This paper was published in *Combinatorica* **24** (2004), 271-286. To the best of my knowledge this is the final version as it was submitted to the publisher. – NH

# Partition Theorems for Left and Right Variable Words

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**Abstract**. In 1984 T. Carlson and S. Simpson established an infinitary extension of the Hales-Jewett Theorem in which the leftmost letters of all but one of the words were required to be variables. (We call such words *left variable* words.) In this paper we extend the Carlson-Simpson result for left variable words, prove a corresponding result about *right variable* words, and determine precisely the extent to which left and right variable words can be combined in such extensions. The results mentioned so far all involve a finite alphabet. We show that the results for left variable words do not extend to words over an infinite alphabet, but that the results for right variable words do extend.

## 1. Introduction

Our story begins with the Hales-Jewett Theorem. Given $k \in \mathbb{N}$, let $\mathcal{W}_k$ denote the free semigroup on the alphabet $\{1, 2, \ldots, k\}$. That is, $\mathcal{W}_k$ consists of all "words with letters from $\{1, 2, \ldots, k\}$" (i.e. functions whose domain is an initial segment of $\mathbb{N}$ and whose range is contained in $\{1, 2, \ldots, k\}$) with the operation of concatenation. (We are taking $\mathbb{N}$ to be the set of positive integers. We let $\omega = \mathbb{N} \cup \{0\}$.) A *variable word* over $\mathcal{W}_k$ is a word on the alphabet $\{1, 2, \ldots, k\} \cup \{v\}$ in which $v$ occurs, where $v$ is a "variable" not in $\{1, 2, \ldots, k\}$. Given a variable word $w$ over $\mathcal{W}_k$, and $t \in \{1, 2, \ldots, k\}$, $w(t)$ has its obvious meaning, namely the result of replacing all occurrences of $v$ with $t$.

**1.1 Theorem (Hales-Jewett)**. *Let $k, r \in \mathbb{N}$ and let $\mathcal{W}_k = \bigcup_{i=1}^{r} A_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a variable word $w$ over $\mathcal{W}_k$ such that $\{w(t) : t \in \{1, 2, \ldots, k\}\} \subseteq A_i$.*

**Proof**. [6]. □

Notice that, in the Hales-Jewett Theorem, one cannot expect that the variable $v$ will

[1] This author acknowledges support received from the National Science Foundation via grant DMS-0070593.

[2] This author acknowledges support received from the National Science Foundation via a post doctoral fellowship administered by the University of Maryland.

occur as either the leftmost or rightmost letter of $w$. Indeed, suppose that $k \geq 2$. Let

$$\begin{array}{rcl} B_1 & = & \{w \in \mathcal{W}_k : \text{the leftmost letter of } w \text{ is } 1\}\,, \\ B_2 & = & \{w \in \mathcal{W}_k : \text{the leftmost letter of } w \text{ is not } 1\}\,, \\ C_1 & = & \{w \in \mathcal{W}_k : \text{the rightmost letter of } w \text{ is } 1\}\,, \text{ and} \\ C_2 & = & \{w \in \mathcal{W}_k : \text{the rightmost letter of } w \text{ is not } 1\}\,. \end{array}$$

Then let $A_1 = B_1 \cap C_1$, $A_2 = B_1 \cap C_2$, $A_3 = B_2 \cap C_1$, and $A_4 = B_2 \cap C_2$. Then no variable word satisfying the conclusion of Theorem 1.1 has the variable $v$ as either its leftmost or rightmost letter.

**1.2 Definition**. Let $k \in \mathbb{N}$.

(a) A *left variable word over* $\mathcal{W}_k$ is a variable word over $\mathcal{W}_k$ whose leftmost letter is the variable $v$.

(b) A *right variable word over* $\mathcal{W}_k$ is a variable word over $\mathcal{W}_k$ whose rightmost letter is the variable $v$.

In 1984, T. Carlson and S. Simpson established an infinitary extension of the Hales-Jewett Theorem which involved left variable words. (When $F$ is a finite nonempty subset of $\mathbb{N}$, by $\prod_{m \in F} a_m$, we mean the product in increasing order of indices.)

**1.3 Theorem (Carlson-Simpson)**. *Let $k, r \in \mathbb{N}$ and let $\mathcal{W}_k = \bigcup_{i=1}^r A_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle w_m \rangle_{m=1}^\infty$ of variable words over $\mathcal{W}_k$ such that for every $m > 1$, $w_m$ is a left variable word, and for every $n \in \mathbb{N}$ and every $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$, $\prod_{m=1}^n w_m\big(f(m)\big) \in A_i$.*

**Proof**. [3, Theorem 6.3]. □

Notice that the Carlson-Simpson Theorem easily implies the corresponding result in which the leftmost letter of each word is required *not* to be variable.

**1.4 Corollary**. *Let $k, r \in \mathbb{N}$ and let $\mathcal{W}_k = \bigcup_{i=1}^r A_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle u_m \rangle_{m=1}^\infty$ of variable words over $\mathcal{W}_k$ such that for every $m$, the leftmost letter of $w_m$ is not the variable $v$, and for every $n \in \mathbb{N}$ and every $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$, $\prod_{m=1}^n u_m\big(f(m)\big) \in A_i$.*

**Proof**. For each $m$, let $u_m = w_{2m-1}(1) w_{2m}(v)$. (By $w_{2m}(v)$, we mean of course simply $w_{2m}$.) □

In 1988 and 1989, T. Carlson and (independently) H. Furstenberg and Y. Katznelson established the following extension of Corollary 1.4, in which one is not restricted to products of initial segments of $\mathbb{N}$. (Given a set $A$, we denote the set of finite nonempty subsets of $A$ by $\mathcal{P}_f(A)$.)

**1.5 Theorem (Carlson and Furstenberg-Katznelson)**. *Let $k, r \in \mathbb{N}$ and let $\mathcal{W}_k = \bigcup_{i=1}^{r} A_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle u_m \rangle_{m=1}^{\infty}$ of variable words over $\mathcal{W}_k$ such that for every $F \in \mathcal{P}_f(\mathbb{N})$ and every $f : F \to \{1, 2, \ldots, k\}$, $\prod_{m \in F} u_m\big(f(m)\big) \in A_i$.*

**Proof**. [5, Theorem 2.5] or [2, Theorem 12] applied to $\vec{e} = \overline{1} = (1, 1, 1 \ldots)$. $\square$

Notice that one cannot hope to extend Theorem 1.3 in exactly the same way that Corollary 1.4 is extended by Theorem 1.5 because of the observation already made that one can easily prevent having a left variable word $w$ with $w(1)$ and $w(2)$ in the same cell of a partition. However, it was recently established that almost as much can be done.

**1.6 Theorem**. *Let $k, r \in \mathbb{N}$ and let $\mathcal{W}_k = \bigcup_{i=1}^{r} A_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle w_m \rangle_{m=1}^{\infty}$ of variable words over $\mathcal{W}_k$ such that for every $m > 1$, $w_m$ is a left variable word, and for every $F \in \mathcal{P}_f(\mathbb{N})$ with $\min F = 1$ and every $f : F \to \{1, 2, \ldots, k\}$, $\prod_{m \in F} w_m\big(f(m)\big) \in A_i$.*

**Proof**. [8, Theorem 2.3]. $\square$

In Section 2 of this paper we generalize Theorem 1.6 in a way that allows us to avoid requiring that $\min F = 1$, derive a similar result for right variable words, and then establish a common generalization of both of these extensions. (Notice that results for right variable words are not a consequence of a simple right-left switch. Any such switch involves writing products in decreasing order of indices.) We also give an example showing that our results are the best possible.

In Section 3 we investigate extensions of these results to the free semigroup on infinitely many letters, showing that the natural extension for right variable words is valid, but that the extension for left variable words is impossible.

We shall utilize extensively the algebraic structure of the Stone-Čech compactification $\beta S$ of a (discrete) semigroup $(S, \cdot)$. We take the points of $\beta S$ to be the ultrafilters on $S$, the principal ultrafilters being identified with the points of $S$. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of $\beta S$.

There is a natural extension of the operation $\cdot$ of $S$ to $\beta S$ making $\beta S$ a compact right topological semigroup with $S$ contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$. The operation is characterized as follows. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. (The notation is not intended to suggest that $x$ has an inverse.) See [7] for an elementary introduction to the semigroup $\beta S$.

Any compact Hausdorff right topological semigroup $(T,\cdot)$ has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of $T$, each of which is closed [7, Theorem 2.8], and any compact right topological semigroup contains idempotents. Since the minimal left ideals are themselves compact right topological semigroups, this says in particular that there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of $T$ determined by $p \leq q$ if and only if $p = p \cdot q = q \cdot p$. An idempotent $p$ is minimal with respect to this order if and only if $p \in K(T)$ [7, Theorem 1.59]. Such an idempotent is called simply "minimal"

**1.7 Definition**. Let $(S,\cdot)$ be an infinite discrete semigroup. A set $A \subseteq S$ is *central* if and only if there is some minimal idempotent $p$ such that $A \in p$.

Central sets were introduced by Furstenberg [4] and defined in terms of notions of topological dynamics. These sets enjoy very strong combinatorial properties. (See [4, Proposition 8.21] or [7, Chapter 14].) See [7, Theorem 19.27] for a proof of the equivalence of the definition above with the original dynamical definition.

We shall state our results in terms of an arbitrary central subset of $\mathcal{W}_k$. Partition results automatically follow because, given a minimal idempotent $p \in \beta\mathcal{W}_k$ and given $r \in \mathbb{N}$, if $\mathcal{W}_k = \bigcup_{i=1}^{r} A_i$, then some $A_i \in p$ and is therefore central.

We would like to thank the referees for constructive reports.

## 2. Left and Right Variable Words over Finite Alphabets

We work throughout this section with a fixed finite alphabet $\{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$.

**2.1 Definition**. Let $k \in \mathbb{N}$.

(a) $Y = \times_{j=1}^{k} \beta\mathcal{W}_k$.

(b) $I = \big\{\big(w(1), w(2), \ldots, w(k)\big) : w \text{ is a variable word over } \mathcal{W}_k\big\}$.

(c) $J = \big\{\big(w(1), w(2), \ldots, w(k)\big) : w \text{ is a left variable word over } \mathcal{W}_k\big\}$.

(d) $H = \big\{\big(w(1), w(2), \ldots, w(k)\big) : w \text{ is a right variable word over } \mathcal{W}_k\big\}$.

(e) $E = I \cup \big\{\big(w, w, \ldots, w\big) : w \in \mathcal{W}_k\big\}$.

We denote the closures of $I$, $J$, $H$, and $E$ in $Y$ by $\overline{I}$, $\overline{J}$, $\overline{H}$, and $\overline{E}$ respectively.

The proof of the following lemma is an elaboration of the method of proof introduced by Furstenberg and Katznelson in [5] in the context of enveloping semigroups.

**2.2 Lemma**. *Let $p$ be any minimal idempotent in $\beta\mathcal{W}_k$. Then $\overline{p} = (p, p, \ldots, p) \in \overline{I}$. Also there exist minimal idempotents $q_1, q_2, \ldots, q_k$ and $r_1, r_2, \ldots, r_k$ in $\beta\mathcal{W}_k$ such that*

*(1) $\vec{q} = (q_1, q_2, \ldots, q_k) \in \overline{J}$;*

*(2) $\vec{r} = (r_1, r_2, \ldots, r_k) \in \overline{H}$;*

*(3) for all $j \in \{1, 2, \ldots, k\}$, $pq_j = p$ and $q_j p = q_j$;*

*(4) for all $j \in \{1, 2, \ldots, k\}$, $pr_j = r_j$ and $r_j p = p$;*

*(5) for all $i, j \in \{1, 2, \ldots, k\}$, $q_i q_j = q_i$; and*

*(6) for all $i, j \in \{1, 2, \ldots, k\}$, $r_j r_i = r_i$.*

**Proof**. By [7, Theorems 2.22 and 4.17] we have that $\overline{E}$ is a subsemigroup of $Y$, $\overline{I}$ is an ideal of $\overline{E}$, $\overline{J}$ is a right ideal of $\overline{E}$, and $\overline{H}$ is a left ideal of $\overline{E}$.

By [7, Theorem 2.23], $K(Y) = \times_{t=1}^{k} K(\beta\mathcal{W}_k)$. We claim that

$$(*) \qquad \text{if } s \in K(\beta\mathcal{W}_k), \text{ then } (s, s, \ldots, s) \in \overline{E}\,.$$

To see this, let $U$ be a neighborhood of $(s, s, \ldots, s)$ and pick $A \in s$ such that $\overline{A}^{\,k} \subseteq U$. Pick $w \in A$. Then $(w, w, \ldots, w) \in U \cap E$.

Thus we have that $\overline{E} \cap K(Y) \neq \emptyset$ so by [7, Theorem 1.65], $K(Y) \cap \overline{E} = K(\,\overline{E}\,)$. Since $\overline{I}$ is an ideal of $\overline{E}$, we have that $\overline{p} \in K(Y) \cap \overline{E} = K(\,\overline{E}\,) \subseteq \overline{I}$.

Since $\overline{p}$ is a minimal idempotent in $\overline{E}$, pick (by [7, Theorem 2.8]) a minimal left ideal $L$ and a minimal right ideal $R$ of $\overline{E}$ such that $\overline{p} \in L \cap R$. Pick, by [7, Corollary 2.6 and Theorem 2.7], a minimal right ideal $R'$ and a minimal left ideal $L'$ of $\overline{E}$ such that $R' \subseteq \overline{J}$ and $L' \subseteq \overline{H}$. Then by [7, Theorem 2.7], $L \cap R'$ and $L' \cap R$ are groups. Let $\vec{q} = (q_1, q_2, \ldots, q_k)$ be the identity of $L \cap R'$ and let $\vec{r} = (r_1, r_2, \ldots, r_k)$ be the identity of $L' \cap R$.

By [7, Lemma 1.30], we have that $\vec{q}$ and $\overline{p}$ are right identities for $L$ and $\vec{r}$ and $\overline{p}$ are left identities for $R$. Thus conclusions (1), (2), (3), and (4) hold.

Conclusion (5) follows from conclusion (3) and conclusion (6) follows from conclusion (4). (For example, given $i, j \in \{1, 2, \ldots, k\}$, $q_i q_j = (q_i p) q_j = q_i (p q_j) = q_i p = q_i$.) $\square$

Notice that, while the coordinates of $\overline{p}$ are all equal, none of the coordinates of the vectors $\vec{q}$ or $\vec{r}$ produced in Lemma 2.2 can be equal. (For example, $\{w \in \mathcal{W}_k :$ the rightmost letter of $w$ is $1\} \in r_1$ while $\{w \in \mathcal{W}_k :$ the rightmost letter of $w$ is $2\} \in r_2$.)

Observe that Lemma 2.2 says in particular that $\{p, q_1, q_2, \ldots, q_k\}$ and $\{p, r_1, r_2, \ldots, r_k\}$ are subsemigroups of $\beta\mathcal{W}_k$.

**2.3 Lemma**. *Let $S$ be a discrete semigroup and let $T$ be a finite subsemigroup of $\beta S$. For $A \subseteq S$, let $A^\dagger = \{x \in A : x^{-1}A \in \bigcap T\}$. If $A \in \bigcap T$, then*

*(1) $A^\dagger \in \bigcap T$ and*

*(2) for each $x \in A^\dagger$, $x^{-1}A^\dagger \in \bigcap T$.*

**Proof**. For conclusion (1), we have that $A^\dagger = A \cap \bigcap_{t \in T}\{x \in S : x^{-1}A \in t\}$ so it suffices to let $t, s \in T$ and show that $\{x \in S : x^{-1}A \in t\} \in s$. Since $st \in T$ we have that $A \in st$ and so $\{x \in S : x^{-1}A \in t\} \in s$ as required.

To establish conclusion (2), let $x \in A^\dagger$ and let $B = x^{-1}A$. Then $B \in \bigcap T$ and so, by conclusion (1), $B^\dagger \in \bigcap T$. It thus suffices to show that $B^\dagger \subseteq x^{-1}A^\dagger$. So let $y \in B^\dagger$. Then $y \in B$ and so $xy \in A$. Also, $(xy)^{-1}A = y^{-1}(x^{-1}A) = y^{-1}B \in \bigcap T$. □

Theorems 2.4 and 2.5 are both corollaries of Theorem 2.7. However, the individual proofs are simpler, and we feel it is also instructive to see how the two proofs are combined to yield Theorem 2.7, so we present them both.

**2.4 Theorem**. *Let $k \in \mathbb{N}$ and let $B$ be any central subset of $\mathcal{W}_k$. There exists a sequence $\langle w_n \rangle_{n=1}^\infty$ of variable words over $\mathcal{W}_k$ such that*
*(1) for each $n \in \omega$, $w_{2n+1}$ is a right variable word and*
*(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $f : F \to \{1, 2, \ldots, k\}$, if $\max F$ is even, then $\prod_{m \in F} w_m\big(f(m)\big) \in B$.*

**Proof**. Pick an idempotent $p \in K(\beta\mathcal{W}_k)$ such that $B \in p$ and let $r_1, r_2, \ldots, r_k$ be as in Lemma 2.2. Let $M = \{x \in \mathcal{W}_k : x^{-1}B \in p\}$. Since $p = pp = r_1p = r_2p = \ldots = r_kp$, we have that $M \in p \cap \bigcap_{j=1}^k r_j$. Let $M^\dagger = \{x \in M : x^{-1}M \in p \cap \bigcap_{j=1}^k r_j\}$. Let $N_1 = M^\dagger$. Then by Lemma 2.3, $N_1 \in \bigcap_{j=1}^k r_j$ so that $\overline{N_1}^{\,k}$ is a neighborhood of $\vec{r}$, and hence $N_1{}^k \cap H \neq \emptyset$. Pick a right variable word $w_1$ such that

$$\big(w_1(1), w_1(2), \ldots, w_1(k)\big) \in N_1{}^k .$$

Let $n \in \mathbb{N}$ and assume that we have chosen variable words $w_1, w_2, \ldots, w_n$ satisfying the following induction hypotheses.

(a) If $l \in \{1, 2, \ldots, n\}$ is odd, then $w_l$ is a right variable word.

(b) If $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$ and $f : F \to \{1, 2, \ldots, k\}$, then $\prod_{m \in F} w_m\big(f(m)\big) \in M^\dagger$ and, if $\max F$ is even, then $\prod_{m \in F} w_m\big(f(m)\big) \in B$.

Let $L = \big\{\prod_{m \in F} w_m\big(f(m)\big) : \emptyset \neq F \subseteq \{1, 2, \ldots, n\}$ and $f : F \to \{1, 2, \ldots, k\}\big\}$ and let $N_{n+1} = M^\dagger \cap \bigcap_{x \in L} x^{-1}M^\dagger$. By assumption $L \subseteq M^\dagger$ and so, by Lemma 2.3, $N_{n+1} \in p \cap \bigcap_{j=1}^k r_j$.

Assume first that $n$ is even. Then $\overline{N_{n+1}}^{\,k}$ is a neighborhood of $\vec{r}$ and so $N_{n+1}{}^k \cap H \neq \emptyset$. Pick a right variable word $w_{n+1}$ such that $\big(w_{n+1}(1), w_{n+1}(2), \ldots, w_{n+1}(k)\big) \in N_{n+1}{}^k$.

Next assume that $n$ is odd, and let $P_{n+1} = N_{n+1} \cap B \cap \bigcap_{x \in L} x^{-1}B$. Then $\overline{P_{n+1}}^{\,k}$ is a neighborhood of $\overline{p}$ and so $P_{n+1}{}^k \cap I \neq \emptyset$. Pick a variable word $w_{n+1}$ such that $\big(w_{n+1}(1), w_{n+1}(2), \ldots, w_{n+1}(k)\big) \in P_{n+1}{}^k$.

Hypothesis (a) clearly holds. To verify hypothesis (b), let $\emptyset \neq F \subseteq \{1,2,\ldots,n+1\}$ and let $f : F \to \{1,2,\ldots,k\}$. If $n+1 \notin F$, the conclusion holds by assumption, so assume that $n+1 \in F$. If $F = \{n+1\}$ we have that $w_{n+1}\big(f(n+1)\big) \in M^\dagger$ and if $n+1$ is even, $w_{n+1}\big(f(n+1)\big) \in B$. So assume that $G = F \backslash \{n+1\} \neq \emptyset$. Let $x = \prod_{m\in G} w_m\big(f(m)\big)$. Then $x \in L$ so $w_{n+1}\big(f(n+1)\big) \in x^{-1}M^\dagger$ and if $n+1$ is even, $w_{n+1}\big(f(n+1)\big) \in x^{-1}B$. $\square$

**2.5 Theorem**. *Let $k \in \mathbb{N}$ and let $B$ be any central subset of $\mathcal{W}_k$. There exists a sequence $\langle w_n\rangle_{n=1}^\infty$ of variable words over $\mathcal{W}_k$ such that*

*(1) for each $n \in \mathbb{N}$, $w_{2n}$ is a left variable word and*

*(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $f : F \to \{1,2,\ldots,k\}$, if $\min F$ is odd, then $\prod_{m\in F} w_m\big(f(m)\big) \in B$.*

**Proof**. Pick $p \in K(\beta\mathcal{W}_k)$ such that $B \in p$ and let $q_1, q_2, \ldots, q_k$ be as in Lemma 2.2. Given $A \in p \cap \bigcap_{j=1}^k q_j$, let $A^\ddagger = \{x \in A : x^{-1}A \in p \cap \bigcap_{j=1}^k q_j\}$. Then by Lemma 2.3, $A^\ddagger \in p \cap \bigcap_{j=1}^k q_j$ and for each $x \in A^\ddagger$, $x^{-1}A^\ddagger \in p \cap \bigcap_{j=1}^k q_j$.

Let $C_1 = \{x \in B : x^{-1}B \in p \cap \bigcap_{j=1}^k q_j\}$. Since $p = pp = pq_1 = pq_2 = \ldots = pq_k$, $C_1 \in p$ and so $\overline{C_1}^{\,k}$ is a neighborhood of $\overline{p}$ and thus $C_1{}^k \cap I \neq \emptyset$. Pick a variable word $w_1$ such that $\big(w_1(1), w_1(2), \ldots, w_1(k)\big) \in C_1{}^k$.

Let $D_2 = \bigcap_{t=1}^k w_1(t)^{-1}B$ and note that $D_2 \in p \cap \bigcap_{j=1}^k q_j$. Let $E_2 = D_2{}^\ddagger$. Then $\overline{E_2}^{\,k}$ is a neighborhood of $\vec{q} = (q_1, q_2, \ldots, q_k)$ and thus $E_2{}^k \cap J \neq \emptyset$. Pick a left variable word $w_2$ such that $\big(w_2(1), w_2(2), \ldots, w_2(k)\big) \in E_2{}^k$.

Let $n \geq 2$ and assume that we have chosen variable words $w_1, w_2, \ldots, w_n$. Assume also that for odd $l \in \{1,2,\ldots,n\}$ we have chosen $C_l \in p$ and that for even $l \in \{1,2,\ldots,n\}$ we have chosen $D_l \in p \cap \bigcap_{j=1}^k q_j$ so that the following induction hypotheses are satisfied.

(a) If $l \in \{1,2,\ldots,n\}$ is even, then $w_l$ is a left variable word and $D_l \subseteq \bigcap_{t=1}^k w_{l-1}(t)^{-1}B$.

(b) If $l \in \{1,2,\ldots,n\}$ is odd and $t \in \{1,2,\ldots,k\}$, then $w_l(t) \in C_l \subseteq C_1$.

(c) If $\emptyset \neq F \subseteq \{1,2,\ldots,n\}$, $f : F \to \{1,2,\ldots,k\}$, and $2l \leq \min F$, then $\prod_{m\in F} w_m\big(f(m)\big) \in D_{2l}{}^\ddagger$ and, if $\min F$ is odd, then $\prod_{m\in F} w_m\big(f(m)\big) \in B$.

For $l \in \mathbb{N}$ with $2l \leq n$, let

$$G_{l,n} = \left\{\textstyle\prod_{m\in F} w_m\big(f(m)\big) : \emptyset \neq F \subseteq \{2l, 2l+1, \ldots, n\} \text{ and } f : F \to \{1,2,\ldots,k\}\right\}.$$

If $n = 2s$, let $C_{n+1} = C_{n-1} \cap \bigcap_{l=1}^s (D_{2l}{}^\ddagger \cap \bigcap_{x\in G_{l,n}} x^{-1}D_{2l}{}^\ddagger)$. By induction hypothesis (c), if $x \in G_{l,n}$, then $x \in D_{2l}{}^\ddagger$ and so, by Lemma 2.3, $x^{-1}D_{2l}{}^\ddagger \in p$. Therefore $\overline{C_{n+1}}^{\,k}$ is a neighborhood of $\overline{p}$. So we may pick a variable word $w_{n+1}$ such that $\big(w_{n+1}(1), w_{n+1}(2), , \ldots, w_{n+1}(k)\big) \in C_{n+1}{}^k$.

If $n = 2s+1$, let $D_{n+1} = D_{n-1} \cap \bigcap_{t=1}^{k} w_n(t)^{-1}B$. By hypothesis (b), we have that each $w_n(t) \in C_1$ and so $D_{n+1} \in p \cap \bigcap_{j=1}^{k} q_j$. Let $E_{n+1} = {D_{n+1}}^{\ddagger} \cap \bigcap_{l=1}^{s} ({D_{2l}}^{\ddagger} \cap \bigcap_{x \in G_{l,n}} x^{-1}{D_{2l}}^{\ddagger})$. Then $E_{n+1} \in p \cap \bigcap_{j=1}^{k} q_j$. Pick a left variable word $w_{n+1}$ such that

$$\big(w_{n+1}(1), w_{n+1}(2), , \ldots, w_{n+1}(k)\big) \in {E_{n+1}}^{k} .$$

Hypotheses (a) and (b) are satisfied immediately. To verify hypothesis (c), let $\emptyset \neq F \subseteq \{1, 2, \ldots, n+1\}$, let $f : F \to \{1, 2, \ldots, k\}$, and, if $\min F > 1$, let $2l \leq \min F$. If $n+1 \notin F$, the conclusions hold by assumption, so assume that $n+1 \in F$. Assume first that $F = \{n+1\}$. If $n+1$ is odd, then $w_{n+1}\big(f(n+1)\big) \in C_{n+1} \subseteq {D_{2l}}^{\ddagger} \cap B$. If $n+1$ is even, then $w_{n+1}\big(f(n+1)\big) \in E_{n+1} \subseteq {D_{2l}}^{\ddagger}$. Now assume that $|F| \geq 2$, let $K = F \backslash \{n+1\}$, and let $x = \prod_{m \in K} w_m\big(f(m)\big)$. If $\min F > 1$, then $x \in G_{l,n}$ and so $w_{n+1}\big(f(n+1)\big) \in x^{-1}{D_{2l}}^{\ddagger}$. Therefore $\prod_{m \in F} w_m\big(f(m)\big) \in {D_{2l}}^{\ddagger}$. Finally, let $a = \min F$ and assume that $a = 2s-1$. Let $P = F \backslash \{a\}$. Then we have established that $\prod_{m \in P} w_m\big(f(m)\big) \in {D_{2s}}^{\ddagger}$. And $D_{2s} \subseteq w_a\big(f(a)\big)^{-1}B$, so $\prod_{m \in F} w_m\big(f(m)\big) \in B$. □

When we deal with products including both left and right variable words, we shall be concerned with the need to separate their occurrences.

**2.6 Definition**. Let $\langle w_n \rangle_{n=1}^{\infty}$ be a sequence of variable words and let $F \in \mathcal{P}_f(\mathbb{N})$.

(a) The set $F$ *strongly separates* $\langle w_n \rangle_{n \in F}$ if and only if whenever $i$ and $j$ are members of $F$ such that $w_i$ is a right variable word and $w_j$ is a left variable word, there is some $l \in F$ such that $l$ is between $i$ and $j$ and $w_l$ is neither a left variable word nor a right variable word.

(b) The set $F$ *weakly separates* $\langle w_n \rangle_{n \in F}$ if and only if whenever $i$ and $j$ are members of $F$ such that $w_i$ is a right variable word, $w_j$ is a left variable word, and $i < j$, there is some $l \in F$ such that $l$ is between $i$ and $j$ and $w_l$ is neither a left variable word nor a right variable word.

**2.7 Theorem**. *Let $k \in \mathbb{N}$ and let $B$ be any central subset of $\mathcal{W}_k$. There exists a sequence $\langle w_n \rangle_{n=1}^{\infty}$ of variable words over $\mathcal{W}_k$ such that*

*(1) for each $n \in \omega$, $w_{4n+1}$ is a right variable word;*

*(2) for each $n \in \omega$, $w_{4n+3}$ is a left variable word;*

*(3) for each $n \in \omega$, $w_{4n+2}$ and $w_{4n+4}$ are neither left nor right variable words; and*

*(4) for all $F \in \mathcal{P}_f(\mathbb{N})$, if $F$ strongly separates $\langle w_n \rangle_{n \in F}$, $\min F \not\equiv 3 \pmod 4$, $\max F \not\equiv 1 \pmod 4$, and $f : F \to \{1, 2, \ldots, k\}$, then $\prod_{m \in F} w_m\big(f(m)\big) \in B$.*

**Proof**. Pick $p \in K(\beta\mathcal{W}_k)$ such that $B \in p$. Now $\mathcal{W}_k = \bigcup_{i=1}^{k} \bigcup_{j=1}^{k} \{w \in \mathcal{W}_k :$ the leftmost letter of $W$ is $i$ and the rightmost letter of $w$ is $j\}$, and consequently one of these sets must be in $p$. We may therefore assume that for all $u, w \in B$, the leftmost letters of $u$ and $w$ are the same and the rightmost letters of $u$ and $w$ are the same. In particular, if $w$ is a variable word with $\{w(1), w(2), \ldots, w(k)\} \subseteq B$, then $w$ is neither a left variable word nor a right variable word.

Let $q_1, q_2, \ldots, q_k$ and $r_1, r_2, \ldots, r_k$ be as in Lemma 2.2. Let $M = \{x \in \mathcal{W}_k : x^{-1}B \in p\}$. Since $p = pp = r_1 p = r_2 p = \ldots = r_k p$, we have that $M \in p \cap \bigcap_{j=1}^k r_j$. Let $M^\dagger = \{x \in M : x^{-1}M \in p \cap \bigcap_{j=1}^k r_j\}$.

We shall choose the sequence $\langle w_n \rangle_{n=1}^\infty$ inductively. Given $n > 1$, if we have chosen $\langle w_m \rangle_{m=1}^n$, we shall let

$$L_n = \big\{\textstyle\prod_{m\in F} w_m\big(f(m)\big) : \emptyset \neq F \subseteq \{1, 2, \ldots, n\} \text{ and } f : F \to \{1, 2, \ldots, k\}\big\} \cap M^\dagger .$$

We shall also be choosing for each $n \equiv 3 \pmod 4$, a set $D_n \in p \cap \bigcap_{j=1}^k q_j$. We shall let ${D_n}^\ddagger = \{x \in D_n : x^{-1}D_n \in p \cap \bigcap_{j=1}^k q_j\}$. Then, given $l$ and $n$, if $4l - 1 \leq n$ and we have chosen $\langle w_m \rangle_{m=1}^n$, we shall let

$$\begin{aligned} G_{l,n} \quad = \quad & \big\{\textstyle\prod_{m\in F} w_m\big(f(m)\big) : \emptyset \neq F \subseteq \{4l-1, 4l, \ldots, n\} \text{ and } f : F \to \{1, 2, \ldots, k\}\big\} \\ & \cap {D_{4l-1}}^\ddagger . \end{aligned}$$

Let $N_1 = M^\dagger$. Then $\overline{N_1}^{\,k}$ is a neighborhood of $\vec{r} = (r_1, r_2, \ldots, r_k)$ and $\vec{r} \in c\ell_Y(H)$ so pick a right variable word $w_1$ such that $\big(w_1(1), w_1(2), \ldots, w_1(k)\big) \in {N_1}^k$.

Let $B_2 = B \cap M^\dagger \cap \bigcap_{x\in L_1} (x^{-1}M^\dagger \cap x^{-1}B)$. Since $L_1 \subseteq M^\dagger$ we have that $B_2 \in p$. Let $C_2 = \{x \in B_2 : x^{-1}B_2 \in p \cap \bigcap_{j=1}^k q_j\}$. Since $p = pp = pq_1 = pq_2 = \ldots = pq_k$, we have that $C_2 \in p$ and so $\overline{C_2}^{\,k}$ is a neighborhood of $\overline{p}$. Since $\overline{p} \in c\ell_Y(I)$, pick a variable word $w_2$ such that $\big(w_2(1), w_2(2), \ldots, w_2(k)\big) \in {C_2}^k$. Since $C_2 \subseteq B$, $w_2$ is neither a left variable word nor a right variable word.

Let $D_3 = \bigcap_{t=1}^k w_2(t)^{-1}B_2$. Then $D_3 \in \bigcap_{j=1}^k q_j$. Let $E_3 = {D_3}^\ddagger$. Then $\overline{E_3}^{\,k}$ is a neighborhood of $\vec{q} = (q_1, q_2, \ldots, q_k)$ and $\vec{q} \in c\ell_Y(J)$ so pick a left variable word $w_3$ such that $\big(w_3(1), w_3(2), \ldots, w_3(k)\big) \in {E_3}^k$.

Let $n \geq 3$ and assume that we have chosen variable words $w_1, w_2, \ldots, w_n$. Assume also that for $l \in \{1, 2, \ldots, n\}$, if $l$ is even we have chosen $B_l$ and $C_l$ in $p$ with $B_l \subseteq B$ and if $l \equiv 3 \pmod 4$ we have chosen $D_l \in p \cap \bigcap_{j=1}^k q_j$. Assume that the following induction hypotheses are satisfied. (We also assume that the sets we construct are constructed in the same way at each stage.)

(a) If $l \in \{1, 2, \ldots, n\}$ and $l \equiv 1 \pmod 4$, then $w_l$ is a right variable word.

(b) If $l \in \{1, 2, \ldots, n\}$ and $l \equiv 3 \pmod 4$, then $w_l$ is a left variable word.

(c) If $l \in \{1, 2, \ldots, n\}$ is even then $w_l$ is a variable word which is neither a left variable word nor a right variable word and if $t \in \{1, 2, \ldots, k\}$, then $w_l(t) \in C_l$ and $C_l = \{x \in B_l : x^{-1}B_l \in p \cap \bigcap_{j=1}^k q_j\}$.

(d) If $\emptyset \neq F \subseteq \{1,2,\ldots,n\}$, $f : F \to \{1,2,\ldots,k\}$, $l \in \mathbb{N}$ with $4l-1 \leq \min F < 4l+3$, and for all $m \in F$, $m \not\equiv 1 \pmod 4$, then $\prod_{m\in F} w_m\big(f(m)\big) \in {D_{4l-1}}^{\ddagger}$.

(e) If $\emptyset \neq F \subseteq \{1,2,\ldots,n\}$, $f : F \to \{1,2,\ldots,k\}$, $a = \min F$ is even, and for all $m \in F$, $m \not\equiv 1 \pmod 4$, then $\prod_{m\in F} w_m\big(f(m)\big) \in B_a$.

(f) If $\emptyset \neq F \subseteq \{1,2,\ldots,n\}$, $f : F \to \{1,2,\ldots,k\}$, $\min F \not\equiv 3 \pmod 4$, and $F$ strongly separates $\langle w_m\rangle_{m\in F}$, then $\prod_{m\in F} w_m\big(f(m)\big) \in M^{\dagger}$. If also $\max F \not\equiv 1 \pmod 4$, then $\prod_{m\in F} w_m\big(f(m)\big) \in B$.

Notice that $w_1(1) \in L_n$ and for each $l \in \mathbb{N}$ with $4l-1 \leq n$, $w_{4l-1}(1) \in G_{l,n}$, so these sets are nonempty.

Assume first that $n$ is odd, and pick $s$ such that $4s-1 \leq n < 4s+3$. Let

$$\begin{aligned} B_{n+1} = \quad & B_{n-1} \cap \textstyle\bigcap_{t=1}^{k} w_{n-1}(t)^{-1}B_{n-1} \cap \bigcap_{x\in L_n}(x^{-1}M^{\dagger} \cap x^{-1}B) \cap \\ & \textstyle\bigcap_{l=1}^{s}\big({D_{4l-1}}^{\ddagger} \cap \bigcap_{x\in G_{l,n}} x^{-1}{D_{4l-1}}^{\ddagger}\big)\,. \end{aligned}$$

By hypothesis (c) we have that $w_{n-1}(t)^{-1}B_{n-1} \in p$ for each $t \in \{1,2,\ldots,k\}$. Since $L_n \subseteq M^{\dagger}$ and each $G_{l,n} \subseteq {D_{4l-1}}^{\ddagger}$, we have by Lemma 2.3 that $B_{n+1} \in p$. Let $C_{n+1} = \{x \in B_{n+1} : x^{-1}B_{n+1} \in p \cap \bigcap_{j=1}^{k} q_j\}$. Since $p = pp = pq_1 = pq_2 = \ldots = pq_k$, $C_{n+1} \in p$. Pick a variable word $w_{n+1}$ such that $\big(w_{n+1}(1), w_{n+1}(2), \ldots, w_{n+1}(k)\big) \in {C_{n+1}}^{k}$. Since $C_{n+1} \subseteq B$ we have that $w_{n+1}$ is neither a left variable word nor a right variable word.

Now assume that $n = 4s$ for some $s \in \mathbb{N}$. Let $N_{n+1} = M^{\dagger} \cap \bigcap_{x\in L_n} x^{-1}M^{\dagger}$. Then $\overline{N_{n+1}}^{\,k}$ is a neighborhood of $\vec{r}$ so pick a right variable word $w_{n+1}$ such that $\big(w_{n+1}(1), w_{n+1}(2), \ldots, w_{n+1}(k)\big) \in {N_{n+1}}^{k}$.

Finally assume that $n = 4s+2$ for some $s \in \mathbb{N}$. Let

$$D_{n+1} = D_{n-3} \cap \textstyle\bigcap_{t=1}^{k}\big(w_n(t)^{-1}B_n \cap w_{n-2}(t)^{-1}B_{n-2}\big)\,.$$

By hypothesis (c) we have that $D_{n+1} \in p \cap \bigcap_{j=1}^{k} q_j$. Let

$$E_{n+1} = {D_{n+1}}^{\ddagger} \cap \textstyle\bigcap_{l=1}^{s} \bigcap_{x\in G_{l,n}} x^{-1}{D_{4l-1}}^{\ddagger}\,.$$

Since each $G_{l,n} \subseteq {D_{4l-1}}^{\ddagger}$, we have by Lemma 2.3 that $E_{n+1} \in p \cap \bigcap_{j=1}^{k} q_j$. Then $\overline{E_{n+1}}^{\,k}$ is a neighborhood of $\vec{q}$ so pick a left variable word $w_{n+1}$ such that $\big(w_{n+1}(1), w_{n+1}(2), \ldots, w_{n+1}(k)\big) \in {E_{n+1}}^{k}$.

Hypotheses (a), (b), and (c) are satisfied directly. To verify hypothesis (d), assume that $\emptyset \neq F \subseteq \{1,2,\ldots,n+1\}$, $f : F \to \{1,2,\ldots,k\}$, $4l-1 \leq \min F < 4l+3$, and for all $m \in F$, $m \not\equiv 1 \pmod 4$. If $n+1 \notin F$, the conclusion holds by assumption, so we assume that $n+1 \in F$. Assume that $F = \{n+1\}$. If $n+1$ is even, we have $w_{n+1}\big(f(n+1)\big) \in C_{n+1} \subseteq B_{n+1} \subseteq {D_{4l-1}}^{\ddagger}$. If $n+1$ is odd, in which case $n+1 = 4l-1$, we have $w_{n+1}\big(f(n+1)\big) \in E_{n+1} \subseteq {D_{4l-1}}^{\ddagger}$.

To conclude the verification of hypothesis (d), assume that $|F| \geq 2$. Let $K = F\backslash\{n{+}1\}$, and let $x = \prod_{m\in K} w_m\big(f(m)\big)$. By assumption, $x \in D_{4l-1}{}^{\ddagger}$ and so $x \in G_{l,n}$. If $n+1$ is even, we have $w_{n+1}\big(f(n+1)\big) \in C_{n+1} \subseteq x^{-1}D_{4l-1}{}^{\ddagger}$. If $n+1$ is odd, we have $w_{n+1}\big(f(n+1)\big) \in E_{n+1} \subseteq x^{-1}D_{4l-1}{}^{\ddagger}$. Thus, in either case, $\prod_{m\in F} w_m\big(f(m)\big) \in D_{4l-1}{}^{\ddagger}$.

To verify hypothesis (e), assume that $\emptyset \neq F \subseteq \{1,2,\ldots,n\}$, $f : F \to \{1,2,\ldots,k\}$, $a = \min F$ is even, and for all $m \in F$, $m \not\equiv 1 \pmod 4$. We may assume that $n+1 \in F$. If $F = \{n+1\}$, we have $w_{n+1}\big(f(n+1)\big) \in C_{n+1} \subseteq B_{n+1}$, so assume that $|F| \geq 2$ and let $T = F\backslash\{a\}$. Pick $s$ and $l$ in $\mathbb{N}$ such that $4s-1 \leq \min T < 4s+3$ and either $a = 4l$ or $a = 4l+2$. If $l < s$ we have $\prod_{m\in T} w_m\big(f(m)\big) \in D_{4s-1}{}^{\ddagger} \subseteq D_{4l+3} \subseteq w_a\big(f(a)\big)^{-1}B_a$. So assume that $l = s$. Then since $4l+1 \notin F$ we have $a = 4l$ and $\min T = 4l+2$. Thus $\prod_{m\in T} w_m\big(f(m)\big) \in B_{4l+2} \subseteq w_{4l}\big(f(4l)\big)^{-1}B_{4l}$.

To verify hypothesis (f), assume that $\emptyset \neq F \subseteq \{1,2,\ldots,n\}$, $f : F \to \{1,2,\ldots,k\}$, $\min F \not\equiv 3 \pmod 4$, and $F$ strongly separates $\langle w_m\rangle_{m\in F}$. Again, we may assume that $n+1 \in F$. Assume first that $F = \{n+1\}$. If $n+1$ is even, we have that $w_{n+1}\big(f(n+1)\big) \in C_{n+1} \subseteq B_{n+1} \subseteq B_2 \subseteq M^{\dagger} \cap B$. If $n+1$ is odd, in which case $n+1 \equiv 1 \pmod 4$, we have $w_{n+1} \in N_{n+1} \subseteq M^{\dagger}$.

Now assume that $|F| \geq 2$. Assume first that $n{+}1 \not\equiv 3 \pmod 4$, let $K = F\backslash\{n{+}1\}$, and let $x = \prod_{m\in K} w_m\big(f(m)\big)$. Then $K$ strongly separates $\langle w_m\rangle_{m\in K}$ and $\min K \not\equiv 3 \pmod 4$, so by assumption $x \in M^{\dagger}$ and so $x \in L_n$. If $n+1 \equiv 1 \pmod 4$, then $w_{n+1}\big(f(n{+}1)\big) \in N_{n+1} \subseteq x^{-1}M^{\dagger}$ and so $\prod_{m\in F} w_m\big(f(m)\big) \in M^{\dagger}$. If $n+1$ is even, then $w_{n+1}\big(f(n+1)\big) \in C_{n+1} \subseteq x^{-1}M^{\dagger} \cap x^{-1}B$ and so $\prod_{m\in F} w_m\big(f(m)\big) \in M^{\dagger} \cap B$.

Finally assume that $n+1 \equiv 3 \pmod 4$. Since $F$ strongly separates $\langle w_m\rangle_{m\in F}$ and $\min F \not\equiv 3 \pmod 4$, some member of $F$ is even. Let $b = \max\{t \in F : t \text{ is even}\}$. Let $P = \{t \in F : t \geq b\}$ and notice that for all $m \in P$, $m \not\equiv 1 \pmod 4$. Then by hypothesis (e), we have $\prod_{m\in P} w_m\big(f(m)\big) \in B_b$. If $P = F$, then since $B_b \subseteq B_2 \subseteq M^{\dagger} \cap B$, we are done. Assume then that $P \neq F$ and let $K = F\backslash P$. Let $x = \prod_{m\in K} w_m\big(f(m)\big)$. Then $K$ strongly separates $\langle w_m\rangle_{m\in K}$ and $\min K \not\equiv 3 \pmod 4$ so by assumption $x \in M^{\dagger}$ and thus $x \in L_{b-1}$. Therefore

$$\prod_{m\in P} w_m\big(f(m)\big) \in B_b \subseteq x^{-1}M^{\dagger} \cap x^{-1}B$$

and so $\prod_{m\in F} w_m\big(f(m)\big) \in M^{\dagger} \cap B$. $\square$

We see that as an immediate consequence of Theorem 2.7, we do not need to demand that there be a separator between a left variable word and a following right variable word. (The reason for this lies in the fact that, if $\overline{p}$, $\vec{q}$, and $\vec{r}$ are as in Lemma 2.2, then $\vec{q}\,\vec{r} = \vec{q}\,\overline{p}\,\vec{r}$.)

**2.8 Corollary**. *Let $k \in \mathbb{N}$ and let $B$ be any central subset of $\mathcal{W}_k$. There exists a sequence*

$\langle u_n\rangle_{n=1}^\infty$ *of variable words over* $\mathcal{W}_k$ *such that*

*(1) for each* $n\in\omega$, $u_{3n+1}$ *is a right variable word;*

*(2) for each* $n\in\omega$, $u_{3n+3}$ *is a left variable word;*

*(3) for each* $n\in\omega$, $u_{3n+2}$ *is neither a left nor right variable word; and*

*(4) for all* $F\in\mathcal{P}_f(\mathbb{N})$, *if* $F$ *weakly separates* $\langle u_n\rangle_{n\in F}$, $\min F\not\equiv 3 \pmod 3$, $\max F\not\equiv 1 \pmod 3$, *and* $f:F\to\{1,2,\ldots,k\}$, *then* $\prod_{m\in F} u_m\big(f(m)\big)\in B$.

**Proof**. Let $\langle w_n\rangle_{n=1}^\infty$ be as guaranteed by Theorem 2.7. For $n\in\omega$, let $u_{3n+1}=w_{4n+1}$, $u_{3n+2}=w_{4n+2}$, and $u_{3n+3}=w_{4n+3}w_{4n+4}$. Then conclusions (1), (2), and (3) hold immediately.

Let $F\in\mathcal{P}_f(\mathbb{N})$ be given such that $F$ weakly separates $\langle u_n\rangle_{n\in F}$, $\min F\not\equiv 3 \pmod 3$, and $\max F\not\equiv 1 \pmod 3$, and let $f:F\to\{1,2,\ldots,k\}$. Let

$$\begin{array}{rcl} G &=& \{4n+1 : n\in\omega \text{ and } 3n+1\in F\}\cup \\ && \{4n+2 : n\in\omega \text{ and } 3n+2\in F\}\cup \\ && \{4n+3 : n\in\omega \text{ and } 3n+3\in F\}\cup \\ && \{4n+4 : n\in\omega \text{ and } 3n+3\in F\}\,. \end{array}$$

Then $G$ strongly separates $\langle w_n\rangle_{n\in G}$, $\min G\not\equiv 3 \pmod 4$, and $\max G\not\equiv 1 \pmod 4$. Define $g:G\to\{1,2,\ldots,k\}$ by $g(4n+1)=f(3n+1)$, $g(4n+2)=f(3n+2)$, and $g(4n+3)=g(4n+4)=f(4n+3)$. Then $\prod_{m\in F} u_m\big(f(m)\big)=\prod_{m\in G} w_m\big(g(m)\big)\in B$. $\square$

We conclude this section by observing that it is necessary to separate right variable words from following left variable words.

**2.10 Theorem**. *Assume that* $k\geq 2$. *Then there is a two cell partition of* $\mathcal{W}_k$ *such that there do not exist a right variable word* $w_1$ *and a left variable word* $w_2$ *such that both*

*(a)* $w_1(1)w_2(1)$ *and* $w_1(1)w_2(2)$ *lie in the same cell of the partition and*

*(b)* $w_1(2)w_2(1)$ *and* $w_1(2)w_2(2)$ *lie in the same cell of the partition.*

**Proof**. For $w\in\mathcal{W}_k$, let $\varphi(w)$ count the number of occurrences in $w$ of a 1 followed immediately by a 2. For $i\in\{1,2\}$, let $A_i=\{w\in\mathcal{W}_k : \varphi(w)\equiv i \pmod 2\}$. Suppose that we have $i,j\in\{1,2\}$ such that $w_1(1)w_2(1), w_1(1)w_2(2)\in A_i$ and $w_1(2)w_2(1), w_1(2)w_2(2)\in A_j$.

Then $\varphi\big(w_1(1)\big)+\varphi\big(w_2(1)\big)=\varphi\big(w_1(1)w_2(1)\big)\equiv i\equiv\varphi\big(w_1(1)w_2(2)\big)=\varphi\big(w_1(1)\big)+1+\varphi\big(w_2(2)\big)$, so that $\varphi\big(w_2(1)\big)\not\equiv\varphi\big(w_2(2)\big) \pmod 2$.

But also $\varphi\big(w_1(2)\big)+\varphi\big(w_2(1)\big)=\varphi\big(w_1(2)w_2(1)\big)\equiv j\equiv\varphi\big(w_1(2)w_2(2)\big)=\varphi\big(w_1(2)\big)+\varphi\big(w_2(2)\big)$, so that $\varphi\big(w_2(1)\big)\equiv\varphi\big(w_2(2)\big) \pmod 2$, a contradiction. $\square$

## 3. Extensions to Infinite Alphabets

In this section we shall deal with possible extensions of our earlier results to $\mathcal{W}_\infty$, the free semigroup over a countably infinite alphabet, which we take to be $\mathbb{N}$. The kind of extension we are concerned with is illustrated by the following (already known) generalization of Theorem 1.5. (The fact that some cell $B$ of any finite partition of $\mathcal{W}_\infty$ satisfies the conclusion of Theorem 3.1 is a consequence of [2, Theorem 15].)

**3.1 Theorem**. *Let $B$ be a central subset of $\mathcal{W}_\infty$. There exists a sequence $\langle u_m\rangle_{m=1}^\infty$ of variable words over $\mathcal{W}_\infty$ such that for every $F \in \mathcal{P}_f(\mathbb{N})$ and every $f : F \to \mathbb{N}$ for which $f(n) \le n$ for all $n \in F$, $\prod_{m\in F} u_m\big(f(m)\big) \in B$.*

**Proof**. This is an immediate consequence of the non-commutative Central Sets Theorem [7, Theorem 14.15]. To apply that theorem, let for each $l, n \in \mathbb{N}$, $y_{l,n} = l$. Let the sequences $\langle m(n)\rangle_{n=1}^\infty$, $\langle \vec{a_n}\rangle_{n=1}^\infty$, and $\langle \vec{H}_n\rangle_{n=1}^\infty$ be as guaranteed by that theorem. For each $n \in \mathbb{N}$, let

$$u_n = \Big(\prod_{i=1}^{m(n)} a_{n,i} \cdot v^{|H_{n,i}|}\Big) \cdot a_{n,m(n)+1}\,. \qquad \square$$

The reader wishing to verify Theorem 3.1 without going through the proof of the non-commutative Central Sets Theorem should note that Theorem 3.1 is a corollary to Theorem 3.5 below. We see now that no similar extension of Theorem 2.5 is possible.

**3.2 Theorem**. *There exists a two cell partition $\{A_1, A_2\}$ of $\mathcal{W}_\infty$ such that there do not exist $i \in \{1, 2\}$ and a sequence $\langle w_n\rangle_{n=1}^\infty$ of variable words such that*

*(1) for each $n > 1$, $w_n$ is a left variable word, and*

*(2) for each $n > 1$ and each $l \in \{1, 2, \ldots, n\}$, $w_1(1)w_n(l) \in A_i$.*

**Proof**. For $w \in \mathcal{W}_\infty$, let $\varphi(w)$ count the number of occurrences of $l$ as the $l^{\text{th}}$ letter. Thus, if $w = a_1a_2\cdots a_m$ where each $a_i$ is a letter, $\varphi(w) = |\{l \in \{1, 2, \ldots, m\} : a_l = l\}|$. For $i \in \{1, 2\}$, let $A_i = \{w \in \mathcal{W}_\infty : \varphi(w) \equiv i \pmod 2\}$.

Suppose we have $i \in \{1, 2\}$ and a sequence $\langle w_n\rangle_{n=1}^\infty$ of variable words such that

(1) for each $n > 1$, $w_n$ is a left variable word, and

(2) for each $n > 1$ and each $l \in \{1, 2, \ldots, n\}$, $w_1(1)w_n(l) \in A_i$.

Let $l$ be the length of $w_1$ and let $n = l+1$. Then $\varphi\big(w_1(1)w_n(l+1)\big) = \varphi\big(w_1(1)w_n(l)\big) + 1$, which is a contradiction. $\square$

Of course Theorem 3.2 also provides a counterexample to the corresponding assertion involving central sets, since necessarily one of $A_1$ and $A_2$ must be central. In fact both $A_1$ and $A_2$ are central. To see this, notice that $13\mathcal{W}_\infty$ is a right ideal of $\mathcal{W}_\infty$ and thus $\overline{13\mathcal{W}_\infty}$ is a right ideal of $\beta\mathcal{W}_\infty$ by [7, Corollary 4.18]. Consequently there is a minimal idempotent $p$ in $\overline{13\mathcal{W}_\infty}$. Now consider the isomorphism $\tau$ from $\mathcal{W}_\infty$ to itself which interchanges the letters 1 and 2 and leaves all other letters fixed. Then the continuous extension $\widetilde{\tau} : \beta\mathcal{W}_\infty \to \mathcal{W}_\infty$ is an isomorphism by [7, Corollary 4.22] and thus $\widetilde{\tau}(p)$ is also a minimal idempotent. Given any $w \in 13\mathcal{W}_\infty$, $\varphi(w) = \varphi\big(\tau(w)\big) + 1$. Consequently $A_1 \in p$ if and only if $A_2 \in \widetilde{\tau}(p)$. (We are grateful to Dona Strauss for providing us with this simple argument.)

An attempt to modify the proof of Theorem 2.5 to apply to $\mathcal{W}_\infty$ must, of course, fail. The reason is that the set $A^\ddagger$ produced there would now refer to membership in infinitely many ultrafilters. A similar objection about $M^\dagger$ prevents the direct extension of the proof of Theorem 2.4. However, we are able to change the proof somewhat. We shall need a modification of Lemma 2.2.

**3.3 Definition**.
(a) $Y = \times_{j=1}^{\infty} \beta\mathcal{W}_\infty$.
(b) $I = \big\{(w(1), w(2), w(3), \ldots) : w$ is a variable word over $\mathcal{W}_\infty\big\}$.
(c) $H = \big\{(w(1), w(2), w(3), \ldots) : w$ is a right variable word over $\mathcal{W}_\infty\big\}$.
(d) $E = I \cup \big\{(w, w, w, \ldots) : w \in \mathcal{W}_\infty\big\}$.

The entire analogue of Lemma 2.2 remains valid, but we shall only need a small part.

**3.4 Lemma**. *Let $p$ be any minimal idempotent in $\beta\mathcal{W}_\infty$. Then $\overline{p} = (p, p, p, \ldots) \in \overline{I}$. Also there exist idempotents $r_1, r_2, r_3, \ldots$ in $\beta\mathcal{W}_\infty$ such that*
*(1) $\vec{r} = (r_1, r_2, r_3, \ldots) \in \overline{H}$ and*
*(2) for all $j \in \mathbb{N}$, $r_j p = p$.*

**Proof**. By [7, Theorems 2.22 and 4.17] we have that $\overline{E}$ is a subsemigroup of $Y$, $\overline{I}$ is an ideal of $\overline{E}$, and $\overline{H}$ is a left ideal of $\overline{E}$.

By [7, Theorem 2.23], $K(Y) = \times_{t\in\mathbb{N}} K(\beta\mathcal{W}_\infty)$. We claim that

$$(*) \qquad \text{if } s \in K(\beta\mathcal{W}_\infty), \text{ then } (s, s, s, \ldots) \in \overline{E}\,.$$

To see this, let $U$ be a neighborhood of $(s, s, s, \ldots)$, pick $k \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, k\}$, pick $A_i \in s$ such that $\bigcap_{i=1}^{k} {\pi_i}^{-1}[\,\overline{A_i}\,] \subseteq U$. (Here $\pi_i$ is the projection from $Y$ onto its $i^{\text{th}}$ coordinate.) Pick $w \in \bigcap_{i=1}^{k} A_i$. Then $(w, w, w, \ldots) \in U \cap E$.

Thus we have that $\overline{E} \cap K(Y) \neq \emptyset$ so by [7, Theorem 1.65], $K(Y) \cap \overline{E} = K(\,\overline{E}\,)$. Since $\overline{I}$ is an ideal of $\overline{E}$, we have that $\overline{p} \in K(Y) \cap \overline{E} = K(\,\overline{E}\,) \subseteq \overline{I}$.

Since $\overline{p}$ is a minimal idempotent in $\overline{E}$, pick (by [7, Theorem 2.8]) a minimal right ideal $R$ of $\overline{E}$ such that $\overline{p} \in R$. Pick, by [7, Corollary 2.6], a minimal left ideal $L$ of $\overline{E}$ such that

$L \subseteq \overline{H}$. Then $L \cap R$ is a group. Let $\vec{r} = (r_1, r_2, r_3, \ldots)$ be the identity of $L \cap R$. By [7, Lemma 1.30], we have that $\vec{r}$ and $\overline{p}$ are left identities for $R$. $\square$

**3.5 Theorem**. *Let $B$ be any central subset of $\mathcal{W}_\infty$. There exists a sequence $\langle w_n \rangle_{n=1}^\infty$ of variable words over $\mathcal{W}_\infty$ such that*

*(1) for each $n \in \omega$, $w_{2n+1}$ is a right variable word and*

*(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $f : F \to \mathbb{N}$, if $f(n) \leq n$ for each $n \in F$ and* $\max F$ *is even, then $\prod_{m \in F} w_m\big(f(m)\big) \in B$.*

**Proof**. Pick an idempotent $p \in K(\beta\mathcal{W}_\infty)$ such that $B \in p$ and let $r_1, r_2, r_3, \ldots$ be as in Lemma 3.4. Let $M = \{x \in \mathcal{W}_\infty : x^{-1}B \in p\}$. Since $p = pp = r_1 p = r_2 p = r_3 p = \ldots$, we have that $M \in p \cap \bigcap_{j=1}^\infty r_j$. In particular, ${\pi_1}^{-1}[\,\overline{M}\,]$ is a neighborhood of $\vec{r}$ and so ${\pi_1}^{-1}[\,\overline{M}\,] \cap H \neq \emptyset$. Pick a right variable word $w_1$ such that $w_1(1) \in M$.

Now let $n \in \mathbb{N}$ and assume that we have chosen variable words $w_1, w_2, \ldots, w_n$ such that

(1) if $l \in \{1, 2, \ldots, n\}$ is odd, then $w_l$ is a right variable word;

(2) if $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$ and $f : F \to \mathbb{N}$ so that $f(l) \leq l$ for each $l \in F$, then $\prod_{m \in F} w_m\big(f(m)\big) \in M$; and

(3) if $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$, $f : F \to \mathbb{N}$ so that $f(l) \leq l$ for each $l \in F$, and $\max F$ is even, then $\prod_{m \in F} w_m\big(f(m)\big) \in B$.

Let $L = \{\prod_{m \in F} w_m\big(f(m)\big) : \emptyset \neq F \subseteq \{1, 2, \ldots, n\}$ and $f : F \to \mathbb{N}$ so that $f(l) \leq l$ for each $l \in F\}$. Given $x \in L$, we have that $x \in M$ and so $x^{-1}B \in p = pp = r_1 p = r_2 p = r_3 p = \ldots$, and hence

$$x^{-1}M = \{y \in \mathcal{W}_\infty : xy \in M\} = \{y \in \mathcal{W}_\infty : y^{-1}(x^{-1}B) \in p\} \in p \cap \textstyle\bigcap_{j=1}^\infty r_j\,.$$

Let $D = M \cap \bigcap_{x \in L} x^{-1}M$.

Assume first that $n$ is even. Then we have $\bigcap_{i=1}^{n+1} {\pi_i}^{-1}[\,\overline{D}\,]$ is a neighborhood of $\vec{r}$, so pick a right variable word $w_{n+1}$ such that $\{w_{n+1}(1), w_{n+1}(2), \ldots, w_{n+1}(n+1)\} \subseteq D$. Then hypotheses (1) and (2) are satisfied.

Now assume that $n$ is odd. Let $E = D \cap B \cap \bigcap_{x \in L} x^{-1}B$. Since $L \subseteq M$, we have that $E \in p$. Thus we have $\bigcap_{i=1}^{n+1} {\pi_i}^{-1}[\,\overline{E}\,]$ is a neighborhood of $\overline{p}$, so pick a variable word $w_{n+1}$ such that $\{w_{n+1}(1), w_{n+1}(2), \ldots, w_{n+1}(n+1)\} \subseteq E$. Then all hypotheses are satisfied. $\square$

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