This article was published in *Proc. Amer. Math. Soc.* **130** (2002), 2559-2567. To the best of my knowledge this is the final version as submitted to the publishers. – NH

One Sided Ideals and Carlson's Theorem

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May 14, 2002

Abstract

Using left ideals, right ideals, and the smallest two sided ideal in a compact right topological semigroup, we derive an extension of the Main Lemma to Carlson's Theorem. This extension involves an infinite sequence of variable words over a finite alphabet, some of which are required to have the variable as the first letter and others of which are required to have the variable as the last letter.

1 Introduction

In 1988 T. Carlson published a Ramsey Theoretic result [2, Theorem 2] which has as corollaries many earlier results in Ramsey Theory. (See [8, Section 18.4] for a relatively short presentation of Carlson's Theorem and some of its consequences.) Experience suggests that Carlson's "Main Lemma" [2, Lemma 5.9] implies those Ramsey Theoretic corollaries of his theorem in

*This author acknowledges support received from the National Science Foundation (USA) via grant DMS-0070593.

†This author acknowledges support received from the National Science Foundation via a post doctoral fellowship administered by the University of Maryland.

Mathematics Subject Classifications: Primary 05D10; Secondary 22A15, 22A30, 54D30 which a finite collection of finite objects is partitioned into finitely many classes. We shall state this Main Lemma after introducing some necessary terminology.

For $k \in \mathbb{N}$, the set of positive integers, let $\mathcal{W}_k$ be the free semigroup with identity e on the alphabet $\{1, 2, \ldots, k\}$. That is, $\mathcal{W}_k$ consists of all "words with letters from $\{1, 2, \ldots, k\}$" (i.e. functions whose domain is an initial segment of $\mathbb{N}$ and whose range is contained in $\{1, 2, \ldots, k\}$) together with the empty word, with the operation of concatenation. A *variable word* over $\mathcal{W}_k$ is a word on the alphabet $\{1, 2, \ldots, k\} \cup \{v\}$ in which v occurs, where v is a "variable" not in $\{1, 2, \ldots, k\}$. Given a variable word w over $\mathcal{W}_k$, and $t \in \{1, 2, \ldots, k\}$, $w(t)$ has its obvious meaning, namely the result of replacing all occurrences of v with t. (There is a potential conflict here with the formal viewpoint which takes w to be a function. If we have occasion to need the value of the function w at t we will denote it as w_t.)

Definition 1.1 Let $k \in \mathbb{N}$ and let $\langle w_n \rangle_{n=1}^{\infty}$ be a sequence of variable words over $\mathcal{W}_k$. The sequence $\langle t_n \rangle_{n=1}^{\infty}$ is a *variable reduction* of $\langle w_n \rangle_{n=1}^{\infty}$ if and only if there exist an increasing function $g : \mathbb{N} \to \mathbb{N}$ and a function $f : \mathbb{N} \to \{1, 2, \ldots, k\} \cup \{v\}$ such that

(1) $g(1) = 1$,
(2) for each $n \in \mathbb{N}$, $v \in f[\{g(n), g(n) + 1, \ldots, g(n + 1) - 1\}]$, and
(3) for each $n \in \mathbb{N}$, $t_n = \prod_{i=g(n)}^{g(n+1)-1} w_i(f(i))$, where the product is in increasing order of indices.

Theorem 1.2 (Carlson) *Let $k \in \mathbb{N}$, let the set of variable words over $\mathcal{W}_k$ be partitioned into finitely many classes, and let $\langle w_n \rangle_{n=1}^{\infty}$ be a sequence of variable words. Then there exists a variable reduction $\langle t_n \rangle_{n=1}^{\infty}$ of $\langle w_n \rangle_{n=1}^{\infty}$ such that all expressions of the form $\prod_{i=1}^{n} t_i(f(i))$, where $n \in \mathbb{N}$, $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\} \cup \{v\}$, and $v \in \mathrm{range}(f)$, lie in the same cell of the partition.*

Proof. [2, Lemma 5.9]. □

Somewhat earlier, T. Carlson and S. Simpson had established a similar result which partitioned $\mathcal{W}_k$ rather than the variable words over $\mathcal{W}_k$ and required that most of the variable words used must have v as the leftmost letter. Such words are *left variable* words. Similarly, a *right variable* word must have the variable v as its rightmost letter.

Theorem 1.3 (Carlson-Simpson) *Let $k \in \mathbb{N}$ and let $\mathcal{W}_k$ be partitioned into finitely many classes. Then there exists a sequence $\langle w_n \rangle_{n=1}^{\infty}$ of variable words over $\mathcal{W}_k$ such that for every $n > 1$, w_n is a left variable word, and all expressions of the form $\prod_{i=1}^{n} w_i(f(i))$, where $n \in \mathbb{N}$ and $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$, lie in the same cell of the partition.*

Proof. [3, Theorem 6.3]. □

In [1], V. Bergelson, A. Blass, and the first author of the current paper established a generalization of Theorem 1.2 by utilizing the algebraic structure of the Stone-Čech compactification of a discrete semigroup. They could not, however, extend by these methods Theorem 1.3. The reasons involve the algebraic constructs used in the proof, which we pause now to introduce.

Given a discrete semigroup $(S, \cdot)$, we take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

There is a natural extension of the operation $\cdot$ of S to βS making βS a compact right topological semigroup with S contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$. The operation is characterized by the fact that for any p and q in βS and any $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. See [8] for an elementary introduction to the semigroup βS.

Any compact Hausdorff right topological semigroup $(T, \cdot)$ has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T, each of which is closed, and is also the union of all of the minimal right ideals of T [8, Theorem 2.8]. Given any minimal left ideal L and any minimal right ideal R, $L \cap R$ is a group (and in particular has an idempotent) [8, Theorem 2.7]. There is a partial ordering of the idempotents of T determined by $p \leq q$ if and only if $p = p \cdot q = q \cdot p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$ [8, Theorem 1.59]. Such an idempotent is called simply "minimal".

Members of minimal idempotents in βS are the *central* subsets of S. Central sets were introduced by H. Furstenberg in [4] and defined in terms of notions of topological dynamics. These sets enjoy very strong combinatorial properties. (See [4, Proposition 8.21] or [8, Chapter 14].) See [8, Theorem

19.27] for a proof of the equivalence of the definition of "central" in terms of βS with the original dynamical definition.

The basic algebraic fact used in [1] (as well as several other papers) is that any two sided ideal in a compact right topological semigroup T contains $K(T)$. While variable words over $\mathcal{W}_k$ yield a two sided ideal in an appropriately chosen compact right topological semigroup, left variable words and right variable words do not.

Left variable words do, however correspond naturally to a right ideal (and right variable words correspond to a left ideal). In [7] we were able to use these natural left and right ideals to obtain a generalization of Theorem 1.3 involving both left and right variable words. In the current paper we use similar left and right ideals to obtain a generalization of Theorem 1.2 which involves both left and right variable words.

2 Extending Carlson's Main Lemma

We shall have throughout this section a fixed $k \in \mathbb{N}$. We begin with the following simple lemma from semigroup theory.

Lemma 2.1 *Let S be a semigroup which has a minimal left ideal which contains an idempotent. Let L be a left ideal of S, let R be a right ideal of S, and let e be any idempotent in S. There is a minimal idempotent $m \in Le \cap eR$. Necessarily $m \leq e$.*

Proof. Notice that if $m \in Le$, then $me = m$ and if $m \in eR$, then $em = m$. Thus any idempotent $m \in Le \cap eR$ satisfies $m \leq e$.

By [8, Lemma 1.57] S has a minimal right ideal which contains an idempotent. Thus, by [8, Corollary 1.47] we may presume that L is a minimal left ideal and R is a minimal right ideal. By [8, Theorem 1.46] Le is a minimal left ideal and eR is a minimal right ideal and so, by [8, Theorem 1.61] $Le \cap eR$ is a group. Let m be the identity of $Le \cap eR$. □

Given a set X, we denote the set of finite nonempty subsets of X by $\mathcal{P}_f(X)$.

Lemma 2.2 *Let S be a discrete semigroup, let $A \subseteq \times_{j=1}^{k+1} S$, and let $Z = \times_{j=1}^{k+1} \beta S$. Assume that $(y_1, y_2, \ldots, y_{k+1}) \in c\ell_Z(A)$, $C \subseteq S$, $W \in \mathcal{P}_f(S)$, and*

$L \in \mathcal{P}_f(\beta S)$. *Then there exists* $(a_1, a_2, \ldots, a_{k+1}) \in A$ *such that, for all* $l \in L$, *all* $u \in W$, *and all* $i \in \{1, 2, \ldots, k+1\}$,

$$(ua_i)^{-1}C \in l \Leftrightarrow u^{-1}C \in y_i l \,.$$

Proof. For $l \in L$, $u \in W$, and $i \in \{1, 2, \ldots, k+1\}$, let

$$C_{i,u,l} = \begin{cases} C & \text{if } C \in uy_i l \\ S \backslash C & \text{if } C \notin uy_i l \,. \end{cases}$$

Then $\overline{C_{i,u,l}}$ is a neighborhood of $uy_i l = \lambda_u(\rho_l(y_i))$ so pick a member $U_{i,u,l}$ of y_i such that $\lambda_u[\rho_l[\overline{U_{i,u,l}}]] \subseteq \overline{C_{i,u,l}}$.

For $i \in \{1, 2, \ldots, k+1\}$, let $N_i = \bigcap_{l \in L} \bigcap_{u \in W} U_{i,u,l}$. Pick $(a_1, a_2, \ldots, a_{k+1}) \in A \cap \times_{i=1}^{k+1} N_i$. □

Definition 2.3 (a) $Y = (\times_{j=1}^{k} \beta \mathcal{W}_k) \times \beta \mathcal{W}_{k+1}$.

(b) $Z = \times_{j=1}^{k+1} \beta \mathcal{W}_{k+1}$.

(c) $I = \{(w(1), w(2), \ldots, w(k+1)) : w$ is a variable word over $\mathcal{W}_k\}$.

(d) $J = \{(w(1), w(2), \ldots, w(k+1)) : w$ is a left variable word over $\mathcal{W}_k\}$.

(e) $H = \{(w(1), w(2), \ldots, w(k+1)) : w$ is a right variable word over $\mathcal{W}_k\}$.

(f) $E = I \cup \{(w, w, \ldots, w) : w \in \mathcal{W}_k\}$.

Notice that each of I, J, H, and E are contained in $(\times_{j=1}^{k} \mathcal{W}_k) \times \mathcal{W}_{k+1}$, and consequently their closures in Y and in Z are identical.

Lemma 2.4 $\overline{E}$ *is a subsemigroup of* Y, $\overline{I}$ *is an ideal of* $\overline{E}$, $\overline{J}$ *is a right ideal of* $\overline{E}$, *and* $\overline{H}$ *is a left ideal of* $\overline{E}$.

Proof. [8, Theorems 2.22 and 4.17]. □

The proof of the following lemma uses an idea from [1]. This lemma is needed so that in Theorem 2.9 the cell of the partition $\mathcal{F}$ can be guaranteed to be central in $\mathcal{W}_{k+1}$. This fact is significant because, as we have noted, central sets are guaranteed to contain substantial combinatorial structures.

Lemma 2.5 $\overline{E} \cap K(Y) \neq \emptyset$ *and so*

$$K(\overline{E}\,) = \overline{E} \cap \Big((\times_{j=1}^{k} K(\beta \mathcal{W}_k)) \times K(\beta \mathcal{W}_{k+1}) \Big) = \overline{E} \cap K(Y) \,.$$

Proof. Let s be a minimal idempotent in $\beta\mathcal{W}_k$ and pick by [8, Theorem 1.60 and Corollary 2.6] an idempotent p in $\mathcal{W}_{k+1}$ such that $p \leq s$. Let $\vec{p} = (s, s, \ldots, s, p)$.

We claim that it suffices to show that $\vec{p} \in \overline{I}$. Indeed, assume that we have done so. We have that $\vec{p} \in (\times_{j=1}^{k} K(\beta\mathcal{W}_k)) \times K(\beta\mathcal{W}_{k+1})$ and by [8, Theorem 2.23] $K(Y) = (\times_{j=1}^{k} K(\beta\mathcal{W}_k)) \times K(\beta\mathcal{W}_{k+1})$. Thus we have that $\vec{p} \in \overline{I} \cap K(Y) \subseteq \overline{E} \cap K(Y)$. It then follows from [8, Theorem 1.65] that $K(\overline{E}) = \overline{E} \cap K(Y)$.

To see that $\vec{p} \in \overline{I}$, let $A \in s$ and $B \in p$ be given. We need to show that $(\overline{A}^{k} \times \overline{B}) \cap I \neq \emptyset$. For each $t \in \{1, 2, \ldots, k\}$ define $g_t : \mathcal{W}_{k+1} \to \mathcal{W}_k$ by letting $g_t(w)$ be the word obtained by replacing each occurrence of $k+1$ by an occurrence of t and let $\widetilde{g}_t : \beta\mathcal{W}_{k+1} \to \beta\mathcal{W}_k$ be its continuous extension. Let $t \in \{1, 2, \ldots, k\}$ be given. By [8, Corollary 4.22], $\widetilde{g}_t$ is a homomorphism and so $\widetilde{g}_t(p) \leq \widetilde{g}_t(s)$. Since g_t is the identity on $\mathcal{W}_k$, we have that $\widetilde{g}_t(s) = s$, and thus $\widetilde{g}_t(p) \leq s$. Since s is minimal, we therefore have that $\widetilde{g}_t(p) = s$.

Since $\overline{A}$ is a neighborhood of $\widetilde{g}_t(p)$ for each $t \in \{1, 2, \ldots, k\}$, we have that $\bigcap_{t=1}^{k} {g_t}^{-1}[A] \in p$. Also, since $\beta\mathcal{W}_{k+1} \backslash \beta\mathcal{W}_k$ is an ideal of $\beta\mathcal{W}_{k+1}$ and p is minimal in $\beta\mathcal{W}_{k+1}$, we have that $\mathcal{W}_{k+1} \backslash \mathcal{W}_k \in p$. Pick

$$u \in (\mathcal{W}_{k+1} \backslash \mathcal{W}_k) \cap B \cap \textstyle\bigcap_{t=1}^{k} {g_t}^{-1}[A]$$

and let w be the variable word over $\mathcal{W}_k$ which results from replacing each occurrence of $k+1$ by v. Then $w(k+1) = u$ and for each $t \in \{1, 2, \ldots, k\}$ we have that $g_t(u) = w(t)$. Consequently $(w(1), w(2), \ldots, w(k+1)) \in I \cap (A^k \times B)$ as required. $\square$

The proof of the following lemma uses an old idea of H. Furstenberg and Y. Katznelson in [5]. There is some redundancy in this lemma. For example, (5) and (13) both tell us that $p \cdot q_{k+1} = p$.

Lemma 2.6 *Let s be any minimal idempotent in $\beta\mathcal{W}_k$. There exist minimal idempotents $p, q_{k+1}, r_{k+1} \in \beta\mathcal{W}_{k+1}$ and $q_1, q_2, \ldots, q_k, r_1, r_2, \ldots, r_k \in \beta\mathcal{W}_k$ such that*

(1) $\vec{p} = (s, s, \ldots, s, p) \in \overline{I}$;
(2) $\vec{q} = (q_1, q_2, \ldots, q_{k+1}) \in \overline{J}$;
(3) $\vec{r} = (r_1, r_2, \ldots, r_{k+1}) \in \overline{H}$;
(4) $p = p \cdot s = s \cdot p$;
(5) $\vec{p} \cdot \vec{q} = \vec{p}$ *and* $\vec{q} \cdot \vec{p} = \vec{q}$;

(6) $\vec{p}\cdot\vec{r}=\vec{r}$ *and* $\vec{r}\cdot\vec{p}=\vec{p}$*;*

(7) for $i\in\{1,2,\ldots,k+1\}$ *and* $j\in\{1,2,\ldots,k\}$*,* $q_i\cdot q_j=q_i$*;*

(8) for $i\in\{1,2,\ldots,k+1\}$ *and* $j\in\{1,2,\ldots,k\}$*,* $r_j\cdot r_i=r_i$*;*

(9) $s\cdot q_{k+1}=p$*;*

(10) $r_{k+1}\cdot s=p$*;*

(11) $q_{k+1}\cdot s=q_{k+1}$*;*

(12) $s\cdot r_{k+1}=r_{k+1}$*;*

(13) for $j\in\{1,2,\ldots,k+1\}$*,* $p\cdot q_j=p$*;*

(14) for $j\in\{1,2,\ldots,k+1\}$*,* $r_j\cdot p=p$*;*

(15) for $j\in\{1,2,\ldots,k\}$*,* $q_j\cdot q_{k+1}=q_j\cdot p$*; and*

(16) for $j\in\{1,2,\ldots,k\}$*,* $r_{k+1}\cdot r_j=p\cdot r_j$*.*

Proof. Let $\overline{s}=(s,s,\ldots,s)$. Then $\overline{s}\in\overline{E}$. By [8, Corollary 2.6 and Theorem 2.7], every left ideal of $\overline{E}$ contains a minimal left ideal and every right ideal of $\overline{E}$ contains a minimal right ideal. Pick a minimal left ideal L of $\overline{E}$ with $L\subseteq\overline{H}$ and a minimal right ideal R of $\overline{E}$ with $R\subseteq\overline{J}$. Pick by Lemma 2.1 a minimal idempotent $\vec{p}\leq\overline{s}$ in $\overline{E}$ with $\vec{p}\in L\overline{s}\cap\overline{s}R$.

Now $\vec{p}=(p_1,p_2,\ldots,p_{k+1})$. Since $\vec{p}\leq\overline{s}$, we have that for each $t\in\{1,2,\ldots,k+1\}$, $p_t\leq s$. Since s is minimal in $\beta\mathcal{W}_k$, we have in particular that $p_t=s$ for $t\in\{1,2,\ldots,k\}$. Let $p=p_{k+1}$. Then we have $\vec{p}=(s,s,\ldots,s,p)$. Since $\vec{p}\in K(\overline{E}\,)$ and, by Lemma 2.5 $K(\overline{E}\,)=\overline{E}\cap(\times_{j=1}^{k}K(\beta\mathcal{W}_k)\times K(\beta\mathcal{W}_{k+1}))$, we have that p is minimal in $\beta\mathcal{W}_{k+1}$. Since $p\leq s$, we have that (4) holds. Since $\vec{p}$ is minimal in $\overline{E}$ and $\overline{I}$ is an ideal of $\overline{E}$, we have that $\vec{p}\in\overline{I}$. That is, (1) holds.

Since $\vec{p}\in L\overline{s}\cap\overline{s}R$, pick $\vec{m}\in L$ and $\vec{n}\in R$ such that $\vec{p}=\vec{m}\,\overline{s}=\overline{s}\,\vec{n}$. Then $p=m_{k+1}s=sn_{k+1}$ and for $j\in\{1,2,\ldots,k\}$, $s=m_js=sn_j$. Also $m_{k+1}p=m_{k+1}sp=pp=p$ and $pn_{k+1}=p$. For $j\in\{1,2,\ldots,k\}$, we have that $m_jp=m_jsp=sp=p$ and $pn_j=p$. Let $\vec{q}=\vec{n}\,\vec{p}$ and let $\vec{r}=\vec{p}\,\vec{m}$. Now $q_{k+1}q_{k+1}=n_{k+1}pn_{k+1}p=n_{k+1}pp=n_{k+1}p=q_{k+1}$. The fact that q_j is an idempotent for each $j\in\{1,2,\ldots,k\}$ follows from (7) which we shall verify below. Likewise, for each $j\in\{1,2,\ldots,k+1\}$, $r_jr_j=r_j$.

Since $\vec{n}\in R$ and $\vec{m}\in L$ we have $\vec{q}\in R\subseteq\overline{J}$ and $\vec{r}\in R\subseteq\overline{H}$. Thus (2) and (3) hold. Also $\vec{q}\in R\subseteq K(\overline{E}\,)\subseteq\times_{j=1}^{k}K(\beta\mathcal{W}_k)\times K(\beta\mathcal{W}_{k+1})$, so we have that q_{k+1} is minimal in $\beta\mathcal{W}_{k+1}$ and q_j is minimal in $\beta\mathcal{W}_k$ for each $j\in\{1,2,\ldots,k\}$. Similarly, r_{k+1} is minimal in $\beta\mathcal{W}_{k+1}$ and r_j is minimal in $\beta\mathcal{W}_k$ for each $j\in\{1,2,\ldots,k\}$.

We proceed to verify the odd numbered statements from (5) through (15), the corresponding even numbered statements being analogous.

Since $\vec{p} = \overline{s}\,\vec{n}$ and $pn_{k+1} = p$, we have that $\vec{p} = \vec{p}\,\vec{n}$. Therefore we have $\vec{p}\,\vec{q} = \vec{p}\,\vec{n}\,\vec{p} = \vec{p}\,\vec{p} = \vec{p}$ and $\vec{q}\,\vec{p} = \vec{n}\,\vec{p}\,\vec{p} = \vec{n}\,\vec{p} = \vec{q}$. Thus (5) holds.

To verify statement (7), first let $i, j \in \{1, 2, \ldots, k\}$. Then $q_i q_j = n_i s q_j = n_i s = q_i$. Now let $j \in \{1, 2, \ldots, k\}$. Then $q_{k+1} q_j = n_{k+1} p n_j s = n_{k+1} p s = n_{k+1} p = q_{k+1}$.

Since $s q_{k+1} = s n_{k+1} p = pp = p$ we have that (9) holds. For statement (11), we have $q_{k+1} s = n_{k+1} p s = n_{k+1} p = q_{k+1}$. To verify (13), let $j \in \{1, 2, \ldots, k\}$. Then $p q_j = p n_j s = ps = p$. From (5) we know that $p q_{k+1} = p$. To verify (15), let $j \in \{1, 2, \ldots, k\}$. Then $q_j q_{k+1} = n_j s n_{k+1} p = n_j p p = n_j p = n_j s p = q_j p$. □

In Theorem 2.9 we shall be choosing a sequence of variable words $\langle w_n \rangle_{n=1}^{\infty}$ such that whenever $n \equiv 1 \pmod 3$, w_n is a right variable word and whenever $n \equiv 0 \pmod 3$, w_n is a left variable word. We shall expect certain products of these words to lie in specified cells of finite partitions of $\mathcal{W}_k$ and of $\mathcal{W}_{k+1} \backslash \mathcal{W}_k$. We clearly cannot have the first word of such a product be a left variable word nor can we have the last word be a right variable word, since one may divide $\mathcal{W}_k$ according to the first or last letter. Nor can we allow a right variable word to be followed by a left variable word, since that allows manipulation of adjacent occurrences of letters. (See [7] for a fuller discussion of these points.)

Definition 2.7 A set $F \in \mathcal{P}_f(\mathbb{N})$ is *allowable* if and only if $\max F \not\equiv 1 \pmod 3$, $\min F \not\equiv 0 \pmod 3$, and for all $i < j$ in F, if $i \equiv 1 \pmod 3$ and $j \equiv 0 \pmod 3$, then there exists $h \in F$ such that $i < h < j$ and $h \equiv 2 \pmod 3$.

Notice that if in the following lemma F is allowable, then conclusion (1) or (2) applies.

Lemma 2.8 *Let $F \in \mathcal{P}_f(\mathbb{N})$ and assume that $\max F \not\equiv 1 \pmod 3$ and for all $i < j$ in F, if $i \equiv 1 \pmod 3$ and $j \equiv 0 \pmod 3$, then there exists $h \in F$ such that $i < h < j$ and $h \equiv 2 \pmod 3$. Let s, p, $\vec{p} = (p_1, p_2, \ldots, p_{k+1})$, $\vec{q} = (q_1, q_2, \ldots, q_{k+1})$, and $\vec{r} = (r_1, r_2, \ldots, r_{k+1})$ be as in Lemma 2.6 (so that $p_{k+1} = p$ and $p_j = s$ for $j \in \{1, 2, \ldots, k\}$). For $f : F \to \{1, 2, \ldots, k+1\}$ and*

$n \in F$, *define*

$$\phi(f,n) = \begin{cases} q_{f(n)} & \text{if } n \equiv 0 \pmod 3 \\ r_{f(n)} & \text{if } n \equiv 1 \pmod 3 \\ p_{f(n)} & \text{if } n \equiv 2 \pmod 3 . \end{cases}$$

(1) If $\min F \not\equiv 0 \pmod 3$ *and* $k+1 \notin \mathrm{range}(f)$, *then* $\prod_{n\in F} \phi(f,n) = s$.

(2) If $\min F \not\equiv 0 \pmod 3$ *and* $k+1 \in \mathrm{range}(f)$, *then* $\prod_{n\in F} \phi(f,n) = p$.

(3) If $\min F \equiv 0 \pmod 3$ *and* $k+1 \notin \mathrm{range}(f)$, *then* $\prod_{n\in F} \phi(f,n) \in \{q_1, q_2, \ldots, q_k\}$.

(4) If $\min F \equiv 0 \pmod 3$ *and* $k+1 \in \mathrm{range}(f)$, *then* $\prod_{n\in F} \phi(f,n) \in \{q_{k+1}\} \cup \{q_1 p, q_2 p, \ldots, q_k p\}$.

Proof. We proceed by induction on $|F|$. If $F = \{m\}$, then $m \equiv 0 \pmod 3$ or $m \equiv 2 \pmod 3$. If $m \equiv 0 \pmod 3$, then $\phi(f,m)$ is $q_{f(m)}$ and if $m \equiv 2 \pmod 3$, then $\phi(f,m)$ is $p_{f(m)}$.

Now assume that $|F| > 1$, let $m = \min F$, and let $G = F\backslash\{m\}$. Let $l = \min G$. Then the value of $\prod_{n\in G} \phi(f,n)$ is determined by $f_{|G}$ and the congruence class of l using the induction hypothesis. The conclusions then follow from Lemma 2.6 and the fact that one cannot have both $m \equiv 1 \pmod 3$ and $l \equiv 0 \pmod 3$. □

Notice that in the following theorem, which is our main result, the fact that A can be any central subset of $\mathcal{W}_k$ yields a stronger result than choosing some A out of a given finite partition of $\mathcal{W}_k$, because for any finite partition, some cell must be central. Notice also that one cannot reverse the roles of A and B by taking B to be an arbitrary central set in $\mathcal{W}_{k+1}$ and picking A out of a finite partition of $\mathcal{W}_k$. Indeed, let $R = (k+1)\mathcal{W}_{k+1}$. Then R is a right ideal of $\mathcal{W}_{k+1}$ and so by [8, Corollary 4.18] $\overline{R}$ is a right ideal of $\beta\mathcal{W}_{k+1}$, so that R is central. Given the sequence $\langle w_n\rangle_{n=1}^{\infty}$ of variable words over $\mathcal{W}_k$ as chosen in Theorem 2.9 one cannot have $w_1(1)w_2(k+1) \in R$.

Theorem 2.9 *Let A be a central subset of $\mathcal{W}_k$ and let $\mathcal{F}$ be a finite partition of $\mathcal{W}_{k+1}\backslash\mathcal{W}_k$. Then there exist $B \in \mathcal{F}$ such that B is central in $\mathcal{W}_{k+1}$ and a sequence $\langle w_n\rangle_{n=1}^{\infty}$ of variable words over $\mathcal{W}_k$ such that*

(1) for each $n \in \mathbb{N}$, if $n \equiv 1 \pmod 3$, *then w_n is a right variable word;*

(2) for each $n \in \mathbb{N}$, if $n \equiv 0 \pmod 3$, *then w_n is a left variable word; and*

(3) for every allowable $F \in \mathcal{P}_f(\mathbb{N})$ and every $h : F \to \{1, 2, \ldots, k+1\}$, if $k+1 \in h[F]$, then $\prod_{n\in F} w_n(h(n)) \in B$, and if $k+1 \notin h[F]$, then $\prod_{n\in F} w_n(h(n)) \in A$.

Proof. Pick a minimal idempotent $s \in \beta\mathcal{W}_k$ such that $A \in p$. Pick $\vec{p} = (p_1, p_2, \ldots, p_{k+1})$, $\vec{q} = (q_1, q_2, \ldots, q_{k+1})$, and $\vec{r} = (r_1, r_2, \ldots, r_{k+1})$ as guaranteed by Lemma 2.6. Notice that, with p as in Lemma 2.6 we have $p_{k+1} = p$ and for $j \in \{1, 2, \ldots, k\}$, $p_j = s$. Since p is minimal in $\mathcal{W}_{k+1}$ and $\mathcal{W}_{k+1} \backslash \mathcal{W}_k$ is an ideal of $\mathcal{W}_{k+1}$, we have that $\mathcal{W}_{k+1} \backslash \mathcal{W}_k \in p$. Pick $B \in \mathcal{F}$ such that $B \in p_{k+1}$. Let $C = A \cup B$ and notice that $\vec{p} \in c\ell_Z(\times_{j=1}^{k+1} C)$.

Let $L = \{e, p, s, q_1, q_2, \ldots, q_{k+1}, q_1 p, q_2 p, \ldots, q_{k+1} p\}$. For notational convenience, let $p_0 = q_0 = r_0 = e$. Also, if w is a variable word, we let $w(0) = e$. If $n \in \mathbb{N}$, $f : \{1, 2, \ldots, 3n\} \to \{0, 1, \ldots, k+1\}$, and $i \in \{1, 2, \ldots, 3n\}$, define $\phi(f, i)$ as in Lemma 2.8 (with the obvious extension of the definition when $f(i) = 0$). Given $\theta \in L$, we say that the pair (f, θ) is *admissible* if and only if for every $j \in \{1, 2, \ldots, 3n\}$, we have $(\prod_{i=j}^{3n} \phi(f, j))\theta \in L$. Notice that, if $n > 1$, (f, θ) is admissible, and g is the restriction of f to $\{1, 2, \ldots, 3n-3\}$, then $(g, r_{f(3n-2)} p_{f(3n-1)} q_{f(3n)} \theta)$ is also admissible.

We construct the sequence $\langle w_n \rangle_{n=1}^{\infty}$ inductively, three terms at a time. Let $W_0 = \{e\}$ and for each $i \in \mathbb{N}$, as soon as we have chosen w_i, let $W_i = \{\prod_{j=1}^{i} w_j(f(j)) : \ f : \{1, 2, \ldots, i\} \to \{0, 1, \ldots, k+1\}\}$.

Let $n \in \mathbb{N} \cup \{0\}$ and assume that we have chosen w_i for all $i \in \mathbb{N}$ with $i \leq 3n$ (if any) so that, if $n \geq 1$, $\theta \in L$, $f : \{1, 2, \ldots, 3n\} \to \{0, 1, \ldots, k+1\}$, and (f, θ) is admissible, then

$$(*) \left(\prod_{i=1}^{3n} w_i(f(i))\right)^{-1} C \in \theta \Leftrightarrow C \in \left(\prod_{i=0}^{n-1} r_{f(3i+1)} p_{f(3i+2)} q_{f(3i+3)}\right) \theta \, .$$

Now $\vec{r} \in \overline{H}$, so pick by Lemma 2.2 a right variable word w_{3n+1} over $\mathcal{W}_k$ such that for all $l \in L$, all $u \in W_{3n}$, and all $t \in \{1, 2, \ldots, k+1\}$,

$$(a)(u w_{3n+1}(t))^{-1} C \in l \Leftrightarrow u^{-1} C \in r_t l \, .$$

Since $w_{3n+1}(0) = e = r_0$ we have also that (a) holds for $t = 0$.

Since $\vec{p} \in \overline{I}$, pick by Lemma 2.2 (and the observation above about the case $t = 0$) a variable word w_{3n+2} over $\mathcal{W}_k$ such that for all $l \in L$, all $u \in W_{3n+1}$, and all $t \in \{0, 1, \ldots, k+1\}$,

$$(b)(u w_{3n+2}(t))^{-1} C \in l \Leftrightarrow u^{-1} C \in p_t l \, .$$

Since $\vec{q} \in \overline{J}$, pick by Lemma 2.2 a left variable word w_{3n+3} over $\mathcal{W}_k$ such that for all $l \in L$, all $u \in W_{3n+2}$, and all $t \in \{0, 1, \ldots, k+1\}$,

$$(c)(u w_{3n+3}(t))^{-1} C \in l \Leftrightarrow u^{-1} C \in q_t l \, .$$

Now let $\theta \in L$, let $f : \{1, 2, \ldots, 3n+3\} \to \{0, 1, \ldots, k+1\}$, and assume that (f, θ) is admissible. We verify that $(*)$ holds. Notice that, since (f, θ) is admissible, we have that

$$\{q_{f(3n+3)}\theta, p_{f(3n+2)}q_{f(3n+3)}\theta, r_{f(3n+1)}p_{f(3n+2)}q_{f(3n+3)}\theta\} \subseteq L\,.$$

(In the following, if $n = 0$ we interpret $\prod_{i=1}^{3n} w_i(f(i))$ as e.)

$$\begin{aligned}
& \left(\textstyle\prod_{i=1}^{3n+3} w_i(f(i))\right)^{-1} C \in \theta \\
\Leftrightarrow\quad & \left(\left(\textstyle\prod_{i=1}^{3n} w_i(f(i))\right) w_{3n+1}(f(3n+1))w_{3n+2}(f(3n+2))\right)^{-1} C \in q_{f(3n+3)}\theta \\
\Leftrightarrow\quad & \left(\left(\textstyle\prod_{i=1}^{3n} w_i(f(i))\right) w_{3n+1}(f(3n+1))\right)^{-1} C \in p_{f(3n+2)}q_{f(3n+3)}\theta \\
\Leftrightarrow\quad & \left(\textstyle\prod_{i=1}^{3n} w_i(f(i))\right)^{-1} C \in r_{f(3n+1)}p_{f(3n+2)}q_{f(3n+3)}\theta \\
\Leftrightarrow\quad & C \in \left(\textstyle\prod_{i=0}^{n} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)}\right)\theta\,.
\end{aligned}$$

Here the first three double implications hold by (c), (b), and (a) respectively. If $n = 0$, the last double implication is a tautology. Otherwise, it is a consequence of the induction hypothesis.

The construction being complete, we claim that the sequence $\langle w_n\rangle_{n=1}^{\infty}$ is as required. The first two conclusions are immediate. So let $F \in \mathcal{P}_f(\mathbb{N})$ be allowable and let $h : F \to \{1, 2, \ldots, k+1\}$. Choose the least $n \in \mathbb{N}$ such that $F \subseteq \{1, 2, \ldots, 3n\}$. Define $f : \{1, 2, \ldots, 3n\} \to \{0, 1, \ldots, k+1\}$ by

$$f(i) = \begin{cases} h(i) & \text{if } i \in F \\ 0 & \text{if } i \in \{1, 2, \ldots, 3n\}\backslash F \end{cases}$$

Then we have immediately that $\prod_{i\in F} w_i(h(i)) = \prod_{i=1}^{3n} w_i(f(i))$.

By Lemma 2.8 we have that the pair (f, e) is admissible. Recall that we identify e with the principal ultrafilter generated by e. Thus by $(*)$, we have that

$$\textstyle\prod_{i=1}^{3n} w_i(f(i)) \in C \Leftrightarrow C \in \prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)}\,.$$

Again by Lemma 2.8 we have that

$$\textstyle\prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)} = \begin{cases} s & \text{if } k+1 \notin \text{range}(f) \\ p & \text{if } k+1 \in \text{range}(f)\,. \end{cases}$$

In either case, we have that $C \in \prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)}$, and so $\prod_{i=1}^{3n} w_i(f(i)) \in C$. If $k+1 \notin \text{range}(h)$, then $\prod_{i=1}^{3n} w_i(f(i)) \in C \cap \mathcal{W}_k = A$. If $k+1 \in \text{range}(h)$, then $\prod_{i=1}^{3n} w_i(f(i)) \in C\backslash\mathcal{W}_k = B$. $\square$

We conclude by showing that Carlson's Main Lemma is a consequence of Theorem 2.9.

Corollary 2.10 (Carlson) *Let $k \in \mathbb{N}$, let the set of variable words over $\mathcal{W}_k$ be partitioned into finitely many classes, and let $\langle w_n \rangle_{n=1}^{\infty}$ be a sequence of variable words. Then there exists a variable reduction $\langle t_n \rangle_{n=1}^{\infty}$ of $\langle w_n \rangle_{n=1}^{\infty}$ such that all expressions of the form $\prod_{i=1}^{n} t_i(f(i))$, where $n \in \mathbb{N}$, $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\} \cup \{v\}$, and $v \in \text{range}(f)$, lie in the same cell of the partition.*

Proof. Let $\mathcal{W}_{k,v}$ be the set of variable words over $\mathcal{W}_k$, let $\mathcal{F}$ be a finite partition of $\mathcal{W}_{k,v}$, and let $\langle w_n \rangle_{n=1}^{\infty}$ be a sequence of variable words. Define a function $\varphi : \mathcal{W}_{k+1} \backslash \mathcal{W}_k \to \mathcal{W}_{k,v}$ as follows. If $u = a_1 a_2 \cdots a_l$, where each $a_i \in \{1, 2, \ldots, k+1\}$, let $b_i = a_i$ if $a_i \neq k+1$ and $b_i = v$ if $a_i = k+1$; then let $\varphi(u) = \prod_{i=1}^{l} w(b_i)$.

Let $\mathcal{F} = \{\varphi^{-1}[H] : H \in \mathcal{F}\}$. Then $\mathcal{F}$ is a finite partition of $\mathcal{W}_{k+1} \backslash \mathcal{W}_k$. Let A be any central subset of $\mathcal{W}_k$ and pick $B \in \mathcal{F}$ and a sequence $\langle w_n' \rangle_{n=1}^{\infty}$ of variable words as guaranteed by Theorem 2.9. Pick $H \in \mathcal{F}$ such that $B = \varphi^{-1}[H]$.

For each $n \in \mathbb{N}$, let l_n be the length of w_{3n-1}' and write $w_{3n-1}' = a_{n,1} a_{n,2} \cdots a_{n,l_n}$ where each $a_{n,i} \in \{1, 2, \ldots, k\} \cup \{v\}$. (For this corollary, we are avoiding both left and right variable words.) Let $\alpha_0 = 0$ and for $n \in \mathbb{N}$, let $\alpha_n = \sum_{m=1}^{n} l_m$. For each $n \in \mathbb{N}$, let $t_n = \prod_{m=1}^{l_n} w_{\alpha_{n-1}+m}(a_{n,m})$. Then $\langle t_n \rangle_{n=1}^{\infty}$ is a variable reduction of $\langle w_n \rangle_{n=1}^{\infty}$.

To see that $\langle t_n \rangle_{n=1}^{\infty}$ is as required, let $n \in \mathbb{N}$ and let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\} \cup \{v\}$ with $v \in \text{range}(f)$. Define $g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k+1\}$ by $g(i) = f(i)$, if $f(i) \neq v$ and $g(i) = k+1$ if $f(i) = v$. Then $\prod_{i=1}^{n} w_{3i-1}'(g(i)) \in B$ and so $\prod_{i=1}^{n} t_i(f(i)) = \varphi\left(\prod_{i=1}^{n} w_{3i-1}'(g(i))\right) \in H$. $\square$

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