\mathcal{S}}F_Y$.

For $x\in S$, let $A_x=\{m\in\{1,2,\ldots,n\}:(\exists f\in F)(fx\in C_m)\}$. Given $x\in S$ and $Y\in\mathcal{S}$, one may pick $f\in F_Y$ such that $fx\in C_Y$ so there is some $m\in Y$ such that $fx\in C_m$ so we have $A_x\cap Y\neq\emptyset$.

We claim that for all $x\in S$, $A_x\in\mathcal{T}$. So let $x\in S$ and suppose that $C_{A_x}$ is not thick in $(S,\cdot)$. Let $V=\{1,2,\ldots,n\}\setminus A_x$. We have $S\setminus C_{A_x}$ is syndetic and $S\setminus C_{A_x}\subseteq C_V$ so $V\in\mathcal{S}$ and thus $A_x\cap V\neq\emptyset$, a contradiction.

Let $\mathcal{R}=\{A_x:x\in S\}$. Since $\mathcal{T}\subseteq\mathcal{P}(\{1,2,\ldots,n\})$, $\mathcal{R}$ is finite. Enumerate $\mathcal{R}$ as $Y_1,Y_2,\ldots,Y_k$. Since $\mathcal{R}\subseteq\mathcal{T}$, conclusion (i) is immediate. To verify (ii), let $x\in S$. Pick $l\in\{1,2,\ldots,k\}$ such that $A_x=Y_l$. By the definition of $A_x$, we have for all $m\in Y_l$, there is some $f\in F$ with $fx\in C_m$. $\square$

**Theorem 2.6.** *Let $n\in\mathbb{N}$ and let $S=\bigcup_{i=1}^n C_i$. There exist $y\in S$ and $m\in\{1,2,\ldots,n\}$ such that $\{x\in S:\{x,y,x+y,xy\}\subseteq C_m\}$ is infinite.*

*Proof.* Pick $k,Y_1,Y_2,\ldots,Y_k$, and $F$ as guaranteed by Lemma 2.5. As before, for $Y\subseteq\{1,2,\ldots,n\}$, let $C_Y=\bigcup_{m\in Y}C_m$. Pick an invariant mean $\lambda$ on $(\mathbb{Q},+)$ and pick $z\in F$. For $A\subseteq\mathbb{Q}$, let $d(A)=\lambda(\chi_A)$. We claim that

$(*)$ $\quad (\forall x \in S)(\exists l \in \{1,2,\ldots,k\})(\exists (f_1,f_2,\ldots,f_n) \in F^n)$
$(\forall m \in Y_l)(f_m x \in C_m)$.

To see this, let $x \in S$. Pick by Lemma 2.5(ii), $l \in \{1,2,\ldots,k\}$ such that $(\forall m \in Y_l)(\exists f \in F)(fx \in C_m)$. So given $m \in Y_l$, pick $f_m \in F$ such that $f_m x \in C_m$. If $m \in \{1,2,\ldots,n\} \setminus Y_l$, let $f_m = z$.

Define $\Psi : S \to \{1,2,\ldots,k\} \times F^n$ by choosing $\Psi(x) = (l, f_1, f_2, \ldots, f_n)$ where $(\forall m \in Y_l)(f_m x \in C_m)$. (This is a choice, since for $m \in Y_l$ there may be many possible choices for $f_m$.) Let $K = k \cdot |F|^n + 1$ and note that $K > |\Psi[S]|$,.

Pick $N \in \mathbb{N}$ large enough so that, given any sequence $\langle \vec{v}_j \rangle_{j=1}^N$ in $\{1,2,\ldots,k\} \times F^n$, there exist $i$ and $j$ with $1 < i < j-1 < N-2$ such that $\vec{v}_i = \vec{v}_j$. Let $s = \binom{N}{2} \cdot |F|$.

We now choose inductively $\langle \alpha_j \rangle_{j=1}^N$, $\langle \alpha'_j \rangle_{j=1}^N$, and $\langle r_j \rangle_{j=1}^N$ with each $\alpha_j > 0$, each $\alpha'_j > 0$, and each $r_j \in \mathbb{N}$. Let $\alpha_1 = \frac{1}{K}$. By Lemma 2.3 pick $r_1 \in \mathbb{N}$ and $\alpha'_1 > 0$ such that for any $R \in \mathcal{P}_f(S)$ with $|R| \le s$ and $A \subseteq S$ such that $d(A) > \alpha_1$, one has

$$\{x \in S : d\big(A \cap \textstyle\bigcap_{q \in R}(A - qx)\big) > \alpha'_1\} \text{ is } IP^*_{r_1}\,.$$

Given $j \in \{1,2,\ldots,N-1\}$ and having chosen $\alpha_j$, $\alpha'_j$, and $r_j$, let $\alpha_{j+1} = \frac{\alpha'_j}{K}$. Again using Lemma 2.3 pick $r_{j+1} \in \mathbb{N}$ and $\alpha'_{j+1} > 0$ such that for any $R \in \mathcal{P}_f(S)$ with $|R| \le s$ and $A \subseteq S$ such that $d(A) > \alpha_{j+1}$, one has

$$\{x \in S : d\big(A \cap \textstyle\bigcap_{q \in R}(A - qx)\big) > \alpha'_{j+1}\} \text{ is } IP^*_{r_{j+1}}\,.$$

Let $r = \max\big\{r_j : j \in \{1,2,\ldots,N\}\big\}$. If $j \in \{1,2,\ldots,N\}$ and a set is $IP^*_{r_j}$, then it is $IP^*_r$.

By Lemma 2.4, pick $\big\langle \langle S_{l,i} \rangle_{l=1}^k \big\rangle_{i=1}^{N-1}$ such that

(1) for $l \in \{1,2,\ldots,k\}$ and $i \in \{1,2,\ldots,N-1\}$
 (a) $S_{l,i}$ is a finite $IP_r$ set in $(\mathbb{Q},+)$ and
 (b) $S_{l,i} \subseteq C_{Y_l}$; and
(2) for $1 \le i \le j < N$ and $l_i, l_{i+1}, \ldots, l_j \in \{1,2,\ldots,k\}$,
 $S_{l_i,i} \cdot S_{l_{i+1},i+1} \cdots S_{l_j,j} \subseteq C_{Y_{l_i}}$.

Let $Q_1 = \{\frac{1}{f} : f \in F\}$. We define

(I) $A_1, A_2, \ldots, A_N$, subsets of $S$,
(II) $Q_1, Q_2, \ldots Q_N$, finite nonempty subsets of $S$,
(III) tuples $(l_1, f_{1,1}, f_{2,1}, \ldots, f_{n,1}), \ldots, (l_N, f_{1,N}, f_{2,N}, \ldots, f_{n,N})$ in $\{1,2,\ldots,k\} \times F^n$, and
(IV) $y_1, y_2, \ldots, y_{N-1}$ in $S$ such that for $j \in \{1,2,\ldots,N-1\}$,
 (1) $A_{j+1} \subseteq A_j \cap \bigcap_{q \in Q_j}(A_j - qy_j)$,

(2) $A_{j+1} \subseteq \{x \in A_j : \Psi(xy_1y_2\cdots y_j) = (l_{j+1}, f_{1,j+1}, f_{2,j+1}, \ldots, f_{n,j+1})\}$,
(3) $y_j \in S_{l_j,j}$,
(4) $d(A_j) > \alpha_j$, and
(5) if $j > 1$, then $Q_j = \{\frac{y_iy_{i+1}\cdots y_{j-1}}{fy_1y_2\cdots y_{i-1}} : 1 \leq i < j \text{ and } f \in F\}$.

Now $|\Psi[S]| < K$ so $\alpha_1 = \frac{1}{K} < \frac{1}{|\Psi[S]|}$. If for each $\vec{v} \in \{1,2,\ldots,k\} \times F^n$ we had $d(\Psi^{-1}[\{\vec{v}\}]) \leq \alpha_1$ we would have $d(S) \leq \frac{|\Psi[S]|}{K} < 1$, so we can pick $(l_1, f_{1,1}, f_{2,1}, \ldots, f_{n,1}) \in \{1,2,\ldots,k\} \times F^n$ such that $d\big(\Psi^{-1}[\{(l_1, f_{1,1}, f_{2,1}, \ldots, f_{n,1})\}]\big) > \alpha_1$ and let

$$A_1 = \Psi^{-1}[\{(l_1, f_{1,1}, f_{2,1}, \ldots, f_{n,1})\}]\,.$$

Since $|Q_1| < s$ and $d(A_1) > \alpha_1$, we have that

$$\{x \in S : d\big(A_1 \cap \textstyle\bigcap_{q\in Q_1}(A_1 - qx)\big) > \alpha_1'\} \text{ is } IP^*_{r_1}\,,$$

hence is $IP^*_r$. Since $S_{l_1,1}$ is an $IP_r$ set, we can pick $y_1 \in S_{l_1,1}$ such that $d\big(A_1 \cap \bigcap_{q\in Q_1}(A_1 - qy_1)\big) > \alpha_1'$ and let $A_1' = A_1 \cap \bigcap_{q\in Q_1}(A_1 - qy_1)$.

We claim that there is some $\vec{v} \in \{1,2,\ldots,k\} \times F^n$ such that

$$d(\{x \in A_1' : \Psi(xy_1) = \vec{v}\}) > \alpha_2 = \frac{\alpha_1'}{K}\,.$$

If instead for each $\vec{v} \in \{1,2,\ldots,k\} \times F^n$ one has

$$d(\{x \in A_1' : \Psi(xy_1) = \vec{v}\}) \leq \alpha_2\,,$$

then $d(A_1') < \alpha_2 \cdot |\Psi[S]| < \frac{\alpha_1'}{K} \cdot K = \alpha_1'$, a contradiction. So pick $(l_2, f_{1,2}, f_{2,2}, \ldots, f_{n,2}) \in \{1,2,\ldots,k\} \times F^n$ such that

$$d(\{x \in A_1' : \Psi(xy_1) = (l_2, f_{1,2}, f_{2,2}, \ldots, f_{n,2})\}) > \alpha_2$$

and let $A_2 = A_1' \cap \{x \in S : \Psi(xy_1) = (l_2, f_{1,2}, f_{2,2}, \ldots, f_{n,2})\})$. Let $Q_2 = \{\frac{y_1}{f} : f \in F\}$.

Let $j \in \{2,3,\ldots,N-1\}$ and assume we have constructed $A_1, A_2, \ldots, A_j$, $Q_1, Q_2, \ldots, Q_j$, and $y_1, y_2, \ldots, y_{j-1}$ as required. Now $|Q_j| < s$ and $d(A_j) > \alpha_j$ so $\{x \in S : d\big(A_j \cap \bigcap_{q\in Q_j}(A_j - qx)\big) > \alpha_j'\}$ is $IP^*_{r_j}$ so is $IP^*_r$. Since $S_{l_j,j}$ is $IP_r$, we may pick $y_j \in S_{l_j,j}$ such that $d\big(A_j \cap \bigcap_{q\in Q_j}(A_j - qy_j)\big) > \alpha_j'$ and let $A_{j+1}' = A_j \cap \bigcap_{q\in Q_j}(A_j - qy_j)$. Let $Q_{j+1} = \{\frac{y_iy_{i+1}\cdots y_j}{fy_1y_2\cdots y_{i-1}} : 1 \leq i < j+1 \text{ and } f \in F\}$.

We claim that there is some $\vec{v} \in \{1,2,\ldots,k\} \times F^n$ such that $d(\{x \in A_{j+1}' : \Psi(xy_1\cdots y_j) = \vec{v}\}) > \alpha_{j+1} = \frac{\alpha_j'}{K}$. If instead for each $\vec{v} \in \{1,2,\ldots,k\} \times F^n$ one has $d(\{x \in A_{j+1}' : \Psi(xy_1\cdots y_j) = \vec{v}\}) \leq \alpha_{j+1}$, then $d(A_{j+1}') \leq \alpha_{j+1} \cdot |\Psi[S]| < \frac{\alpha_j'}{K} \cdot K = \alpha_j'$, a contradiction. So pick $(l_{j+1}, f_{1,j+1}, f_{2,j+1}, \ldots, f_{n,j+1}) \in \{1,2,\ldots,k\} \times F^n$ such that $d(\{x \in$

$A'_{j+1} : \Psi(xy_1 \cdots y_j) = (l_{j+1}, f_{1,j+1}, f_{2,j+1}, \ldots, f_{n,j+1})\}) > \alpha_{j+1}$ and let $A_{j+1} = A'_{j+1} \cap \{x \in S : \Psi(xy_1 \cdots y_j) = (l_{j+1}, f_{1,j+1}, f_{2,j+1}, \ldots, f_{n,j+1})\})$.

The construction is complete. By our choice of $N$, we may pick $i$ and $j$ such that $1 < i < j-1 < N-2$ and $(l_i, f_{1,i}, f_{2,i}, \ldots, f_{n,i}) = (l_j, f_{1,j}, f_{2,j}, \ldots, f_{n,j})$ and let $(l, f_1, f_2, \ldots, f_n) = (l_i, f_{1,i}, f_{2,i}, \ldots, f_{n,i})$. Let $y = y_i y_{i+1} \cdots y_{j-1}$. We have for each $t \in \{i, i+1, \ldots, j-1\}$ that $y_t \in S_{l_t,t}$ so

$$y \in S_{l_i,i} \cdot S_{l_{i+1},i+1} \cdots S_{l_{j-1},j-1} \subseteq C_{Y_{l_i}} = C_{Y_l} = \bigcup_{m \in Y_l} C_m$$

so pick $m \in Y_l$ such that $y \in C_m$.

We will show now that for any $x' \in A_j$, if $x = f_m x' y_1 \cdots y_{i-1}$, then $\{x, y, x+y, xy\} \subseteq C_m$. So let $x' \in A_j$ and let $x = f_m x' y_1 \cdots y_{i-1}$. Since $x' \in A_j$, by (IV)(2), $\Psi(x' y_1 \cdots y_{j-1}) = (l, f_1, f_2, \ldots, f_n)$ so $f_m x' y_1 \cdots y_{j-1} \in C_m$ so $xy = (f_m x' y_1 \cdots y_{i-1})(y_i \cdots y_{j-1}) \in C_m$. Also $x' \in A_i$ so by (IV)(2), $\Psi(x' y_1 \cdots y_{i-1}) = (l, f_1, f_2, \ldots, f_n)$ so $x = f_m x' y_1 \cdots y_{i-1} \in C_m$. Finally,

$$x' \in A_j \subseteq A_{j-1} \cap \bigcap_{q \in Q_{j-1}} (A_{j-1} - q y_{j-1}) \subseteq A_i \cap \bigcap_{q \in Q_{j-1}} (A_i - q y_{j-1}) .$$

Let $q = \frac{y_i \cdots y_{j-2}}{f_m y_1 \cdots y_{i-1}} \in Q_{j-1}$. Then $x' + q y_{j-1} = x' + \frac{y_i \cdots y_{j-1}}{f_m y_1 \cdots y_{i-1}} \in A_i$ so by (IV)(2), $\Psi\big((x' + q y_{j-1}) \cdot y_1 \cdots y_{i-1}\big) = (l, f_1, \ldots f_n)$. That is, $\Psi(x' y_1 \cdots y_{i-1} + \frac{y_i \cdots y_{j-1}}{f_m}) = (l, f_1, \ldots f_n)$ so $f_m x' y_1 \cdots y_{i-1} + y_i \cdots y_{j-1} \in C_m$. That is, $x + y \in C_m$. $\square$

## 3. Exponential Triples

In this section we present the very simple proof by Mauro Di Nasso and Mariaclara Ragosta [44] of the result of Sahasrabudhe [114] that for any finite coloring of $\mathbb{N} \setminus \{1\}$, there exist $a$ and $b$ such that $\{a, b, b^a\}$ is monochromatic. We also present their infinitary extension showing that for any finite coloring of $\mathbb{N} \setminus \{1\}$ there exists a sequence $\langle b_n \rangle_{n=1}^{\infty}$ such that $\{b_n : n \in \mathbb{N}\} \cup \{b_{n+1}^{b_n} : n \in \mathbb{N}\}$ is monochromatic. (The infinitary result is new in [44].)

The proofs use the operation $*$ on $\mathbb{N}$ defined by $n * m = 2^n m$. That operation is not associative, but by [72, Theorem 4.1], there is a unique binary operation on $\beta\mathbb{N}$, which we also denote by $*$, such that for each $n \in \mathbb{N}$, $\lambda_n : \beta\mathbb{N} \to \beta\mathbb{N}$ is continuous and for each $p \in \beta\mathbb{N}$, $\rho_p : \beta\mathbb{N} \to \beta\mathbb{N}$ is continuous. (Here as usual, $\lambda_n(q) = n * q$ and $\rho_p(q) = q * p$.) Given $n \in \mathbb{N}$, $q \in \beta\mathbb{N}$, and $A \subseteq \mathbb{N}$, if $A \in n * q$, then there is some $B \in q$ such that $\lambda_n[\,\overline{B}\,] \subseteq \overline{A}$ so $\{m \in \mathbb{N} : n * m \in A\} \in q$; that is $(2^n)^{-1} A \in q$. Then, given $p, q \in \beta\mathbb{N}$ and $A \subseteq \mathbb{N}$, if $A \in p * q$, then there is some $C \in p$ such that $\rho_q[\,\overline{C}\,] \subseteq \overline{A}$ so $\{n \in \mathbb{N} : n * q \in \overline{A}\} \in p$ and thus $\{n \in \mathbb{N} : (2^n)^{-1} A \in q\} \in p$.

**Lemma 3.1.** *There exists $p \in \beta\mathbb{N}$ such that for all $A \in p$ and every $l \in \mathbb{N}$, there exist $b, c \in \mathbb{N}$ such that $\{b, c, b+c, b+2c, \ldots, b+lc\} \subseteq A$.*

*Proof.* Let $l \in \mathbb{N}$ and let

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & l \end{pmatrix} .$$

Then $M$ satisfies the first entries condition so by [72, Theorem 15.24] has images in any central set. Thus one my let $p$ be any minimal idempotent in $(\beta\mathbb{N}, +)$. □

Alternatively, one can prove Lemma 3.1 by invoking Rado's Theorem [72, Theorem 15.20] with an appropriately chosen matrix and [72, Theorem 3.11].

The existence of monochromatic exponential triples is a special case of the infinitary theorem that we will prove (Corollary 3.5), but the proof for triples is very simple, so we present it first.

**Theorem 3.2.** *Let $p \in \beta\mathbb{N}$ be as guaranteed by Lemma 3.1. For each $A \in p * p$, there exist $x$ and $y$ in $\mathbb{N} \setminus \{1\}$ such that $\{x, y, 2^x y\} \subseteq A$.*

*Proof.* Let $A \in p * p$ and let $A' = \{n \in \mathbb{N} : (2^n)^{-1}A \in p\}$. Then $A' \in p$. Pick $a \in A'$. Pick (with $l = 2^a$) $b$ and $c$ in $\mathbb{N}$ such that $\{b, c, b + 2^a c\} \subseteq (2^a)^{-1}A \cap A'$. Then $(2^b)^{-1}A \cap (2^{b+2^a c})^{-1}A \in p$ so pick $d \in (2^b)^{-1}A \cap (2^{b+2^a c})^{-1}A$. Let $x = 2^a c$ and $y = 2^b d$. Since $c \in (2^a)^{-1}A$, we have $x \in A$. Since $d \in (2^b)^{-1}A$, we have $y \in A$. Since $d \in (2^{b+2^a c})^{-1}A$, we have $2^x y = 2^{2^a c} 2^b d = 2^{b+2^a c} d \in A$. □

**Corollary 3.3.** *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^r C_i$. There exist $i \in \{1, 2, \ldots, r\}$ and $a, b \in \mathbb{N} \setminus \{1\}$ such that $\{a, b, b^a\} \subseteq C_i$.*

*Proof.* Pick $p \in \beta\mathbb{N}$ as guaranteed by Lemma 3.1. For $i \in \{1, 2, \ldots, r\}$ let $D_i = \{n : 2^n \in C_i\}$ and pick $i \in \{1, 2, \ldots, r\}$ such that $D_i \in p * p$. Pick $x, y \in \mathbb{N}$ such that $\{x, y, 2^x y\} \subseteq D_i$. Let $a = 2^x$ and $b = 2^y$. Then immediately $a \in C_i$ and $b \in C_i$. Also $b^a = (2^y)^{2^x} = 2^{2^x y} \in C_i$. □

Now we turn our attention to the infinitary result of Di Nasso and Ragosta.

**Theorem 3.4.** *Let $p \in \beta\mathbb{N}$ be as guaranteed by Lemma 3.1. For each $A \in p * p$, there exists an increasing sequence $\langle a_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ with the* *property that for all $i, j, k$ in $\mathbb{N}$ with $i < 2j$ and $2j + 1 < k$, if $x = a_{2j}2^{a_i}$ and $y = a_k 2^{a_{2j+1}}$, then $\{x, y, 2^x y\} \subseteq A$.*

*Proof.* Let $A \in p * p$ and let $A' = \{n \in \mathbb{N} : (2^n)^{-1}A \in p\}$. Then $A' \in p$. Pick $a_1 \in A'$ and let $A_1 = A' \cap (2^{a_1})^{-1}A$. Pick $a_2$ and $a_3$ such that $\{a_3, a_2, a_3 + 2^{a_1}a_2\} \subseteq A_1$. Consequently, $2^{a_1}a_2, 2^{a_1}a_3 \in A$ and $(2^{a_2})^{-1}A$, $(2^{a_3})^{-1}A$, and $(a_3 + 2^{a_1}a_2)^{-1}A$ are in $p$.

Let $A_2 = A' \cap (2^{a_1})^{-1}A \cap (2^{a_2})^{-1}A \cap (2^{a_3})^{-1}A \cap (a_3 + 2^{a_1}a_2)^{-1}A$. Then $A_2 \in p$ so pick $a_4$ and $a_5$ in $A_2$ such that $a_5 + ta_4 \in A_2$ for each $t \in \{2^{a_1}, 2^{a_2}, 2^{a_3}\}$. Then $2^{a_i}a_4$ and $2^{a_i}a_5$ are in $A$ for $i \in \{1, 2, 3\}$, $a_4 2^{a_3+2^{a_1}a_2}$ and $a_5 2^{a_3+2^{a_1}a_2}$ are in $A$, and all of $(2^{a_4})^{-1}A$, $(2^{a_5})^{-1}A$, $(2^{a_5+2^{a_1}a_4})^{-1}A$, $(2^{a_5+2^{a_2}a_4})^{-1}A$, and $(2^{a_5+2^{a_3}a_4})^{-1}A$ are in $p$.

Now let $n \geq 3$ and assume that $a_1, a_2, \ldots, a_{2n-1}$ have been chosen satisfying the following induction hypotheses.

(1) $a_i \in A'$ for every $i \leq 2n - 1$;
(2) $a_{2j+1} + 2^{a_i}a_{2j} \in A'$ for all $i < 2j < 2n - 1$;
(3) $2^{a_i}a_k \in A$ for all $i < k \leq 2n - 1$ except when $k = i + 1$ is odd; and
(4) $a_k 2^{a_{2j+1}+2^{a_i}a_{2j}} \in A$ for all $i, j, k \in \mathbb{N}$ such that $i < 2j$ and $2j+1 < k \leq 2n - 1$.

Let $A_n = A' \cap \bigcap_{i=1}^{2n-1}(2^{a_i})^{-1}A \cap \bigcap\{(2^{a_{2j+1}+2^{a_i}a_{2j}})^{-1}A : 1 \leq i < 2j < 2n - 1\}$. By hypotheses (1) and (2), $A_n \in p$. Pick $a_{2n}$ and $a_{2n+1}$ in $A_n$ such that $a_{2n+1} + ta_{2n} \in A_n$ for each $t \in \{2^{a_1}, 2^{a_2}, \ldots, 2^{a_{2n-1}}\}$. Then, all hypotheses are satified for $a_1, a_2, \ldots, a_{2n+1}$. Indeed,

(1) $a_{2n}, a_{2n+1} \in A'$, and hence $a_i \in A'$ for every $i \leq 2n + 1$;
(2) $a_{2n+1} + 2^{a_i}a_{2n} \in A'$ for every $i \leq 2n-1$, and hence $a_{2j+1} + 2^{a_i}a_{2j} \in A'$ for all $i < 2j < 2n + 1$;
(3) $a_{2n} \in (2^{a_i})^{-1}A$ for every $i \leq 2n - 1$ and $a_{2n+1} \in (2^{a_i})^{-1}A$ for every $i \leq 2n - 1$ (but in general $2^{a_{2n}}a_{2n+1} \notin A$), and hence $2^{a_i}a_k \in A$ for all $i < k \leq 2n + 1$ except when $k = i + 1$ is odd; and
(4) $a_{2n} \in (2^{a_{2j+1}+2^{a_i}a_{2j}})^{-1}A$ whenever $i < 2j < 2n - 1$ and $a_{2n+1} \in (2^{a_{2j+1}+2^{a_i}a_{2j}})^{-1}A$ whenever $i < 2j < 2n - 1$, and hence $a_k 2^{a_{2j+1}+2^{a_i}a_{2j}} \in A$ for all $i, j, k$ such that $i < 2j$ and $2j + 1 < k \leq 2n + 1$.

Given $i, j, k \in \mathbb{N}$ with $i < 2j$ and $2j + 1 < k$, let $x = a_{2j}2^{a_i}$ and $y = a_k 2^{a_{2j+1}}$. By (3), $x$ and $y$ are in $A$. And $2^x y = a_k 2^{a_{2j+1}+2^{a_i}a_{2j}} \in A$ by (4). $\square$

**Corollary 3.5.** *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{t=1}^{r} C_t$. There exist $t \in \{1, 2, \ldots, r\}$ and an infinite sequence $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ with the property that* *for all $i, j, k$ in $\mathbb{N}$ with $i < 2j$ and $2j+1 < k$, if $a = 2^{a_{2j}2^{a_i}}$ and $b = 2^{a_k 2^{a_{2j+1}}}$, then $\{a, b, b^a\} \subseteq C_t$.*

*In particular, if for each $n \in \mathbb{N}$, $b_n = 2^{a_{2n}2^{a_{2n-1}}}$, then for each $n$, $\{b_n, b_{n+1}, (b_{n+1})^{b_n}\} \subseteq C_t$.*

*Proof.* Pick $p \in \beta\mathbb{N}$ as guaranteed by Lemma 3.1. For $t \in \{1, 2, \ldots, r\}$ let $D_t = \{n : 2^n \in C_t\}$ and pick $t \in \{1, 2, \ldots, r\}$ such that $D_t \in p * p$. Pick a sequence $\langle a_n \rangle_{n=1}^{\infty}$ as guaranteed by Theorem 3.4 for $D_t$. Let $i, j, k$ in $\mathbb{N}$ be given with $i < 2j$ and $2j+1 < k$, let $x = a_{2j}2^{a_i}$, let $y = a_k 2^{a_{2j+1}}$, let $a = 2^x$, and let $b = 2^y$. Then $\{x, y, 2^x y\} \subseteq D_t$ and $b^a = 2^{2^x y}$ so $\{a, b, b^a\} \subseteq C_t$.

Now let $n \in \mathbb{N}$, let $i = 2n-1$, let $j = n$, and let $k = 2n+2$. Then $2^{a_{2j}2^{a_i}} = 2^{a_{2n}2^{a_{2n-1}}} = b_n$ and $2^{a_k 2^{a_{2j+1}}} = 2^{a_{2n+2}2^{a_{2n+1}}} = b_{n+1}$ so $\{b_n, b_{n+1}, (b_{n+1})^{b_n}\} \subseteq C_t$. □

## 4. Polynomials

In recent years there have been important advances in the study of the Ramsey theoretic properties of polynonials. We are grateful to Vitaly Bergelson for providing us with several references for this section.

Perhaps the earliest Ramsey theoretic result involving polynomials is the following result of Sárközy and Furstenberg.

**Theorem 4.1.** *Let $p$ be a polynomial taking on integer values at the integers with $p(0) = 0$ and let $A \subseteq \mathbb{Z}$ have positive upper Banach density. Then there exist distinct $x$ and $y$ in $A$ and $z \in \mathbb{Z}$ such that $x - y = p(z)$.*

*Proof.* [52, Proposition 3.19(b)]. (In that proof, Furstenberg says that it was proved independently by Sárközy, without citing a reference. It is probably in [116].) □

See Bergelson's survey [6] for substantial information on early polynomial theorems in Ramsey theory. Another early result involving polynomials is the following theorem due to Bergelson and McCutcheon.

**Theorem 4.2.** *Let $j \in \mathbb{N}$, let $p : \mathbb{Z}^j \to \mathbb{Z}$ be a polynomial such that $p(\overline{0}) = 0$, let $A$ be a subset of $\mathbb{N}$ with positive upper Banach density, let $F \in \mathcal{P}_f(\mathbb{Z}^j)$, and for each $i \in \{1, 2, \ldots, j\}$, let $\langle x_n^{(i)} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. There exist $u \in \mathbb{Z}$ and $\alpha \in \mathcal{P}_f(\mathbb{N})$ such that for each $\vec{z} \in F$, $u + p(z_1 x_\alpha^{(1)}, z_2 x_\alpha^{(2)}, \ldots, z_j x_\alpha^{(j)}) \in A$, where for $i \in \{1, 2, \ldots, j\}$, $x_\alpha^{(i)} = \sum_{t \in \alpha} x_t^{(i)}$.*

*Proof.* [15, Theorem 0.10]. □

In this section we prove a recent result of Bergelson, Johnson, and Moriera [10] involving multi-variable polynomials and some simple consequences thereof. We will present this result as Corollary 4.8 below.

**Definition 4.3.** Let $j \in \mathbb{N}$ and let $f : \mathbb{Z}^j \to \mathbb{Z}$. Then $f$ is an *integral polynomial* provided $f$ is a polynomial with zero constant term. (Equivalently, $f(\vec{0}\,) = 0$.)

We begin with a self contained proof of Theorem 4.4, a version of [12, Corollary 8.8], which was derived in [12] as a consequence of the difficult Polynomial Hales-Jewett Theorem. The proof of Theorem 4.4 is based on the proof of [58, Theorem 3.6], which was the $j = 1$ case.

**Theorem 4.4.** *Let $j \in \mathbb{N}$ and let $u = u + u \in \beta(\mathbb{N}^j)$. If $R$ is a finite set of integral polynomials from $\mathbb{Z}^j$ to $\mathbb{Z}$, $A$ is a piecewise syndetic subset of $\mathbb{N}$, and $L$ is a minimal left ideal of $\beta\mathbb{N}$ such that $\overline{A} \cap L \neq \emptyset$, then*

$$\{\vec{x} \in \mathbb{N}^j : \overline{A} \cap L \cap \textstyle\bigcap_{p\in R} \overline{-p(\vec{x}\,) + A} \neq \emptyset\} \in u\,.$$

*Proof.* For each $n \in \mathbb{N}$, let $T_n = \{\vec{v} \in \omega^j$ such that $\sum_{i=1}^{j} v_i = n\}$. Given $p$, an integral polynomial of degree $l > 0$ from $\mathbb{Z}^j \to \mathbb{Z}$, for each $n \in \{1, 2, \ldots, l\}$ there is a unique $\psi_{p,n} : T_n \to \mathbb{Z}$ such that $\psi_{p,l}[T_l] \neq \{0\}$ and for each $\vec{x} \in \mathbb{Z}^j$, $p(\vec{x}\,) = \sum_{n=1}^{l} \sum_{\vec{v}\in T_n} \psi_{p,n}(\vec{v}\,) \prod_{i=1}^{j} x_i^{v_i}$.

Let $\mathcal{R} = \{R : R$ is a finite set of integral polynomials from $\mathbb{Z}^j$ to $\mathbb{Z}\}$. Recall that $\bigoplus_{i=1}^{\infty} \omega$ is the set of all sequences in $\omega$ with finitely many nonzero coordinates. Order $\bigoplus_{i=1}^{\infty} \omega$ lexicographically based on the largest coordinate on which elements differ, denoting this order by $<$. Define $\varphi : \mathcal{R} \to \bigoplus_{i=1}^{\infty} \omega$ as follows. For $R \in \mathcal{R}$ and $l \in \mathbb{N}$, let $J_{R,l} = \{\psi_{p,l} : p \in R$ and $\deg p = l\}$. Let $\varphi(R) = (w_1, w_2, w_3, \ldots)$ where for each $l \in \mathbb{N}$, $w_l = |J_{R,l}|$.

For $l \in \mathbb{N}$ and $p \in R$ with degree $l$, Let $p^\sharp$ denote the polynomial obtained from $p$ be deleting all the terms of degree less than $l$. Notice that $w_l = |\{p^\sharp : p \in R \text{ and } \deg p = l\}|$.

As an example, let $j = 3$ and let $R = \{p, q, r, s\}$, where for $\vec{x} \in \mathbb{Z}^j$,

$$\begin{aligned} p(\vec{x}) &= x_1^2 x_2 - x_1 x_2 x_3 + 3x_2^2\,, \\ q(\vec{x}) &= x_1^2 x_2 - x_1 x_2 x_3 + 2x_1^2 - 3x_3\,, \\ r(\vec{x}) &= -4x_1^3 + 2x_1^2\,, \text{ and} \\ s(\vec{x}) &= -7x_2^3 x_3\,. \end{aligned}$$

Since $j = 3$, we have, for instance, that $T_2 = \{(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}$. Also $\psi_{s,4}(0,3,1) = -7$, and $\psi_{s,4}[T_4 \setminus \{(0,3,1)\}] = \{0\}$ so $J_{R,4} = \{\psi_{s,4}\}$ and thus $w_4 = 1$. And $\psi_{p,3}(2,1,0) = \psi_{q,3}(2,1,0) = 1$, $\psi_{p,3}(1,1,1) = \psi_{q,3}(1,1,1) = -1$, $\psi_{p,3}[T_3 \setminus \{(2,1,0),(1,1,1)\}] = \psi_{q,3}[T_3 \setminus \{(2,1,0),(1,1,1)\}] = \{0\}$, $\psi_{r,3}(3,0,0) = -4$, and

$\psi_{r,3}[T_3 \setminus \{(3,0,0)\}] = \{0\}$, so $J_{R,3} = \{\psi_{p,3}, \psi_{q,3}, \psi_{r,3}\} = \{\psi_{p,3}, \psi_{r,3}\}$ so $w_3 = 2$.

We now claim that

(*) *If $R \in \mathcal{R}$, $R \neq \emptyset$, $\overline{0} \notin R$, $f \in R$ of smallest degree, $F \in \mathcal{P}_f(\mathbb{Z}^j)$, for $\vec{x} \in F$ and $p \in R$, $g(p,\vec{x}\,) : \mathbb{Z}^j \to \mathbb{Z}$ is defined by $g(p,\vec{x}\,)(\vec{y}\,) = p(\vec{x} + \vec{y}\,) - p(\vec{x}\,) - f(\vec{y})$, and $S = \{g(p,\vec{x}\,) : p \in R \text{ and } \vec{x} \in F\}$, then $S \in \mathcal{R}$ and $\varphi(S) < \varphi(R)$.*

To verify (*), assume $R$, $f$, $F$, and $S$ are as specified. Trivially $S \in \mathcal{R}$. Let $m = \deg f$, let $\varphi(R) = (w_1, w_2, w_3, \ldots)$, and let $\varphi(S) = (w'_1, w'_2, w'_3, \ldots)$. We claim that for $l > m$, $w'_l = w_l$ and that $w'_m = w_m - 1$. So assume first that $l > m$. Let $p \in R$ of degree $l$, let $\vec{x} \in F$, and let $r = g(p,\vec{x})$. Then $\deg r = l$ and corresponding degree $l$ coefficients of $p$ and $r$ are equal. That is, $\psi_{r,l} = \psi_{p,l}$, so $J_{S,l} = J_{R,l}$ and so $w'_l = w_l$.

Now assume that $l = m$. Let $p \in R$ of degree $m$, let $\vec{x} \in F$, and let $r = g(p,\vec{x}\,)$. Then for each $\vec{v} \in T_m$, $\psi_{r,m}(\vec{v}\,) = \psi_{p,m}(\vec{v}\,) - \psi_{f,m}(\vec{v}\,)$. Let $\vec{c} = \psi_{f,m}$. If $p = f$, then all degree $m$ coefficients of $r$ are 0. So $J_{S,m} = \big\{\vec{z} - \vec{c} : \vec{z} \in J_{R,m} \setminus \{\vec{c}\}\big\}$ and thus $|J_{S,m}| = |J_{R,m}| - 1$ so (*) is established.

We continue with the example above, in which case $m = 3$ and $f$ could be any one of $p$, $q$, or $r$. Say $f = r$. Then the degree 3 terms in $g(p,\vec{x}\,)(\vec{y}\,)$ are $(x_1+y_1)^2(x_2+y_2) - (x_1+y_1)(x_2+y_2)(x_3+y_3) - x_1^2x_2 + x_1x_2x_3 + 4y_1^3 = y_1^2y_2 - y_1y_2y_3 + 4y_1^3 + h(\vec{y}\,)$ where $h$ is a polynomial of degree 2 in $\vec{y}$ with coefficients in $\mathbb{Z}$ involving the constants $x_1$, $x_2$, and $x_3$.

Suppose the theorem is false and pick $R$ such that $\varphi(R)$ is minimal among all counterexamples. Notice that $R \neq \emptyset$ and $R \neq \{\overline{0}\}$ because the statement is trivially true for both of these sets. We may in fact assume that $\overline{0} \notin R$ because $R \setminus \{\overline{0}\}$ is also a counterexample and $\varphi(R\setminus\{\overline{0}\}) = \varphi(R)$.

Pick a piecewise syndetic subset $A$ of $\mathbb{N}$ and a minimal left ideal $L$ of $\beta\mathbb{N}$ such that $\overline{A} \cap L \neq \emptyset$ and

$$\{\vec{x} \in \mathbb{N}^j : \overline{A} \cap L \cap \textstyle\bigcap_{p\in R} \overline{-p(\vec{x}\,) + A} \neq \emptyset\} \notin u\,.$$

(We know there is a minimal left ideal $L$ of $\beta\mathbb{N}$ such that $\overline{A} \cap L \neq \emptyset$ because $\overline{A} \cap K(\beta\mathbb{N}) \neq \emptyset$ and $K(\beta\mathbb{N})$ is the union of all of the minimal left ideals of $\beta\mathbb{N}$.)

Let $D = \mathbb{N}^j \setminus \{\vec{x} \in \mathbb{N}^j : \overline{A} \cap L \cap \bigcap_{p\in R} \overline{-p(\vec{x}\,) + A} \neq \emptyset\}$ and note that $D \in u$. Let $D^\star = \{\vec{y} \in D : -\vec{y} + D \in u\}$ so that by [72, Lemma 4.14], whenever $\vec{y} \in D^\star$, $-\vec{y} + D^\star \in u$. Notice also that $L$ is in fact a left ideal of $\beta\mathbb{Z}$. (It is an easy exercise, which is [72, Exercise 4.3.5], that $\mathbb{N}^*$ is a left ideal of $\beta\mathbb{Z}$ so [72, Lemma 1.43(c)] applies.)

Pick $f \in R$ of smallest degree. For $\vec{x} \in \mathbb{Z}^j$ and $p \in R$, let $g(p, \vec{x}\,)$ be as in (*). Pick $q_0 \in \overline{A} \cap L$ and let $B = \{x \in \mathbb{N} : -x + A \in q_0\}$. By [72, Lemma 4.39] $B$ is syndetic, so pick $H \in \mathcal{P}_f(\mathbb{N})$ such that $\mathbb{N} = \bigcup_{t \in H}(-t + B)$. Pick $t_0 \in H$ such that $-t_0 + B \in q_0$ and let $C_0 = -t_0 + B$. Since $C_0 \in q_0$, $\overline{C_0} \cap L \neq \emptyset$.

Let $S_0 = \{g(p, \vec{0}\,) : p \in R\}$ and let $E_0 = \{\vec{x} \in \mathbb{N}^j : \overline{C_0} \cap L \cap \bigcap_{p \in S_0} \overline{-p(\vec{x}\,) + C_0} \neq \emptyset\}$. By (*), $S_0 \in \mathcal{R}$ and $\varphi(S_0) < \varphi(R)$ so $E_0 \in u$. Pick $\vec{y_1} \in E_0 \cap D^\star$ and pick $r_1 \in \overline{C_0} \cap L \cap \bigcap_{p \in S_0} \overline{-p(\vec{y_1}\,) + C_0}$. Let $q_1 = -f(\vec{y_1}) + r_1$ and note that, since $L$ is a left ideal of $\beta\mathbb{Z}$, $q_1 \in L$. Pick $t_1 \in H$ such that $-t_1 + B \in q_1$.

Inductively, assume that we have $m \in \mathbb{N}$ and have chosen $\langle q_i \rangle_{i=0}^m$ in $L$, $\langle t_i \rangle_{i=0}^m$ in $H$, and $\langle \vec{y_i}\,\rangle_{i=1}^m$ in $\mathbb{N}^j$ such that

(1) for $j \in \{0, 1, \ldots, m\}$, $-t_j + B \in q_j$,
(2) for $l \in \{1, 2, \ldots, m\}$, $\vec{y_l} + \vec{y_{l+1}} + \ldots + \vec{y_m} \in D^\star$, and
(3) for $l \in \{0, 1, \ldots, m-1\}$ and $p \in R$,
$-\big(t_l + p(\vec{y_{l+1}} + \vec{y_{l+2}} + \ldots + \vec{y_m}\,)\big) + B \in q_m$.

Hypotheses (1) and (2) trivially hold for $m = 1$. To verify hypothesis (3), let $p \in R$. We need to show that $-\big(t_0 + p(\vec{y_1}\,)\big) + B \in q_1$. Now $r_1 + g(p, \vec{0}\,)(\vec{y_1}) \in \overline{C_0}$ and so $-t_0 + B \in r_1 + g(p, \vec{0})(\vec{y_1}\,) = r_1 + p(\vec{y_1}\,) - f(\vec{y_1}\,) = q_1 + p(\vec{y_1}\,)$ as required.

Now let $G_m = \big\{\{\vec{y_{l+1}} + \vec{y_{l+2}} + \ldots + \vec{y_m}\,\} : l \in \{0, 1, \ldots, m-1\}\big\} \cup \{\vec{0}\,\}$ and let $S_m = \{g(p, \vec{x}\,) : p \in R \text{ and } \vec{x} \in G_m\}$. Let $C_m = (-t_m + B) \cap \bigcap_{p \in R} \bigcap_{l=0}^{m-1} \big(-\big(t_l + p(\vec{y_{l+1}} + \vec{y_{l+2}} + \ldots + \vec{y_m}\,)\big) + B\big)$. Then by hypotheses (1) and (3), $C_m \in q_m$ and so $\overline{C_m} \cap L \neq \emptyset$. Let $E_m = \{\vec{x} \in \mathbb{N}^j : \overline{C_m} \cap L \cap \bigcap_{p \in S_m} \overline{-p(\vec{x}\,) + C_m} \neq \emptyset\}$. By (*), $S_m \in \mathcal{R}$ and $\varphi(S_m) < \varphi(R)$ so $E_m \in u$. By hypothesis (2), for each $l \in \{1, 2, \ldots, m\}$, $-(\vec{y_l} + \vec{y_{l+1}} + \ldots + \vec{y_m}\,) + D^\star \in u$. Pick $\vec{y_{m+1}} \in E_m \cap \bigcap_{l=1}^m -(\vec{y_l} + \vec{y_{l+1}} + \ldots + \vec{y_m}\,) + D^\star$ and pick $r_{m+1} \in \overline{C_m} \cap L \cap \bigcap_{p \in S_m} \overline{-p(\vec{y_{m+1}}\,) + C_m}$. Let $q_{m+1} = -f(\vec{y_{m+1}}\,) + r_{m+1}$ and note that $q_{m+1} \in L$. Pick $t_{m+1} \in H$ such that $-t_{m+1} + B \in q_{m+1}$.

Hypotheses (1) and (2) hold directly. To verify hypothesis (3), let $l \in \{0, 1, \ldots, m\}$ and let $p \in R$. Assume first that $l = m$. Then $r_{m+1} + g(p, \vec{0}\,)(\vec{y_{m+1}}\,) \in \overline{C_m}$ and so $-t_m + B \in r_{m+1} + g(p, \vec{0}\,)(\vec{y_{m+1}}\,) = r_{m+1} + p(\vec{y_{m+1}}\,) - f(\vec{y_{m+1}}\,) = q_{m+1} + p(\vec{y_{m+1}}\,)$ so that $-\big(t_m + p(\vec{y_{m+1}}\,)\big) + B \in q_{m+1}$ as required.

Now assume that $l < m$, let $\vec{x} = \vec{y_{l+1}} + \vec{y_{l+2}} + \ldots + \vec{y_m}$, and notice that $\vec{x} \in G_m$. Then $r_{m+1} + g(p, \vec{x}\,)(\vec{y_{m+1}}\,) \in \overline{C_m} \subseteq \overline{-\big(t_l + p(\vec{x}\,)\big) + B}$ and so $-\big(t_l + p(\vec{x}\,)\big) + B \in r_{m+1} + g(p, \vec{x}\,)(\vec{y_{m+1}}\,) = r_{m+1} + p(\vec{x} + \vec{y_{m+1}}\,) - p(\vec{x}\,) - f(\vec{y_{m+1}}\,) = q_{m+1} + p(\vec{x} + \vec{y_{m+1}}\,) - p(\vec{x}\,)$. Thus $-\big(t_l + p(\vec{x} + \vec{y_{m+1}})\big) + B \in q_{m+1}$ as required.

The induction being complete we may choose $l < m$ such that $t_l = t_m$, because $H$ is finite. Let $\vec{y} = \vec{y_{l+1}} + \vec{y_{l+2}} + \ldots + \vec{y_m}$. By hypothesis (2), $\vec{y} \in D^\star$. We have that $(-t_m + B) \cap \bigcap_{p\in R} \big(-\big(t_m + p(\vec{y}\,)\big) + B\big) \in q_m$ so pick $a \in (-t_m + B) \cap \bigcap_{p\in R} \big(-\big(t_m + p(\vec{y}\,)\big) + B\big)$. Let $r = a + t_m + q_0$ and notice that $r \in \overline{A} \cap L \cap \bigcap_{p\in R} \overline{-p(\vec{y}\,) + A}$, contradicting the fact that $\vec{y} \in D$. □

**Corollary 4.5.** *Let $j \in \mathbb{N}$, let $R$ be a finite set of integral polynomials from $\mathbb{Z}^j$ to $\mathbb{Z}$, let $A$ be a piecewise syndetic subset of $\mathbb{N}$, and let $\langle \vec{y_n} \rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}^j$. There exist $a \in \mathbb{N}$ and $\alpha \in \mathcal{P}_f(\mathbb{N})$ such that for every $f \in R$, $a + f(\sum_{n\in\alpha} \vec{y_n}) \in A$.*

*Proof.* By [72, Lemma 5.11] pick an idempotent $u \in \bigcap_{m=1}^\infty \overline{FS(\langle \vec{y_n} \rangle_{n=m}^\infty)}$. Pick a minimal left ideal $L$ of $\beta\mathbb{N}$ such that $L \cap \overline{A} \neq \emptyset$. Let $B = \{\vec{x} \in \mathbb{N}^j : \overline{A} \cap L \cap \bigcap_{f\in R} \overline{-f(\vec{x}\,) + A} \neq \emptyset\}$. Then $FS(\langle \vec{y_n} \rangle_{n=m}^\infty) \in u$ and by Theorem 4.4, $B \in u$, so pick $\alpha \in \mathcal{P}_f(\mathbb{N})$ such that $\sum_{n\in\alpha} \vec{y_n} \in B$. Pick $a \in \overline{A} \cap L \cap \bigcap_{f\in R} \overline{-f(\sum_{n\in\alpha} \vec{y_n}\,) + A}$. □

The proof of the following theorem is adapted from the proof of [10, Proposition 4.10].

**Theorem 4.6.** *Let $j \in \mathbb{N}$, let $R$ be a finite set of integral polynomials from $\mathbb{Z}^j$ to $\mathbb{Z}$, let $p$ be an idempotent in $c\ell K(\beta\mathbb{N})$, let $A \in p$, and let $\langle \vec{y_n} \rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}^j$. There exist sequences $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ and $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and letting $\vec{z_n} = \sum_{t\in H_n} \vec{y_t}$, for each $f \in R$ and each $\beta \in \mathcal{P}_f(\mathbb{N})$, we have $\sum_{n\in\beta} x_n + f(\sum_{n\in\beta} \vec{z_n}) \in A$.*

*Proof.* Let $A^\star = \{n \in A : -n + A \in p\}$. By [72, Lemma 4.14], for all $n \in A^\star$, $-n + A^\star \in p$. By Corollary 4.5 pick $x_1 \in \mathbb{N}$ and $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in R$, $x_1 + f(\sum_{t\in H_1} \vec{y_t}) \in A^\star$ and let $\vec{z_1} = \sum_{t\in H_1} \vec{y_t}$.

Let $n \in \mathbb{N}$, and assume we have chosen $x_1, x_2, \ldots, x_n$ in $\mathbb{N}$, $H_1, H_2, \ldots, H_n$ in $\mathcal{P}_f(\mathbb{N})$, and $\vec{z_1}, \vec{z_2}, \ldots, \vec{z_n}$ in $\mathbb{N}^j$ such that

(1) if $k \in \{1, 2, \ldots, n-1\}$, then $\max H_k < \min H_{k+1}$;
(2) if $k \in \{1, 2, \ldots, n\}$, then $\vec{z}_k = \sum_{t\in H_k} \vec{y_t}$; and
(3) if $\emptyset \neq \beta \subseteq \{1, 2, \ldots, n\}$ and $f \in R$, then $\sum_{t\in\beta} x_t + f(\sum_{t\in\beta} \vec{z_t}) \in A^\star$.

Let $D = \big\{ \sum_{t\in\beta} x_t + f(\sum_{t\in\beta} \vec{z_t}) : \emptyset \neq \beta \subseteq \{1, 2, \ldots, n\}\big\}$ and let $C = A^\star \cap \bigcap_{w\in D}(-w + A^\star)$. For $f \in R$ and $\emptyset \neq \beta \subseteq \{1, 2, \ldots, n\}$ define a polynomial $g(f, \beta) : \mathbb{Z}^j \to \mathbb{Z}$ by $g(f, \beta)(\vec{v}) = f(\sum_{t\in\beta} \vec{z}_t + \vec{v}\,) - f(\sum_{t\in\beta} \vec{z_t})$. Let $\Phi = R \cup \big\{g(f, \beta) : f \in R \text{ and } \emptyset \neq \beta \subseteq \{1, 2, \ldots, n\}\big\}$. Let $d = \max H_n$.

By Corollary 4.5 applied to the sequence $\langle \vec{y_t} \rangle_{t=d+1}^\infty$, pick $x_{n+1} \in \mathbb{N}$ and $H_{n+1} \in \mathcal{P}_f(\mathbb{N})$ with $\min H_{n+1} > d$ such that for all $g \in \Phi$, $x_{n+1} +$

$g(\sum_{t\in H_{n+1}} \vec{y_t}) \in C$ and let $\vec{z_{n+1}} = \sum_{t\in H_{n+1}} \vec{y_t}$. We claim that $x_{n+1}$, $H_{n+1}$, and $\vec{z_{n+1}}$ are as required.

Conclusions (1) and (2) hold directly. So let $f \in R$ and nonempty $\beta \subseteq \{1,2,\ldots,n+1\}$ be given. If $\max\beta \le n$, then conclusion (3) holds by assumption. So assume $\max\beta = n+1$. If $\beta = \{n+1\}$, then (3) holds because $R \subseteq \Phi$. So assume $\{n+1\} \subsetneq \beta$ and let $\gamma = \beta\setminus\{n+1\}$. Then $g(f,\gamma) \in \Phi$ so $x_{n+1} + g(f,\gamma)(\sum_{t\in H_{n+1}} \vec{y_t}) \in C \subseteq -\big(\sum_{t\in\gamma} x_t + f(\sum_{t\in\gamma} \vec{z_t}\,)\big) + A^\star$ so $\sum_{t\in\gamma} x_t + f(\sum_{t\in\gamma} \vec{z_t}\,) + x_{n+1} + f(\sum_{t\in\gamma} \vec{z_t} + \sum_{t\in H_{n+1}} \vec{y_t}\,) - f(\sum_{t\in\gamma} \vec{z_t}\,) \in A^\star$. That is $\sum_{t\in\beta} x_t + f(\sum_{t\in\beta} \vec{z_t}) \in A^\star$. □

**Theorem 4.7.** *Let $m \in \mathbb{N}$, for $j \in \{1,2,\ldots,m\}$ let $\Gamma_j$ be the set of integral polynomials from $\mathbb{Z}^j$ to $\mathbb{Z}$. Let $p$ be an idempotent in $c\ell K(\beta\mathbb{N})$. For $j \in \{1,2,\ldots,m\}$, let $F_j \in \mathcal{P}_f(\Gamma_j)$ and for $j \in \{0,1,\ldots,m\}$, let $c_j \in \mathbb{N}$. For any $A \in p$, there exists a sequence $\langle \vec{s_n}\rangle_{n=1}^\infty$ in $\mathbb{N}^{m+1}$ such that for each $\alpha \in \mathcal{P}_f(\mathbb{N})$, if $\vec{r}_\alpha = \langle r_{\alpha,0}, r_{\alpha,1},\ldots,r_{\alpha,m}\rangle = \sum_{t\in\alpha}\langle s_{t,0}, s_{t,1},\ldots,s_{t,m}\rangle$, then $c_0 r_{\alpha,0} \in A$ and for $j \in \{1,2,\ldots,m\}$ and $f \in F_j$, $f(r_{\alpha,0}, r_{\alpha,1},\ldots,r_{\alpha,j-1}) + c_j r_{\alpha,j} \in A$.*

*Proof.* We proceed by induction on $m$, so assume first that $m = 1$. We may presume that $\overline{0} \in F_1$. Since $p$ is an idempotent, by [72, Lemma 6.6], $c_0\mathbb{N} \in p$. Pick a sequence $\langle l_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $FS(\langle l_n\rangle_{n=1}^\infty) \subseteq A \cap c_0\mathbb{N}$ and for each $n$ let $y_n = \frac{l_n}{c_0}$. Then given $\alpha \in \mathcal{P}_f(\mathbb{N})$, $c_0\sum_{t\in\alpha} y_t \in A$.

Now pick sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle H_n\rangle_{n=1}^\infty$ as guaranteed by Theorem 4.6 for $j = 1$, the set $A \cap c_1\mathbb{N} \in p$, the set $F_1$ of polynomials, and the sequence $\langle y_n\rangle_{n=1}^\infty$. Letting $z_n = \sum_{t\in H_n} y_t$, we have for each $\beta \in \mathcal{P}_f(\mathbb{N})$ and each $f \in F_1$, $\sum_{n\in\beta} x_n + f(\sum_{n\in\beta} z_n) \in A \cap c_1\mathbb{N}$. Since $\overline{0} \in F_1$, each $x_n$ is in $c_1\mathbb{N}$. For $n \in \mathbb{N}$, let $s_{n,0} = z_n$ and $s_{n,1} = \frac{x_n}{c_1}$. For $\alpha \in \mathcal{P}_f(\mathbb{N})$, let $\vec{r}_\alpha = \sum_{n\in\alpha}\langle s_{n,0}, s_{n,1}\rangle$. Then $c_o r_{\alpha,0} = c_0\sum_{n\in\alpha} z_n = c_0\sum_{n\in\alpha}\sum_{t\in H_n} y_t = c_0\sum_{t\in\beta} y_t \in A$ where $\beta = \bigcup_{n\in\alpha} H_n$. Also for $f \in F_1$ and $\alpha \in \mathcal{P}_f(\mathbb{N})$, $f(r_{\alpha,0}) + c_1 r_{\alpha,1} = f(\sum_{n\in\alpha} z_n) + \sum_{n\in\alpha} x_n \in A$. So the theorem holds for $m = 1$.

Now let $m \in \mathbb{N}$ and assume the theorem has been proved for $m$. For $j \in \{1,2,\ldots,m+1\}$ let $\Gamma_j$ be the set of integral polynomials from $\mathbb{Z}^j$ to $\mathbb{Z}$. Let $p$ be an idempotent in $c\ell K(\beta\mathbb{N})$. For $j \in \{1,2,\ldots,m+1\}$, let $F_j \in \mathcal{P}_f(\Gamma_j)$ and for $j \in \{0,1,\ldots,m+1\}$, let $c_j \in \mathbb{N}$. We may presume that $\overline{0} \in F_{m+1}$ and we note that $c_{m+1}\mathbb{N} \in p$.

By assumption we have a sequence $\langle \vec{b_n}\rangle_{n=1}^\infty$ in $\mathbb{N}^{m+1}$ such that for each $\alpha \in \mathcal{P}_f(\mathbb{N})$, if $\vec{a}_\alpha = \langle a_{\alpha,0}, a_{\alpha,1},\ldots,a_{\alpha,m}\rangle = \sum_{t\in\alpha}\langle b_{t,0}, b_{t,1},\ldots,b_{t,m}\rangle$, then $c_0 a_{\alpha,0} \in A$ and for $j \in \{1,2,\ldots,m\}$ and $f \in F_j$, $f(a_{\alpha,0}, a_{\alpha,1},\ldots,a_{\alpha,j-1}) + c_j a_{\alpha,j} \in A$.

Now pick sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle H_n\rangle_{n=1}^\infty$ as guaranteed by Theorem 4.6 for $j = m+1$, the set $A \cap c_{m+1}\mathbb{N} \in p$, the set $F_{m+1}$ of polynomials, and the sequence $\langle \vec{b_n}\rangle_{n=1}^\infty$. For $n\in\mathbb{N}$, let $\vec{z_n}=\sum_{t\in H_n}\vec{b_t}$. Then for $f\in F_{m+1}$ and $\alpha\in\mathcal{P}_f(\mathbb{N})$, $\sum_{n\in\alpha}x_n+f(\sum_{n\in\alpha}\vec{z_n}\,)\in A\cap c_{m+1}\mathbb{N}$. For $n\in\mathbb{N}$, let $s_{n,m+1}=\frac{x_n}{c_{m+1}}$ and $\langle s_{n,0},s_{n,1},\ldots,s_{n,m}\rangle=\vec{z_n}$. For $\alpha\in\mathcal{P}_f(\mathbb{N})$ let $\vec{r_\alpha}=\langle r_{\alpha,0},r_{\alpha,1},\ldots,r_{\alpha,m+1}\rangle=\sum_{t\in\alpha}\langle s_{t,0},s_{t,1},\ldots,s_{t,m+1}\rangle$. We shall show that $\langle\vec{r_\alpha}\rangle_{\alpha\in\mathcal{P}_f(\mathbb{N})}$ is as required. So let $\alpha\in\mathcal{P}_f(\mathbb{N})$ and let $\beta=\bigcup_{n\in\alpha}H_n$. Then $c_0r_{\alpha,0}=c_0\sum_{n\in\alpha}s_{n,0}=c_0\sum_{n\in\alpha}z_{n,0}=c_0\sum_{n\in\alpha}\sum_{t\in H_n}b_{t,0}=c_0\sum_{t\in\beta}b_{t,0}=c_0a_{\beta,0}\in A$.

Now let $j\in\{1,2,\ldots,m+1\}$ and let $f\in F_j$. We need that $c_jr_{\alpha,j}+f(r_{\alpha,0},r_{\alpha,1},\ldots,r_{\alpha,j-1})\in A$. Assume first that $j=m+1$. Then $c_{m+1}r_{\alpha,m+1}+f(r_{\alpha,0},r_{\alpha,1},\ldots,r_{\alpha,m})=\sum_{n\in\alpha}x_n+f(\sum_{n\in\alpha}\vec{z_n}\,)\in A$.

Finally, assume that $j\le m$. Then $c_jr_{\alpha,j}+f(r_{\alpha,0},r_{\alpha,1},\ldots,r_{\alpha,j-1})=$
$c_j\sum_{n\in\alpha}s_{n,j}+f(\sum_{n\in\alpha}\langle s_{n,0},s_{n,1},\ldots,s_{n,j-1}\rangle)=$
$c_ja_{\beta,j}+f(a_{\beta,0},a_{\beta,1},\ldots,a_{\beta,j-1})\in A$. □

The following is the main Ramsey theoretic result of this section.

**Corollary 4.8.** *Let $m\in\mathbb{N}$, for $j\in\{1,2,\ldots,m\}$ let $F_j$ be a finite set of integral polynomials from $\mathbb{Z}^j$ to $\mathbb{Z}$, and for $j\in\{0,1,\ldots,m\}$, let $c_j\in\mathbb{N}$. If $\mathbb{N}$ is finitely colored, there exist a color class $A$ and a sequence $\langle\vec{s_n}\rangle_{n=1}^\infty$ in $\mathbb{N}^{m+1}$ such that for each $\alpha\in\mathcal{P}_f(\mathbb{N})$, if $\vec{r}_\alpha=\langle r_{\alpha,0},r_{\alpha,1},\ldots,r_{\alpha,m}\rangle=\sum_{t\in\alpha}\langle s_{t,0},s_{t,1},\ldots,s_{t,m}\rangle$, then $c_0r_{\alpha,0}\in A$ and for $j\in\{1,2,\ldots,m\}$ and $f\in F_j$, $f(r_{\alpha,0},r_{\alpha,1},\ldots,r_{\alpha,j-1})+c_jr_{\alpha,j}\in A$.*

*Proof.* Given an idempotent $p\in c\ell K(\beta\mathbb{N})$ and a finite coloring of $\mathbb{N}$, pick a color class $A$ in $p$ and apply Theorem 4.7. (Members of idempotents in $c\ell K(\beta\mathbb{N})$ are known as *quasicentral* sets – a weaker notion than central. See [68].) □

The following corollary is probably easier to understand. In the authors' words from [10], it involves a "chain of configurations" of the form $\{x,y,x+f(y)\}$. This corollary is [10, Corollary 1.11].

**Corollary 4.9.** *Let $m\in\mathbb{N}$ and let $f_1,f_2,\ldots,f_m$ be integral polynomials from $\mathbb{Z}$ to $\mathbb{Z}$. For any finite coloring of $\mathbb{N}$ there exist $y_0,y_1,\ldots,y_m$ and $x_1,x_2,\ldots,x_m$ all of the same color such that for each $j\in\{1,2,\ldots,m\}$, $x_j=y_j+f_j(y_{j-1})$.*

*Proof.* For $j\in\{1,2,\ldots,m\}$ let $c_j=1$ and let $F_j=\{\,\overline{0},g_j\}$ where $g_j(y_0,y_1,\ldots,y_{j-1})=f_j(y_{j-1})$ and apply Corollary 4.8. The conclusion follows when $\alpha$ is a singleton. □

We conclude this section with the statements of two recent Ramsey theoretic results about more general polynomials. (We will not prove these results, and they will not be used later in this paper.) The first, due to Bergelson and Robertson, extends the definition of polynomials to apply to functions into finite dimensional vector spaces over countable fields.

**Definition 4.10.** Let $F$ be a countable field, let $W$ be a finite dimensional vector space over $F$, and let $n \in \mathbb{N}$.

(a) A function $q : F^n \to F$ is a *monomial* if and only if there exist $a \in F$ and $(d_1, d_2, \ldots, d_n) \in \omega^n \setminus \{\overline{0}\}$ such that for $\vec{x} \in F^n$, $q(\vec{x}\,) = a x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$.

(b) A function $p : F^n \to W$ is a *polynomial* if and only if for $\vec{x} \in F^n$, $p(\vec{x}\,)$ is a linear combination of vectors with monomial coefficients.

**Definition 4.11.** Let $(G, +)$ be an abelian group and let $r \in \mathbb{N}$.

(a) A subset $A$ of $G$ is $\mathrm{IP}_r^*$ if and only if whenever $x_1, x_2, \ldots, x_r \in G$, there exists $\emptyset \neq \alpha \subseteq \{1, 2, \ldots, r\}$ such that $\sum_{n \in \alpha} x_n \in A$.

(b) A subset $A$ of $G$ is $\mathrm{AIP}_r^*$ if and only if there exist subsets $B$ and $C$ of $G$ such that $B$ is $\mathrm{IP}_r^*$, $C$ has zero upper Banach density, and $A = B \setminus C$.

Any $\mathrm{AIP}_r^*$ set is quite large. For example, if $G = \mathbb{Z}$, and $A$ is $\mathrm{AIP}_r^*$, then $A$ is a member of any minimal idempotent in $\beta\mathbb{Z}$.

**Theorem 4.12.** *Let $F$ be a countable field, let $W$ be a finite dimensional vector space over $F$, let $(X, \mathcal{B}, \mu)$ be a probability space, let $T$ be an action of the additive group of $W$ on $(X, \mathcal{B}, \mu)$, let $n \in \mathbb{N}$, let $p : F^n \to W$ be a polynomial, let $B \in \mathcal{B}$, and let $\epsilon > 0$. Then there is some $r \in \mathbb{N}$ such that $\{\vec{u} \in F^n : \mu(B \cap T^{p(\vec{u}\,)} B) > \mu(B)^2 - \epsilon\}$ is $\mathrm{AIP}_r^*$.*

*Proof.* [19, Theorem 1.2]. □

As noted in [19], Theorem 4.12 is a strengthening of [105, Corollary 5], a result of McCutcheon and Windsor.

The last result that we will state is another result from [10]. It uses an extension of the notion of polynomial to apply to functions from one countable commutative group to another.

**Definition 4.13.** Let $H$ and $G$ be countable abelian groups and let $f : H \to G$. Then $f$ is of *polynomial type of degree 0* if and only if it is constant. For $d \in \mathbb{N}$, $f$ is of *polynomial type of degree $d$* if and only if $f$ is not of polynomial type of degree $d-1$ and for every $h \in H$, the function defined by $x \mapsto f(x+h) - f(x)$ is of polynomial type of degree $c$ for some $c < d$. The function $f$ is *of polynomial type* if and only if it is of polynomial type of degree $d$ for some $d \in \omega$.

Notice that the trivial homomorphism from $H$ to $G$ is of polynomial type of degree 0 and any other homomorphism from $H$ to $G$ is of polynomial type of degree 1. In particular, the following theorem applies if $F$ is any finite set of homomorphisms.

**Theorem 4.14.** *Let $j \in \mathbb{N}$, let $G$ be a countable abelian group, let $F$ be a finite family of functions of polynomial type from $G^j$ to $G$ such that $f(\overline{0}\,) = 0$ for each $f \in F$, let $A$ be a piecewise syndetic subset of $G$, and let $\langle \vec{y_n}\,\rangle_{n=1}^{\infty}$ be a sequence in $G^j$. There exist $a \in A$ and $\alpha \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + f(\sum_{t\in\alpha} \vec{y_t}\,) \in A$.*

*Proof.* [10, Corollary 2.12]. □

**Part 3. Structure of $\beta S$**

## 5. Elements of finite order in $\beta\mathbb{N}$ and continuous homomorphisms into $\mathbb{N}^*$

In this section we present Zelenyuk's proof [132] that for each $n \in \mathbb{N}$, there exists an element of order $n$ in $\mathbb{N}^*$, and consequently there is a continuous homomorphism $\varphi : \beta\mathbb{N} \to \mathbb{N}^*$ such that $|\varphi[\beta\mathbb{N}]| = n$.

We start with the simpler proof of the $n = 2$ version.

For $x \in \mathbb{N}$, we denote the binary support of $x$ by $\mathrm{supp}(x)$. That is, $x = \sum_{t\in\mathrm{supp}(x)} 2^t$.

**Theorem 5.1.** *There exist $p \in c\ell K(\beta\mathbb{N}) \setminus K(\beta\mathbb{N})$ and $q \in K(\beta\mathbb{N})$ such that $p + p = q = q + q = q + p = p + q$.*

*Proof.* Let $I$ be an infinite subset of $\omega$ such that $\omega \setminus I$ is also infinite. Let $Y = \{x \in \mathbb{N} : \mathrm{supp}(x) \subseteq I\}$ and let $T = \overline{Y} \cap \mathbb{H}$. It is routine to verify that $T$ is a compact subsemigroup of $\mathbb{H}$ and that $\mathbb{H} \setminus T$ is an ideal of $\mathbb{H}$ so that $T \cap K(\mathbb{H}) = \emptyset$. By [72, Theorem 1.65], $K(\mathbb{H}) = \mathbb{H} \cap K(\beta\mathbb{N})$, so we have that $T \cap K(\beta\mathbb{N}) = \emptyset$.

Let $X = \{x \in \mathbb{N} : \mathrm{supp}(x) \cap I \neq \emptyset\}$, define $\tau : X \to I$ by $\tau(x) = \max(\mathrm{supp}(x) \cap I)$, and let $\widetilde{\tau} : \beta X \to \beta\omega$ be the continuous extension of $\tau$. The restriction of $\tau$ to $\{2^k : k \in I\}$ is a bijection onto $I$ so the restriction of $\widetilde{\tau}$ to $\overline{\{2^k : k \in I\}}$ is a homeomorphism onto $\overline{I}$. We claim that

$(*)$ if $u \in \beta\mathbb{N}$ and $v \in \overline{X} \cap \mathbb{H}$, then $u + v \in \overline{X}$ and $\widetilde{\tau}(u + v) = \widetilde{\tau}(v)$.

To verify $(*)$, let $u \in \beta\mathbb{N}$ and $v \in \overline{X} \cap \mathbb{H}$. We claim that $\mathbb{N} \subseteq \{x \in \mathbb{N} : -x + X \in v\}$ so let $x \in \mathbb{N}$. Let $m = \max\mathrm{supp}(x) + 1$. Then $2^m\mathbb{N} \cap X \in v$ and $2^m\mathbb{N} \cap X \subseteq -x + X$, so $X \in u + v$. To see that $\widetilde{\tau}(u + v) = \widetilde{\tau}(v)$ we show that $\widetilde{\tau} \circ \rho_v$ is constantly equal to $\widetilde{\tau}(v)$ on $\mathbb{N}$. So let $x \in \mathbb{N}$ and let $m = \max\mathrm{supp}(x) + 1$. Then $\widetilde{\tau} \circ \lambda_x$ and $\widetilde{\tau}$ agree on $2^m\mathbb{N}$ so agree at $v$.

Pick a minimal right ideal $R$ of $T$. By [72, Exercise 3.4.3(b)] we may pick an injective strongly discrete sequence $\langle r_j\rangle_{j=0}^{\infty}$ in $\{2^k : k \in I\}^*$. For $j \in \omega$ pick a minimal left ideal $L_j$ of $T$ with $L_j \subseteq T + r_j$. Let $e_j$ be the identity of $R \cap L_j$. By [72, Theorem 1.60] pick an idempotent $f \in K(\mathbb{H})$ such that $f < e_0$. Let $D = \{f + e_j : j \in \omega\}$.

Now $\widetilde{\tau}$ is a homeomorphism on $\overline{\{2^k : k \in I\}}$ so $\langle \widetilde{\tau}(r_j)\rangle_{j=0}^{\infty}$ is an injective strongly discrete sequence in $\overline{I}$. By $(*)$, for each $j \in \omega$, $\widetilde{\tau}[L_j] = \{\widetilde{\tau}(r_j)\}$ so $\widetilde{\tau}(f + e_j) = \widetilde{\tau}(e_j) = \widetilde{\tau}(r_j)$, so $\langle \widetilde{\tau}(f + e_j)\rangle_{j=0}^{\infty}$ is an injective strongly discrete sequence in $\overline{I}$. In particular, $D$ is infinite.

Pick $w \in c\ell(D) \setminus D$. We claim that $w$ is right cancelable in $\beta\mathbb{N}$. So suppose not. Then by [72, Theorem 8.18] we may pick $v \in \mathbb{N}^*$ such that $w = v + w$. Let $D' = \{u \in D : \widetilde{\tau}(u) \neq \widetilde{\tau}(w)\}$. Since $\widetilde{\tau}$ is injective on $D$, $|D \setminus D'| \leq 1$. Then $w \in c\ell(D') \cap c\ell(\mathbb{N} + w)$ so by [72, Theorem 3.40] either

(i) there exists $k \in \mathbb{N}$ such that $k + w \in c\ell(D')$ or
(ii) there exists $u \in D'$ such that $u \in \beta\mathbb{N} + w$.

Case (i) is out since $w \in \mathbb{H}$ and $c\ell(D') \subseteq \mathbb{H}$ while for any $k \in \mathbb{N}$, $(k + \mathbb{H}) \cap \mathbb{H} = \emptyset$. So pick $u \in D'$ such that $u \in \beta\mathbb{N} + w$. Then by $(*)$, $\widetilde{\tau}(u) = \widetilde{\tau}(w)$, so $u \notin D'$.

Let $p = e_0 + w$. Now $D \subseteq K(\mathbb{H}) \subseteq K(\beta\mathbb{N})$ so $w \in c\ell\big(K(\beta\mathbb{N})\big)$ and by [72, Theorem 4.44], $c\ell\big(K(\beta\mathbb{N})\big)$ is an ideal of $\beta\mathbb{N}$ so $p \in c\ell\big(K(\beta\mathbb{N})\big)$. To see that $p \notin K(\beta\mathbb{N})$, suppose instead that $p \in K(\beta\mathbb{N})$. Pick a minimal right ideal $V$ of $\beta\mathbb{N}$ such that $p \in V$. Pick an idempotent $u \in V$. Then $V = u + \beta\mathbb{N}$ so by [72, Lemma 1.30], $p = u + p$ so $e_0 + w = u + e_0 + w$. Since $w$ is right cancelable, $e_0 = u + e_0$ so $e_0 \in K(\beta\mathbb{N})$, while $e_0 \in T$, a contradiction.

Given $x \in R = e_0 + T$, we claim that $\rho_x$ is constantly equal to $f + x$ on $D$ so that $w + x = f + x$. To see this, note that for $j \in \omega$, $x = e_0 + x$ so $e_j + x = e_j + e_0 + x = e_0 + x = x$ so $f + e_j + x = f + x$ as claimed. In particular $w + e_0 = f + e_0 = f$. Let $q = f + w$.

Then $p + p = e_0 + w + e_0 + w = e_0 + f + w = f + w = q$. And $q + q = f + w + f + w = f + w + e_0 + f + w = f + f + f + w = f + w = q$. Also $q + p = f + w + e_0 + w = f + f + w = f + w = q$ and $p + q = p + p + p = q + p = q$. $\square$

In [72] immediately after Corollary 8.31 we noted that we did not know whether it was possible for the sum of two elements of $\beta\mathbb{N} \setminus K(\beta\mathbb{N})$ to be in $K(\beta\mathbb{N})$. This question is answered by Theorem 5.1 since $p \notin K(\beta\mathbb{N})$ and $p + p \in K(\beta\mathbb{N})$.

Ordinarily if $n \in \mathbb{N}$ and $p \in \beta\mathbb{N}$, by $np$ we would mean $n \cdot p$, that is multiplication in the semigroup $(\beta\mathbb{N}, \cdot)$. However, in the statement and proof of the next theorem, by $np$ we mean the sum of $p$ with itself $n$ times. Recall that for $m, n \in \mathbb{N}$, $m \vee n = \max\{m, n\}$.

**Theorem 5.2.** *Let $n \in \mathbb{N} \setminus \{1\}$. There exists $p \in c\ell K(\beta\mathbb{N}) \setminus K(\beta\mathbb{N})$ such that $p$, $2p$, $\ldots$, $np$ are all distinct, $(n+1)p = np$, and $np \in K(\beta\mathbb{N})$.*

*Proof.* The case $n = 2$ is Theorem 5.1. We will assume that $n \geq 3$. For $i \in \{0, 1, \ldots, n\}$ pick a set $I_i$ with $\emptyset = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = \omega$ such that for each $i \in \{1,2,\ldots,n\}$, $|I_i \setminus I_{i-1}| = \omega$. Define $h : \mathbb{N} \to \{1,2,\ldots,n\}$ by, for $x \in \mathbb{N}$, $h(x) = \min\{i \in \{1,2,\ldots,n\} : \mathrm{supp}(x) \subseteq I_i\}$ and note that $h(x) = \max\{i \in \{1,2,\ldots,n\} : \mathrm{supp}(x) \cap (I_i \setminus I_{i-1}) \neq \emptyset\}$. Let $\widetilde{h} : \beta\mathbb{N} \to \{1,2,\ldots,n\}$ be the continuous extension of $h$. We claim that

$(*)$ if $u \in \beta\mathbb{N}$ and $v \in \mathbb{H}$, then $\widetilde{h}(u+v) = \widetilde{h}(u) \vee \widetilde{h}(v)$.

To verify $(*)$, let $u \in \beta\mathbb{N}$ and $v \in \mathbb{H}$ be given. We will show that $\widetilde{h} \circ \rho_v$ and $\rho_{\widetilde{h}(v)} \circ \widetilde{h}$ agree on $\mathbb{N}$ so let $x \in \mathbb{N}$ and let $m = \max \mathrm{supp}(x) + 1$. We show that $\widetilde{h} \circ \lambda_x$ and $\lambda_{h(x)} \circ \widetilde{h}$ agree on $2^m\mathbb{N}$, so let $y \in 2^m\mathbb{N}$. Then $\mathrm{supp}(x+y) = \mathrm{supp}(x) \cup \mathrm{supp}(y)$ so $h(x+y) = \max\{i \in \{1,2,\ldots,n\} : \mathrm{supp}(x+y) \cap (I_i \setminus I_{i-1}) \neq \emptyset\} = h(x) \vee h(y)$.

For $i \in \{1,2,\ldots,n\}$, let $T_i = \widetilde{h}^{-1}[\{1,2,\ldots,i\}] \cap \mathbb{H}$. By $(*)$, the restriction of $\widetilde{h}$ to $\mathbb{H}$ is a homomorphism onto $(\{1,2,\ldots,n\},\vee)$ and each $T_i$ is a compact subsemigroup of $\mathbb{H}$. Further, for each $i \in \{1,2,\ldots,n\}$, $\widetilde{h}[K(T_i)] = \{i\}$ by [72, Exercaise 1.7.3]. Thus, if $i \in \{1,2,\ldots,n-1\}$, then $T_i \cap K(T_{i+1}) = \emptyset$. Note that $T_n = \mathbb{H}$ and by [72, Lemma 6.8 and Theorem 1.65], $K(\mathbb{H}) = K(\beta\mathbb{N}) \cap \mathbb{H}$.

For $i \in \{1,2,\ldots,n\}$, let $X_i = \{x \in \mathbb{N} : \mathrm{supp}(x) \cap (I_i \setminus I_{i-1}) \neq \emptyset\}$, and define $\tau_i : X_i \to \omega$ by for $x \in X_i$, $\tau_i(x) = \max\big(\mathrm{supp}(x) \cap (I_i \setminus I_{i-1})\big)$. Let $\widetilde{\tau}_i : \overline{X_i} \to \beta\omega$ be the continuous extension of $\tau_i$.

For $k \in I_i \setminus I_{i-1}$, $2^k \in X_i$ and $\tau_i(2^k) = k$, so the restriction of $\widetilde{\tau}_i$ to $\overline{\{2^k : k \in I_i \setminus I_{i-1}\}}$ is a homeomorphism onto $\overline{I_i \setminus I_{i-1}}$.

We now claim that for $i \in \{1,2,\ldots,n\}$,

(1) if $u \in \beta\mathbb{N}$ and $v \in \overline{X_i} \cap \mathbb{H}$, then $u+v \in \overline{X_i}$ and $\widetilde{\tau}_i(u+v) = \widetilde{\tau}_i(v)$ and
(2) if $v \in \overline{X_i}$ and $w \in \mathbb{H} \setminus \overline{X_i}$, then $v+w \in \overline{X_i}$ and $\widetilde{\tau}_i(v+w) = \widetilde{\tau}_i(v)$.

To verify (1), let $u \in \beta\mathbb{N}$ and $v \in \overline{X_i} \cap \mathbb{H}$. To see that $X_i \in u+v$, we show that $\mathbb{N} \subseteq \{x \in \mathbb{N} : -x + X_i \in v\}$. So let $x \in \mathbb{N}$ and let $m = \max \mathrm{supp}(x) + 1$. Then $X_i \cap 2^m\mathbb{N} \subseteq -x + X_i$.

To see that $\widetilde{\tau}_i(u+v) = \widetilde{\tau}_i(v)$ we show that $\widetilde{\tau}_i \circ \rho_v$ is contstantly equal to $\widetilde{\tau}_i(v)$ on $\mathbb{N}$. So let $x \in \mathbb{N}$ and let $m = \max \mathrm{supp}(x) + 1$. Then $\widetilde{\tau}_i \circ \lambda_x$ and $\widetilde{\tau}_i$ agree on $2^m\mathbb{N} \cap X_i$.

To verify (2), let $v \in \overline{X_i}$ and $w \in \mathbb{H} \setminus \overline{X_i}$. To see that $X_i \in u+v$, we show that $X_i \subseteq \{x \in \mathbb{N} : -x + X_i \in w\}$ so let $x \in X_i$ and let $m = \max \mathrm{supp}(x) + 1$. Then $2^m\mathbb{N} \subseteq -x + X_i$.

To see that $\widetilde{\tau}_i(v+w) = \widetilde{\tau}_i(v)$, we show that $\widetilde{\tau}_i \circ \rho_w$ and $\widetilde{\tau}_i$ agree on $X_i$. So let $x \in X_i$ and let $m = \max \mathrm{supp}(x) + 1$. Then $\widetilde{\tau}_i \circ \lambda_x$ is constantly equal to $\tau_i(x)$ on $2^m\mathbb{N} \setminus X_i$.

We note that for $i \in \{1, 2, \ldots, n\}$, $K(T_i) \subseteq \overline{X_i} \cap \mathbb{H}$. To see this, let $i \in \{1, 2, \ldots, n\}$ and let $v \in K(T_i)$. Then $\widetilde{h}(v) = i$ so pick $B \in v$ such that $\widetilde{h}[\,\overline{B}\,] \subseteq \{i\}$. Then $B \subseteq X_i$. Note also that if $i \geq 2$, then $T_{i-1} \subseteq \mathbb{H} \setminus \overline{X_i}$.

We now construct idempotents $e_1 > e_2 > \ldots > e_n$ with each $e_i \in K(T_i)$ and for $i \in \{1, 2, \ldots, n-1\}$ a right zero semigroup $\{e_{i,j} : j \in \omega\} \subseteq K(T_i)$ with $e_{i,0} = e_i$ such that for each $i \in \{1, 2, \ldots, n-1\}$,

(i) if $i \geq 2$, then for each $j \in \omega$, $e_{i,j} < e_{i-1}$ and

(ii) for $j < k$ in $\omega$, $\widetilde{\tau}_i(e_{i,j}) \neq \widetilde{\tau}_i(e_{i,k})$ and $\widetilde{\tau}_i(e_i) \notin c\ell\{\widetilde{\tau}_i(e_{i,j}) : j \in \mathbb{N}\}$.

Pick a minimal right ideal $R_1$ of $T_1$ By [72, Exercise 3.4.3(b)], pick an injective strongly discrete sequence $\langle r_{1,j} \rangle_{j=0}^{\infty}$ in $\{2^k : k \in I_1\}^*$. For $j \in \omega$, choose a minimal left ideal $L_{1,j}$ of $T_1$ such that $L_{1,j} \subseteq T_1 + r_{1,j}$, let $e_{1,j}$ be the identity of $R_1 \cap L_{1,j}$, and let $e_1 = e_{1,0}$.

Given $j \in \omega$, $e_{1,j} \in \beta\mathbb{N} + r_{1,j}$ and $r_{1,j} \in \overline{X_1} \cap \mathbb{H}$ so by (1) above, $\widetilde{\tau}_1(e_{1,j}) = \widetilde{\tau}_1(r_{1,j})$. Since $\widetilde{\tau}_1$ is a homeomorphism on $\overline{\{2^k : k \in I_1\}}$, we have $\widetilde{\tau}_1(r_{1,0}) \notin c\ell\{\widetilde{\tau}_1(r_{1,j}) : j \in \mathbb{N}\}$ so $\widetilde{\tau}_1(e_1) \notin c\ell\{\widetilde{\tau}_i(e_{1,j}) : j \in \mathbb{N}\}$.

Now let $i \in \{2, 3, \ldots, n-1\}$ and assume we have done the construction for $i-1$. Pick a minimal right ideal $R_i$ of $T_i$ with $R_i \subseteq e_{i-1} + T_i$. Pick an injective strongly discrete sequence $\langle r_{i,j} \rangle_{j=0}^{\infty}$ in $\{2^k : k \in I_i \setminus I_{i-1}\}^*$. For $j \in \omega$ pick a minimal left ideal $L_{i,j}$ of $T_i$ with $L_{i,j} \subseteq T_i + r_{i,j} + e_{i-1}$ and let $e_{i,j}$ be the identity of $R_i \cap L_{i,j}$.

Now for $j \in \omega$, $r_{i,j} \in \overline{X_i}$ and $e_{i-1} \in T_{i-1} \subseteq \mathbb{H} \setminus \overline{X_i}$ so by (2) above, $\widetilde{\tau}_i(r_{i,j} + e_{i-1}) = \widetilde{\tau}_i(r_{i,j})$ and by (1) above, $\widetilde{\tau}_i(e_{i,j}) = \widetilde{\tau}_i(r_{i,j} + e_{i-1})$ and thus $\widetilde{\tau}_i(e_{i,j}) = \widetilde{\tau}_i(r_{i,j})$.

Since $\widetilde{\tau}_i$ is a homeomorphism from $\overline{\{2^k : k \in I_i \setminus I_{i-1}\}}$ onto $\overline{I_i \setminus I_{i-1}}$ we have $\langle \widetilde{\tau}_i(e_{i,j}) \rangle_{j=0}^{\infty}$ is an injective strongly discrete sequence in$\overline{I_i \setminus I_{i-1}}$ and $\widetilde{\tau}_i(e_{i,0}) = \widetilde{\tau}_i(r_{i,0}) \notin c\ell\{\widetilde{\tau}_i(r_{i,j}) : j \in \mathbb{N}\} = c\ell\{\widetilde{\tau}_i(e_{i,j}) : j \in \mathbb{N}\}$. Let $e_i = e_{i,0}$. For $j \in \omega$, $e_{i,j} \in (e_{i-1} + \beta\mathbb{N}) \cap (\beta\mathbb{N} + e_{i-1})$, so we have that $e_{i,j} < e_{i-1}$.

Pick a minimal right ideal $R_n$ of $T_n = \mathbb{H}$ with $R_n \subseteq e_{n-1} + T_n$ and pick a minimal left ideal $L_n$ of $T_n$ with $L_n \subseteq T_n + e_{n-1}$. Let $e_n$ be the identity of $R_n \cap L_n$ and note that $e_n < e_{n-1}$.

Let $D_{n-1} = \{e_n + e_{n-1,j} : j \in \mathbb{N}\}$. Given $j \in \mathbb{N}$, $\widetilde{\tau_{n-1}}(e_n + e_{n-1,j}) = \widetilde{\tau_{n-1}}(e_{n-1,j})$, so $D_{n-1}$ is infinite. Pick $q_{n-1} \in \overline{D_{n-1}} \setminus D_{n-1}$. Note that for $j \in \mathbb{N}$, $e_{n-1,j} \in K(T_{n-1}) \subseteq \overline{X_{n-1}} \cap \mathbb{H}$ so by (1) above, $D_{n-1} \subseteq \overline{X_{n-1}} \cap \mathbb{H}$.

Now let $i \in \{1, 2, \ldots, n-2\}$ and assume that $q_{i+1}$ has been chosen. Let $D_i = \{e_{i+1} + q_{i+1} + e_{i,j} : j \in \mathbb{N}\}$ and note that $D_i \subseteq \overline{X_i} \cap \mathbb{H}$. Given $j \in \omega$, $\widetilde{\tau}_i(e_{i+1} + q_{i+1} + e_{i,j}) = \widetilde{\tau}_i(e_{i,j})$, so $\widetilde{\tau}_i$ is injective on $D_i$ and $D_i$ is infinite. Pick $q_i \in \overline{D_i} \setminus D_i$.

We can show that for each $i \in \{1, 2, \ldots, n-1\}$, $q_i$ is right cancelable in $\beta\mathbb{N}$ exactly as in the proof of Theorem 5.1. Note that $e_n \in K(T_n) = K(\mathbb{H}) \subseteq K(\beta\mathbb{N})$ so $D_{n-1} \subseteq K(\beta\mathbb{N})$ and thus $q_{n-1} \in c\ell K(\beta\mathbb{N})$. By [72, Theorem 14.44], $c\ell K(\beta\mathbb{N})$ is an ideal of $\beta\mathbb{N}$ and given $i \in \{1, 2, \ldots, n-2\}$, $D_i \subseteq \beta\mathbb{N} + q_{i+1} + \beta\mathbb{N}$ so $q_i \in c\ell K(\beta\mathbb{N})$.

Let $p = e_1 + q_1$. Then $p \in c\ell K(\beta\mathbb{N})$. To see that $p \notin K(\beta\mathbb{N})$ supose that $p \in K(\beta\mathbb{N})$. Then as in the proof of Theorem 5.1, we have $p = u + p$ for some $u \in K(\beta\mathbb{N})$ so $e_1 + q_1 = u + e_1 + q_1$ so by right cancellation, $e_1 = u + e_1 \in K(\beta\mathbb{N})$ while $e_1 \in T_1 \subseteq \beta\mathbb{N} \setminus K(\beta\mathbb{N})$.

We note that for each $j \in \mathbb{N}$, $e_{n-1,j} + e_{n-1} = e_{n-1}$, so $e_n + e_{n-1,j} + e_{n-1} = e_n + e_{n-1} = e_n$. Thus $\rho_{e_{n-1}}$ is constantly equal to $e_n$ on $D_{n-1}$ so $q_{n-1} + e_{n-1} = e_n$. Also for $i \in \{1, 2, \ldots, n-2\}$ and $j \in \mathbb{N}$, $e_{i+1} + q_{i+1} + e_{i,j} + e_i = e_{i+1} + q_{i+1} + e_i$ so $\rho_{e_i}$ is constantly equal to $e_{i+1} + q_{i+1} + e_i$ on $D_i$ and thus $q_i + e_i = e_{i+1} + q_{i+1} + e_i$.

Now we verify that for $k \in \{1, 2, \ldots, n-2\}$, $e_1 + q_1 + e_k = e_{k+1} + q_{k+1} + e_k$ and $e_1 + q_1 + e_{n-1} = e_n$. First let $k = 1$. Then $e_1 + q_1 + e_1 = e_1 + e_2 + q_2 + e_1 = e_2 + q_2 + e_1$. Now assume that $k \in \{2, 3, \ldots, n-2\}$ and we know that $e_1 + q_1 + e_{k-1} = e_k + q_k + e_{k-1}$. Then $e_1 + q_1 + e_k = e_1 + q_1 + e_{k-1} + e_k = e_k + q_k + e_{k-1} + e_k = e_k + q_k + e_k = e_k + e_{k+1} + q_{k+1} + e_k = e_{k+1} + q_{k+1} + e_k$. Now we have that $e_1 + q_1 + e_{n-2} = e_{n-1} + q_{n-1} + e_{n-2}$ so $e_1 + q_1 + e_{n-1} = e_1 + q_1 + e_{n-2} + e_{n-1} = e_{n-1} + q_{n-1} + e_{n-2} + e_{n-1} = e_{n-1} + q_{n-1} + e_{n-1} = e_{n-1} + e_n = e_n$.

Now we show that for $k \in \{1, 2, \ldots, n-1\}$, $kp = e_k + q_k + e_{k-1} + q_{k-1} + \ldots + e_1 + q_1$ and $np = e_n + q_{n-1} + e_{n-2} + q_{n-2} + \ldots + e_1 + q_1$. In particular this will show that $kp \in c\ell K(\beta\mathbb{N})$ and $np \in K(\beta\mathbb{N})$. For $k = 1$, $kp = p = e_1 + q_1$. Let $k \in \{2, 3, \ldots, n-1\}$ and assume that $(k-1)p = e_{k-1} + q_{k-1} + e_{k-2} + q_{k-2} + \ldots + e_1 + q_1$. Then $kp = e_1 + q_1 + e_{k-1} + q_{k-1} + \ldots + e_1 + q_1 = e_k + q_k + e_{k-1} + q_{k-1} + \ldots + e_1 + q_1$.

In particular, $(n-1)p = e_{n-1} + q_{n-1} + e_{n-2} + q_{n-2} + \ldots + e_1 + q_1$ so $np = e_1 + q_1 + e_{n-1} + q_{n-1} + \ldots + e_1 + q_1 = e_n + q_{n-1} + e_{n-2} + q_{n-2} + \ldots + e_1 + q_1$.

Also $(n+1)p = e_1 + q_1 + e_n + q_{n-1} + e_{n-2} + q_{n-2} + \ldots + e_1 + q_1 = e_1 + q_1 + e_{n-1} + e_n + q_{n-1} + e_{n-2} + q_{n-2} + \ldots + e_1 + q_1 = e_n + e_n + q_{n-1} + e_{n-2} + q_{n-2} + \ldots + e_1 + q_1 = e_n + q_{n-1} + e_{n-2} + q_{n-2} + \ldots + e_1 + q_1 = np$.

To complete the proof, we need to show that $p$, $2p$, $\ldots$,$np$ are all distinct. We have shown that $(n+1)p = np$, so to show that $p$, $2p$, $\ldots$,$np$ are all distinct, it suffices to show that $(n-1)p \neq np$.

We now claim that for each $i \in \{2, 3, \ldots, n-1\}$, $q_i + e_{i-1} = q_i$. For $i = n-1$ we have that for each $j \in \mathbb{N}$, $(e_n + e_{n-1,j}) + e_{n-2} = e_n + (e_{n-1,j} + e_{n-2}) = e_n + e_{n-1,j}$ so $\rho_{e_{n-2}}$ is the identity on $D_{n-1}$ and thus $q_{n-1} + e_{n-2} = q_{n-1}$. For $i \in \{2, 3, \ldots, n-2\}$ we have for each $j \in \mathbb{N}$, $(e_{i+1} + q_{i+1} + e_{i,j}) + e_{i-1} = e_{i+1} + q_{i+1} + (e_{i,j} + e_{i-1}) = e_{i+1} + q_{i+1} + e_{i,j}$ so $\rho_{e_{i-1}}$ is the identity on $D_i$ and thus $q_i + e_{i-1} = q_i$.

Now suppose that $(n-1)p = np$. That is $e_{n-1} + q_{n-1} + e_{n-2} + q_{n-2} + \ldots + e_1 + q_1 = e_n + q_{n-1} + e_{n-2} + q_{n-2} + \ldots + e_1 + q_1$. Then, using the fact just established that for each $i \in \{2, 3, \ldots, n-1\}$, $q_i + e_{i-1} = q_i$, we have that $e_{n-1}+q_{n-1}+q_{n-2}+\ldots+q_1=e_n+q_{n-1}+q_{n-2}+\ldots+q_1$. Then cancelling $q_{n-1}+q_{n-2}+\ldots+q_1$ on the right, we have that $e_{n-1}=e_n$, a contradiction. □

## 6. Subsets of $\beta S$ that are not Borel

We take as is usual (but not, unfortunately, universal) that the Borel subsets of a topological space $X$ are the members of the smallest $\sigma$-algebra of subsets of $X$ that contains the open subsets.

Given a discrete semigroup $(S,\cdot)$, there are many algebraically interesting subsets of $\beta S$. Included are the set of idempotents in $\beta S$, the smallest ideal of $\beta S$, $S^*$, $S^*S^*$, any semiprincipal right ideal of the form $p\beta S$ with $p\in S^*$, any semiprincipal left ideal, minimal right ideals, minimal left ideals, maximal groups in the smallest ideal, the closure of the smallest ideal, and so on. Some of these are automatically compact such as the semiprincipal left ideals (including the mimimal left ideals) and $S^*$. And, of course, the closure of any one of these algebraically interesting subsets is compact.

We present here results from [80] showing that if $S$ is countable and cancellative, then none of the set of idempotents of $\beta S$, the smallest ideal of $\beta S$, $S^*S^*$, or $p\beta S$ for any $p\in S^*$ is Borel. In fact hypotheses weaker than cancellation suffice, though not much weaker. The hypotheses cannot be weakened to left cancellative or right cancellative. If $S$ is a right zero semigroup, then $S$ is left cancellative, $\beta S$ is a right zero semigroup, and $E(\beta S)=K(\beta S)=\beta S$, $S^*S^*=S^*$ and if $r\in S^*$, then $rS^*=S^*$. If $S$ is a left zero semigroup, then $S$ is right cancellative, $\beta S$ is a left zero semigroup, and $E(\beta S)=K(\beta S)=\beta S$, $S^*S^*=S^*$ and if $r\in S^*$, then $rS^*=\{r\}$. Nor can they be weakened to weakly right cancellative and weakly left cancellative as shown by the example $(\mathbb{N},\vee)$, where $x\vee y=\max\{x,y\}$. In this case, for $p,q\in\beta\mathbb{N}$, if $q\in\mathbb{N}^*$, then $p\vee q=q$, while if $q\in\mathbb{N}$ and $p\in\mathbb{N}^*$, then $p\vee q=p$ so $E(\beta\mathbb{N})=\beta\mathbb{N}$, $\mathbb{N}^*\vee\mathbb{N}^*=K(\beta\mathbb{N})=\mathbb{N}^*$, and if $r\in\mathbb{N}^*$, then $r\vee\mathbb{N}^*=\mathbb{N}^*$.

Throughout this section we will assume that $(S,\cdot)$ is a countably infinite weakly left cancellative semigroup. We will assume that $S$ has been ordered in order type $\omega$ and write $s\prec t$ if $s$ precedes $t$ in this ordering.

**Lemma 6.1.** *Every Borel subset of $\beta S$ is the union of at most $\mathfrak{c}$ compact subsets of $\beta S$.*

*Proof.* One may construct the Borel subsets of $\beta S$ as follows. Let $\mathcal{A}_0=\{A\subseteq\beta S: A$ is open or closed in $\beta S\}$. Inductively let $0<\alpha<\omega_1$ and assume $\mathcal{A}_\sigma$ has been defined for all $\sigma<\alpha$. If $\alpha$ is a limit ordinal let $\mathcal{A}_\alpha=\bigcup_{\sigma<\alpha}\mathcal{A}_\sigma$. If $\alpha=\delta+1$, let

$$\mathcal{A}_\alpha=\{\textstyle\bigcup\mathcal{C}:\mathcal{C}\subseteq\mathcal{A}_\delta\text{ and }|\mathcal{C}|\leq\omega\}\cup\{\textstyle\bigcap\mathcal{C}:\emptyset\neq\mathcal{C}\subseteq\mathcal{A}_\delta\text{ and }|\mathcal{C}|\leq\omega\}\,.$$

Then it is routine to verify that $\bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ is the set of Borel subsets of $\beta S$.

Let $\mathcal{D} = \{A \subseteq \beta S : A \text{ is compact}\}$ and let $\mathcal{F} = \{A \subseteq \beta S : (\exists \mathcal{C} \subseteq \mathcal{D})(|\mathcal{C}| \leq \mathfrak{c} \text{ and } A = \bigcup \mathcal{C})$. It suffices to show that for all $\alpha < \omega_1$, $\mathcal{A}_\alpha \subseteq \mathcal{F}$. Since the topology of $\beta S$ has a basis consisting of $\mathfrak{c}$ clopen sets, $\mathcal{A}_0 \subseteq \mathcal{F}$. Let $0 < \alpha < \omega_1$ and assume that for all $\sigma < \alpha$, $\mathcal{A}_\sigma \subseteq \mathcal{F}$. If $\alpha$ is a limit ordinal, then trivially $\mathcal{A}_\alpha \subseteq \mathcal{F}$. So assume that $\alpha = \delta + 1$ and let $\mathcal{C} \subseteq \mathcal{A}_\delta$ such that $0 < |\mathcal{C}| \leq \omega$. Trivially $\bigcup \mathcal{C} \in \mathcal{F}$. To see that $\bigcap \mathcal{C} \in \mathcal{F}$, for each $A \in \mathcal{C}$, pick $\mathcal{E}_A \subseteq \mathcal{D}$ such that $|\mathcal{E}_A| \leq \mathfrak{c}$ and $A = \bigcup \mathcal{E}_A$. It is routine to verify that $\bigcap \mathcal{C} = \bigcup \{\bigcap_{A\in\mathcal{C}} F(A) : F \in \times_{A\in\mathcal{C}} \mathcal{E}_A\}$. Since $|\times_{A\in\mathcal{C}} \mathcal{E}_A| \leq \mathfrak{c}^\omega = \mathfrak{c}$, we are finished. □

**Lemma 6.2.** *There is a sequence $\langle s_n \rangle_{n=1}^\infty$ in $S$ such that for each $n \in \mathbb{N}$,*

(1) *$s_n \prec s_{n+1}$;*
(2) *if $a \preceq s_n$ and $b \preceq s_n$, then $ab \prec s_{n+1}$; and*
(3) *if $a \preceq s_n$ and $ab \preceq s_n$, then $b \prec s_{n+1}$.*

*Proof.* Pick $s_1 \in S$. Let $n \in \mathbb{N}$ and assume $s_n$ has been chosen. Let $A = \{ab : a \preceq s_n \text{ and } b \preceq s_n\} \cup \{b \in S : (\exists a \preceq s_n)(\exists c \preceq s_n)(ab = c)\} \cup \{s_n\}$. Then $A$ is finite. (The second of the three listed sets is finite since $S$ is weakly left cancellative.) Pick $s_{n+1}$ such that for all $b \in A$, $b \prec s_{n+1}$. □

We will assume that we have fixed $\langle s_n \rangle_{n=1}^\infty$ as guaranteed by Lemma 6.2 and let $P = \{s_n : n \in \mathbb{N}\}$. If $\mathbb{N}$ has its natural order, we can take $s_n = 2^n$ for $(\mathbb{N}, +)$ and $s_n = 2^{2^n}$ for $(\mathbb{N}, \cdot)$.

**Definition 6.3.** We define $\tau : S \to \mathbb{N}$ by $\tau(t) = \min\{n \in \mathbb{N} : t \preceq s_n\}$ and let $\widetilde{\tau} : \beta S \to \beta\mathbb{N}$ be its continuous extension.

Note that if $y \in S^*$, then $\widetilde{\tau}(y) \in \mathbb{N}^*$ so $-1 + \widetilde{\tau}(y) \in \mathbb{N}^*$.

**Lemma 6.4.** *Let $x \in \beta S$ and let $y \in S^*$. Then*

$$\widetilde{\tau}(xy) \in \{-1 + \widetilde{\tau}(y), \widetilde{\tau}(y), 1 + \widetilde{\tau}(y)\}.$$

*Proof.* We claim that for every $a \in S$, there exists $m \in \mathbb{N}$ such that if $s_m \prec b$, then $\tau(ab) \in \{-1 + \tau(b), \tau(b), 1 + \tau(b)\}$. To see this, pick $m > 1$ such that $a \prec s_{m-1}$ and assume that $s_m \prec b$. Let $n = \tau(b)$. Then $s_{n-1} \prec b \preceq s_n$ so $m \leq n-1$ and $a \prec s_{n-2}$. By Lemma 6.2 (2), $ab \prec s_{n+1}$. If we had $ab \preceq s_{n-2}$, then by Lemma 6.2 (3) we would have $b \prec s_{n-1}$ so $s_{n-2} \prec ab \prec s_{n+1}$ so $\tau(ab) \in \{n-1, n, n+1\}$.

For each $a \in S$ and $i \in \{-1, 0, 1\}$, let $B_{a,i} = \{b \in S : \tau(ab) = i + \tau(b)\}$. Then $\bigcup_{i=-1}^{1} B_{a,i}$ is cofinite so pick $j(a)$ such that $B_{a,j(a)} \in y$. For $i \in \{-1, 0, 1\}$ let $C_i = \{a \in S : j(a) = i\}$ and pick $i$ such that $C_i \in x$. We claim that $\widetilde{\tau}(xy) = i + \widetilde{\tau}(y)$. For this it suffices to show that $\widetilde{\tau} \circ \rho_y$ is constantly equal to $i + \widetilde{\tau}(y)$ on $C_i$, so let $a \in C_i$. To see that $\widetilde{\tau}(ay) = i + \widetilde{\tau}(y)$, it suffices to show that $\widetilde{\tau} \circ \lambda_a$ and $\lambda_i \circ \widetilde{\tau}$ agree on $B_{a,i}$, where $\lambda_i$ is addition on the left by $i$ in $\beta\mathbb{Z}$.. So let $b \in B_{a,i}$. Then $\tau(ab) = i + \tau(b)$ as required. □

**Lemma 6.5.** *Assume that $S$ is left cancellative and $k \in \mathbb{N} \setminus \{1\}$ such that for any $a, b \in S$, $|\{x \in S : xa = b\}| < k$. Then for any $p, q \in \beta S$, $|\{x \in S : xp = q\}| < k$.*

*Proof.* Let $p, q \in \beta S$ and suppose that $|\{x \in S : xp = q\}| \geq k$. Pick distinct $x_1, x_2, \ldots, x_k$ in $S$ such that $x_i p = q$ for each $i \in \{1, 2, \ldots, k\}$. Define $f : S \to S$ as follows.

(1) If $v \in S \setminus x_1 S$, then $f(v) = (x_1)^2$.
(2) Assume that $v = x_1 u$ for some $u \in S$ and note that since $S$ is left cancellative, there is only one such $u$. Let $f(v) = x_i u$ where $i$ is the first member of $\{2, 3, \ldots, k\}$ such that $x_i u \neq x_1 u$.

Then $f$ has no fixed points so by [72, Lemma 3.33], pick $A_0, A_1, A_2$ such that $S = A_0 \cup A_1 \cup A_2$ and for each $i \in \{0, 1, 2\}$, $A_i \cap f[A_i] = \emptyset$. Pick $i \in \{0, 1, 2\}$ such that $A_i \in x_1 p$. For $j \in \{2, 3, \ldots, k\}$, let $B_j = \{u \in S : f(x_1 u) = x_j u\}$ and pick $j \in \{2, 3, \ldots, k\}$ such that $B_j \in p$. Let $\widetilde{f} : \beta S \to \beta S$ denote the continuous extension of $f$. Then for $u \in B_j$, $f(x_1 u) = x_j u$ so $\widetilde{f} \circ \lambda_{x_1}$ and $\lambda_{x_j}$ agree on a member of $p$ so $\widetilde{f}(x_1 p) = x_j p$. Since $A_i \in x_1 p$, $f[A_i] \in \widetilde{f}(x_1 p) = x_j p = x_1 p$ while $f[A_i] \cap A_i = \emptyset$, a contradiction. □

**Lemma 6.6.** *Assume that $S$ is left cancellative and $k \in \mathbb{N} \setminus \{1\}$ such that for any $a, b \in S$, $|\{x \in S : xa = b\}| < k$. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $S^*$ such that $\widetilde{\tau}$ is injective on $\langle x_n \rangle_{n=1}^{\infty}$ and $\{\widetilde{\tau}(x_n) : n \in \mathbb{N}\}$ is discrete. If $x$ is a cluster point of $\langle x_n \rangle_{n=1}^{\infty}$, then $x \notin S^* S^*$.*

*Proof.* We claim that $\widetilde{\tau}$ is injective on $c\ell\{x_n : n \in \mathbb{N}\}$. Suppose instead we have distinct $p$ and $q$ in $c\ell\{x_n : n \in \mathbb{N}$ such that $\widetilde{\tau}(p) = \widetilde{\tau}(q)$. Pick $A \in p$ and $B \in q$ such that $A \cap B = \emptyset$. Then $\widetilde{\tau}(p) \in c\ell\{\widetilde{\tau}(x_n) : x_n \in \overline{A}\}$ and $\widetilde{\tau}(q) \in c\ell\{\widetilde{\tau}(x_n) : x_n \in \overline{B}\}$. By [72, Theorem 3.40] we can assume without loss of generality that $\{\widetilde{\tau}(x_n) : x_n \in \overline{A}\} \cap c\ell\{\widetilde{\tau}(x_n) : x_n \in \overline{B}\} \neq \emptyset$ so pick $m$ such that $x_m \in \overline{A}$ and $\widetilde{\tau}(x_m) \in c\ell\{\widetilde{\tau}(x_n) : x_n \in \overline{B}\}$. This contradicts the fact that $\{\widetilde{\tau}(x_n) : n \in \mathbb{N}\}$ is discrete.

Now let $x$ be a cluster point of $\langle x_n \rangle_{n=1}^{\infty}$ and suppose that $x = yz$ for some $y$ and $z$ in $S^*$. By Lemma 6.4, $\widetilde{\tau}$ takes on at most 3 values on $\widetilde{\tau}[\beta S z]$. Let $M = \{s \in S : sz \in c\ell_{\beta S}(\{x_n : n \in \mathbb{N}\})\}$. Since $\widetilde{\tau}$ is injective on $\{x_n : n \in \mathbb{N}\}$ and $\{x_n : n \in \mathbb{N}\}$ is discrete, $\widetilde{\tau}$ is injective on $c\ell\{x_n : n \in \mathbb{N}\}$. By Lemma 6.4 $\widetilde{\tau}$ takes on at most three values on $\beta S z$ so by Lemma 6.5, $M$ is finite. So $x$ is in $c\ell_{\beta S}\big((S \setminus M)z\big)$ and in $c\ell_{\beta S}(\{x_n : \widetilde{\tau}(x_n) \notin \{-1 + \widetilde{\tau}(z), \widetilde{\tau}(z), 1 + \widetilde{\tau}(z)\})$. Hence, by [72, Theorem

3.40], there exists $v \in c\ell_{\beta S}(\{x_n : n \in \mathbb{N}\})$ and $s \in S \setminus M$ such that $v = sz$, or else there exists $n \in \mathbb{N}$ such that $\widetilde{\tau}(x_n) \notin \{-1 + \widetilde{\tau}(z), \widetilde{\tau}(z), 1 + \widetilde{\tau}(z)\}$ and $x_n \in \beta S z$. The first possibility is ruled out by the definition of $M$, and the second possibility is ruled out by Lemma 6.4. □

**Lemma 6.7.** *Assume that $S$ is left cancellative and $k \in \mathbb{N} \setminus \{1\}$ such that for any $a, b \in S$, $|\{x \in S : xa = b\}| < k$. Let $D$ be a compact subset of $S^*S^*$. Then $\widetilde{\tau}[D]$ is finite. Consequently for any Borel subset $B$ of $S^*S^*$, $|\widetilde{\tau}[B]| \leq \mathfrak{c}$.*

*Proof.* Supppose not and pick a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $D$ such that $\widetilde{\tau}$ is injective on $\langle x_n \rangle_{n=1}^{\infty}$. We may assume that $\{\widetilde{\tau}(x_n) : n \in \mathbb{N}\}$ is discrete. Pick a cluster point $x$ of $\langle x_n \rangle_{n=1}^{\infty}$. Then $x \in D$ but by Lemma 6.6, $x \notin S^*S^*$.

Now let $B$ be a Borel subset of $S^*S^*$. By Lemma 6.1, there is a set $\mathcal{E}$ of compact subsets of $\beta S$ with $|\mathcal{E}| \leq \mathfrak{c}$ such that $B = \bigcup \mathcal{E}$ so $\widetilde{\tau}[B] = \bigcup_{D \in \mathcal{E}} \widetilde{\tau}[D]$. □

**Theorem 6.8.** *Assume that $S$ is left cancellative and $k \in \mathbb{N} \setminus \{1\}$ such that for any $a, b \in S$, $|\{x \in S : xa = b\}| < k$. Let $T \subseteq S^*$ such that $P^* \subseteq T$. Then $TT$ is not Borel. (In particular $S^*S^*$ is not Borel.)*

*Proof.* For each $n \in \mathbb{N}$, $\tau(s_n) = n$ so $\tau[P] = \mathbb{N}$ and thus $\widetilde{\tau}[P^*] = \mathbb{N}^*$ so that $|\widetilde{\tau}[P^*]| = 2^{\mathfrak{c}}$. It will suffice by Lemma 6.7 to show that $|\widetilde{\tau}[TT]| = 2^{\mathfrak{c}}$.

Pick $x \in P^*$. We will show that $|\widetilde{\tau}[xP^*]| = 2^{\mathfrak{c}}$. For $i \in \{-1, 0, 1\}$, let $B_i = \{y \in P^* : \widetilde{\tau}(xy) = i + \widetilde{\tau}(y)\}$. By Lemma 6.4, $P^* = \bigcup_{i=-1}^{1} B_i$ so $\widetilde{\tau}[P^*] = \bigcup_{i=-1}^{1} \widetilde{\tau}[B_i]$ so pick $i \in \{-1, 0, 1\}$ such that $|\widetilde{\tau}[B_i]| = 2^{\mathfrak{c}}$. Pick a subset $D$ of $B_i$ such that $|D| = 2^{\mathfrak{c}}$ and $\widetilde{\tau}$ is injective on $D$.

Note that, if $y$ and $z$ are distinct members of $D$, then $\widetilde{\tau}(xy) \neq \widetilde{\tau}(xz)$. (Otherwise one has $i + \widetilde{\tau}(y) = \widetilde{\tau}(xy) = \widetilde{\tau}(xz) = i + \widetilde{\tau}(z)$ so by [72, Lemma 8.1], $\widetilde{\tau}(y) = \widetilde{\tau}(z)$.) Thus $|\widetilde{\tau}[TT]| \geq |\widetilde{\tau}[xP^*]| \geq |\widetilde{\tau}[xD]| = 2^{\mathfrak{c}}$. □

Recall that $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}} 2^n\mathbb{N}$ and that we are assuming that for $(\mathbb{N}, +)$, $s_n = 2^n$.

**Corollary 6.9.** *The sets $\mathbb{N}^* + \mathbb{N}^*$ and $\mathbb{H} + \mathbb{H}$ are not Borel in $\beta\mathbb{N}$.*

Recall that for $T \subseteq \beta S$ we let $E(T)$ be the set of idempotents in $T$.

**Corollary 6.10.** *Assume that $S$ is left cancellative and $k \in \mathbb{N} \setminus \{1\}$ such that for any $a, b \in S$, $|\{x \in S : xa = b\}| < k$. Then the following sets are not Borel: $E(\beta S)$, $K(\beta S)$, $p\beta S$ for any $p \in S^*$, and $E(R)$ for any right ideal $R$ of $S^*$.*

*Proof.* We define an equivalence relation $\equiv$ on $\beta S$ by $x \equiv y$ if and only if $\widetilde{\tau}(x) \in \mathbb{Z} + \widetilde{\tau}(y)$. Since $\widetilde{\tau}$ is injective on $P^*$, each equivalence class of $\equiv$

meets $P^*$ in at most countably many points so we may pick $D \subseteq P^*$ such that $|D| = 2^{\mathfrak{c}}$ and if $x$ and $y$ are distinct members of $D$, then $x \not\equiv y$.

Note that $E(S^*) \subseteq S^*S^*$ and, since $S^*S^*$ is an ideal of $\beta S$, $K(\beta S) \subseteq S^*S^*$. We will show that $\left|\widetilde{\tau}\big[E\big(K(\beta S)\big)\big]\right| = 2^{\mathfrak{c}}$ so that neither $K(\beta S)$ nor $E(S^*)$ is Borel by Lemma 6.7. Since $E(\beta S) = E(S) \cup E(S^*)$ and $E(S)$ is countable, this will also show that $E(\beta S)$ is not Borel. For each $p \in D$, there is an idempotent $e_p$ in $K(\beta S) \cap \beta Sp$. Then $\widetilde{\tau}(e_p) = i + \widetilde{\tau}(p)$ for some $i \in \{-1, 0, 1\}$ so $e_p \equiv p$ and thus $\left|\widetilde{\tau}\big[E\big(K(\beta S)\big)\big]\right| = 2^{\mathfrak{c}}$ as required.

Now let $p \in S^*$. Then $pD \subseteq S^*S^*$ and for $q \in D$, $\widetilde{\tau}(pq) = i + \widetilde{\tau}(q)$ for some $i \in \{-1, 0, 1\}$ so $pq \equiv q$ and thus $|\widetilde{\tau}[pD]| = 2^{\mathfrak{c}}$. Thus $|\widetilde{\tau}[p\beta S \cap S^*S^*]| = 2^{\mathfrak{c}}$ so that $p\beta S \cap S^*S^*$ is not Borel. Since $p\beta S \setminus S^*S^*$ is countable, $p\beta S$ is not Borel.

Let $R$ be a right ideal of $S^*$. For every $p \in P^*$, we can choose an idempotent $e_p \in R \cap \beta Sp$. Then $\widetilde{\tau}(e_p) \in \{-1 + \widetilde{\tau}(p), \widetilde{\tau}(p), 1 + \widetilde{\tau}(p)\}$ by Lemma 6.4. So $\widetilde{\tau}[P^*] \subseteq (-1 + \widetilde{\tau}[E(R)]) \cup \widetilde{\tau}[E(R)] \cup (1 + \widetilde{\tau}[(E(R)])$. Since $\widetilde{\tau}$ is injective on $P^*$, $|\widetilde{\tau}[P^*]| = 2^{\mathfrak{c}}$. It follows that $|\widetilde{\tau}[E(R)]| > \mathfrak{c}$. If $E(R)$ were the union of $\mathfrak{c}$ or fewer compact sets, there would be a compact subset $C$ of $E(R)$ for which $\widetilde{\tau}[C]$ is infinite. This contradicts Lemma 6.7. □

**Corollary 6.11.** *Let $T$ be an infinite semigroup. Assume that $T$ is left cancellative and $k \in \mathbb{N} \setminus \{1\}$ such that for any $a, b \in T$, $|\{x \in T : xa = b\}| < k$. Then $E(\beta T)$ is not Borel.*

*Proof.* Let $S$ be an infinite countable subsemigroup of $T$. By Corollary 6.10, $E(\beta S)$ is not Borel. Since $E(\beta S)$ can be identified with $c\ell_{\beta T}(S)$ and $E(c\ell_{\beta T}(S)) = c\ell_{\beta T}(S) \cap E(\beta T)$, $E(\beta T)$ is not Borel. □

**Theorem 6.12.** *Let $L$ be a minimal left ideal of $\beta\mathbb{N}$. Then $E(L)$ is not Borel.*

*Proof.* For $n \in \mathbb{N}$, define $\text{supp}(n)$ by $n = \sum_{i \in \text{supp}(n)} 2^i$ and let $\theta(n) = \min\big(\text{supp}(n)\big)$. Let $\widetilde{\theta} : \beta\mathbb{N} \to \beta\omega$ be the continuous extension of $\theta$. By [72, Theorem 6.15.1], if $\langle q_n \rangle_{n=1}^{\infty}$ is any sequence of idempotents in $\beta\mathbb{N}$ such that $\{\widetilde{\theta}(q_n) : n \in \mathbb{N}\}$ is discrete and $\widetilde{\theta}(q_m) \neq \widetilde{\theta}(q_n)$ if $m$ and $n$ are distinct positive integers, then no cluster point of $\{q_n : n \in \mathbb{N}\}$ can be idempotent.

Assume that $E(L)$ is Borel, so that $E(L)$ is the union of $\mathfrak{c}$ or fewer compact sets by Lemma 6.1. We claim that $|\widetilde{\theta}[E(L)]| = 2^{\mathfrak{c}}$. Let $B = \{2^n : n \in \mathbb{N}\}^*$. By [72, Exercise 3.4.1], $\widetilde{\theta}$ is injective on $B$ and so $|\widetilde{\theta}[B]| = 2^{\mathfrak{c}}$. So to establish the claim it suffices to show that $\widetilde{\theta}[B] \subseteq \widetilde{\theta}[E(L)]$. Let $x \in B$. Pick an idempotent $e \in (x + \beta\mathbb{N}) \cap L$. Then $e = x + y$ for some $y \in \beta\mathbb{N}$. Since $e \in \mathbb{H}$ and $x \in \mathbb{H}$ we have that $y \in \mathbb{H}$. By [72, Lemma 6.8], $\widetilde{\theta}(e) = \widetilde{\theta}(x + y) = \widetilde{\theta}(x)$ and so $|\widetilde{\theta}[E(L)]| = 2^{\mathfrak{c}}$ as required.

Hence there is a compact subset $C$ of $E(L)$ for which $\widetilde{\theta}[C]$ is infinite. Then $C$ contains a sequence $\langle q_n\rangle_{n\in\mathbb{N}}$ for which $\langle\widetilde{\theta}(q_n)\rangle_{n\in\mathbb{N}}$ is an injective discrete sequence. This is a contradiction because by Lemma 6.6 no cluster point of $\langle q_n\rangle_{n\in\mathbb{N}}$ can be in $E(L)$. □

**Corollary 6.13.** *Let $G$ be a countable group which can be algebraically embedded in a compact metrizable topological group. If $L$ is a minimal left ideal of $\beta G$, $E(L)$ is not Borel.*

*Proof.* This follows immedately from Theorem 6.12 and the fact that $\beta G$ contains a subset which is topologically isomorphic to $\mathbb{H}$ and contains all the idempotents of $\beta G$, by [72, Theorem 7.28]. □

**Corollary 6.14.** *Let $(S,+)$ be a countably infinite commutative cancellative semigroup with an identity $0$. If $L$ is a minimal left ideal of $\beta S$, $E(L)$ is not Borel.*

*Proof.* Let $G$ denote the group of differences of $S$. By [72, Lemma 7.29], for every $a\neq 0$ in $G$ there is a homomorphism $h_a : G\to\mathbb{T}$, where $\mathbb{T}$ denotes the circle group written additively, such that $h_a(a)\neq 0$. Let $H=\{h_a : a\in G\setminus\{0\}\}$. Then $\mathbb{T}^H$ is a compact metrizable toplogical group, and the natural mapping of $G$ into $\mathbb{T}^H$ is an injective homomrphism. Hence, by Corollary 6.13, $E(L)$ is not Borel if $L$ denotes any minimal left ideal of $G$. Now $S$ can be regarded as a subset of $G$ by identifying each $s\in S$ with $s-0$. Then $S$ is a thick subset of $G$ because, if $n\in\mathbb{N}$ and $a_1-b_1,a_2-b_2,\ldots a_n-b_n\in G$, where $a_i,b_i\in S$ for every $i\in\{1,2,\ldots,n\}$, then $a_i-b_i+b_1+b_2+\ldots+b_n\in S$ for every $i\in\{1,2,\ldots,n\}$. So $\beta S$ contains a minimal left ideal of $\beta G$, by [72, Theorem 4.48], and hence $K(\beta S)\subseteq K(\beta G)$, by [72, Theorem 1.65].

We claim that every minimal left ideal of $\beta S$ is also a minimal left ideal of $\beta G$. It will then follow from Corollary 6.13 that $E(L)$ is not Borel.

Let $L$ be a minimal left ideal of $\beta S$ and pick $p\in E(L)$. We claim that $\beta G+p\subseteq L=\beta S+p$ for which it suffices that $G+p\subseteq L$. So let $g\in G$ and pick $s,t\in S$ such that $g=s-t$. Let $x$ denote the inverse of $t+p=p+t+p$ in the group $p+\beta S+p$. Then $t+x=t+p+x=p$ so $s+p=s+t+x$ and so $g+p=s-t+p=s+x\in L$. □

## 7. Long increasing $<_R$-chains in $\beta\mathbb{N}$

In this section we will establish the result from [79] that there is a sequence $\langle p_\sigma\rangle_{\sigma<\omega_1}$ of idempotents in $\beta\mathbb{N}$ such that $p_\sigma<_R p_\tau$ whenever $\sigma<\tau<\omega_1$. This result contrasts strongly with the result of Zelenyuk which we will present in Section 8 that there does not exist a sequence $\langle p_n\rangle_{n=1}^\infty$ of idempotents in $\beta\mathbb{N}$ such that $p_n<_L p_{n+1}$ for each $n\in\mathbb{N}$. (If $p$ and $q$ are idempotents in $\beta\mathbb{N}$, then $p<_L q$ if and only if $\beta\mathbb{Z}+p\subsetneq\beta\mathbb{Z}+q$.)

The results of this section through Lemma 7.8 consist of a presentation of some of the details of [72, Exercise 8.5.1].

**Lemma 7.1.** *Let $p \in \beta\mathbb{N}$ such that $p$ is right cancelable in $(\beta\mathbb{N}, +)$. There is a sequence $\langle b_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that for each $k \in \mathbb{N}$, $\{b_n : n \in \mathbb{N}$ and $b_n + k < b_{n+1}\} \in p$.*

*Proof.* This is [72, Lemma 8.27]. □

**Definition 7.2.** Let $p$ be a right cancelable element of $\beta\mathbb{N}$ and let $\langle b_n \rangle_{n=1}^{\infty}$ be as guaranteed by Lemma 7.1.

(a) $T_p = \{b_{n_1} + b_{n_2} + \ldots + b_{n_k}$ : if $k > 1$, then $n_1 < n_2$ and for each $i \in \{2, 3, \ldots, k\}$, $b_{n_i+1} > (1 + 2 + \ldots + b_{n_{i-1}}) + b_{n_i}\}$.

(b) For $n \in \mathbb{N}$, $T_{p,n} = \{b_{n_1} + b_{n_2} + \ldots + b_{n_k} : n_1 > n$, $b_{n_1+1} > 1 + 2 + \ldots + b_n + b_{n_1}$ and if $k > 1$, then $n_1 < n_2$ and for each $i \in \{2, 3, \ldots, k\}$, $b_{n_i+1} > 1 + 2 + \ldots + b_{n_{i-1}} + b_{n_i}\}$.

(c) $T_{p,\infty} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}} T_{p,n}$.

An expression of the form $b_{n_1} + b_{n_2} + \ldots + b_{n_k}$ as in the definition of $T_p$ will be called a *p-sum*. As an example, the requirements for $b_2 + b_5 + b_9$ to be a $p$-sum are that $b_6 > 1 + 2 + \ldots + b_2 + b_5$ and $b_{10} > 1 + 2 + \ldots + b_5 + b_9$.

**Lemma 7.3.** *Let $p$ be a right cancelable element of $\beta\mathbb{N}$ and let $\langle b_n \rangle_{n=1}^{\infty}$ be as guaranteed by Lemma 7.1. Let $a, k, l \in \mathbb{N}$ and assume that $b_{m_1} + \ldots + b_{m_k}$ and $b_{n_1} + \ldots + b_{n_l}$ are p-sums, $b_{m_1+1} > 1 + 2 + \ldots + a + b_{m_1}$, $b_{m_1} > a$, and $a + b_{m_1} + \ldots + b_{m_k} = b_{n_1} + \ldots + b_{n_l}$. Then $l > k$ and, if $i = l - k$, then $a = b_{n_1} + \ldots + b_{n_i}$ and for $j \in \{1, 2, \ldots, k\}$, $b_{m_j} = b_{n_{i+j}}$.*

*Proof.* Suppose the conclusion fails and pick a counterexample with $k+l$ a minimum among all counterexamples. Assume first that $k > 1$ and $l > 1$. We cannot have $m_k = n_l$, for then the equation $a + b_{m_1} + \ldots + b_{m_{k-1}} = b_{n_1} + \ldots + b_{n_{l-1}}$ would provide a smaller counterexample.

If $m_k < n_l$, then $m_k + 1 \leq n_l$, so

$$b_{n_l} \geq b_{m_k+1} > 1+2+\ldots+b_{m_{k-1}}+b_{m_k} \geq a+b_{m_1}+\ldots+b_{m_k} = b_{n_1}+\ldots+b_{n_l}\,,$$

a contradiction. If $n_l < m_k$, then $n_l + 1 \leq m_k$ so

$$b_{m_k} \geq b_{n_l+1} > 1+2+\ldots+b_{n_{l-1}}+b_{n_l} \geq b_{n_1}+\ldots+b_{n_l} = a+b_{m_1}+\ldots+b_{m_k}\,,$$

again a contradiction.

Thus we must have $k = 1$ or $l = 1$.

Case 1. $k = 1$ and $l = 1$. Then $a + b_{m_1} = b_{n_1}$ so $b_{n_1} > b_{m_1}$ and thus $m_1 + 1 \leq n_1$. Therefore $b_{n_1} \geq b_{m_1+1} > 1 + 2 + \ldots + a + b_{m_1} \geq b_{n_1}$, a contradiction.

Case 2. $l = 1$ and $k > 1$. Then $a + b_{m_1} + \ldots + b_{m_k} = b_{n_1}$ so $n_1 \geq m_k + 1$. Therefore $b_{n_1} \geq b_{m_k+1} > 1 + 2 + \ldots + b_{m_{k-1}} + b_{m_k} \geq a + b_{m_1} + \ldots + b_{m_k} = b_{n_1}$, a contradiction.

Case 3. $l > 1$ and $k = 1$. Then $a + b_{m_1} = b_{n_1} + \ldots + b_{n_l}$. If $m_1 > n_l$, then $b_{m_1} \geq b_{n_l+1} > 1 + 2 + \ldots + b_{n_l-1} + b_{n_l} \geq b_{n_1} + \ldots + b_{n_l} = a + b_{m_1}$, a contradiction. If $n_l > m_1$, then $b_{n_l} \geq b_{m_1+1} > 1 + 2 + \ldots + a + b_{m_1} \geq a + b_{m_1} = b_{n_1} + \ldots + b_{n_l}$, a contradiction.

So $m_1 = n_l$ and thus the conclusion of the lemma holds, and we did not have a counterexample. $\square$

**Lemma 7.4.** *Let $p$ be a right cancelable element of $\beta\mathbb{N}$ and let $\langle b_n\rangle_{n=1}^\infty$ be as guaranteed by Lemma 7.1. The expression of an element of $T_p$ as a $p$-sum is unique.*

*Proof.* Suppose that we have $p$-sums $b_{m_1} + \ldots + b_{m_k}$ and $b_{n_1} + \ldots + b_{n_l}$ such that $b_{m_1} + \ldots + b_{m_k} = b_{n_1} + \ldots + b_{n_l}$ but $(m_1, m_2, \ldots, m_k) \neq (n_1, n_2, \ldots, n_l)$ and pick such an example with $k + l$ a minimum among all examples.

Case 1. $k > 1$ and $l > 1$. Then $m_k \neq n_l$ or else the equation $b_{m_1} + \ldots + b_{m_{k-1}} = b_{n_1} + \ldots + b_{n_{l-1}}$ provides a smaller example. So assume without loss of generality that $m_k + 1 \leq n_l$. Then $b_{n_l} \geq b_{m_k+1} > 1 + 2 + \ldots + b_{m_{k-1}} + b_{m_k} \geq b_{m_1} + \ldots + b_{m_k} = b_{n_1} + \ldots + b_{n_l}$, a contradiction.

Case 2. $k = 1$ or $l = 1$. Assume without loss of generality that $k = 1$. If $l = 1$ also, then there was not a counterexample, so $l > 1$. Then $m_1 \geq n_l + 1$ so $b_{m_1} \geq b_{n_l+1} > 1 + 2 + \ldots + b_{n_{l-1}} + b_{n_l} \geq b_{m_1}$, a contradiction. $\square$

**Definition 7.5.** Let $p$ be a right cancelable element of $\beta\mathbb{N}$ and let $\langle b_n\rangle_{n=1}^\infty$ be as guaranteed by Lemma 7.1. Define $\psi_p : T_p \to \mathbb{N}$ by $\psi_p(b_{n_1} + b_{n_2} + \ldots + b_{n_k}) = k$ and let $\widetilde{\psi_p} : c\ell_{\beta\mathbb{N}} T_p \to \beta\mathbb{N}$ be its continuous extension.

**Definition 7.6.** Let $p \in \beta\mathbb{N}$. Then $C_p$ is the smallest compact subsemigroup of $(\beta\mathbb{N}, +)$ with $p$ as a member.

**Theorem 7.7.** *Let $p$ be a right cancelable elment of $\beta\mathbb{N}$. $T_{p,\infty}$ is a compact subsemigroup of $\mathbb{N}^*$, $C_p \subseteq T_{p,\infty}$, the restriction of $\widetilde{\psi_p}$ to $T_{p,\infty}$ is a homomorphism, $\widetilde{\psi_p}(p) = 1$, and $\widetilde{\psi_p}[C_p] = \beta\mathbb{N}$.*

*Proof.* Let $\langle b_n\rangle_{n=1}^\infty$ be as guaranteed for $p$ by Lemma 7.1. For $k \in \mathbb{N}$, let $P_k = \{b_n : n \in \mathbb{N} \text{ and } b_n + k < b_{n+1}\}$. We first claim that for each $n \in \mathbb{N}$, if $k = 1 + 2 + \ldots + b_n$, then $\{b_m \in P_k : m > n\} \subseteq T_{p,n}$. To see this let $b_m \in P_k$ such that $m > n$. Then $b_{m+1} > 1 + 2 + \ldots + b_n + b_m$ so $b_m \in T_{p,n}$. Thus, given $n$, since $\{b_m \in P_k : m > n\} \in p$, we have that $p \in c\ell_{\beta\mathbb{N}} T_{p,n}$. Consequently, $p \in T_{p,\infty}$ and $\widetilde{\psi_p}(p) = 1$.

To see that $T_{p,\infty}$ is a subsemigroup of $\beta\mathbb{N}$, let $m \in \mathbb{N}$ and let $x \in T_{p,m}$. Pick $k \in \mathbb{N}$ and $m_1, m_2, \ldots, m_k$ in $\mathbb{N}$ such that $x = b_{m_1} + \ldots + b_{m_k}$, where $m_1 > m$ , $b_{m_1+1} > 1 + 2 + \ldots + b_m + b_{m_1}$ and if $k > 1$, then $m_1 <$

$m_2$ and for each $i \in \{2,3,\ldots,k\}$, $b_{m_i+1} > 1+2+\ldots+b_{m_{i-1}}+b_{m_i}$. By [72, Theorem 4.20], it suffices to show that $x+T_{p,m_k} \subseteq T_{p,m}$. So let $y \in T_{p,m_k}$. Pick $l \in \mathbb{N}$ and $n_1,n_2,\ldots,n_l$ in $\mathbb{N}$ such that $y = b_{n_1}+\ldots+b_{n_l}$, where $n_1 > m_k$, $b_{n_1+1} > 1+2+\ldots+b_{m_k}+b_{n_1}$ and if $l>1$, then $n_1 < n_2$ and for each $i \in \{2,3,\ldots,l\}$, $b_{n_i+1} > 1+2+\ldots+b_{n_{i-1}}+b_{n_i}$. To see that $x+y \in T_{p,m}$ we need that $b_{m_1}+\ldots+b_{m_k}+b_{n_1}+\ldots+b_{n_l}$ is as in the definition of $T_{p,m}$. If $k>1$, we only need to note that $b_{n_1+1} > 1+2+\ldots+b_{m_k}+b_{n_1}$. If $k=1$, we also need to note that $n_1 > m_k$.

Further, with $x = b_{m_1}+\ldots+b_{m_k}$ and $y = b_{n_1}+\ldots+b_{n_l}$ as in the preceeding paragraph, we have that $\psi_p(x+y) = k+l = \psi_p(x)+\psi_p(y)$, so by [72, Theorem 4.21], the restriction of $\widetilde{\psi_p}$ to $T_{p,\infty}$ is a homomorphism.

Since $p \in T_{p,\infty}$, we have $C_p \subseteq T_{p,\infty}$. Since $D = \{p, p+p, p+p+p, \ldots\} \subseteq C_p$ and $\psi_p[D] = \mathbb{N}$, we have $\widetilde{\psi_p}[C_p] = \beta\mathbb{N}$. $\square$

**Lemma 7.8.** *Let $x \in \beta\mathbb{N}$, let $y \in T_{p,\infty}$, and assume that $x+y \in T_{p,\infty}$. Then $x \in T_{p,\infty}$.*

*Proof.* Suppose that $x \notin T_{p,\infty}$ and pick $r \in \mathbb{N}$ such that $x \notin c\ell_{\beta\mathbb{N}} T_{p,r}$. Let $X = \mathbb{N} \setminus T_{p,r}$ and let $Z = \{a+b_{m_1}+\ldots+b_{m_k} : a \in X,\ b_{m_1}+\ldots+b_{m_k}$ is a $p$-sum, $b_{m_1+1} > 1+2+\ldots+a+b_{m_1}$, and $m_1 > a\}$. We claim that $Z \in x+y$ for which it suffices that $X \subseteq \{a \in \mathbb{N} : -a+Z \in y\}$. So let $a \in X$. We claim that $T_{p,a} \subseteq -a+Z$. To see this, let $b_{m_1}+\ldots+b_{m_k}$ be a $p$-sum in $T_{p,a}$. Then $m_1 > a$ and $b_{m_1+1} > 1+2+\ldots+b_a+b_{m_1} \geq 1+2+\ldots+a+b_{m_1}$. so $a+b_{m_1}+\ldots+b_{m_k} \in Z$ as claimed.

Now $x+y \in T_{p,\infty} \subseteq c\ell_{\beta\mathbb{N}} T_{p,r}$ so pick $w \in Z \cap T_{p,r}$. Since $w \in Z$, pick $a \in X$ and a $p$-sum $b_{m_1}+\ldots+b_{m_k}$ such that $b_{m_1+1} > 1+2+\ldots+a+b_{m_1}$, $m_1 > a$, and $w = a+b_{m_1}+\ldots+b_{m_k}$. Since $w \in T_{p,r}$, pick a $p$-sum $b_{n_1}+b_{n_2}+\ldots+b_{n_l}$ such that $w = b_{n_1}+b_{n_2}+\ldots+b_{n_l}$, $n_1 > r$, $b_{n_1+1} > 1+2+\ldots+b_r+b_{n_1}$ and if $k>1$, then $n_1 < n_2$. By Lemma 7.3, there is some $i < l$ such that $a = b_{n_1}+\ldots+b_{n_i}$, so that $a \in T_{p,r}$, a contradiction. $\square$

**Definition 7.9.** (a) For $n \in \mathbb{N}$, $\mathrm{supp}(n)$ is the finite set $F \subseteq \omega$ such that $n = \sum_{t \in F} 2^t$.

(b) Define $\phi : \mathbb{N} \to \omega$ by $\phi(n) = \max \mathrm{supp}(n)$ and let $\widetilde{\phi} : \beta\mathbb{N} \to \beta\omega$ be its continuous extension.

We write $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}} 2^n\mathbb{N}$. Given any $p \in \beta\mathbb{N}$, $C_p$ is a compact right topological semigroup, so it has a smallest ideal and idempotents minimal in $C_p$.

**Lemma 7.10.** *Assume that $p \in \mathbb{N}^*$, $p$ is right cancelable in $\beta\mathbb{N}$, and $q$ is an idempotent which is minimal in $C_p$. There exist $p' \in C_p \cap \mathbb{H}$ and an* *idempotent $q'$ which is minimal in $C_{p'}$ such that $p'$ is right cancelable in $\beta\mathbb{N}$, $q <_R q'$, and $p' + q = q$.*

*Proof.* By Theorem 7.7, $\widetilde{\psi_p}$ is a homomorphism on $T_{p,\infty}$, $C_p \subseteq T_{p,\infty}$, and $\widetilde{\psi_p}[C_p] = \beta\mathbb{N}$. By [72, Lemma 6.8] if $r \in \beta\mathbb{N}$ and $s \in \mathbb{H}$, then $\widetilde{\phi}(r+s) = \widetilde{\phi}(s)$.

Pick a sequence $\langle D_n\rangle_{n=1}^{\infty}$ of pairwise disjoint infinite subsets of $\mathbb{N}$ and for $n \in \mathbb{N}$, pick $x_n \in \mathbb{N}^*$ such that $\{2^t : t \in D_n\} \in x_n$. Then for each $n$, $D_n \in \widetilde{\phi}(x_n)$ so $\{\widetilde{\phi}(x_n) : n \in \mathbb{N}\}$ is discrete. For each $n \in \mathbb{N}$ pick $y_n \in C_p$ such that $\widetilde{\psi_p}(y_n) = x_n$. Then $C_p + y_n$ is a left ideal of $C_p$ which therefore contains a minimal left ideal of $C_p$ and $q + C_p$ is a minimal right ideal of $C_p$. Recalling that in any compact Hausdorff right topological semigroup, the intersection of a minimal left ideal and a minimal right ideal is a group, we may pick an idempotent $q_n \in (C_p + y_n) \cap (q + C_p)$ and pick $s_n \in C_p$ such that $q_n = s_n + y_n$. Let $p'$ be a cluster point of the sequence $\langle q_n\rangle_{n=1}^{\infty}$. Since by [72, Lemma 6.6] all idempotents of $\beta\mathbb{N}$ are in $\mathbb{H}$, we have that $p' \in C_p \cap \mathbb{H}$.

Let $r = \widetilde{\psi_p}(p')$ and note that $r$ is a cluster point of $\langle\widetilde{\psi_p}(q_n)\rangle_{n=1}^{\infty}$. Note that $\phi[\mathbb{N}] = \omega$; for all $n < \omega$, $\{m \in \mathbb{N} : \phi < n\}$ is finite; and for all $n$ and $k$ in $\mathbb{N}$, if $\phi(n) + 1 < \phi(k)$, then $\phi(n+k) \in \{\phi(k), \phi(k)+1\}$. Also, given $n \in \mathbb{N}$, $\widetilde{\psi_p}(q_n) = \widetilde{\psi_p}(s_n + y_n) = \widetilde{\psi_p}(s_n) + \widetilde{\psi_p}(y_n) = \widetilde{\psi_p}(s_n) + x_n$ and since $x_n \in \mathbb{H}$, $\widetilde{\phi}(\widetilde{\psi}(s_n) + x_n) = \widetilde{\phi}(x_n)$. That is $\widetilde{\phi}\big(\widetilde{\psi_p}(q_n)\big) = \widetilde{\phi}(x_n)$. Since $\{\widetilde{\phi}(x_n) : n \in \mathbb{N}\}$ is discrete and $r$ is a cluster point of $\langle\widetilde{\psi}(q_n)\rangle_{n=1}^{\infty}$, we have by [72, Theorem 6.54.4] with $S = T = \mathbb{N}$, $f = \phi$, and $p_n = \widetilde{\psi_p}(q_n)$, that $(\mathbb{N} + r) \cap (\mathbb{N}^* + \mathbb{N}^*) = \emptyset$.

We claim that $r$ is right cancelable in $\beta\mathbb{N}$. By (9) $\Rightarrow$ (3) of [72, Theorem 8.11] with $S = T = \mathbb{N}$, it suffices to show that for $a \in \mathbb{N}$ and $s \in \beta\mathbb{N} \setminus \{a\}$, $a + r \neq s + r$. If $s \in \mathbb{N}$, this holds by [72, Corollary 8.2]. If $s \in \mathbb{N}^*$, this holds because $(\mathbb{N} + r) \cap (\mathbb{N}^* + \mathbb{N}^*) = \emptyset$.

Next we claim that $p'$ is right cancelable in $\beta\mathbb{N}$. Suppose not and by [72, Theorem 8.18] pick an idempotent $e \in \mathbb{N}^*$ such that $p' = e + p'$. Now $p' \in C_p \subseteq T_{p,\infty}$ so by Lemma 7.8, $e \in T_{p,\infty}$ and thus by Theorem 7.7, $r = \widetilde{\psi_p}(p') = \widetilde{\psi_p}(e) + \widetilde{\psi_p}(p') = \widetilde{\psi_p}(e) + r$ so by [72, Theorem 8.18], $r$ is not right cancelable in $\beta\mathbb{N}$, a contradiction.

For each $n \in \mathbb{N}$, $q_n \in q + C_p$ so $q_n + C_p \subseteq q + C_p$ and, since $q$ is minimal in $C_p$, $q + C_p$ is a minimal right ideal of $C_p$, so $q_n + C_p = q + C_p$ and therefore $q_n + q = q$. That is $\rho_q$ is constantly equal to $q$ on $\{q_n : n \in \mathbb{N}\}$, so $p' + q = q$.

Since $p' \in \{y \in \beta\mathbb{N} : y + q = q\} = \rho_q^{-1}[\{q\}]$ we have $\{y \in \beta\mathbb{N} : y + q = q\}$ is a compact subsemigroup of $\beta\mathbb{N}$ with $p'$ as a member and thus $C_{p'} \subseteq$

$\{y \in \beta\mathbb{N} : y + q = q\}$. Let $q'$ be a minimal idempotent in $C_{p'}$. Then $q' + q = q$ so $q \leq_R q'$. It remains only to show that the inequality is strict.

We show now that $C_r \cap K(\beta\mathbb{N}) = \emptyset$. To this end, we first establish that we may pick a minimal right ideal $R$ of $\beta\mathbb{N}$ such that $r \in c\ell E(R)$, where $E(R)$ is the set of idempotents in $R$. By Theorem 7.7, the restriction of $\widetilde{\psi_p}$ to $C_p$ is a homomorphism onto $\beta\mathbb{N}$ so by [72, Exercise 1.7.3], $\widetilde{\psi_p}[K(C_p)] = K(\beta\mathbb{N})$. Pick a minimal right ideal $R$ of $\beta\mathbb{N}$ such that $\widetilde{\psi_p}(q) \in R$. By [72, Exercise 1.7.3] again, $\widetilde{\psi_p}[q + C_p] = R$. Each $q_n \in q + C_p$ and $p' \in c\ell\{q_n : n \in \mathbb{N}\}$ so $r = \widetilde{\psi_p}(p') \in c\ell\{\widetilde{\psi_p}(q_n) : n \in \mathbb{N}\} \subseteq c\ell E(R)$.

Let $G = \{v \in \beta\mathbb{N} : (\forall u \in R)(v + u = u)\}$. By [72, Lemma 1.30(b)], $E(R) \subseteq G$, so $G$ is a compact subsemigroup of $\beta\mathbb{N}$. We claim that $C_r \subseteq G$ for which it suffices that $r \in G$. To see this, let $u \in R$. We show that $r + u \subseteq u$, so let $A \in (r + u)$ and let $B = \{x \in \mathbb{N} : -x + A \in u\}$. Then $B \in r$ and $r \in c\ell E(R)$ so pick $w \in E(R) \cap \overline{B}$. Then $w + u = u$ so $A \in u$ and thus $r + u = u$ as required.

Now suppose that $C_r \cap K(\beta\mathbb{N}) \neq \emptyset$. We claim that $C_r \cap K(\beta\mathbb{N}) \subseteq R$. To see this, let $w \in C_r \cap K(\beta\mathbb{N})$. Pick a minimal right ideal $R'$ of $\beta\mathbb{N}$ such that $w \in R'$. Pick $u \in R$. Since $C_r \subseteq G$, $w + u = u$ so $R \cap R' \neq \emptyset$ and thus $R' = R$.

Now fix $v \in C_r \cap K(\beta\mathbb{N})$. By Theorem 7.7, the restriction of $\widetilde{\psi_r}$ to $C_r$ is a homomorphism onto $\beta\mathbb{N}$ so by [72, Exercise 1.7.3], $\widetilde{\psi_r}[K(C_r)] = K(\beta\mathbb{N})$. Also $C_r \cap K(\beta\mathbb{N}) = K(C_r)$ by [72, Theorem 1.65]. We claim that $\widetilde{\psi_r}(v)$ is a left identity for $K(\beta\mathbb{N})$ so let $w \in K(\beta\mathbb{N})$ and pick $u \in K(C_r)$ such that $\widetilde{\psi_r}(u) = w$. Then $\widetilde{\psi_r}(v) + w = \widetilde{\psi_r}(v) + \widetilde{\psi_r}(u) = \widetilde{\psi_r}(v + u) = \widetilde{\psi_r}(u) = w$. We thus have that $\widetilde{\psi_r}(v) \in K(\beta\mathbb{N})$ and $\widetilde{\psi_r}(v) + K(\beta\mathbb{N}) = K(\beta\mathbb{N})$ so $\beta\mathbb{N}$ has only one minimal right ideal, while by [72, Theorem 6.9] $\beta\mathbb{N}$ has $2^{\mathfrak{c}}$ minimal right ideals. This contradiction establishes that $C_r \cap K(\beta\mathbb{N}) = \emptyset$.

To finish the proof of the lemma, we will show that $C_{p'} \cap K(C_p) = \emptyset$. This will suffice since then if $q' = q + q'$ we have $q' \in C_{p'} \subseteq C_p$ and $q \in K(C_p)$ so $q' = q + q' \in C_{p'} \cap K(C_p)$.

So suppose we have $s \in C_{p'} \cap K(C_p)$. Then $\widetilde{\psi_p}(s) \in K(\beta\mathbb{N})$. Also, $\widetilde{\psi_p}^{-1}[C_r]$ is a compact semigroup and $p' \in \widetilde{\psi_p}^{-1}[C_r]$ so $C_{p'} \subseteq \widetilde{\psi_p}^{-1}[C_r]$ and thus $\widetilde{\psi_p}(s) \in C_r \cap K(\beta\mathbb{N})$, a contradiction. $\square$

In the proof of the following theorem we shall inductively construct two $\omega_1$ sequences, $\langle p_\sigma \rangle_{\sigma<\omega_1}$ and $\langle q_\sigma \rangle_{\sigma<\omega_1}$ where each $p_\sigma$ is right cancelable in $\beta\mathbb{N}$ and $\langle q_\sigma \rangle_{\sigma<\omega_1}$ is a $<_R$-increasing chain of idempotents, with each $q_\sigma$ being a minimal idempotent in $C_{p_\sigma}$.

**Theorem 7.11.** *Let $p$ be a right cancelable element of $\beta\mathbb{N}$ and let $q$ be a minimal idempotent in $C_p$. There exists a sequence $\langle q_\sigma\rangle_{\sigma<\omega_1}$ of idempotents in $\beta\mathbb{N}$ such that $q_0 = q$ and $q_\sigma <_R q_\delta$ whenever $\sigma < \delta < \omega_1$.*

*Proof.* Let $p_0 = p$ and $q_0 = q$. Let $0 < \alpha < \omega_1$ and assume we have chosen $\langle p_\sigma\rangle_{\sigma<\alpha}$ and $\langle q_\sigma\rangle_{\sigma<\alpha}$ such that

(1) if $0 < \delta < \alpha$, then $p_\delta \in \mathbb{H}$;
(2) if $\delta < \alpha$, then $p_\delta$ is right cancelable in $\beta\mathbb{N}$;
(3) if $\delta < \alpha$, then $q_\delta$ is a minimal idempotent in $C_{p_\delta}$;
(4) if $\delta < \sigma < \alpha$, then $q_\delta <_R q_\sigma$;
(5) if $\delta < \sigma < \alpha$, then $p_\sigma \in C_{p_\delta}$; and
(6) if $\delta < \sigma < \alpha$, then $p_\sigma + q_\delta = q_\delta$.

The hypotheses hold for $\alpha = 1$, all but (2) amd (3), vacuously.

Case 1. $\alpha = \gamma + 1$ for some $\gamma$. By hypotheses (2) and (3) and Lemma 7.10 we may pick $p_\alpha \in C_{p_\gamma} \cap \mathbb{H}$ which is right cancelable in $\beta\mathbb{N}$ and an idempotent $q_\alpha$ which is minimal in $C_{p_\alpha}$ such that $q_\gamma <_R q_\alpha$ and $p_\alpha + q_\gamma = q_\gamma$. One sees immediately that hypotheses (1) through (4) hold at $\alpha + 1$. To verify hypothesis (5), let $\delta < \alpha$. If $\delta = \gamma$, we have $p_\alpha \in C_{p_\delta}$ directly. Otherwise, $p_\gamma \in C_{p_\delta}$ by assumption so $p_\alpha \in C_{p_\gamma} \subseteq C_{p_\delta}$.

To verify hypothesis (6), again if $\delta = \gamma$ we have $p_\alpha + q_\delta = q_\delta$ directly, so assume $\delta < \gamma$. Then $p_\alpha + q_\gamma = q_\gamma$ and, since $q_\delta <_R q_\gamma$, $q_\gamma + q_\delta = q_\delta$ so $p_\alpha + q_\delta = p_\alpha + q_\gamma + q_\delta = q_\gamma + q_\delta = q_\delta$.

Case 2. $\alpha$ is a limit ordinal. Choose a cofinal sequence $\langle\delta(n)\rangle_{n<\omega}$ in $\alpha$ such that $\delta(0) > 0$ and $\delta(n) < \delta(n+1)$ for each $n < \omega$. Let $p_\alpha$ be a cluster point of the sequence $\langle p_{\delta(n)}\rangle_{n<\omega}$. Let $q_\alpha$ be a minimal idempotent in $C_{p_\alpha}$. Since $p_{\delta(n)} \in \mathbb{H}$ for each $n < \omega$, we have $p_\alpha \in \mathbb{H}$.

We claim that $p_\alpha$ is right cancelable in $\beta\mathbb{N}$. Suppose not and by [72, Theorem 8.18] pick an idempotent $e \in \mathbb{N}^*$ such that $p_\alpha = e + p_\alpha$. Then $p_\alpha \in \beta\mathbb{N} + p_\alpha = c\ell_{\beta\mathbb{N}}(\mathbb{N} + p_\alpha)$ and $p_\alpha \in c\ell_{\beta\mathbb{N}}\{p_{\delta(n)} : n < \omega\}$ so by [72, Theorem 3.40], either there is some $n \in \mathbb{N}$ such that $n + p_\alpha \in c\ell_{\beta\mathbb{N}}\{p_{\delta(n)} : n < \omega\}$ or there is some $n < \omega$ such that $p_{\delta(n)} \in \beta\mathbb{N} + p_\alpha$. The first alternative is impossible because $p_\alpha \in \mathbb{H}$ and $\{p_{\delta(n)} : n < \omega\} \subseteq \mathbb{H}$. So pick $n < \omega$ and $x \in \beta\mathbb{N}$ such that $p_{\delta(n)} = x + p_\alpha$. Since $p_{\delta(m)} \in C_{p_{\delta(n)}}$ for all $m > n$ by hypothesis (5), we have $p_\alpha \in C_{p_{\delta(n)}} \subseteq T_{p_{\delta(n)},\infty}$. Since also $p_{\delta(n)} \in T_{p_{\delta(n)},\infty}$, we have by Lemma 7.8 that $x \in T_{p_{\delta(n)},\infty}$. But now, by Theorem 7.7, $1 = \widetilde{\psi_{p_{\delta(n)}}}(p_{\delta(n)}) = \widetilde{\psi_{p_{\delta(n)}}}(x) + \widetilde{\psi_{p_{\delta(n)}}}(p_\alpha)$ which is impossible. Thus hypothesis (2) holds.

Hypothesis (3) holds directly. To verify hypotheses (4), (5), and (6), let $\sigma < \alpha$ and pick $n < \omega$ such tht $\sigma < \delta(n) < \alpha$. For each $m$ with $n < m < \omega$, we have by hypothesis (6) that $p_{\delta(m)} + q_{\delta(n)} = q_{\delta(n)}$ so $p_\alpha + q_{\delta(n)} = q_{\delta(n)}$. Therefore $\{y \in \beta\mathbb{N} : y + q_{\delta(n)} = q_{\delta(n)}\}$ is a compact subsemigroup of $\beta\mathbb{N}$ with $p_\alpha$ as a member so $C_{p_\alpha} \subseteq \{y \in \beta\mathbb{N} : y + q_{\delta(n)} = q_{\delta(n)}\}$. Therefore

$q_\alpha + q_{\delta(n)} = q_{\delta(n)}$ so $q_\sigma <_R q_{\delta(n)} \leq_R q_\alpha$ and we have verified hypothesis (4). Also, for each $m \geq n$ we have $p_{\delta(m)} \in C_{p_\sigma}$ so $p_\alpha \in C_{p_\sigma}$ as required by hypothesis (5). Since for all $m \geq n$, $p_{\delta(m)} + q_\sigma = q_\sigma$, we have $p_\alpha + q_\sigma = q_\sigma$ as required by hypothesis (6). $\square$

## 8. Increasing Principal Left Ideals in $\beta\mathbb{Z}$

In this section we present Yevhen Zelenyuk's proof [133] that there does not exist a sequence of increasing principal left ideals of $(\beta\mathbb{Z}, +)$.

We begin with some notation that will be used throughout the section.

**Definition 8.1.** (a) $W$ is the set of finite nonempty words over the alphabet $\mathbb{N}$. That is, $w \in W$ if and only if there exists $n \in \mathbb{N}$ such that $w : \{1, 2, \ldots, n\} \to \mathbb{N}$.

(b) Given $w \in W$, if the domain of $w$ is $\{1, 2, \ldots, n\}$, then $\ell(w) = n$.

(c) Given an infinite sequence $\langle w_j \rangle_{j=1}^\infty$ in $W$, we say that the sequence is *increasing* if and only if for each $j \in \mathbb{N}$, $\ell(w_j) \geq j$ and the sequence $\langle w_k(j) \rangle_{k=j}^\infty$ is strictly increasing.

(d) Given a finite sequence $\langle w_j \rangle_{j=1}^n$ in $W$, we say that the sequence is *increasing* if and only if for each $j \in \{1, 2, \ldots, n\}$, $\ell(w_j) \geq j$ and the sequence $\langle w_k(j) \rangle_{k=j}^n$ is strictly increasing.

(e) If $n \in \mathbb{N}$ and $\langle w_j \rangle_{j=1}^n$ is an increasing sequence in $W$, then $[w_1, w_2, \ldots, w_n]$ is the word $v \in W$ with $\ell(v) = \ell(w_n)$ such that for $j \in \{1, 2, \ldots, n{-}1\}$, $v(j) = w_j(j)$ and for $j \in \{n, n{+}1, \ldots, \ell(w_n)\}$, $v(j) = w_n(j)$.

Notice that if $w \in W$, then $[w] = w$. Also, if $\langle w_j \rangle_{j=1}^\infty$ is an increasing sequence in $W$, then whenever $j \leq k$ in $\mathbb{N}$, $w_k(j) \geq k - j + 1$.

We will write $w = \alpha_1 \alpha_2 \cdots \alpha_m$ when $m = \ell(w)$ and for each $i \in \{1, 2, \ldots, m\}$, $\alpha_i = w(i)$.

**Lemma 8.2.** *There is a 2-coloring of $W$ such that there does not exist an increasing sequence $\langle w_j \rangle_{j=1}^\infty$ in $W$ such that*

$$\{[w_{j_1}, w_{j_2}, \ldots, w_{j_k}] : k \in \mathbb{N} \text{ and } j_1 < j_2 < \ldots < j_k\}$$

*is monochromatic.*

*Proof.* Given $w \in W$ with $\ell(w) = m > 1$, we define inductively $r(w) \in \mathbb{N}$ and a sequence $s(w) = \langle i_0, i_1, \ldots, i_{r(w)} \rangle$ such that $m = i_0 > i_1 > \ldots > i_{r(w)} = 1$. Let $w = \alpha_1 \alpha_2 \cdots \alpha_m$ and let $i_0 = m$. Assume that $t \in \{0, 1, \ldots, m-1\}$ and $i_t$ has been defined. If $i_t = 1$, let $r(w) = t$. Otherwise, let

$$i_{t+1} = \min\{i \in \{1, 2, \ldots, i_t - 1\} : (\forall j \in \{i, i+1, \ldots, i_t - 1\})(i_t - j \leq \alpha_j)\}.$$

Notice that if $j = i_t - 1$, then $i_t - j = 1 \leq \alpha_j$ so such a choice is always possible.

Let $d_1(w) = i_{r(w)-1} - 1$ and, if $r(w) \geq 2$, let $d_2(w) = i_{r(w)-2} - 1$. We claim that if $\ell(w) > d_1(w) + 1$, then $r(w) \geq 2$ so that $d_2(w)$ is defined. To see this, note that if $r(w) = 1$, then $d_1(w) = i_0 - 1 = \ell(w) - 1$ so $\ell(w) = d_1(w) + 1$.

Define $\chi : W \to \{0, 1\}$ by

$$\chi(w) = \begin{cases} 1 & \text{if } r(w) \text{ is odd} \\ 0 & \text{if } \ell(w) = 1 \text{ or } r(w) \text{ is even.} \end{cases}$$

Suppose we have an increasing sequence $\langle w_j \rangle_{j=1}^{\infty}$ in $W$ such that $\chi$ is constant on $\{[w_{j_1}, w_{j_2}, \ldots, w_{j_k}] : k \in \mathbb{N} \text{ and } j_1 < j_2 < \ldots < j_k\}$. The sequence $\langle w_j \rangle_{j=2}^{\infty}$ is also an increasing sequence, so we may assume that each $\ell(w_j) > 1$.

We claim that $\{d_1(w_j) : j \in \mathbb{N}\}$ is finite. Suppose instead that $\{d_1(w_j) : j \in \mathbb{N}\}$ is infinite. Pick $j$ such that $d_1(w_j) > w_1(1)$ and let $\alpha = w_1(1)$. Let $\beta_1\beta_2 \cdots \beta_m = w_j$, let $w = [w_1, w_j]$, and let $\delta_1\delta_2 \cdots \delta_m = w$. Then $\delta_1 = \alpha$ and for $t \in \{2, 3, \ldots, m\}$, $\delta_t = \beta_t$.

Let $s(w_j) = \langle i_0, i_1, \ldots, i_{r(w_j)} \rangle$ where $m = i_0 > i_1 > \ldots > i_{r(w_j)} = 1$ and if $t \in \{0, 1, \ldots, r(w_j) - 1\}$, then

$$i_{t+1} = \min\{i \in \{1, 2, \ldots, i_t - 1\} : (\forall j \in \{i, i+1, \ldots, i_t - 1\})(i_t - j \leq \beta_j)\}\,.$$

Let $s(w) = \langle i'_0, i'_1, \ldots, i'_{r(w)} \rangle$ where $m = i'_0 > i'_1 > \ldots > i'_{r(w)} = 1$ and if $t \in \{0, 1, \ldots, r(w) - 1\}$, then

$$i'_{t+1} = \min\{i \in \{1, 2, \ldots, i'_t - 1\} : (\forall j \in \{i, i+1, \ldots, i'_t - 1\})(i'_t - j \leq \delta_j)\}\,.$$

Let $r = r(w_j)$. We claim that for $t \in \{0, 1, \ldots, r - 1\}$, $i'_t = i_t$. This is true for $t = 0$. The claim holds if $r = 1$, so assume that $r \geq 2$, let $t \in \{0, 1, \ldots, r - 2\}$ and assume that $i'_t = i_t$. Then

$$i_{t+1} = \min\{i \in \{1, 2, \ldots, i_t - 1\} : (\forall j \in \{i, i+1, \ldots, i_t - 1\})(i_t - j \leq \beta_j)\}\,.$$

Also

$$i'_{t+1} = \min\{i \in \{1, 2, \ldots, i_t - 1\} : (\forall j \in \{i, i+1, \ldots, i_t - 1\})(i_t - j \leq \delta_j)\}\,.$$

Now $t + 1 \leq r - 1$ so $i_{t+1} \geq i_{r-1} = d_1(w_j) + 1 > w_1(1) \geq 2$. So for all $j \in \{i_{t+1}, i_{t+1} + 1, \ldots, i_t - 1\}$, $i_t - j \leq \beta_j$ and if $j = i_{t+1} - 1$, then $i_t - j > \beta_j$ and $j > 1$. So for $j \in \{i_{t+1} - 1, i_{t+1}, \ldots, i_t - 1\}$, $\delta_j = \beta_j$ so $i'_{t+1} = i_{t+1}$.

Now we claim that $i'_r = 2$. Since $i_r = 1$, $i_{r-1} - j \leq \beta_j$ for all $j \in \{1, 2, \ldots, i_{r-1} - 1\}$ and therefore $i'_{r-1} \leq \delta_j$ for all $j \in \{2, 3, \ldots, i'_{r-1} - 1\}$ so that $i'_r \leq 2$. Since also $i_{r-1} - 1 = d_1(w_j) > \alpha = \delta_1$, we have $i'_r = 2$ as claimed. Consequently $r(w) = r(w_j) + 1$ and thus $\chi([w_j]) \neq \chi([w_1, w_j])$. We have established that $\{d_1(w_j) : j \in \mathbb{N}\}$ is finite. Consequently only finitely many $j$ have $\ell(w_j) \leq d_1(w_j) + 1$ so $d_2(w_j)$ is defined for all by finitely many values of $j$. So we may assume that $d_2(w_j)$ is defined for all $j$.

Now we claim that $\{d_2(w_j) : j \in \mathbb{N}\}$ is infinite. Suppose instead that $\{d_2(w_j) : j \in \mathbb{N}\}$ is finite. Recall that whenever $j \leq k$ in $\mathbb{N}$, $w_k(j) \geq k - j + 1$. Let $k = \max\{d_2(w_j) : j \in \mathbb{N}\} + 1$. Let $w = w_k = \beta_1\beta_2 \cdots \beta_m$. Let $s(w) = \langle i_0, i_1, \ldots, i_{r(w)}\rangle$ where $m = i_0 > i_1 > \ldots > i_{r(w)} = 1$ and if $t \in \{0, 1, \ldots, r(w) - 1\}$, then

$$i_{t+1} = \min\{i \in \{1, 2, \ldots, i_t - 1\} : (\forall j \in \{i, i+1, \ldots, i_t - 1\})(i_t - j \leq \beta_j)\}\,.$$

We claim that $i_{r(w)-1} = 1$, which is a contradiction. We need to show that for each $j \in \{1, 2, \ldots, i_{r(w)-2} - 1\}$, $i_{r(w)-2} - j \leq \beta_j = w_k(j)$. So let $j \in \{1, 2, \ldots, i_{r(w)-2} - 1\}$. Then $w_k(j) \geq k - j + 1 \geq d_2(w_k) + 1 - j + 1 = i_{r(w)-2} - j + 1$ so $i_{r(w)-2} - j < w_k(j) = \beta_j$.

So now we have that $\{d_1(w_j) : j \in \mathbb{N}\}$ is finite and $\{d_2(w_j) : j \in \mathbb{N}\}$ is infinite. Pick $j_1$ such that $w_{j_1}(1) \geq \max\{d_1(w_j) : j \in \mathbb{N}\}$. Let $\alpha_1 = w_{j_1}(1)$ and note that $\alpha_1 \geq j_1$. Let $k = \alpha_1 + 1$ so that $k - 1 + 1 > \alpha_1$. Pick $j_2 > j_1$ such that $w_{j_2}(2) > k - 2 + 1$. Given $t \in \{2, 3, \ldots, k - 1\}$ pick $j_{t+1} > j_t$ such that $w_{j_{t+1}}(t + 1) > k - (t + 1) + 1$. If $t = k - 1$, require also that $w_{j_{t+1}}(t + 1) > 3$. For $t \in \{1, 2, \ldots, k\}$, let $\alpha_t = w_{j_t}(t)$. Pick $j_{k+1} > j_k$ such that $d_2(w_{j_{k+1}}) > \alpha_k + k - 1$.

Let $\beta_1\beta_2 \cdots \beta_m = w_{j_{k+1}}$. Let $r = r(w_{j_{k+1}})$. Let $s(w_{j_{k+1}}) = \langle i_0, i_1, \ldots, i_r\rangle$ where $m = i_0 > i_1 > \ldots > i_r = 1$ and if $t \in \{0, 1, \ldots, r - 1\}$, then

$$i_{t+1} = \min\{i \in \{1, 2, \ldots, i_t - 1\} : (\forall j \in \{i, i+1, \ldots, i_t - 1\})(i_t - j \leq \beta_j)\}\,.$$

Now $k - 1 = \alpha_1 = w_{j_1}(1) \geq d_1(w_{j_{k+1}}) = i_{r-1} - 1$ so $i_{r-1} \leq k$. Also $\alpha_k + k - 1 < d_2(w_{j_{k+1}}) = i_{r-2} - 1$ so $i_{r-2} > \alpha_k + k \geq 2 + k$ and $i_{r-2} - k > \alpha_k$.

Let $w = [w_{j_1}, w_{j_2}, \ldots, w_{j_{k+1}}] = \delta_1\delta_2 \cdots \delta_m$. Then $w = \alpha_1\alpha_2 \cdots \alpha_k\beta_{k+1} \cdots \beta_m$. (Since $\ell(w_{j_{k+1}}) \geq j_{k+1} > k$, we have that $m > k$.) Let $s(w) = \langle i'_0, i'_1, \ldots, i'_{r(w)}\rangle$ where $m = i'_0 > i'_1 > \ldots > i'_{r(w)} = 1$ and if $t \in \{0, 1, \ldots, r(w) - 1\}$, then

$$i'_{t+1} = \min\{i \in \{1, 2, \ldots, i'_t - 1\} : (\forall j \in \{i, i+1, \ldots, i'_t - 1\})(i'_t - j \leq \delta_j)\}\,.$$

We claim that

(1) for $t \in \{0, 1, \ldots, r - 2\}$, $i'_t = i_t$,
(2) $i'_{r-1} = k + 1$, and
(3) $i'_r = 2$

so that $r(w) = r(w_{j_{k+1}}) + 1$. This will complete the proof.

To establish (1), note that $i'_0 = m = i_0$. Let $t \in \{0, 1, \ldots, r - 3\}$ and assume that $i'_t = i_t$. Then $t + 1 \leq r - 2$. Let $i = i_{t+1}$. Then for $j \in \{i, i + 1, \ldots, i_t - 1\}$, $i_t - j \leq \beta_j$ and $i_t - (i - 1) > \beta_{i-1}$. Since

$i-1=i_{t+1}-1\geq i_{r-2}-1\geq k+2$ we have that for $j\in\{i,i+1,\ldots,i'_t-1\}$, $i'_t-j\leq\delta_j$ and $i'_t-(i-1)>\delta_{i-1}$. so $i'_{t+1}=i_{t+1}$ as required.

For (2), we have seen that $i_{r-1}\leq k$ so for $j\in\{k+1,k+2,\ldots,i_{r-2}-1\}$, $i_{r-2}-j\leq\beta_j=\delta_j$ and thus $i'_{r-1}\leq k+1$. But also $i_{r-2}-k>\alpha_k=\delta_k$ so $i'_{r-1}=k+1$.

To verify (3), let $t\in\{2,3,\ldots,k\}$. Then $\alpha_t=w_{j_t}(t)>k-t+1$ so $i'_{r-1}-t=k+1-t<\alpha_t$ so $i'_r\leq 2$. But we chose $k>\alpha_1$ so $i'_{r-1}-1=k>\alpha_1$ so $i'_r=2$. □

**Proposition 8.3.** *Assume that there is an increasing sequence of principal left ideals of $\beta\mathbb{Z}$. Then for every finite coloring of $W$, there is an infinite sequence $w_1<w_2<\ldots$ such that the set*

$$\{[w_{j_1},w_{j_2},\ldots,w_{j_k}]:k\in\mathbb{N}\text{ and }1\leq j_1<\ldots<j_k\}$$

*is monochromatic.*

*Proof.* Let $\langle p_n\rangle_{n=0}^\infty$ be a sequence in $\beta\mathbb{Z}$ such that the sequence $\langle\beta\mathbb{Z}+p_n\rangle_{n=0}^\infty$ is strictly increasing. If $p\in\mathbb{Z}$, then $\beta\mathbb{Z}+p=\beta\mathbb{Z}$ so each $p_n\in\mathbb{Z}^*$. Since $\{p_n:n\in\omega\}$ is an infinite Hausdorff space, it contains an infinite strongly discrete subspace, so we may presume that $\{p_n:n\in\omega\}$ is strongly discrete. For each $n\in\omega$, pick $A_n\in p_n$ such that all $A_n$ are pairwise disjoint and $\overline{A_{n+1}}\cap(\beta\mathbb{Z}+p_n)=\emptyset$. Then $x+p_n\notin\overline{A_{n+1}}$ for all $x\in\mathbb{Z}$ and all $n\in\omega$.

For $n\in\omega$, let $X_{n,n}=\{x\in\mathbb{Z}:x+p_n\in\overline{A_n}\}$ and $X_{n+1,n}=\{x\in\mathbb{Z}:x+p_{n+1}\in\overline{A_n}\}$. We note that for each $n\in\omega$, $p_n\in c\ell\{x+p_{n+1}:x\in X_{n+1,n}\}$. To see this, let $B\in p_n$. Since $p_n\in\beta\mathbb{Z}+p_{n+1}=c\ell(\mathbb{Z}+p_{n+1})$ and $B\cap A_n\in p_n$, pick $x\in\mathbb{Z}$ such that $x+p_{n+1}\in\overline{A_n\cap B}$. Then $x\in X_{n+1,n}$ and $x+p_{n+1}\in\overline{B}$.

We shall construct inductively for each $n\in\mathbb{N}$ a sequence $\langle A_{n,j}\rangle_{j=0}^\infty$ of members of $p_n$ and a sequence $\langle x_{n,j}\rangle_{j=1}^\infty$ of members of $\mathbb{Z}$. For $n\in\omega$, let $A_{n,0}=A_n$. (We do not define $x_{0,j}$ for any $j$.) Let $E=\{(x,n):n\in\omega\text{ and }x\in X_{n+1,n}\}$ and let $A=\bigcup_{n=0}^\infty A_n$. Let $\langle e_m\rangle_{m=0}^\infty$ enumerate $E$ and let $\langle a_m\rangle_{m=0}^\infty$ enumerate $A$.

For $m\in\omega$ we inductively choose $k_n^m$ and $Z_n^m$ for each $n\in\omega$ and sequences $\langle A_{n,j}\rangle_{j=1}^{k_n^m}$ and $\langle x_{n,j}\rangle_{j=1}^{k_n^m}$ for each $n\in\mathbb{N}$ satisfying the following induction hypotheses, where

$$C_0^m=\bigcup\{x_{1,j}+A_{1,j}:j\in\{1,2,\ldots,k_1^m\}\text{ and }x_{1,j}\in X_{1,0}\}$$

and if $n\in\mathbb{N}$,

$$\begin{aligned}C_n^m=&\bigcup\{x_{n,j}+A_{n,j}:j\in\{1,2,\ldots,k_n^m\}\text{ and }x_{n,j}\in X_{n,n}\}\cup\\&\bigcup\{x_{n+1,j}+A_{n+1,j}:j\in\{1,2,\ldots,k_{n+1}^m\}\text{ and }x_{n+1,j}\in X_{n+1,n}\}.\end{aligned}$$

(i) For $n\in\mathbb{N}$ and $j\in\{1,2,\ldots,k_n^m\}$, $A_{n,j}\in p_n$ and $A_{n,j}\subseteq A_{n,j-1}$.

(ii) The sets $\{x_{n,j}+A_{n,j} : n \in \mathbb{N}$ and $j \in \{1,2,\ldots,k_n^m\}\}$ are pairwise disjoint and for $n \in \mathbb{N}$ and $j \in \{1,2,\ldots,k_n^m\}$, $(x_{n,j}+A_{n,j}) \cap A_{n,j} = \emptyset$.
(iii) For $n \in \mathbb{N}$ and $j \in \{1,2,\ldots,k_n^m\}$, $x_{n,j} \in X_{n,n-1} \cup X_{n,n}$.
(iv) For $n \in \omega$ and $j \in \{1,2,\ldots,k_{n+1}^m\}$, if $x_{n+1,j} \in X_{n+1,n+1}$, then $x_{n+1,j} + A_{n+1,j} \subseteq A_{n+1,k-1} \setminus A_{n+1,k}$ for some $k \in \{1,2,\ldots,k_{n+1}^m\}$ or $x_{n+1,j} + A_{n+1,j} \subseteq A_{n+1,k_{n+1}^m}$.
(v) For $n \in \omega$ and $j \in \{1,2,\ldots,k_{n+1}^m\}$, if $x_{n+1,j} \in X_{n+1,n}$, then $x_{n+1,j} + A_{n+1,j} \subseteq A_{n,k-1} \setminus A_{n,k}$ for some $k \in \{1,2,\ldots,k_n^m\}$ or $x_{n+1,j} + A_{n+1,j} \subseteq A_{n,k_n^m}$.
(vi) For $n \in \omega$, $Z_n^m$ is a finite subset of $A_n$, $Z_n^m \cap C_n^m = \emptyset$, and $C_n^m \notin p_n$.
(vii) If $m > 0$, then for each $n \in \mathbb{N}$, $A_{n,k_n^m} \cap C_n^{m-1} = \emptyset$.
(viii) If $m > 0$ and $a_m \in A_t$, there exist $l \in \omega$, finite $J_0, J_1, \ldots, J_l \subseteq \mathbb{N}$ with $J_i \neq \emptyset$ if $i > 0$, and $z_m \in Z_{t+l}^m$ such that $a_m = z_m + \sum_{i=0}^{l} \sum_{j \in J_i} x_{t+i,j}$;
  (a) $J_0 = \emptyset$ if and only if $a_m \notin \bigcup\{x_{t,j} + A_{t,j} : j \in \{1,2,\ldots,k_t^{m-1}\}$ and $x_{t,j} \in X_{t,t}\}$;
  (b) $l = 0$ if and only if $-\sum_{j \in J_0} x_{t,j} + a_m \notin \bigcup\{x_{t+1,j} + A_{t+1,j} : j \in \{1,2,\ldots,k_{t+1}^{m-1}\}$ and $x_{t+1,j} \in X_{t+1,t}\}$;
  (c) for each $k \in J_0$, if any, $x_{t,k} \in X_{t,t}$ and $-\sum_{J_0 \ni j < k} x_{t,j} + a_m \in x_{t,k} + A_{t,k}$;
  (d) for each $i \in \{1,2,\ldots,l\}$, if any, and each $k \in J_i$, $-\left(\sum_{J_i \ni j < k} x_{t+i,j} + \sum_{n=0}^{i-1} \sum_{j \in J_n} x_{t+n,j}\right) + a_m \in x_{t+i,k} + A_{t+i,k}$;
  (e) for $i \in \{1,2,\ldots,l\}$, if any, if $j = \min J_i$, then $x_{t+i,j} \in X_{t+i,t+i-1}$ and if $j \in J_i \setminus \{\min J_i\}$, then $x_{t+i,j} \in X_{t+i,t+i}$;
  (f) for $i \in \{1,2,\ldots,l-1\}$, if any, $-\sum_{n=0}^{i} \sum_{j \in J_n} x_{t+n,j} + a_m \in \bigcup\{x_{t+i+1,j} + A_{t+i+1,j} : j \in \{1,2,\ldots,k_{t+i+1}^{m-1}\}$ and $x_{t+i+1,j} \in X_{t+i+1,t+i}\}$;
  (g) if $l > 0$, then $-\sum_{n=0}^{l} \sum_{j \in J_n} x_{t+n,j} + a_m \notin C_{t+l}^{m-1}$;
  (h) $-\sum_{j \in J_0} x_{t,j} + a_m \notin \bigcup\{x_{t,j} + A_{t,j} : j \in \{1,2,\ldots,k_t^{m-1}\}$ and $x_{t,j} \in X_{t,t}\}$;
  (j) for each $i \in \{0,1,\ldots,l\}$, $J_i \subseteq \{1,2,\ldots,k_{t+i}^{m-1}\}$; and
(ix) If $e_m = (x,r)$, then there exist finite $K_0$ with $K_0 = \emptyset$ if $r = 0$ or $m = 0$ and $K_0 \subseteq \{1,2,\ldots,k_r^{m-1}\}$ if $r \geq 1$ and $m \geq 1$ and finite nonempty $K_1 \subseteq \{1,2,\ldots,k_{r+1}^m\}$ such that $x = \sum_{j \in K_0} x_{r,j} + \sum_{j \in K_1} x_{r+1,j}$,
  (a) $K_0 = \emptyset$ if and only if $x + p_{r+1} \notin \bigcup\{\overline{x_{r,j} + A_{r,j}} : j \in \{1,2,\ldots,k_r^m\}$ and $x_{r,j} \in X_{r,r}\}$;

(b) for each $k \in K_0$, if any, $x_{r,k} \in X_{r,r}$ and $-\sum_{K_0 \ni j<k} x_{r,j} + x + p_{r+1} \in \overline{x_{r,k} + A_{r,k}}$;
(c) $-\sum_{j \in K_0} x_{r,j} + x + p_{r+1} \notin \bigcup\{\overline{x_{r,j} + A_{r,j}} : j \in \{1, 2, \ldots, k_r^{m-1}\}$ and $x_{r,j} \in X_{r,r}\}$;
(d) for each $k \in K_1$, $-(\sum_{j \in K_0} x_{r,j} + \sum_{K_1 \ni j<k} x_{r+1,j}) + x + p_{r+1} \in \overline{x_{r+1,k} + A_{r+1,k}}$;
(e) if $s = \min K_1$, then $x_{r+1,s} \in X_{r+1,r}$, and if $K_0 \neq \emptyset$ and $v = \max K_0$, then $x_{r+1,s} + A_{r+1,s} \subseteq A_{r,v}$; and
(f) if $p \in K_1 \setminus \{\min K_1\}$, then $x_{r+1,p} \in X_{r+1,r+1}$.

(x) For each $m \in \omega$ there is at most one $n \in \mathbb{N}$ such that $k_n^{m+1} > k_n^m$ and if $k_n^{m+1} > k_n^m$, then $k_n^{m+1} = k_n^m + 1$.

(Of course, if $k_n^m = 0$, then the sequences $\langle A_{n,j} \rangle_{j=1}^{k_n^m}$ and $\langle x_{n,j} \rangle_{j=1}^{k_n^m}$ are empty.)

First let $m = 0$. We may assume that $a_0 \in A_0$ and that $e_0 = (x_0, 0)$ for some $x_0 \in X_{1,0}$. We have that $x_0 + p_1 \neq p_0$ since if $x_0 + p_1 = p_0$, then $\mathbb{Z} + p_1 = \mathbb{Z} + p_0$ and thus $\beta\mathbb{Z} + p_1 = \beta\mathbb{Z} + p_0$. Pick $D \in p_0 \setminus (x_0 + p_1)$. Pick $B \in p_1$ such that $B \subseteq A_1$, $a_0 \notin x_0 + B$, and $x_0 + B \subseteq A_0 \setminus D$. (One may make the latter two choices since $x_0 + p_1 \in \mathbb{Z}^*$ and so $a_0 \neq x_0 + p_1$ and $x_0 + p_1 \in \overline{A_0 \setminus D}$ and addition on the left by $x_0$ is continuous.) Let $A_{1,1} = B$, $x_{1,1} = x_0$,

$$\text{for } n \in \omega, \text{ let } k_n^0 = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } Z_n^0 = \begin{cases} \{a_0\} & \text{if } n = 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Hypotheses (i) – (v) and (x) can be routinely checked, (iv) being vacuous. For hypothesis (vi) note that $Z_n^0 = \emptyset$ unless $n = 0$ and $C_n^0 = \emptyset$ unless $n = 0$. If $n = 0$, then $C_n^0 = x_0 + B$ and $Z_n^0 = \{a_0\}$. Further, $C_0^0 \cap D = \emptyset$, so $C_0^0 \notin p_0$.

Hypotheses (vii) and (viii) are vacuous.

For hypothesis (ix) let $K_0 = \emptyset$ and $K_1 = \{1\}$. All statements can be routinely checked, (b) and (f) vacuosly.

Now let $m \in \omega$ and assume that the construction has proceeded through $m$. Pick $t \in \omega$ such that $a_{m+1} \in A_t$. We shall construct $l$, $J_0, J_1, \ldots, J_l$ as required by hypothesis (viii) for $m + 1$.

Note for later reference that by hypotheses (i), (iv), and (v), for each $n \in \omega$, $C_n^m \subseteq A_n$. We will regularly use the following fact:

$$(\dagger) \qquad \begin{array}{l} \text{If } t, n \in \omega\,,\ i, j \in \{1, 2, \ldots, k_n^m\}\,, \text{ and} \\ (x_{t,j} + A_{t,j}) \cap A_{t,i} \neq \emptyset\,, \text{ then } i < j\,. \end{array}$$

To verify (†), assume that $t, n \in \omega$, $i, j \in \{1, 2, \ldots, k_n^m\}$, and $(x_{t,j} + A_{t,j}) \cap A_{t,i} \neq \emptyset$. By hypothesis (ii), $(x_{t,j} + A_{t,j}) \cap A_{t,j} = \emptyset$ so we cannot have $A_{t,i} \subseteq A_{t,j}$ so by hypothesis (i), we must have $i < j$.

If $a_{m+1} \notin C_t^m$ let $l = 0$, let $J_0 = \emptyset$, and for $n \in \mathbb{N}$, let

$$Z_n^{m+1} = \begin{cases} Z_t^m \cup \{a_{m+1}\} & \text{if } n = t \\ Z_n^m & \text{otherwise.} \end{cases}$$

Then hypothesis (viii) holds with $z = a_{m+1}$. (Even though $x_{0,j}$ is not defined for any $j$, we take $\sum_{j\in\emptyset} x_{0,j}$ to be 0.) Statements (a), (b), (h), and (j) of hypothesis (viii) hold directly and (c), (d), (e), (f), and (g) are vacuous.

If $a_{m+1} \in \bigcup\{x_{t+1,j} + A_{t+1,j} : j \in \{1, 2, \ldots, k_{t+1}^m\}$ and $x_{t+1,j} \in X_{t+1,t}\}$, let $J_0 = \emptyset$.

Now assume that $a_{m+1} \in \Gamma = \bigcup\{x_{t,j} + A_{t,j} : j \in \{1, 2, \ldots, k_t^m\}$ and $x_{t,j} \in X_{t,t}\}$. We show that there exist $u$ and $j_1, j_2, \ldots, j_u \in \{1, 2, \ldots, k_t^m\}$ such that

(1) if $p \in \{1, 2, \ldots, u\}$, $-\sum_{s=1}^p x_{t,j_s} + a_{m+1} \in A_{t,j_p}$ and
(2) $-\sum_{s=1}^u x_{t,j_s} + a_{m+1} \notin \Gamma$.

By hypothesis (ii), there is a unique $j_1 \in \{1, 2, \ldots, k_t^m\}$ such that $a_{m+1} \in x_{t,j_1} + A_{t,j_1}$. If $-x_{t,j_1} + a \notin \Gamma$, let $u = 1$.

Assume now that $-x_{t,j_1} + a_{m+1} \in \Gamma$ in which case there is a unique $j_2 \in \{1, 2, \ldots, k_t^m\}$ such that $x_{r,j_2} \in X_{r,r}$ and $-x_{t,j_1} + a_{m+1} \in x_{t,j_2} + A_{t,j_2}$, so that $p = 2$ satisfies (1). Let $p > 1$ and assume we have chosen $j_1, j_2, \ldots, j_p$ satisfying (1). Since $-\sum_{s=1}^{p-1} x_{t,j_s} + a_{m+1} \in (x_{t,j_p} + A_{t,j_p}) \cap A_{t,j_{p-1}}$, by (†), $j_p > j_{p-1}$.

If $-\sum_{s=1}^p x_{t,j_s} + a_{m+1} \notin \Gamma$, let $u = p$. Otherwise, let $j_{p+1}$ be the unique member of $\{1, 2, \ldots, k_t^m\}$ such that $x_{r,j_{p+1}} \in X_{r,r}$ and $-\sum_{s=1}^p x_{t,j_s} + a_{m+1} \in x_{t,j_{p+1}} + A_{t,j_{p+1}}$. Since $j_1 < j_2 < \ldots < j_p \le k_t^m$, this process must terminate and we have $u$ and $j_1, j_2, \ldots, j_u \in \{1, 2, \ldots, k_t^m\}$ satisfying (1) and (2). Let $J_0 = \{j_1, j_2, \ldots, j_u\}$ and note that $J_0$ satisfies statements (c) and (j) of hypothesis (viii) and that $-\sum_{j\in J_0} x_{t,j} + a_{m+1} \in A_t$.

If $-\sum_{j\in J_0} x_{t,j} + a_{m+1} \notin C_t^m$ let $l = 0$, let $z = -\sum_{j\in J_0} x_{t,j} + a_{m+1}$, and let $Z_n^{m+1} = \begin{cases} Z_t^m \cup \{z\} & \text{if } n = t \\ Z_n^m & \text{otherwise.} \end{cases}$ Statements (a), (b), (c), (h), and (j) of hypothesis (viii) hold directly and (d), (e), (f) and (g) are vacuous.

Now assume that $-\sum_{j\in J_0} x_{t,j} + a_{m+1} \in C_t^m$. Notice that this holds in particular if $a_{m+1} \in \bigcup\{x_{t+1,j} + A_{t+1,j} : j \in \{1, 2, \ldots, k_{t+1}^m\}$ and $x_{t+1,j} \in X_{t+1,t}\}$, in which case we have let $J_0 = \emptyset$. Then $-\sum_{j\in J_0} x_{t,j} + a_{m+1} \in x_{t+1,k} + A_{t+1,k}$ for some $k \in \{1, 2, \ldots, k_{t+1}^m\}$ such that $x_{t+1,k} \in X_{t+1,t}$. Then $-(\sum_{j\in J_0} x_{t,j} + x_{t+1,k}) + a_{m+1} \in A_{t+1,k}$.

Assume now that we have $s \in \mathbb{N}$ and for $i \in \{1, 2, \ldots, s-1\}$, if any, we have $J_i \subseteq \{1, 2, \ldots, k_{t+i}^m\}$ such that $-\sum_{n=0}^{s-1} \sum_{j\in J_n} x_{t+n,j} + a_{m+1} \in$

$A_{t+s-1,v}$ where $v = \max J_{s-1}$ and have $j(s,1) < j(s,2) < \ldots < j(s,k)$ such that $-(\sum_{n=0}^{s-1}\sum_{j\in J_n} x_{t+n,j} + \sum_{p=1}^{k} x_{t+s,j(s,p)}) + a_{m+1} \in A_{t+s,j(s,k)}$.

If $-(\sum_{n=0}^{s-1}\sum_{j\in J_n} x_{t+n,j} + \sum_{p=1}^{k} x_{t+s,j(s,p)}) + a_{m+1} \in$ $\bigcup\{x_{t+s,j} + A_{t+s,j} : j \in \{1,2,\ldots,k_{t+s}^m\}$ and $x_{t+s,j} \in X_{t+s,t+s}\}$, then pick $j(s,k+1) \in \{1,2,\ldots,k_{t+s}^m\}$ such that $x_{t+s,j(s,k+1)} \in X_{t+s,t+s}$ and $-(\sum_{n=0}^{s-1}\sum_{j\in J_n} x_{t+n,j} + \sum_{p=1}^{k} x_{t+s,j(s,p)}) + a_{m+1} \in$ $x_{t+s,j(s,k+1)} + A_{t+s,j(s,k+1)}$. Note that $-(\sum_{n=0}^{s-1}\sum_{j\in J_n} x_{t+n,j} + \sum_{p=1}^{k+1} x_{t+s,j(s,p)}) + a_{m+1} \in A_{t+s,j(s,k+1)}$.

Since $-(\sum_{n=0}^{s-1}\sum_{j\in J_n} x_{t+n,j} + \sum_{p=1}^{k} x_{t+s,j(s,p)}) + a_{m+1} \in$ $(x_{t+s,j(s,k+1)} + A_{t+s,j(s,k+1)}) \cap A_{t+s,j(s,k)}$, by (†), $j(s,k+1) > j(s,k)$. Since $j(s,1) < j(s,2) < \ldots < j(s,k+1) \leq k_{t+s}^m$ we eventually arrive at $j(s,u) \leq k_{t+s}^m$ such that $-(\sum_{n=0}^{s-1}\sum_{j\in J_n} x_{t+n,j} + \sum_{p=1}^{u} x_{t+s,j(s,p)}) +$ $a_{m+1} \in A_{t+s,j(s,u)} \setminus$ $\bigcup\{x_{t+s,j} + A_{t+s,j} : j \in \{1,2,\ldots,k_{t+s}^m\}$ and $x_{t+s,j} \in X_{t+s,t+s}\}$. Let $J_s = \{j(s,1), j(s,2), \ldots, j(s,u)\}$ and note that $-\sum_{n=0}^{s}\sum_{j\in J_n} x_{t+n,j} +$ $a_{m+1} \in A_{t+s,j(s,u)} \setminus \bigcup\{x_{t+s,j} + A_{t+s,j} : j \in \{1,2,\ldots,k_{t+s}^m\}$ and $x_{t+s,j} \in$ $X_{t+s,t+s}\}$.

If $-\sum_{n=0}^{s}\sum_{j\in J_n} x_{t+n,j} + a_{m+1} \in C_{t+s}^m$, then since $-\sum_{n=0}^{s}\sum_{j\in J_n} x_{t+n,j} + a_{m+1} \notin$ $\bigcup\{x_{t+s,j} + A_{t+s,j} : j \in \{1,2,\ldots,k_{t+s}^m\}$ and $x_{t+s,j} \in X_{t+s,t+s}\}$, we may pick $j(s+1,1) \in \{1,2,\ldots,k_{t+s+1}^m\}$ such that $x_{t+s+1,j(s+1,1)} \in X_{t+s+1,t+s}$ and $-\sum_{n=0}^{s}\sum_{j\in J_n} x_{t+n,j} + a_{m+1} \in x_{t+s+1,j(s+1,1)} + A_{t+s+1,j(s+1,1)}$ so that $-(\sum_{n=0}^{s}\sum_{j\in J_n} x_{t+n,j} + x_{t+s+1,j(s+1,1)}) + a_{m+1} \in A_{t+s+1,j(s+1,1)}$.

By hypothesis (x) and the definition of $k_n^0$ for $n \in \mathbb{N}$ we have that $|\bigcup_{y=0}^{m}\bigcup_{n=0}^{\infty}\{1,2,\ldots,k_n^y\}| \leq m+1$ so this construction must halt. So we have some $s \in \mathbb{N}$ so that $-\sum_{n=0}^{s}\sum_{j\in J_n} x_{t+n,j} + a_{m+1} \notin C_{t+s}^m$. We then let $l = s$, let $z = -\sum_{i=0}^{l}\sum_{j\in J_i} x_{t+i,j} + a_{m+1}$ and, for $n \in \omega$, let $Z_n^{m+1} = \begin{cases} Z_{t+l}^m \cup \{z\} & \text{if } n = t+l \\ Z_n^m & \text{otherwise.} \end{cases}$ All of the statements of hypothesis (viii) can be routinely verified.

Notice that in any event, $Z_n^{m+1} \cap C_n^m = \emptyset$.

Now let $e_{m+1} = (x,r)$. We shall construct $K_0$ and $K_1$ as required by hypothesis (ix) for $m+1$. For the construction of $K_0$, let

$$\Gamma = \bigcup\{\overline{x_{r,j} + A_{r,j}} : j \in \{1,2,\ldots,k_r^m\} \text{ and } x_{r,j} \in X_{r,r}\}\,.$$

We have that $x + p_{r+1} \in \overline{A_r}$. If $x + p_{r+1} \notin \Gamma$, in particular if $r = 0$, let $K_0 = \emptyset$. Notice that in this event, statements (a), (b), and (c) of hypothesis (ix) are satisfied.

Now assume that $x + p_{r+1} \in \Gamma$. We show that there exist $u$ and $j_1, j_2, \ldots, j_u \in \{1, 2, \ldots, k_r^m\}$ such that

(1) if $p \in \{1, 2, \ldots, u\}$, $-\sum_{s=1}^{p} x_{r,j_s} + x + p_{r+1} \in \overline{A_{r,j_p}}$ and
(2) $-\sum_{s=1}^{u} x_{r,j_s} + x + p_{r+1} \notin \Gamma$.

We have that $x + p_{r+1} \in \overline{x_{r,j_1} + A_{r,j_1}}$ for some (necessarily unique) $j_1 \in \{1, 2, \ldots, k_r^m\}$ such that $x_{r,j_1} \in X_{r,r}$. Since $\overline{x_{r,j_1} + A_{r,j_1}} = x_{r,j_1} + \overline{A_{r,j_1}}$ we have that $-x_{r,j_1} + x + p_{r+1} \in \overline{A_{r,j_1}}$. If $-x_{r,j_1} + x + p_{r+1} \notin \Gamma$, let $u = 1$ and note that (1) and (2) are satisfied.

Assume now that $-x_{r,j_1} + x + p_{r+1} \in \Gamma$ in which case there is a unique $j_2 \in \{1, 2, \ldots, k_r^m\}$ such that $-x_{r,j_1} + x + p_{r+1} \in \overline{x_{r,j_2} + A_{r,j_2}}$, so that $p = 2$ satisfies (1). Let $p > 1$ and assume we have chosen $j_1, j_2, \ldots, j_p$ satisfying (1). Since $-\sum_{s=1}^{p} x_{r,j_s} + x + p_{r+1} \in \overline{A_{r,j_p}}$, we have that $-\sum_{s=1}^{p-1} x_{r,j_s} + x + p_{r+1} \in \overline{x_{r,j_p} + A_{r,j_p}} \cap \overline{A_{r,j_{p-1}}}$ so by (†), $j_p > j_{p-1}$.

If $-\sum_{s=1}^{p} x_{r,j_s} + x + p_{r+1} \notin \Gamma$, let $u = p$. Otherwise, let $j_{p+1}$ be the unique member of $\{1, 2, \ldots, k_r^m\}$ such that $-\sum_{s=1}^{p} x_{t,j_s} + x + p_{r+1} \in \overline{x_{r,j_{p+1}} + A_{r,j_{p+1}}}$. Since $j_1 < j_2 < \ldots < j_p \leq k_r^m$, this process must terminate and we have $u$ and $j_1, j_2, \ldots, j_u \in \{1, 2, \ldots, k_r^m\}$ satisfying (1) and (2). Let $K_0 = \{j_1, j_2, \ldots j_u\}$.

Statements (a), (b), and (c) of hypothesis (ix) hold.

To complete the construction, we consider two cases for the construction of $K_1$ for hypothesis (ix). If $K_0 = \emptyset$, let $v = 0$. If $K_0 \neq \emptyset$, let $v = \max K_0$.

Case 1: $-\sum_{j \in K_0} x_{r,j} + x + p_{r+1} \notin$

$$\bigcup \{\overline{x_{r+1,j} + A_{r+1,j}} : j \in \{1, 2, \ldots, k_{r+1}^m\} \text{ and } x_{r+1,j} \in X_{r+1,r}\}\,.$$

Let $x' = -\sum_{j \in K_0} x_{r,j} + x$. We have established that $x' + p_{r+1} \notin \bigcup \{\overline{x_{r,j} + A_{r,j}} : j \in \{1, 2, \ldots, k_r^m\} \text{ and } x_{r,j} \in X_{r,r}\}$ so we have $x' + p_{r+1} \notin \overline{C_r^m}$. If $K_0 = \emptyset$, then $x' + p_{r+1} = x + p_{r+1} \in \overline{A_r} = \overline{A_{r,0}}$. If $K_0 \neq \emptyset$, then by statement (b) of hypothesis (ix), $x' + p_{r+1} \in \overline{A_{r,v}}$. Therefore, $x' + p_{r+1} \in \overline{A_{r,v} \setminus C_r^m}$. Since $x' + p_{r+1} \neq p_r$, pick $D \in p_r \setminus (x' + p_{r+1})$. Note that by hypothesis (i), $A_{r,v} \subseteq A_r = A_{r,k_r^m} \cup \bigcup_{j=1}^{k_r^m} (A_{r,j-1} \setminus A_{r,j})$. Therefore either $A_{r,k_r^m} \in x' + p_{r+1}$ or there is some $j \in \{1, 2, \ldots, k_r^m\}$ such that $(A_{r,j-1} \setminus A_{r,j}) \in x' + p_{r+1}$. We have that $Z_{r+1}^{m+1}$ is finite and by hypothesis (vi), $C_{r+1}^m \notin p_{r+1}$, so $(Z_{r+1}^{m+1} \cup C_{r+1}^m) \notin p_{r+1}$. Also $D \notin x' + p_{r+1}$, $C_r^m \notin x' + p_{r+1}$, and $Z_{r+1}^{m+1}$ is finite so $(D \cup C_r^m \cup Z_r^{m+1}) \notin x' + p_{r+1}$. Pick $B \in p_{r+1}$ such that $B \subseteq A_{r+1,k_{r+1}^m}$, $B \cap (C_{r+1}^m \cup Z_{r+1}^{m+1}) = \emptyset$, $(x' + B) \cap (D \cup C_r^m \cup Z_r^{m+1}) = \emptyset$, $x' + B \subseteq A_{r,v}$, and either $x' + B \subseteq A_{r,k_r^m}$ or there is some $j \in \{1, 2, \ldots, k_r^m\}$ such that $x' + B \subseteq A_{r,j-1} \setminus A_{r,j}$. Let $A_{r+1,k_{r+1}^{m+1}} = B$,

$x_{r+1,k^{m+1}_{r+1}} = x'$, and for $n \in \omega$, let $k^{m+1}_n = \left\{ \begin{array}{rl} k^m_{r+1}+1 & \text{if } n = r+1 \\ k^m_n & \text{otherwise,} \end{array} \right.$
and let $K_1 = \{k^{m+1}_{r+1}\}$.

We verify that all hypotheses hold for $m+1$. If $n \neq r+1$, then hypothesis (i) holds by assumption. It holds for $n = r+1$ by construction.

To verify hypothesis (ii) we need to show that $(x_{r+1,k^{m+1}_{r+1}} + A_{r+1,k^{m+1}_{r+1}}) \cap A_{r+1,k^{m+1}_{r+1}} = \emptyset$, that is that $(x'+B)\cap B = \emptyset$, and that for each $j \in \{1,2,\ldots,k^m_{r+1}\}$, $(x_{r+1,j}+A_{r+1,j}) \cap (x'+B) = \emptyset$. We have that $(x'+B)\cap B = \emptyset$ since $x'+B \subseteq A_{r,v} \subseteq A_r$ and $B \subseteq A_{r+1,k^m_{r+1}} \subseteq A_{r+1}$. For the other conclusion, let $j \in \{1,2,\ldots,k^m_{r+1}\}$ be given. By hypothesis (iii), either $x_{r+1,j} \in X_{r+1,r}$ or $x_{r+1,j} \in X_{r+1.r+1}$. In the former case, $x_{r+1,j} + A_{r+1,j} \subseteq C^m_r$ and $(x'+B)\cap C^m_r = \emptyset$. In the latter case, by hypothesis (iv), $x_{r+1,j}+A_{r+1,j} \subseteq A_{r+1}$ while $x'+B \subseteq A_{r,v} \subseteq A_r$.

Hypothesis (iii) holds because $x'+p_{r+1} \in \overline{A_r}$ so $x_{r+1,k^{m+1}_{r+1}} \in X_{r+1,r}$.

Hypothesis (iv) holds because it holds at $m$ and hypothesis (v) holds directly.

For hypothesis (vi), we have already noted that $Z^{m+1}_n \cap C^m_n = \emptyset$. Also

$$C^{m+1}_n = \left\{ \begin{array}{rl} C^m_r \cup (x'+B) & \text{if } n = r \\ C^m_n & \text{otherwise.} \end{array} \right.$$

Since $(x'+B)\cap Z^{m+1}_r = \emptyset$ we have $Z^{m+1}_r \cap C^{m+1}_r = \emptyset$. Since $C^m_r \notin p_r$ and $(x'+B)\cap D = \emptyset$, we have $C^{m+1}_r \notin p_r$.

The new part of hypothesis (vii) says that $A_{r+1,k^{m+1}_{r+1}} \cap C^m_{r+1} = \emptyset$, which is true.

We have already verified hypothesis (viii).

We have noted that statements (a), (b), and (c) of hypothesis (ix) hold. We have that

$$-\textstyle\sum_{j\in K_0} +x+p_{r+1} = x'+p_{r+1} \in x'+\overline{B} = \overline{x_{r+1,k^{m+1}_{r+1}} + A_{r+1,k^{m+1}_{r+1}}}$$

so statement (d) holds.

We have already noted that $x_{r+1,k^{m+1}_{r+1}} \in X_{r+1,r}$ so, since $x'+B \subseteq A_{r,v}$, statement (e) holds. Statement (f) holds directly.

Hypothesis (x) holds directly.

Case 2: $-\sum_{j\in K_0} x_{r,j} + x + p_{r+1} \in \overline{x_{r+1,k_1} + A_{r+1,k_1}}$ for some $k_1 \in \{1,2,\ldots,k^m_{r+1}\}$ with $x_{r+1,k_1} \in X_{r+1,r}$.

We note that, as long as $k_1 = \min K_1$, then statement (e) of hypothesis (ix) holds. To see this, let $v = \max K_0$. we have directly that $x_{r+1,k_1} \in X_{r+1,r}$ and that $-\sum_{j\in K_0} x_{r,j} + x + p_{r+1} \in \overline{x_{r+1,k_1} + A_{r+1,k_1}}$. By statement (b) of hypothesis (ix), $-\sum_{j\in K_0} x_{r,j} + x + p_{r+1} \in \overline{A_{r,v}}$. Therefore

$(x_{r+1,k_1} + A_{r+1,k_1}) \cap A_{r,u} \neq \emptyset$ so by hypothesis (v), $x_{r+1,k_1} + A_{r+1,k_1} \subseteq A_{r,v}$.

We show that there exist $w$ and $k_1, k_2, \ldots, k_w$ in $\{1, 2, \ldots, k^m_{r+1}\}$ such that

(1) if $p \in \{1, 2, \ldots, w\}$, then
$-(\sum_{j \in K_0} x_{r,j} + \sum_{s=1}^{p} x_{r+1,k_s}) + x + p_{r+1} \in \overline{A_{r+1,k_p}}$ and

(2) $-(\sum_{j \in K_0} x_{r,j} + \sum_{s=1}^{w} x_{r+1,k_s}) + x + p_{r+1} \notin \overline{C^m_{r+1}}$.

Note that (1) holds for $p = 1$. Assume we have $p$ and $k_1, k_2, \ldots, k_p$ satisfying (1). If (2) holds for $w = p$, let $w = p$. Now assume that

$$-(\textstyle\sum_{j \in K_0} x_{r,j} + \sum_{s=1}^{p} x_{r+1,k_s}) + x + p_{r+1} \in \overline{C^m_{r+1}}\,.$$

Notice that $-(\sum_{j \in K_0} x_{r,j} + \sum_{s=1}^{p} x_{r+1,k_s}) + x + p_{r+1} \notin \bigcup\{\overline{x_{r+2,j} + A_{r+2,j}} : j \in \{1, 2, \ldots, k^m_{r+2}\}$ and $x_{r+2,j} \in X_{r+2,r+1}\}$. (If it were, we would have some $u \in \{1, 2, \ldots, k^m_{r+2}\}$ such that

$$-(\textstyle\sum_{j \in K_0} x_{r,j} + \sum_{s=1}^{p} x_{r+1,k_s} + x_{r+2,u}) + x + p_{r+1} \in \overline{A_{r+2,u}}$$

while for all $y \in \mathbb{Z}$, $y + p_{r+1} \notin \overline{A_{r+2}}$.) Thus it must be that $-(\sum_{j \in K_0} x_{r,j} + \sum_{s=1}^{p} x_{r+1,k_s}) + x + p_{r+1} \in \overline{x_{r+1,k_{p+1}} + A_{r+1,k_{p+1}}}$ for some $k_{p+1} \in \{1, 2, \ldots, k^m_{r+1}\}$ with $x_{r+1,k_{p+1}} \in X_{r+1,r+1}$.

Since $(x_{r+1,k_{p+1}} + A_{r+1,k_{p+1}}) \cap A_{r+1,k_p} \neq \emptyset$ we have by (†) that $k_{p+1} > k_p$. Let $K_1' = \{k_1, k_2, \ldots, k_w\}$.

If $x = \sum_{j \in K_0} x_{r,j} + \sum_{j \in K_1'} x_{r+1,j}$, let $K_1 = K_1'$ and for each $n \in \mathbb{N}$, let $k^{m+1}_n = k^m_n$.

Hypotheses (i) – (v) and (vii) hold because they held at $m$. Given $n \in \mathbb{N}$ we have noted that $Z^{m+1}_n \cap C^m_n = \emptyset$ and we have that $C^{m+1}_n = C^m_n$ so hypothesis (vi) holds. We have verified hypothesis (viii). We have noted that statements (a), (b), and (c) of hypothesis (ix) hold, and statement (d) follows from (1). Since $k_1 = \min K_1$ we have shown that statement (e) holds. Statement (f) holds directly as does hypothesis (x).

Finally, assume that $x' = x - (\sum_{j \in K_0} x_{r,j} + \sum_{j \in K_1'} x_{r+1,j}) \neq 0$. We have $x' + p_{r+1} \in \overline{A_{r+1} \setminus C^m_{r+1}}$. Since $x' \neq 0$, $x' + p_{r+1} \neq p_{r+1}$ so pick $D \in p_{r+1} \setminus (x' + p_{r+1})$. Note that $A_{r+1} = A_{r+1,k^m_{r+1}} \cup \bigcup_{j=1}^{k^m_{r+1}} (A_{r+1,j-1} \setminus A_{r+1,j})$. Pick $B \in p_{r+1}$ such that $B \subseteq A_{r+1,k^m_{r+1}} \cap D$, $B \cap C^m_{r+1} = \emptyset$, $x' + B \subseteq A_{r+1} \setminus (D \cup C^m_{r+1} \cup Z^{m+1}_{r+1})$, and either $(x' + B) \subseteq A_{r+1,k^m_{r+1}}$ or there is some $j \in \{1, 2, \ldots, k^m_{r+1}\}$ such that $(x' + B) \subseteq (A_{r+1,j-1} \setminus A_{r+1,j})$. Let $k^{m+1}_n = \begin{cases} k^m_{r+1} + 1 & \text{if } n = r+1 \\ k^m_n & \text{otherwise,} \end{cases}$, let $A_{r+1,k^{m+1}_{r+1}} = B$, let $x_{r+1,k^{m+1}_{r+1}} = x'$, and let $K_1 = K_1' \cup \{k^{m+1}_{r+1}\}$.

Hypothesis (i) holds directly. The newly introduced set of the form $x_{n,j} + A_{n,j}$ is $x' + B$. Since $B \subseteq D$ and $(x' + B) \cap D = \emptyset$ we have that $(x' + B) \cap A_{r+1,k_{r+1}^{m+1}} = \emptyset$. If $j \in \{1, 2, \ldots, k_{r+1}^m\}$ and $x_{r+1,j} \in X_{r+1,r+1}$, then $x_{r+1,j} + A_{r+1,j} \subseteq C_{r+1}^m$ so, since $(x' + B) \cap C_{r+1}^m = \emptyset$, we have that $(x_{r+1,j}+A_{r+1,j}) \cap (x'+B) = \emptyset$. If $j \in \{1, 2, \ldots, k_{r+1}^m\}$ and $x_{r+1,j} \in X_{r+1,r}$, then by hypothesis (v), $x_{r+1,j} + A_{r+1,j} \subseteq A_r$ so, since $x' + B \subseteq A_{r+1}$, we have that $(x_{r+1,j} + A_{r+1,j}) \cap (x' + B) = \emptyset$. Thus hypothesis (ii) holds.

Since $x' + p_{r+1} \in \overline{A_{r+1}}$, $x_{r+1,k_{r+1}^{m+1}} \in X_{r+1,r+1}$ so hypothesis (iii) holds. Hypothesis (iv) holds directly and hypothesis (v) holds because it holds at $m$.

For hypothesis (vi), we have already noted that $Z_n^{m+1} \cap C_n^m = \emptyset$. Also

$$C_n^{m+1} = \begin{cases} C_{r+1}^m \cup (x' + B) & \text{if } n = r + 1 \\ C_n^m & \text{otherwise.} \end{cases}$$

Since $(x' + B) \cap Z_{r+1}^{m+1} = \emptyset$ we have $Z_{r+1}^{m+1} \cap C_{r+1}^{m+1} = \emptyset$. Since $C_{r+1}^m \notin p_{r+1}$ and $(x' + B) \cap D = \emptyset$, we have $C_{r+1}^{m+1} \notin p_{r+1}$.

The new part of hypothesis (vii) says that $A_{r+1,k_{r+1}^{m+1}} \cap C_{r+1}^m = \emptyset$, which is true.

We have already noted that hypothesis (viii) holds.

We have noted that statements (a), (b), and (c) of hypothesis (ix) hold, and since $k_1 = \min K_1$, statement (e) holds. We have verified that statement (d) holds for $k \in K_1'$. The assertion for $k = k_{r+1}^{m+1}$ is that $-(\sum_{j\in K_0} x_{r,j} + \sum_{j\in K_1'} x_{r+1,j}) + x + p_{r+1} = x' + p_{r+1} \in \overline{x' + B}$. Statement (f) holds directly as does hypothesis (x).

This completes the inductive construction.

We now show that for each $n \in \mathbb{N}$, $\lim_{m\to\infty} k_n^m = \infty$. At the same time, we show that for each $n \in \mathbb{N}$, $\{j \in \mathbb{N} : x_{n,j} \in X_{n,n-1}\}$ is infinite. We proceed by induction on $n$. So let $n = 1$ and let $m_0$ be given. We will show that there exists $m$ such that $k_1^m > k_1^{m_0}$. If $1 \leq j \leq k_1^{m_0}$, then $x_{1,j} + A_{1,j} \notin p_0$. (If $x_{1,j} \in X_{1,1}$, then by hypothesis (iv) $x_{1,j} + A_{1,j} \subseteq A_1$. If $x_{1,j} \in X_{1,0}$, then $x_{1,j} + A_{1,j} \subseteq C_0^{m_0}$ and by hypothesis (vi), $C_0^{m_0} \notin p_0$.) Since $p_0 \in c\ell(\{x + p_1 : x \in X_{1,0}\}$ pick $x \in X_{1,0}$ such that $x + p_1 \notin \bigcup_{j=1}^{k_1^{m_0}} \overline{x_{1,j} + A_{1,j}}$. Pick $m$ such that $e_m = (x, 0)$. Pick $K_0 = \emptyset$ and $K_1 \subseteq \{1, 2, \ldots, k_1^m\}$ as guaranteed by hypothesis (ix). Let $t = \min K_1$. Then by statement (d) of hypothesis (ix), $x + p_1 \in \overline{x_{1,t} + A_{1,t}}$ so $t > k_1^{m_0}$. By statement (e) of hypothesis (ix), $x_{n,t} \in X_{n,n-1}$.

Now assume $n \geq 2$ and $\lim_{m\to\infty} k_{n-1}^m = \infty$. Let $m_0$ be given. We claim that we can pick $s$ such that for $1 \leq j \leq k_n^{m_0}$, $A_{n-1,s} \cap (x_{n,j} + A_{n,j}) = \emptyset$. Indeed, if $x_{n,j} \in X_{n,n}$, then by hypothesis (iv), $x_{n,j} + A_{n,j} \subseteq A_n$. If $x_{n,j} \in$

$X_{n,n-1}$, then $x_{n,j} + A_{n,j} \subseteq C^{m_0}_{n-1}$ and by hypothsis (vii), $A_{n-1,k^{m_0+1}_{n-1}} \cap C^{m_0}_{n-1} = \emptyset$. So, letting $s = k^{m_0+1}_{n-1}$ we get that $A_{n-1,s} \cap (x_{n,j} + A_{n,j}) = \emptyset$ for $1 \leq j \leq k^{m_0}_n$. Now $p_{n-1} \in (\beta\mathbb{Z} + p_n) = c\ell(\mathbb{Z} + p_n)$ so pick $x \in \mathbb{Z}$ such that $x + p_n \in \overline{A_{n-1,s}}$ and notice that $x \in X_{n,n-1}$. Pick $m$ such that $(x, n-1) = e_m$.

Pick $K_0$ and $K_1$ as guaranteed by hypothesis (ix) for $x$ and let $t = \min K_1$. If $K_0 = \emptyset$, then by statement (e) of hypothesis (ix), $x_{n,t} \in X_{n,n-1}$ and by statement (b), $x + p_n \in \overline{x_{n,t} + A_{n,t}}$ so $t > k^{m_0}_n$.

Now assume that $K_0 \neq \emptyset$. Let $i = \min K_0$ and let $v = \max K_0$. Then $x + p_n \in \overline{x_{n-1,i} + A_{n-1,i}} \cap \overline{A_{n-1,s}}$ so by (†), $i > s$. Then by statement (e) of hypothesis (ix), $x_{n,t} + A_{n,t} \subseteq A_{n-1,v}$ and $A_{n-1,v} \subseteq A_{n-1,s}$ because $v \geq i > s$. Therefore $t > k^{m_0}_n$. We have completed the proof that $\lim_{m\to\infty} k^m_n = \infty$.

For $r \in \omega$, let $C_r = \bigcup_{m=0}^{\infty} C^m_r$. By hypotheses (iv) and (v), $C_r \subseteq A_r$.

Given $(x, r) \in E$, we call the sum $x = \sum_{j\in K_0} x_{r,j} + \sum_{j\in K_1} x_{r+1,j}$ guaranteed by hypothesis (ix) the $X_r$-*decomposition* of $x$. We claim that each $x \in X_{r+1,r}$ has a unique $X_r$-decomposition. So let $x \in X_{r+1,r}$ and pick $m \in \omega$ such that $e_m = (x, r)$. Suppose we have $(K_0, K_1)$ and $(K'_0, K'_1)$ as in the statement of hypothesis (ix). We show first that $K_0 = K'_0$. If $r = 0$ or $m = 0$, this is immediate so assume $r \in \mathbb{N}$ and $m \in \mathbb{N}$, suppose $K_0 \neq K'_0$, and let $k = \min(K_0 \bigtriangleup K'_0)$. Assume without loss of generality that $k \in K_0$. Then $k \in \{1, 2, \ldots, k^{m-1}_r\}$. Let $L = \{j \in K_0 : j < k\} = \{j \in K'_0 : j < k\}$. By statement (b)

$$(*1) \qquad -\textstyle\sum_{j\in L} x_{r,j} + x + p_{r+1} \in \overline{x_{r,k} + A_{r,k}} \text{ and } x_{r,k} \in X_{r,r}.$$

Assume first that $K'_0 \neq L$ and let $k' = \min(K'_0 \setminus L)$. Then by statement (b), $-\sum_{j\in L} x_{r,j} + x + p_{r+1} \in \overline{x_{r,k'} + A_{r,k'}}$. This contradicts (*1) by hypothesis (ii) at $m - 1$.

Now assume that $K'_0 = L$. By statement (c), $-\sum_{j\in L} x_{r,j} + x + p_{r+1} \notin \{\overline{x_{r,j} + A_{r,j}} : j \in \{1, 2, \ldots, k^{m-1}_r\}$ and $x_{r,j} \in X_{r,r}\}$. This contradicts (*1). Thus we have established that $K_0 = K'_0$.

Suppose $K_1 \neq K'_1$, and let $k = \min(K_1 \bigtriangleup K'_1)$. Assume without loss of generality that $k \in K_1$. Then $k \in \{1, 2, \ldots, k^m_{r+1}\}$. Let $L = \{j \in K_1 : j < k\} = \{j \in K'_1 : j < k\}$. By statement (d)

$$(*2) \qquad -(\textstyle\sum_{j\in K_0} x_{r,j} + \sum_{j\in L} x_{r+1,j}) + x + p_{r+1} \in \overline{x_{r+1,k} + A_{r+1,k}}.$$

Assume first that $K'_1 \neq L$ and let $k' = \min(K'_1 \setminus L)$. Then by statement (d), $-(\sum_{j\in K_0} x_{r,j} + \sum_{j\in L} x_{r+1,j}) + x + p_{r+1} \in \overline{x_{r+1,k'} + A_{r+1,k'}}$. This contradicts (*2).

Now assume that $K_1' = L$. Then $x = \sum_{j\in K_0} x_{r,j} + \sum_{j\in L} x_{r+1,j}$ so by (*2), $p_{r+1} = -x + x + p_{r+1} \in \overline{x_{r+1,k} + A_{r+1,k}}$. But by hypothesis (ii), $(x_{r+1,k} + A_{r+1,k}) \cap A_{r+1,k} = \emptyset$ and by hypothesis (i), $A_{r+1,k} \in p_{r+1}$ so $(x_{r+1,k} + A_{r+1,k}) \notin p_{r+1}$. This completes the proof of the uniqueness of the $X_r$-decomposition.

We call the sum $a = z + \sum_{i=0}^{l} \sum_{j\in J_i} x_{t+i,j}$ guaranteed by hypothesis (viii) the $A$-*decomposition* of $a$.

We show now that the $A$-dcomposition is unique in the following strong sense. Given $t \in \mathbb{N}$ and $a \in A_t$, pick $m \in \omega$ such that $a = a_m$. We had $a_0 \in A_0$ so $m > 0$. Assume we have $l$, $J_0, J_1, \ldots, J_l$, and $z$ satisfying hypothesis (viii). Suppose also that we have $l' \in \omega$, finite subsets $J_0', J_1', \ldots, J_{l'}'$ of $\mathbb{N}$, and $z'$ such that $J_i' \neq \emptyset$ if $i > 0$, $a_m = z' + \sum_{i=0}^{l'} \sum_{j\in J_i'} x_{t+i,j}$ and

- (a') $J_0' = \emptyset$ if and only if $a_m \notin \bigcup\{x_{t,j} + A_{t,j} : j \in \mathbb{N} \text{ and } x_{t,j} \in X_{t,t}\}$;
- (b') $l' = 0$ if and only if $-\sum_{j\in J_0'} x_{t,j} + a_m \notin \bigcup\{x_{t+1,j} + A_{t+1,j} : j \in \mathbb{N} \text{ and } x_{t+1,j} \in X_{t+1,t}\}$;
- (c') for each $k \in J_0'$, if any, $x_{t,k} \in X_{t,t}$ and $-\sum_{J_0'\ni j<k} x_{t,j} + a_m \in x_{t,k} + A_{t,k}$;
- (d') for each $i \in \{1, 2, \ldots, l'\}$, if any, and each $k \in J_i'$, $-\Big(\sum_{J_i'\ni j<k} x_{t+i,j} + \sum_{n=0}^{i-1} \sum_{j\in J_n'} x_{t+n,j}\Big) + a_m \in x_{t+i,k} + A_{t+i,k}$;
- (e') for $i \in \{1, 2, \ldots, l'\}$, if any, if $j = \min J_i'$, then $x_{t+i,j} \in X_{t+i,t+i-1}$ and if $j \in J_i' \setminus \{\min J_i'\}$, then $x_{t+i,j} \in X_{t+i,t+i}$;
- (f') for $i \in \{1, 2, \ldots, l'-1\}$, if any, $-\sum_{n=0}^{i} \sum_{j\in J_n'} x_{t+n,j} + a_m \in \bigcup\{x_{t+i+1,j} + A_{t+i+1,j} : j \in \mathbb{N} \text{ and } x_{t+i+1,j} \in X_{t+i+1,t+i}\}$;
- (g') if $l' > 0$, then $-\sum_{n=0}^{l} \sum_{j\in J_n'} x_{t+n,j} + a_m \notin C_{t+l}$; and
- (h') $-\sum_{j\in J_0'} x_{t,j} + a_m \notin \bigcup\{x_{t,j} + A_{t,j} : j \in \mathbb{N} \text{ and } x_{t,j} \in X_{t,t}\}$.

We shall show that $l = l'$, $z = z'$, and for each $s \in \{0, 1, \ldots, l\}$, $J_s = J_s'$.

In the proof we will frequently encounter a situation where we have some $j_0 > k_n^{m-1}$ and $x_{n,j_0}$ and $A_{n,j_0}$ were constructed at a stage after $m$. In that situation, by the fact that $\lim\limits_{m\to\infty} k_n^m = \infty$ and hypothesis (x) we have some $m' \geq m-1$ such that $j_0 = k_n^{m'+1} = k_n^{m'} + 1$. Then one had $e_{m'+1} = (x, n-1)$ for some $x \in X_{n,n-1}$ and either

$$x_{n,j_0} \in X_{n,n-1} \text{ and } (x_{n,j_0} + A_{n,j_0}) \cap Z_{n-1}^{m'+1} = \emptyset \tag{‡1}$$

or

$$x_{n,j_0} \in X_{n,n} \text{ and } (x_{n,j_0} + A_{n,j_0}) \cap Z_{n}^{m'+1} = \emptyset. \tag{‡2}$$

(Condition (‡1) happened under Case 1 for the construction of $K_1$ and (‡2) happened under Case 2 for the construction of $K_1$.)

In this proof all references to statement (a), (b), and so on refer to the statements of hypothesis (viii). We show first that $J_0 = \emptyset$ if and only if $J_0' + \emptyset$. By statements (a) and (a') it is immediate that if $J_0 \neq \emptyset$ then $J_0' \neq \emptyset$. So suppose that $J_0 = \emptyset$ and $J_0' \neq \emptyset$. Then we have $a_m \in x_{t,j_0} + A_{t,j_0}$ for some $j_0 > k_t^{m-1}$ such that $x_{t,j_0} \in X_{t,t}$. Then we have $m' \geq m-1$ such that $j_0 = k_t^{m'+1} = k_t^{m'} + 1$ and $e_{m'+1} = (x, t-1)$ for some $x \in X_{t,t-1}$. Since $x_{t,j_0} \in X_{t,t}$, we have by (‡2) that $(x_{t,j_0} + A_{t,j_0}) \cap Z_t^{m'+1} = \emptyset$. But since $J_0 = \emptyset$, $z = a_m \in Z_t^m \subseteq Z_t^{m'+1}$ so $(x_{t,j_0} + A_{t,j_0}) \cap Z_t^{m'+1} \neq \emptyset$, a contradiction.

Thus we have that $J_0 = \emptyset$ if and only if $J_0' = \emptyset$. To see that $J_0 = J_0'$, suppose instead that we have $j_0 = \min(J_0 \triangle J_0')$. Let $L = \{j \in J_0 : j < j_0\} = \{j \in J_0' : j < j_0\}$. Assume first that $j_0 \in J_0$. Then by statement (c), $-\sum_{j \in L} x_{t,j} + a_m \in x_{t,j_0} + A_{t,j_0}$. If $J_0' \neq L$, let $k = \min J_0' \setminus L$. Then by statement (c'), $-\sum_{j \in L} x_{t,j} + a_m \in x_{t,k} + A_{t,k}$, contradicting hypothesis (ii). So we have $J_0' = L$ and by statment (h'), $-\sum_{j \in L} x_{t,j} + a_m \notin x_{t,j_0} + A_{t,j_0}$, a contradiction.

Thus we must have $j_0 \in J_0'$ so by statement (c'), $x_{t,j_0} \in X_{t,t}$ and $-\sum_{j \in L} x_{t,j} + a_m \in x_{t,j_0} + A_{t,j_0}$. If $J_0 \neq L$, let $k = \min J_0 \setminus L$. Then by statement (c), $-\sum_{j \in L} x_{t,j} + a_m \in x_{t,k} + A_{t,k}$, contradicting hypothesis (ii). So we have $J_0 = L$ and by statment (h), $-\sum_{j \in L} x_{t,j} + a_m \notin \bigcup\{x_{t,j} + A_{t,j} : j \in \{1, 2, \ldots, k_t^{m-1}\}$ and $x_{t,j} \in X_{t,t}\}$. So we must have that $j_0 > k_t^{m-1}$ and we have $m' \geq m-1$ such that $j_0 = k_t^{m'+1} = k_t^{m'} + 1$ and $e_{m'+1} = (x, t-1)$ for some $x \in X_{t,t-1}$. Since $x_{t,j_0} \in X_{t,t}$, we have by (‡2) that $(x_{t,j_0} + A_{t,j_0}) \cap Z_t^{m'+1} = \emptyset$. If $l = 0$, we have $-\sum_{j \in L} x_{t,j} + a_m = z \in Z_t^m \subseteq Z_t^{m'+1}$, a contradiction. So $l > 0$. Let $k = \min J_1$. Then by statement (d), we have $-\sum_{j \in L} x_{t,j} + a_m \in x_{t+1,k} + A_{t+1,k}$, contradicting hypothesis (ii). So we have shown that $J_0 = J_0'$.

Now we show that $l = 0$ if and only if $l' = 0$. By statements (b) and (b'), it is immediate that if $l > 0$, then $l' > 0$. So suppose that $l = 0$ and $l' > 0$. By statements (b) and (b') we have $-\sum_{j \in J_0'} x_{t,j} + a_m \in x_{t+1,j_0} + A_{t+1,j_0}$ and $x_{t+1,j_0} \in X_{t+1,t}$ for some $j_0 > k_{t+1}^{m-1}$. So we have some $m' \geq m-1$ such that $j_0 = k_{t+1}^{m'+1} = k_{t+1}^{m'} + 1$ and $e_{m'+1} = (x, t)$ for some $x \in X_{t+1,t}$. Since $x_{t+1,j_0} \in X_{t+1,t}$, we have by (‡1) that $(x_{t+1,j_0} + A_{t+1,j_0}) \cap Z_t^{m'+1} = \emptyset$. Since $l = 0$, we have that $z = -\sum_{j \in J_0} x_{t,j} + a_m \in Z_t^m \subseteq Z_t^{m'+1}$, a contradiction. We have established that $l = 0$ if and only if $l' = 0$. If $l = 0$, then $z = -\sum_{j \in J_0} x_{t,j} + a_m = z'$ and we are done.

Assume that $\min\{l,l'\}>0$. Let $0<s\leq\min\{l,l'\}$ and assume that for $i\in\{0,1,\ldots,s-1\}$, $J_i=J'_i$. Suppose that $J_s\neq J'_s$, let $j_0=\min(J_s\bigtriangleup J'_s)$, and let $L=\{j\in J_0:j<j_0\}=\{j\in J'_0:j<j_0\}$. By statement (d) or (d'),

$$-(\textstyle\sum_{j\in L}x_{t+s,j}+\sum_{i=0}^{s-1}\sum_{j\in J_i}x_{t+i,j})+a_m\in x_{t+s,j_0}+A_{t+s,j_0}\,.$$

Assume first that $j_0\in J_s$. If $J'_s\neq L$, let $k=\min J'_s$. By statement (d'), $-(\sum_{j\in L}x_{t+s,j}+\sum_{i=0}^{s-1}\sum_{j\in J_i}x_{t+i,j})+a_m\in x_{t+s,k}+A_{t+s,k}$, contradicting hypothesis (ii). So we have $J'_s=L$. If $s<l'$, then by statement (f'), $-(\sum_{j\in L}x_{t+s,j}+\sum_{i=0}^{s-1}\sum_{j\in J_i}x_{t+i,j})+a_m\in x_{t+s+1,j}+A_{t+s+1,j}$ for some $j$, again contradicting hypothesis (ii). So we must have $s=l'$. Then $L\neq\emptyset$ so by statement (e'), $x_{t+s,j_0}\in X_{t+s,t+s}$ so that by statement (g'), $-(\sum_{j\in L}x_{t+s,j}+\sum_{i=0}^{s-1}\sum_{j\in J_i}x_{t+i,j})+a_m\notin C_{t+l'}$. But $x_{t+s,j_0}+A_{t+s,j_0}\in C_{t+s}=C_{t+l'}$, a contradiction.

So we must have $j_0\in J'_s$. If $J_s\neq L$, let $k=\min J_s$. By statement (d), $-(\sum_{j\in L}x_{t+s,j}+\sum_{i=0}^{s-1}\sum_{j\in J_i}x_{t+i,j})+a_m\in x_{t+s,k}+A_{t+s,k}$, contradicting hypothesis (ii). So we have $J_s=L$. If $s<l$, then by statement (f), $-(\sum_{j\in L}x_{t+s,j}+\sum_{i=0}^{s-1}\sum_{j\in J_i}x_{t+i,j})+a_m\in x_{t+s+1,j}+A_{t+s+1,j}$ for some $j$, again contradicting hypothesis (ii). So we must have $s=l$. Then $L\neq\emptyset$ so by statement (e), $x_{t+s,j_0}\in X_{t+s,t+s}$ so that by statement (g), $-(\sum_{j\in L}x_{t+s,j}+\sum_{i=0}^{s-1}\sum_{j\in J_i}x_{t+i,j})+a_m\notin C^{m-1}_{t+l}$. Since $x_{t+s,j_0}+A_{t+s,j_0}\in C_{t+s}$, we must have that $j_0>k^{m-1}_{t+s}$. So we have some $m'\geq m-1$ such that $j_0=k^{m'+1}_{t+s}=k^{m'}_{t+s}+1$ and $e_{m'+1}=(x,t+s-1)$ for some $x\in X_{t+1,t}$. Since $x_{t+s,j_0}\in X_{t+s,t+s}$ we have $(x_{t+s,j_0}+A_{t+s,j_0})\cap Z^{m'+1}_{t+s}=\emptyset$. But $x_{t+s,j_0}+A_{t+s,j_0}=a_m-\sum_{i=0}^{l}\sum_{j\in J_i}x_{t+i,j}=z\in Z^m_{t+l}\subseteq Z^{m'+1}_{t+l}$, a contradiction.

So we have established that for all $s\leq\min\{l,l'\}$, $J_s=J'_s$. It remains only to show that $l=l'$ since then $z=z'$ follows. Suppose first that $l'<l$. Then by statement (g'), $-\sum_{i=0}^{l'}\sum_{j\in J_i}x_{t+i,j}+a_m\notin C_{t+l'}$ while by statement (f), $-\sum_{i=0}^{l'}\sum_{j\in J_i}x_{t+i,j}+a_m\in x_{t+l'+1,j_0}+A_{t+l'+1,j_0}$ for some $j$ and $x_{t+l'+1,j_0}+A_{t+l'+1,j_0}\subseteq C_{t+l'}$, a contradiction.

Finally, suppose that $l<l'$. By statement (g), $-\sum_{i=0}^{l}\sum_{j\in J_i}x_{t+i,j}+a_m\notin C^{m-1}_{t+l}$ while by statement (f'),

$$-\textstyle\sum_{i=0}^{l}\sum_{j\in J_i}x_{t+i,j}+a_m\in x_{t+l+1,j_0}+A_{t+l+1,j_0}$$

for some $j_0$ such that $x_{t+l+1,j_0}\in X_{t+l+1,t+l}$. Then we must have that $j_0>k^{m-1}_{t+l+1}$. So we have some $m'\geq m-1$ such that $j_0=k^{m'+1}_{t+l+1}=k^{m'}_{t+l+1}+1$ and $e_{m'+1}=(x,t+l)$ for some $x\in X_{t+l+1,t+l}$.

Then by (‡1), $(x_{t+l+1,j_0} + A_{t+l+1,j_0}) \cap Z^{m'+1}_{t+l} = \emptyset$ while $a_m - \sum_{i=0}^{l} \sum_{j\in J_i} x_{t+i,j} = z \in Z^m_{t+l} \subseteq Z^{m'+1}_{t+l}$, a contradiction.

This completes the proof that the $A$-decomposition of $a$ satisfies the strong uniqueness property.

For $x \in X_{n+1,n}$, if $x = \sum_{j\in K_0} x_{n,j} + \sum_{j\in K_1} x_{n+1,j}$ is its $X_n$-decomposition, let $\theta_n(x) = \min K_1$ and let $\phi_n(x) = \max K_1$. Then we may choose $F_n(x) \in p_{n+1}$ such that $F_n(x) \subseteq A_{n+1,\phi_n(x)}$, for each $k \in K_0$, if any, $-\sum_{K_0\ni j<k} x_{n,j} + x + F_n(x) \subseteq x_{n,k} + A_{n,k}$, and for each $k \in K_1$, $-(\sum_{j\in K_0} x_{n,j} + \sum_{K_1\ni j<k} x_{n+1,j}) + x + F_n(x) \subseteq x_{n+1,k} + A_{n+1,k}$.

Next we claim that for $n \in \omega$ and $j \in \mathbb{N}$, if $x_{n+1,j} \in X_{n+1,n+1}$, then $x_{n+1,j} + A_{n+1,j} \subseteq A_{n+1,k-1} \setminus A_{n+1,k}$ for some $k \in \mathbb{N}$. So let $n \in \omega$ and $j \in \mathbb{N}$ be given and assume that $x_{n+1,j} \in X_{n+1,n+1}$. Since $\lim\limits_{m\to\infty} k^{\infty}_{n+1} = \infty$, by hypothesis (x) we may pick the largest $m$ such that $k^m_{n+1} = j$, so that $k^{m+1}_n = k^m_n + 1$. By hypothesis (iv) either $x_{n+1,j} + A_{n+1,j} \subseteq A_{n+1,k-1} \setminus A_{n+1,k}$ for some $k \in \{1, 2, \ldots, k^m_{n+1}\}$ or $x_{n+1,j} + A_{n+1,j} \subseteq A_{n+1,k^m_{n+1}}$. In the first case, we are done, so assume the latter. We have that $x_{n+1,j} + A_{n+1,j} \subseteq C^m_{n+1}$. By hypothesis (vii), $A_{n+1,k^{m+1}_{n+1}} \cap C^m_{n+1} = \emptyset$. Since $k^{m+1}_{n+1} = k^m_{n+1} + 1$, we then have that $x_{n+1,j} + A_{n+1,j} \subseteq A_{n+1,k^m_{n+1}} \setminus A_{n+1,k^m_{n+1}+1}$.

Now we show that for $n, j \in \mathbb{N}$, if $x_{n+1,j} \in X_{n+1,n}$, then $x_{n+1,j} + A_{n+1,j} \subseteq A_{n,k-1} \setminus A_{n,k}$ for some $k \in \mathbb{N}$. So assume $x_{n+1,j} \in X_{n+1,n}$. Pick $m$ such that $k^m_{n+1} \geq j$ and $k^{m+1}_n = k^m_n + 1$. By hypothesis (v) either $x_{n+1,j} + A_{n+1,j} \subseteq A_{n,k-1} \setminus A_{n,k}$ for some $k \in \{1, 2, \ldots, k^m_n\}$ or $x_{n+1,j} + A_{n+1,j} \subseteq A_{n,k^m_n}$. In the first case we are done, so assume the latter. We have that $x_{n+1,j}+A_{n+1,j} \subseteq C^m_n$. By hypothesis (vii), $A_{n,k^{m+1}_n} \cap C^m_n = \emptyset$. Since $k^{m+1}_n = k^m_n + 1$, we then have that $x_{n+1,j} + A_{n+1,j} \subseteq A_{n,k^m_n} \setminus A_{n,k^m_n+1}$.

We now observe that if $n \in \omega$, $x \in X_{n+1,n}$, and $a \in F_n(x)$, then $l(a + x) \geq 1$. to see this, let $x = \sum_{j\in K_0} x_{n,j} + \sum_{j\in K_1} x_{n+1,j}$ be the $X_n$-decomposition of $x$. Since $a \in F_n(x) \subseteq A_{n+1,\phi_n(x)}$, $a \in A_{n+1}$. Let $a = z + \sum_{i=0}^{l} \sum_{j\in J_i} x_{n+1+i,j}$ be the $A$-decomposition of $a$. I claim that if $J_0 \neq \emptyset$, then $\phi_n(x) = \max K_1 < \min J_0$. So let $k = \min J_0$. Then by statement (c) of hypothesis (viii), $a \in x_{n+1,k} + A_{n+1,k}$ so that $(x_{n+1,k} + A_{n+1,k}) \cap A_{n+1,\phi_n(x)} \neq \emptyset$ so by (†), $\phi_n(x) < k$. Thus we have that $x + a = z + \sum_{j\in K_0} x_{n,j} + \sum_{j\in K_1\cup J_0} x_{n+1,j} + \sum_{i=2}^{l+1} \sum_{j\in J_{i-1}} x_{n+i,j}$. Let $l' = l + 1$, $J'_0 = K_0$, $J'_1 = K_1 \cup J_0$, and for $i \in \{2, 3, \ldots, l + 1\}$, if any, let $J'_i = J_{i-1}$. It is then routine to establish that $l'$ and $J'_0, J'_1, \ldots, J'_{l'}$

satisfy statements (a') – (h') so that by the strong uniqueness of the $A$-decomposition, $x + a = z + \sum_{i=0}^{l'} \sum_{j \in J_i'} x_{n+i,j}$ is the $A$-decomposition of $x + a$ so that $l(x + a) = l' \geq 1$.

For $n \in \mathbb{N}$, let $D_n = \{a \in A_n : l(a) > 0\}$. Let $n \in \mathbb{N}$. We claim that $D_n \subseteq C_n$ and $D_n \in p_n$. To see that $D_n \subseteq C_n$, note that if $a \in A_n \setminus C_n$, then the $A$-decomposition of $a$ is $a = a$, so that $l(a) = 0$. To see that $D_n \in p_n$, suppose instead that $\mathbb{Z} \setminus D_n \in p_n$. Recall that $p_n \in c\ell\{x + p_{n+1} : x \in X_{n+1,n}\}$ so pick $x \in X_{n+1,n}$ such that $x + p_{n+1} \in \overline{\mathbb{Z} \setminus D_n}$. Pick $B \in p_{n+1}$ such that $x + B \subseteq \mathbb{Z} \setminus D_n$. We have that $F_n(x) \in p_{n+1}$ so pick $a \in F_n(x) \cap B$. Then as we saw above, $l(a + x) > 0$ so $a + x \in D_n$, a contradiction.

Define $f : D_1 \to W$ as follows. Given $a \in D_1$, let the $A$-decomposition of $a$ be $a = z + \sum_{i=0}^{l} \sum_{j \in J_i} x_{1+i,j}$. Let $\alpha_s = \min J_s$ for each $s \in \{1, 2, \ldots, l\}$ and let $f(a) = \alpha_1 \alpha_2 \cdots \alpha_l$. Assume that $W$ has been finitely colored, and pick $B_1 \in p_1$ such that $B_1 \subseteq D_1$ and $f[B_1]$ is monochromatic. We shall show that there is an infinite sequence $w_1 < w_2 < \ldots$ such that the set $\{[w_{j_1}, w_{j_2}, \ldots, w_{j_k}] : k \in \mathbb{N} \text{ and } 1 \leq j_1 < \ldots < j_k\} \subseteq f[B_1]$. That will complete the proof of the proposition.

We claim that given any $k \in \mathbb{N}$ and $n \geq 2$, there is some $v \in \mathbb{N}$ such that $(x_{n,j} + A_{n,j}) \cap A_{n-1,v} = \emptyset$ for each $j \in \{1, 2, \ldots, k\}$. For this, it suffices to show that for each $j \in \mathbb{N}$, there exists $v \in \mathbb{N}$ such that $(x_{n,j} + A_{n,j}) \cap A_{n-1,v} = \emptyset$, so let $j \in \mathbb{N}$. If $x_{n,j} \in X_{n,n}$, then $x_{n,j} + A_{n,j} \subseteq A_n$ and $A_n \cap A_{n-1} = \emptyset$. So assume that $x_{n,j} \in X_{n,n-1}$. Then we have shown that there is some $v$ such that $x_{n,j} + A_{n,j} \subseteq A_{n-1,v-1} \setminus A_{n-1,v}$ so $(x_{n,j} + A_{n,j}) \cap A_{n-1,v} = \emptyset$.

For each $j \in \mathbb{N}$, let $B_{1,j} = B_1$. We let $i \geq 2$ and assume we have chosen a sequence $\langle B_{i-1,j} \rangle_{j=i-1}^{\infty}$ of members of $p_{i-1}$ such that $B_{i-1,j+1} \subseteq B_{i-1,j}$ for each $j \geq i - 1$.

We construct a sequence $\langle y_{i,j} \rangle_{j=i-1}^{\infty}$ in $X_{i,i-1}$ and a decreasing sequence $\langle B_{i,j} \rangle_{j=i-1}^{\infty}$ of members of $p_i$ such that $y_{i,j} + B_{i,j} \subseteq B_{i-1,j}$, $B_{i,j} \subseteq F_{i-1}(y_{i,j})$, and $\theta_{i-1}(y_{i,j}) < \theta_{i-1}(y_{i,j+1})$ for each $j \geq i - 1$.

Since $p_{i-1} \in c\ell\{x + p_i : x \in X_{i,i-1}\}$ we may pick $y_{i,i-1} \in X_{i,i-1}$ such that $y_{i,i-1} + p_i \in \overline{B_{i-1,i-1}}$. Then $F_{i-1}(y_{i,i-1}) \in p_i$ so pick $B_{i,i-1} \in p_i$ such that $y_{i,i-1} + B_{i,i-1} \subseteq B_{i-1,i-1}$ and $B_{i,i-1} \subseteq F_{i-1}(y_{i,i-1})$.

Now assume that $\langle y_{i,j} \rangle_{j=i-1}^{k}$ and $\langle B_{i,j} \rangle_{j=i-1}^{k}$ have been chosen. We have $v$ such that $(x_{i,j} + A_{i,j}) \cap A_{i-1,v} = \emptyset$ for each $j \in \{1, 2, \ldots, \theta_{i-1}(y_{i,k})\}$. Pick $y_{i,k+1} \in X_{i,i-1}$ such that $y_{i,k+1} + p_i \in \overline{B_{i-1,k+1} \cap A_{i-1,v}}$. Then $F_{i-1}(y_{i,k+1}) \in p_i$ so pick $B_{i,k+1} \in p_i$ such that $B_{i,k+1} \subseteq B_{i,k}$, $y_{i,k+1} + B_{i,k+1} \subseteq (B_{i-1,k+1} \cap A_{i-1,v})$, and $B_{i,k+1} \subseteq F_{i-1}(y_{i,k+1})$.

We need to show that $\theta_{i-1}(y_{i,k+1}) > \theta_{i-1}(y_{i,k})$. So let

$$y_{i,k+1} = \sum_{j\in K_0} x_{i-1,j} + \sum_{j\in K_{i-1}} x_{i,j}$$

be the $X$-decomposition of $y_{i,k+1}$. Let $u = \min K_1 = \theta_{i-1}(y_{i,k+1})$. If $K_0 = \emptyset$, then $y_{i,k+1} + p_i \in \overline{x_{i,u} + A_{i,u}}$ so $(x_{i,u} + A_{i,u}) \cap A_{i-1,v} \neq \emptyset$ so $u > \theta_{i-1}(y_{i,k})$ as required. So assume that $K_0 \neq \emptyset$, let $s = \min K_0$, and let $t = \max K_0$. Then $y_{i,k+1} + p_i \in \overline{x_{i-1,s} + A_{i-1,s}}$ so $(x_{i-1,s} + A_{i-1,s}) \cap A_{i-1,v} \neq \emptyset$ while $(x_{i-1,s} + A_{i-1,s}) \cap A_{i-1,s} = \emptyset$ so $s > v$. Now $-\sum_{K_0\ni j<t} x_{i-1,j} + y_{i,k+1} + p_i \in \overline{x_{i-1,t} + A_{i-1,t}}$ so $-\sum_{j\in K_0} x_{i-1,j} + y_{i,k+1} + p_i \in \overline{A_{i-1,t}}$ and $-\sum_{j\in K_0} x_{i-1,j} + y_{i,k+1} + p_i \in \overline{x_{i-1,u} + A_{i-1,u}}$ so $(x_{i-1,u} + A_{i-1,u}) \cap A_{i-1,t} \neq \emptyset$. And $t \geq s > v$ so $A_{i-1,t} \subseteq A_{i-1,v}$. Therefore $(x_{i-1,u} + A_{i-1,u}) \cap A_{i-1,v} \neq \emptyset$ so $u > \theta_{i-1}(y_{i,k})$ as required.

We then have that for all $n \geq 2$ and $j_2 \leq j_3 \leq \ldots \leq j_n$ with each $j_i \geq i-1$, $y_{2,j_2} + \ldots + y_{n,j_n} + B_{n,j_n} \subseteq B_1$.

We now claim that if $n \geq 2$ and $j_2 \leq j_3 \leq \ldots \leq j_n$ with each $j_i \geq i-1$, $b \in B_{n,j_n}$, for each $i \in \{2,3,\ldots,n\}$, the $X$-decomposition of $y_{i,j_i}$ is $y_{i,j_i} = \sum_{j\in I_{i-1}} x_{i-1,j} + \sum_{j\in J_i} x_{i,j}$, and the $A$-decomposition of $b$ is

$$b = z + \sum_{j\in I_n} x_{n,j} + \sum_{i=1}^{l} \sum_{j\in J_{n+i}} x_{n+i,j}\,,$$

then the $A$-decomposition of $d = y_{2,j_2} + \ldots + y_{n,j_n} + b$ is

$$d = z + \sum_{j\in I_1} x_{1,j} + \sum_{i=2}^{n} \sum_{j\in J_i\cup I_i} x_{i,j} + \sum_{i=1}^{l} \sum_{j\in J_{n+i}} x_{n+i,j}$$

and for each $i \in \{2,3,\ldots,n\}$, either $I_i = \emptyset$ or $\min I_i > \max J_i$ (so that $\min(J_i \cup I_i) = \min J_i = \theta_{i-1}(y_{i,j_i})$.

We show first that

$$y_{n,j_n} + b = z + \sum_{j\in I_{n-1}} x_{n-1,j} + \sum_{j\in J_n\cup I_n} x_{n,j} + \sum_{i=1}^{l} \sum_{j\in J_{n+i}} x_{n+i,j}$$

is the $A$-decomposition of $y_{n,j_n} + b$. We claim that either $I_n = \emptyset$ or $\min I_n > \max J_n$ so that the equation holds. Suppose instead that $k = \min I_n \leq \max J_n = \phi_{n-1}(y_{n,j_n})$. Then $b \in x_{n,k} + A_{n,k}$ while $b \in B_{n,j_n} \subseteq F_{n-1}(y_{n,j_n}) \subseteq A_{n,\phi_{n-1}(y_{n,j_n})} \subseteq A_{n,k}$ and $(x_{n,k} + A_{n,k}) \cap A_{n,k} = \emptyset$. To see that

$$y_{n,j_n} + b = z + \sum_{j\in I_{n-1}} x_{n-1,j} + \sum_{j\in J_n\cup I_n} x_{n,j} + \sum_{i=1}^{l} \sum_{j\in J_{n+i}} x_{n+i,j}$$

is the $A$-decomposition of $y_{n,j_n} + b$, we need that for $k \in I_{n-1}$, if any,

$$-\sum_{I_{n-1}\ni j<k} x_{n-1,j} + y_{n,j_n} + b \in x_{n-1,k} + A_{n-1,k}$$

and for $k \in J_n$, $-(\sum_{j\in I_{n-1}} x_{n-1,j} + \sum_{J_n\ni j<k} x_{n,j}) + y_{n,j_n} + b \in x_{n,k} + A_{n,k}$. Both of these statements hold because $b \in B_{n,j_n} \subseteq F_{n-1}(y_{n,j_n})$. For $k \in I_n$, if any, we need that $-(\sum_{j\in I_{n-1}} x_{n-1,j} + \sum_{j\in J_n} x_{n,j} + \sum_{I_n\ni j<k} x_{n,j}) + y_{n,j_n} + b \in x_{n,k} + A_{n,k}$, that is, that $-\sum_{I_n\ni j<k} x_{n,j} + b \in x_{n,k} + A_{n,k}$, which holds. Similarly, the remainder of the requirements for $y_{n,j_n} + b$

follow from the corresponding requrements for $b$ and the fact that $y_{n,j_n} = \sum_{j\in I_{n-1}} x_{n-1,j} + \sum_{j\in J_n} x_{n,j}$.

Now let $2 < r \leq n$ and assume we have shown that the $A$-decomposition of $d = y_{r,j_r} + \ldots + y_{n,j_n} + b$ is

$$d = z + \sum_{j\in I_{r-1}} x_{r-1,j} + \sum_{i=r}^{n} \sum_{j\in J_i\cup I_i} x_{i,j} + \sum_{i=1}^{l} \sum_{j\in J_{n+i}} x_{n+i,j}$$

and for each $i \in \{r,\ldots,n\}$, either $I_i = \emptyset$ or $\min I_i > \max J_i$. We have that $y_{r-1,j_{r-1}} = \sum_{j\in I_{r-2}} x_{r-2,j} + \sum_{j\in J_{r-1}} x_{r-1,j}$. Then exactly as before, we show that either $I_{r-1} = \emptyset$ or $\min I_{r-1} > \max J_{r-1}$ so that the equation holds. And one shows in the same way as before that the required conditions to verify that it is the $A$-decomposition hold.

Having determined the $A$-decomposition of $y_{2,j_2} + \ldots + y_{n,j_n} + b$, we have that $f(y_{2,j_2} + \ldots + y_{n,j_n} + b) = \theta_1(y_{2,j_2}) \cdots \theta_{n-1}(y_{n,j_n})\alpha_{n+1} \cdots \alpha_{n+l}$ where $\alpha_{n+s} = \min J_{n+s}$ for $s \in \{1,2,\ldots,l\}$.

For every $n \geq 2$ pick $b_n \in B_{n,n}$, let $a_n = y_{2,n} + y_{3,n} + \ldots + y_{n,n} + b_n$, and let $w_n = f(a_n)$. For each $n \geq 2$, let $J_{n,n}, J_{n,n+1}, \ldots, J_{n,l_n}$ be the finite sets from the $A$-decomposition of $b_n$ and for $s \in \{1,2,\ldots,l_n\}$, let $\alpha_{n,n+s} = \min J_{n,n+s}$. Then $w_n = \theta_1(y_{2,n}) \cdots \theta_{n-1}(y_{n,n})\alpha_{n,n+1} \cdots \alpha_{n,n+l_n}$. Clearly $w_2 < w_3 < \ldots$.

Let $2 \leq j_2 < \ldots < j_k = n$ be given and let $w = [w_{j_2},\ldots,w_{j_k}]$ and $a = y_{2,j_2} + \ldots + y_{k,j_k} + y_{k+1,n} + \ldots + y_{n,n} + b_n$. Then

$$\begin{aligned} w &= \theta_1(y_{2,j_2}) \cdots \\ &\quad \theta_{k-1}(y_{k,j_k})\theta_k(y_{k+1,n}) \cdots \theta_{n-1}(y_{n,n})\alpha_{n,n+1} \cdots \alpha_{n,n+l_n} \text{ and} \\ f(a) &= \theta_1(y_{2,j_2}) \cdots \\ &\quad \theta_{k-1}(y_{k,j_k})\theta_k(y_{k+1,n}) \cdots \theta_{n-1}(y_{n,n})\alpha_{n,n+1} \cdots \alpha_{n,n+l_n}\,, \end{aligned}$$

so $w = f(a)$ and since $a \in B_1$, $w \in f[B_1]$. □

**Theorem 8.4.** *There does not exist an increasing sequence of principal left ideals in* $(\beta\mathbb{Z},+)$.

*Proof.* Lemma 8.2 and Proposition 8.3. □

In [133, Remark 6], the author notes that if $q \notin \mathbb{N}^* + \mathbb{N}^*$, for $n \in \omega$, $p_n = -n + q$, and $L_n = \{p_n\} \cup (\beta\mathbb{N} + p_n)$, then $\langle L_n\rangle_{n=0}^{\infty}$ is a strictly increasing sequence of principal left ideals of $\beta\mathbb{N}$. We conclude this section by noting that the same result holds under the weaker assumption that $q$ is right cancelable in $\beta\mathbb{N}$, and as a consequence of Theorem 8.4, any such sequence in $\beta\mathbb{N}$ must be generated by infinitely many right cancelable elements.

**Theorem 8.5.** *Let $q$ be a right cancelable element of $\beta\mathbb{N}$, for each $n \in \omega$, let $p_n = -n + q$, and let $L_n = \{p_n\} \cup (\beta\mathbb{N} + p_n)$. Then $\langle L_n\rangle_{n=0}^{\infty}$ is a strictly increasing sequence of principal left ideals of $\beta\mathbb{N}$.*

*Proof.* Let $n \in \omega$. Then $p_n = 1+p_{n+1}$ so $L_n \subseteq L_{n+1}$. Suppose $L_{n+1} \subseteq L_n$ and pick $x \in \beta\mathbb{N}$ such that $1+p_{n+1} = x+p_n$. Then $1-n-1+q = x-n+q$ so $1+q = 1+x+q$ so $1 = 1+x$, a contradiction. □

**Lemma 8.6.** *Let $p \in \mathbb{N}^*$. Then $p$ is not right cancelable in $\beta\mathbb{N}$ if and only if $\beta\mathbb{N}+p = \beta\mathbb{Z}+p$.*

*Proof.* For the necessity, assume that $p$ is not right cancelable in $\beta\mathbb{N}$. By [72, Theorem 8.18], pick $u \in \mathbb{N}^*$ such that $p = u+p$. To see that $\beta\mathbb{Z}+p \subseteq \beta\mathbb{N}+p$, let $q \in \beta\mathbb{Z}$. Then $q+p = q+u+p$ and by [72, Exercise 4.3.5], $q+u \in \mathbb{N}^*$, so $q+p \in \beta\mathbb{N}+p$.

For the sufficiency assume that $\beta\mathbb{N}+p = \beta\mathbb{Z}+p$. Then $-1+p \in \beta\mathbb{N}+p$ so pick $x \in \beta\mathbb{N}$ such that $-1+p = x+p$. Then $2-1+p = 2+x+p$. If $p$ were right cancelable, we would have $x = -1$, a contradiction. □

**Corollary 8.7.** *Let $\langle p_n \rangle_{n=0}^{\infty}$ be a sequence such that $\langle L_n \rangle_{n=0}^{\infty}$ is strictly increasing, where $L_n = \{p_n\} \cup (\beta\mathbb{N}+p_n)$. Then $\{n \in \omega : p_n$ is not right cancelable in $\beta\mathbb{N}\}$ is finite.*

*Proof.* Suppose not. Then by passing to a subsequence we may presume that each $p_n$ is not right cancelable in $\beta\mathbb{N}$ so that by Lemma 8.6, $L_n = \beta\mathbb{Z}+p_n$. This contradicts Theorem 8.4. □

We include an extensive bibliography listing all of the papers that we are aware of dealing with the algebraic structure of the Stone-Čech compactification of a discrete semigroup or the combinatorial applications of that structure that were published since the publication of [72]. Except for papers cited in this current paper we do not duplicate items in the bibliography of [72].

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