This paper was published in *Topology Proceedings* **24** (1999), 199-221. To the best of my knowledge, this is the final version as it was submitted to the publisher.

Weak VIP Systems in Commutative Semigroups

Neil Hindman

and

Randall McCutcheon[1]

Abstract. Let $(G,+)$ be a (discrete) commutative semigroup. VIP systems in G are polynomial type generalizations of IP systems, (i.e., sets of finite sums). We provide a self contained algebraic proof, using the algebraic structure of the Stone-Čech compactification βG of G, of a partition theorem about finite sets of VIP systems in abelian groups which had been previously derived as a consequence of the Polynomial Hales-Jewett Theorem due to V. Bergelson and A. Leibman. We also establish an infinitary version of this result valid in arbitrary commutative semigroups.

1. Introduction

We denote by $\mathcal{F}$ the set $\mathcal{P}_f(\mathbb{N})$ of non-empty finite subsets of the set $\mathbb{N}$ of positive integers. We let $\omega = \mathbb{N} \cup \{0\}$, and for a set A and a cardinal κ, we write $[A]^\kappa = \{B \subseteq A : |B| = \kappa\}$. If $(S,+)$ is a commutative semigroup, an IP system in S is a sequence $\langle n_\alpha \rangle_{\alpha \in \mathcal{F}}$ satisfying $n_{\alpha \cup \beta} = n_\alpha + n_\beta$ whenever $\alpha \cap \beta = \emptyset$. Equivalently, there exists a sequence $\langle x_i \rangle_{i=1}^\infty$ in S such that $n_\alpha = \sum_{i \in \alpha} x_i$ for all $\alpha \in \mathcal{F}$.

In [2], the following polynomial version of van der Waerden's theorem is proved:

Theorem A. *Let $k, r \in \mathbb{N}$, let $\mathbb{N} = \bigcup_{i=1}^r C_i$, and suppose that for each $i \in \{1, 2, \ldots, k\}$, $p_i(x) \in \mathbb{Q}[x]$ is a polynomial with $p_i[\mathbb{Z}] \subseteq \mathbb{Z}$ and $p_i(0) = 0$. If $\langle n_\alpha \rangle_{\alpha \in \mathcal{F}}$ is any IP-system in $\mathbb{N}$, then there exist $a \in \mathbb{N}$, $\alpha \in \mathcal{F}$, and $j \in \{1, 2, \ldots, r\}$ such that*

$$\{a, a + p_1(n_\alpha), a + p_2(n_\alpha), \ldots, a + p_k(n_\alpha)\} \subseteq C_j \, .$$

The "$\mathcal{F}$-sequences" $\langle p(n_\alpha) \rangle_{\alpha \in \mathcal{F}}$ appearing in Theorem A are examples of *VIP systems* in $\mathbb{Z}$. We shall be dealing with these in some generality, so we introduce some special notation.

[1] This author acknowledges support received from the National Science Foundation via a post doctoral fellowship administered by the University of Maryland.

1.1 Definition. Let $l \in \omega$ and let $t \in \mathbb{N}$. Then $\Psi_l = \{\alpha \in \mathcal{F} : \min \alpha > l\}$ and $M_{l,t} = \{\alpha \in \Psi_l : |\alpha| = t\}$.

1.2 Definition. Let $(G, +)$ be a commutative semigroup. If G has an identity, denote that identity by 0, in which case $G \cup \{0\} = G$. Otherwise, let $G \cup \{0\}$ be G together with an adjoined identity. Then f is a *VIP system* in G if and only if there exist $l(f) \in \omega$ and m_f such that

(1) $f : \Psi_{l(f)} \to G \cup \{0\}$,

(2) $m_f : \Psi_{l(f)} \to G \cup \{0\}$,

(3) there exists $d \in \mathbb{N}$ such that for all $t > d$ and all $\varphi \in M_{l(f),t}$, $m_f(\varphi) = 0$, and

(4) for all $\alpha \in \Psi_{l(f)}$, $f(\alpha) = \sum_{\emptyset \neq \varphi \subseteq \alpha} m_f(\varphi)$.

Clearly, if f is a VIP system in G, then $l(f)$ is uniquely determined. It is also easy to see by induction on $|\alpha|$ that if G is a group, then m_f is also uniquely determined.

VIP systems were introduced (for groups) in [1]. A related notion, that of *polynomial mapping*, was introduced in [3, Definition 8.1]. Polynomial mappings map all finite subsets (including the empty set) of an arbitrary set to a semigroup. VIP systems correspond exactly to polynomial mappings whose value at the empty set is 0.

We remarked above that if $p(x) \in \mathbb{Q}[x]$ with $p[\mathbb{Z}] \subseteq \mathbb{Z}$ and $p(0) = 0$ and $\langle n_\alpha \rangle_{\alpha \in \mathcal{F}}$ is an IP system in $\mathbb{N}$, then $\langle p(n_\alpha) \rangle_{\alpha \in \mathcal{F}}$ is a VIP system in $\mathbb{Z}$. To see how this is verified, consider an example. Let $p(x) = x^2 - 3x$ and for each $\alpha \in \mathcal{F}$, let $n_\alpha = \sum_{i \in \alpha} y_i$. Then, given $\alpha \in \mathcal{F}$,

$$p(n_\alpha) = (\sum_{i \in \alpha} y_i)^2 - 3 \sum_{i \in \alpha} y_i = \sum_{i \in \alpha} {y_i}^2 + \sum_{\{i,j\} \in [\alpha]^2} 2 y_i y_j - 3 \sum_{i \in \alpha} y_i \,.$$

For $i \in \mathbb{N}$, let $m(\{i\}) = {y_i}^2 - 3y_i$, for $\{i, j\} \in [\mathbb{N}]^2$, let $m(\{i, j\}) = 2y_i y_j$, and for $\varphi \in \mathcal{F}$ with $|\varphi| > 2$, let $m(\varphi) = 0$. Then for each $\alpha \in \mathcal{F}$, $p(n_\alpha) = \sum_{\emptyset \neq \varphi \subseteq \alpha} m(\varphi)$.

In the event that $(G, +)$ is an abelian group, there is another characterization of VIP systems which we shall want to emulate.

1.3 Lemma. *Let $(G, +)$ be an abelian group, let $l \in \mathbb{N}$, and let $f : \Psi_l \to G$. Then f is a VIP system in G if and only if there exists $d \in \mathbb{N}$ such that whenever $\alpha_0, \alpha_1, \ldots, \alpha_d$ are pairwise disjoint members of Ψ_l one has*

$$\sum_{\substack{B \subseteq \{\alpha_0, \ldots, \alpha_d\} \\ B \neq \emptyset,\ |B| \text{ even}}} f(\bigcup B) \;=\; \sum_{\substack{B \subseteq \{\alpha_0, \ldots, \alpha_d\} \\ |B| \text{ odd}}} f(\bigcup B)\,.$$

Proof. [8, Proposition 2.5]. □

The first of the two primary objectives in this paper is to provide an algebraic proof of the following extension of Theorem A.

Theorem B. *Let $(G,+)$ be an abelian group, let $r \in \mathbb{N}$, and let R be a finite set of VIP systems in G. If $G = \bigcup_{i=1}^r C_i$, then there exist $a \in G$, $\alpha \in \mathcal{F}$, and $j \in \{1,2,\ldots,r\}$ such that $\{a\} \cup \{a + f(\alpha) : f \in R\} \subseteq C_j$.*

A version of Theorem B for polynomial mappings was derived from the more powerful Polynomial Hales-Jewett Theorem in [3, Theorem 8.6]. Our proof of Theorem B, which is much simpler, utilizes the algebraic structure of βG, as do the other results of this paper. Here βG is the Stone-Čech compactification of the discrete space G, where $(G,+)$ is only assumed to be a semigroup. We take the points of βG to be the ultrafilters on G. Given $A \subseteq G$, $\overline{A} = \{p \in G : A \in p\}$. Then $\{\overline{A} : A \subseteq G\}$ is a basis for the open sets of βG as well as a basis for the closed sets.

The operation $+$ extends to βG in such a way that $(\beta G,+)$ is a right topological semigroup (meaning the function $\rho_p : \beta G \to \beta G$ is continuous for each $p \in \beta G$ where $\rho_p(q) = q + p$) with G contained in its topological center (meaning the function $\lambda_x : \beta G \to \beta G$ is continuous for each $x \in G$ where $\lambda_x(q) = x + q$). Given $p, q \in \beta G$ and $A \subseteq G$, one has that $A \in p + q$ if and only if $\{x \in G : -x + A \in q\} \in p$. (Here, since we are only assuming that G is a semigroup, $-x + A = \{y \in G : x + y \in A\}$.) See [7] for an elementary introduction to the algebra and topology of βG, as well as for any unfamiliar algebraic facts mentioned here.

The reader should be cautioned that, in spite of the fact that we denote the operation of βG by the same symbol used to denote the operation of G, in this case $+$, the operation in βG is not likely to be commutative. In fact, if G is left cancellative, the center of βG is equal to the center of G [7, Theorem 6.54]. If, as we shall assume throughout this paper, the semigroup G is commutative, then regardless of any cancellation assumptions, if $x \in G$, then x commutes with any member of βG [7, Theorem 4.23].

Like any compact right topological semigroup βG has a smallest two sided ideal $K(\beta G)$, which is the union of all minimal right ideals of βG as well as the union of all minimal left ideals of βG [7, Theorem 2.8]. A subset C of G is said to be *central* if and only if there is an idempotent $p \in K(\beta G)$ such that $C \in p$. (The notion was introduced by Furstenberg in [4] for subsets of the semigroup $(\mathbb{N},+)$, using a different but equivalent definition.)

An infinitary result, more general than Theorem B, is proved in [6], where a weaker notion of "VIP system" was defined for cancellative semigroups. Recall that any such semigroup has a "group of quotients" G. When we speak of the group of quotients, we shall assume that $S \subseteq G$ so that $G = \{x - y : x, y \in S\}$. The following is [6, Corollary 3.14]. (Given α and δ in $\mathcal{F}$, we write $\alpha < \delta$ provided $\max \alpha < \min \delta$.)

Theorem C. *Let $(S, +)$ be a commutative cancellative semigroup, let C be a central set in S, and let R be a finite set of VIP systems in the group of quotients of S. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in S and $\langle \alpha_n \rangle_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\alpha_n < \alpha_{n+1}$ for each n and for every $F \in \mathcal{F}$ and every $p \in R$, if $\gamma = \bigcup_{t \in F} \alpha_t$, then $\sum_{t \in F} a_t \in C$ and $\sum_{t \in F} a_t + p(\gamma) \in C$.*

We note that the case of Theorem C corresponding to $S = \mathbb{Z}^n$ with all the VIP systems as IP systems is Furstenberg's Central Sets Theorem ([4, Proposition 8.21]).

Our second primary objective in this paper is to prove a version of Theorem C valid for general (i.e. not necessarily cancellative) commutative semigroups. Along the way, we introduce the notion of weak VIP system in non cancellative semigroups and obtain what we think is the most general version of the polynomial van der Waerden theorem for semigroups.

2. VIP Systems in Commutative Groups

We provide in this section an algebraic proof of a generalization of Theorem B from the introduction. This proof is completely self contained, except that we appeal to several fundamental facts about βG from [7]. The proof we present is based on the algebraic proof in [5] of the Polynomial van der Waerden Theorem of V. Bergelson and A. Leibman [2], which was in turn based on their original proof.

2.1 Definition. Let $(G, +)$ be an abelian group and let f be a VIP system in G.

(a) The *degree* of f is defined by $\deg(\overline{0}) = 0$ and if $f \neq \overline{0}$, then $\deg(f) = \max\{d \in \mathbb{N} :$ there exists $\varphi \in M_{l(f),d}$ such that $m_f(\varphi) \neq 0$.

(b) The *strong degree* of f is defined by $\mathrm{stdeg}(f) = 0$ if there exists $k \in \mathbb{N}$ such that $f(\alpha) = 0$ for all $\alpha \in \Psi_k$ and otherwise $\mathrm{stdeg}(f) = \max\{t \in \mathbb{N} :$ for all $k \geq l(f)$ there exists $\alpha \in M_{k,t}$ such that $m_f(\alpha) \neq 0\}$.

Notice that trivially $\mathrm{stdeg}(f) \leq \deg(f)$.

2.2 Lemma. *Let $(G, +)$ be an abelian group and let g and f be VIP systems in G. Define $g - f$ by $l(g - f) = \max\{l(g), l(f)\}$ and for $\alpha \in \Psi_{l(g-f)}$, $(g - f)(\alpha) = g(\alpha) - f(\alpha)$. Then $g - f$ is a VIP system in G. If $d = \mathrm{stdeg}(g) = \deg(g) > \deg(f)$, then $\mathrm{stdeg}(g - f) = \deg(g - f) = d$ and whenever $l \geq l(g - f)$, $\langle m_{g-f}(\varphi) \rangle_{\varphi \in M_{l,d}} = \langle m_g(\varphi) \rangle_{\varphi \in M_{l,d}}$.*

Proof. Define $m_{g-f} : \Psi_{l(g-f)} \to G$ by $m_{g-f}(\alpha) = m_g(\alpha) - m_f(\alpha)$. It is then routine to verify all of the conclusions. $\square$

2.3 Lemma. *Let $(G, +)$ be an abelian group and let p be a VIP system in G such that $d = \operatorname{stdeg}(p) = \deg(p)$. Let $\alpha \in \Psi_{l(p)}$ and define $h : \Psi_{\max \alpha} \to G$ by $h(\beta) = p(\alpha \cup \beta) - p(\alpha)$. Then h is a VIP system in G, $\operatorname{stdeg}(h) = \deg(h) = d$ and whenever $l \geq l(h)$, $\langle m_h(\varphi)\rangle_{\varphi \in M_{l,d}} = \langle m_p(\varphi)\rangle_{\varphi \in M_{l,d}}$.*

Proof. We have that $l(h) = \max \alpha$. Define $m_h : \Psi_{l(h)} \to G$ by $m_h(\varphi) = \sum_{\psi \subseteq \alpha} m_p(\varphi \cup \psi)$. (Notice that we are allowing $\psi = \emptyset$.) Observe now two facts:

(1) If $t > d$, $\varphi \in M_{l(h),t}$, and $\psi \subseteq \alpha$, then $|\varphi \cup \alpha| > d$ and so $m_p(\varphi \cup \psi) = 0$.

(2) If $\varphi \in M_{l(h),d}$, $\psi \subseteq \alpha$, and $m_p(\varphi \cup \psi) \neq 0$, then $\psi = \emptyset$, and in particular $m_h(\varphi) = m_p(\varphi)$.

In particular, conditions (1), (2), and (3) of Definition 1.2 hold. To verify condition (4), let $\beta \in \Psi_{l(h)}$. Then

$$\begin{aligned} h(\beta) &= p(\alpha \cup \beta) - p(\alpha) \\ &= \sum_{\emptyset \neq \varphi \subseteq \alpha \cup \beta} m_p(\varphi) - \sum_{\emptyset \neq \varphi \subseteq \alpha} m_p(\varphi) \\ &= \sum_{\emptyset \neq \varphi \subseteq \beta} \sum_{\psi \subseteq \alpha} m_p(\varphi \cup \psi) \\ &= \sum_{\emptyset \neq \varphi \subseteq \beta} m_h(\varphi)\,. \end{aligned}$$

By observation (1) we have $\deg(h) \leq d$ and of course $\operatorname{stdeg}(h) \leq \deg(h)$. Suppose that $\operatorname{stdeg}(h) < d$ and pick $l \geq l(h)$ such that for all $\varphi \in M_{l,d}$, $m_h(\varphi) = 0$. Since $\operatorname{stdeg}(p) = d$, pick $\varphi \in M_{l,d}$ such that $m_p(\varphi) \neq 0$. By observation (2), this is a contradiction. $\square$

2.4 Definition. Let $(G, +)$ be an abelian group.

(a) $\mathcal{R} = \{R : R$ is a finite set of VIP systems in $G\}$.

(b) Order $\bigoplus_{i=1}^{\infty} \omega$ lexicographically based on the largest coordinate on which elements differ, denoting this order by $<$. Define $\theta : \mathcal{R} \to \bigoplus_{i=1}^{\infty} \omega$ by

$$\theta(R) = (w_1, w_2, w_3, \ldots)$$

where for each $i \in \mathbb{N}$, and $l \geq \max\{l(p) : p \in R\}$,

$$F_{l,i} = \{\langle m_p(\varphi)\rangle_{\varphi \in M_{l,i}} : p \in R \text{ and } \operatorname{stdeg}(p) = i\} \text{ and}$$

$$w_i = \min\left\{|F_{l,i}| : l \geq \max\{l(p) : p \in R\}\right\}.$$

Notice that $\bigoplus_{i=1}^{\infty} \omega$ is well ordered by the lexicographic order.

The definition of θ is admittedly daunting at first glance, so let us pay attention to what w_i does (and does not) count. A first approximation is that w_i counts the number of elements of R with strong degree equal to i. But this is far wide of the mark. Given a VIP system p of strong degree i one may view the function $\langle m_p(\varphi)\rangle_{\varphi\in M_{l(p),i}}$ as the "leading coefficient" of p. A much closer approximation to w_i is that it counts the number of distinct functions occurring as leading coefficients of members of R with strong degree equal to i. For each $l \in \mathbb{N}$ with $l \geq \max\{l(p) : p \in R\}$, if we let $R_l = \{p_{|\Psi_l} : p \in R\}$, then what w_i is actually counting is the eventually constant (as l approaches infinity) number of leading coefficients of members of R_l with strong degree equal to i.

2.5 Lemma. *Let $(G, +)$ be an abelian group and let $R \in \mathcal{R}$ such that $R \neq \emptyset$, $\overline{0} \notin R$, and* $\mathrm{stdeg}(p) = \deg(p)$ *for every $p \in R$. Pick f in R of minimal degree and let T be a finite subset of $\bigcap_{p\in R} \Psi_{l(p)}$. For $\alpha \in T$ and $p \in R$, define $g_{p,\alpha} : \Psi_{\max \alpha} \to G$ by $g_{p,\alpha}(\beta) = p(\alpha \cup \beta) - p(\alpha) - f(\beta)$. Let*

$$S = \{g_{p,\alpha} : p \in R \textit{ and } \alpha \in T\} \cup \{p - f : p \in R\}\,.$$

Then $S \in \mathcal{R}$ and $\theta(S) < \theta(R)$.

Proof. By Lemmas 2.2 and 2.3 we have that $S \in \mathcal{R}$.

For $i \in \mathbb{N}$ and $l \geq \max\{l(p) : p \in R\}$, let

$$F_{l,i} = \{\langle m_p(\varphi)\rangle_{\varphi\in M_{l,i}} : p \in R \text{ and } \mathrm{stdeg}(p) = i\}\,.$$

For $i \in \mathbb{N}$ and $l \geq \max\{l(h) : h \in S\}$, let

$$F'_{l,i} = \{\langle m_h(\varphi)\rangle_{\varphi\in M_{l,i}} : h \in S \text{ and } \mathrm{stdeg}(h) = i\}\,.$$

Let $\theta(R) = (w_1, w_2, w_3 \ldots)$ and let $\theta(S) = (u_1, u_2, u_3, \ldots)$. Notice that, if $k < l$, then $|F_{l,i}| \leq |F_{k,i}|$ and $|F'_{l,i}| \leq |F'_{k,i}|$. Consequently, for each i we may pick l such that $w_i = |F_{l,i}|$ and $u_i = |F'_{l,i}|$.

Let $d = \deg(f)$. We claim that for each $i > d$, $w_i = u_i$, and that $u_d < w_d$. To establish the first assertion, let $i > d$ and pick l such that $w_i = |F_{l,i}|$ and $u_i = |F'_{l,i}|$. We claim that $F_{l,i} = F'_{l,i}$. To see that $F_{l,i} \subseteq F'_{l,i}$, let $p \in R$ with $\mathrm{stdeg}(p) = i$. Then $p - f \in S$ and by Lemma 2.2, $\mathrm{stdeg}(p - f) = i$ and $\langle m_{p-f}(\varphi)\rangle_{\varphi\in M_{l,i}} = \langle m_p(\varphi)\rangle_{\varphi\in M_{l,i}}$.

To see that $F'_{l,i} \subseteq F_{l,i}$, let $h \in S$ with $\mathrm{stdeg}(h) = i$. If $h = p - f$ for some $p \in R$, then necessarily, $\deg(p) = i$ and we have already seen that $\langle m_{p-f}(\varphi)\rangle_{\varphi\in M_{l,d}} = \langle m_p(\varphi)\rangle_{\varphi\in M_{l,d}}$. Thus we may assume that we have some $p \in R$ and some $\alpha \in T$ such that $h = g_{p,\alpha}$. Define $q : \Psi_{\max \alpha} \to G$ by $q(\beta) = p(\alpha \cup \beta) - p(\alpha)$. Then $h = q - f$.

By Lemmas 2.2 and 2.3, $\mathrm{stdeg}(q) = \deg(q) = \deg(p)$ and $\mathrm{stdeg}(h) = \deg(h) = \deg(q)$. Thus, again by Lemmas 2.2 and 2.3 $\langle m_h(\varphi)\rangle_{\varphi\in M_{l,i}} = \langle m_q(\varphi)\rangle_{\varphi\in M_{l,i}} = \langle m_p(\varphi)\rangle_{\varphi\in M_{l,i}}$.

To complete the proof, we show that $u_d < w_d$. Again, pick l such that $w_d = |F_{l,d}|$ and $u_d = |F'_{l,d}|$. Let

$$H = \{\langle m_p(\varphi)\rangle_{\varphi\in M_{l,d}} : p \in R\,,\ \mathrm{stdeg}(p) = d, \text{ and } \mathrm{stdeg}(p - f) = d\}$$

and note that $H \subsetneq F_{l,d}$ because $\langle m_f(\varphi)\rangle_{\varphi\in M_{l,d}} \in F_{l,d}\backslash H$. (It may very well be that there is some $p \in R\backslash\{f\}$ such that $\langle m_p(\varphi)\rangle_{\varphi\in M_{l,d}} = \langle m_f(\varphi)\rangle_{\varphi\in M_{l,d}}$, but for such p, $\mathrm{stdeg}(p - f) < d$.)

We now define $\tau : H \to F'_{l,d}$ and show that $\tau[H] = F'_{l,d}$. Given $p \in R$ with $\mathrm{stdeg}(p) = \mathrm{stdeg}(p - f) = d$, let

$$\tau(\langle m_p(\varphi)\rangle_{\varphi\in M_{l,d}}) = \langle m_p(\varphi) - \langle m_f(\varphi)\rangle_{\varphi\in M_{l,d}}\,.$$

Since $p-f \in S$ and for $\varphi \in M_{l,d}$, $m_{p-f}(\varphi) = m_p(\varphi) - \langle m_f(\varphi)$, we have that $\tau[H] \subseteq F'_{l,d}$.

Now let $h \in S$ such that $\mathrm{stdeg}(h) = d$. If $h = p - f$ for some $p \in R$, we have by Lemma 2.2 and the fact that $\deg(f)$ is minimal that $\mathrm{stdeg}(p) = d$. Thus $\langle m_p(\varphi)\rangle_{\varphi\in M_{l,d}} \in H$ and $\tau(\langle m_p(\varphi)\rangle_{\varphi\in M_{l,d}}) = \langle m_h(\varphi)\rangle_{\varphi\in M_{l,d}}$.

Thus we assume that $h = g_{p,\alpha}$ for some $p \in R$ and some $\alpha \in T$. We have already seen that if $\deg(p) = i > d$, then $\deg(h) = i$. So $\deg(p) = d$. Define $q : \Psi_{\max\alpha} \to G$ by $q(\beta) = p(\alpha\cup\beta) - p(\alpha)$. By Lemma 2.3, $\mathrm{stdeg}(q) = \deg(q) = d$ and $\langle m_q(\varphi)\rangle_{\varphi\in M_{l,d}} = \langle m_p(\varphi)\rangle_{\varphi\in M_{l,d}}$. Since $h = q - f$, we have

$$\begin{aligned}\langle m_h(\varphi)\rangle_{\varphi\in M_{l,d}} &= \langle m_q(\varphi) - m_f(\varphi)\rangle_{\varphi\in M_{l,d}}\\ &= \langle m_p(\varphi) - m_f(\varphi)\rangle_{\varphi\in M_{l,d}}\\ &= \tau(\langle m_p(\varphi)\rangle_{\varphi\in M_{l,d}})\,.\end{aligned}$$ □

2.6 Lemma. *Let $(G, +)$ be an abelian group and let $R \in \mathcal{R}$. There exists $k \geq \max\{l(p) : p \in R\}$ such that, if $R' = \{f_{|\Psi_k} : f \in R\}$, then for all $f \in R$,* $\mathrm{stdeg}(f_{|\Psi_k}) = \deg(f_{|\Psi_k}) = \mathrm{stdeg}(f)$ *and $\theta(R') = \theta(R)$.*

Proof. For $i \in \mathbb{N}$ and $l \geq \max\{l(p) : p \in R\}$, let

$$F_{l,i} = \{\langle m_p(\varphi)\rangle_{\varphi\in M_{l,i}} : p \in R \text{ and } \mathrm{stdeg}(p) = i\}\,.$$

Let $\theta(R) = (w_1, w_2, w_3, \ldots)$ and pick m such that $w_i = 0$ for all $i > m$. As in the proof of Lemma 2.5, for each $i \in \{1, 2, \ldots, m\}$, pick $l(i)$ such that $w_i = |F_{l(i),i}|$. Let $k = \max\big\{l(i) : i \in \{1, 2, \ldots, m\}\big\}$. □

2.7 Definition. $\mathbb{H} = \bigcap_{k=1}^{\infty} \overline{\Psi_k}$.

In [7] $\mathbb{H}$ is defined to be the subset $\bigcap_{k=1}^{\infty} \overline{\mathbb{N}2^k}$ of $(\beta\mathbb{N}, +)$. It is easy to see, essentially as in [7, Theorem 6.15], that $\mathbb{H}$ as a subsemigroup of $(\beta\mathcal{F}, \uplus)$ is homeomorphic and isomorphic to $\bigcap_{k=1}^{\infty} \overline{\mathbb{N}2^k}$.

Recall that a subset A of a semigroup $(S, +)$ is *piecewise syndetic* if and only if there is some $H \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ with $F + x \subseteq \bigcup_{t \in H}(-t + A)$.

2.8 Theorem. *Let $(G, +)$ be an abelian group, let $R \in \mathcal{R}$, let $v \uplus v = v \in \mathbb{H}$, let A be a piecewise syndetic subset of G, and let L be a minimal left ideal of βG such that $\overline{A} \cap L \neq \emptyset$. If $\gamma = \max\{l(p) : p \in R\}$, then*

$$\{\alpha \in \Psi_\gamma : \overline{A} \cap L \cap \textstyle\bigcap_{p \in R} \overline{-p(\alpha) + A} \neq \emptyset\} \in v\,.$$

Proof. Suppose not, and pick R such that $\theta(R)$ is minimal among all counterexamples. Notice that $R \neq \emptyset$ and $R \neq \{\overline{0}\}$ because the statement is trivially true for both of these sets. By Lemma 2.6, we may presume that for all $p \in R$, $\text{stdeg}(p) = \deg(p)$. (Given that R is a counterexample, so is the set R' produced in Lemma 2.6.) We may also assume that $\overline{0} \notin R$ because $R \backslash \{\overline{0}\}$ is also a counterexample and $\theta(R \backslash \{\overline{0}\}) = \theta(R)$.

Pick $v = v \uplus v$, a piecewise syndetic subset A of G, and a minimal left ideal L of βG such that $\overline{A} \cap L \neq \emptyset$ but

$$\{\alpha \in \Psi_\gamma : \overline{A} \cap L \cap \textstyle\bigcap_{p \in R} \overline{-p(\alpha) + A} \neq \emptyset\} \notin v$$

where $\gamma = \max\{l(p) : p \in R\}$. Let

$$D = \Psi_\gamma \backslash \{\alpha \in \Psi_\gamma : \overline{A} \cap L \cap \textstyle\bigcap_{p \in R} \overline{-p(\alpha) + A} \neq \emptyset\}$$

and note that $D \in v$. By [7, Lemma 4.14], if $D^\star = \{x \in D : -x + D \in v\}$, then $D^\star \in v$ and for all $x \in D^\star$, $-x + D^\star \in v$. (Here $-x + D = \{\alpha \in \mathcal{F} : \alpha \cup x \in D\}$.)

Pick $f \in R$ of smallest degree. For $\alpha \in \Psi_\gamma$ and $p \in R$, define $g_{p,\alpha} : \Psi_{\max \alpha} \to G$ by $g_{p,\alpha}(\beta) = p(\alpha \cup \beta) - p(\alpha) - f(\beta)$.

Pick $q_0 \in \overline{A} \cap L$ and let $B = \{x \in G : -x + A \in q_0\}$. Then by [7, Theorem 4.39], B is syndetic so pick $H \in \mathcal{P}_f(G)$ such that $G = \bigcup_{t \in H}(-t + B)$. Pick $t_0 \in H$ such that $-t_0 + B \in q_0$ and let $C_0 = -t_0 + B$. Since $C_0 \in q_0$, $\overline{C_0} \cap L \neq \emptyset$.

Let $S_0 = \{p - f : p \in R\}$ and let $E_0 = \{\alpha \in \Psi_\gamma : \overline{C_0} \cap L \cap \bigcap_{p \in S_0} \overline{-p(\alpha) + C_0} \neq \emptyset\}$. By Lemma 2.5, $S_0 \in \mathcal{R}$ and $\theta(S_0) < \theta(R)$ so $E_0 \in v$. Pick $\alpha_1 \in E_0 \cap D^\star$ and pick $r_1 \in \overline{C_0} \cap L \cap \bigcap_{p \in S_0} \overline{-p(\alpha_1) + C_0}$. Let $q_1 = -f(\alpha_1) + r_1$ and note that $q_1 \in L$. Pick $t_1 \in H$ such that $-t_1 + B \in q_1$.

Inductively, assume that we have $m \in \mathbb{N}$ and have chosen $\langle q_i \rangle_{i=0}^m$ in L, $\langle t_i \rangle_{i=0}^m$ in H, and $\langle \alpha_i \rangle_{i=1}^m$ in Ψ_γ such that

(1) for $j \in \{0,1,\ldots,m\}$, $-t_j + B \in q_j$,

(2) for $l \in \{1,2,\ldots,m-1\}$, if any, $\alpha_l < \alpha_{l+1}$,

(3) for $l \in \{1,2,\ldots,m\}$, $\alpha_l \cup \alpha_{l+1} \cup \ldots \cup \alpha_m \in D^\star$, and

(4) for $l \in \{0,1,\ldots,m-1\}$ and $p \in R$, $-\big(t_l + p(\alpha_{l+1} \cup \alpha_{l+2} \cup \ldots \cup \alpha_m)\big) + B \in q_m$.

Hypotheses (1) and (3) hold trivially for $m = 1$ and hypothesis (2) is vacuous there. Hypothesis (4) says that for all $p \in R$, $-\big(t_0 + p(\alpha_1)\big) + B \in q_1$. So let $p \in R$ be given. Now $p - f \in S_0$ so $r_1 + p(\alpha_1) - f(\alpha_1) \in \overline{C_0}$ and so $-t_0 + B \in r_1 + p(\alpha_1) - f(\alpha_1)$ and so $-\big(t_0 + p(\alpha_1)\big) + B \in r_1 - f(\alpha_1) = q_1$ as required.

Now let $T_m = \big\{\{\alpha_{l+1} \cup \alpha_{l+2} \cup \ldots \cup \alpha_m\} : l \in \{0,1,\ldots,m-1\}\big\}$ and let $S_m = \{g(p,\alpha) : p \in R \text{ and } \alpha \in T_m\} \cup \{p - f : p \in R\}$. By Lemma 2.5, $S_m \in \mathcal{R}$ and $\theta(S_m) < \theta(R)$. Let

$$C_m = (-t_m + B) \cap \bigcap_{p\in R} \bigcap_{l=0}^{m-1} \big(-\big(t_l + p(\alpha_{l+1} \cup \alpha_{l+2} \cup \ldots \cup \alpha_m)\big) + B\big)\,.$$

Then by hypotheses (1) and (4), $C_m \in q_m$ and so $C_m \cap L \neq \emptyset$. Consequently the statement of the current theorem is valid for S_m and C_m.

Let $\delta = \max\{l(p) : p \in S_m\}$ and let

$$E_m = \{\alpha \in \Psi_\delta : \overline{C_m} \cap L \cap \bigcap_{p\in S_m} \overline{-p(\alpha) + C_m} \neq \emptyset\}\,.$$

Then $E_m \in v$ and also $\Psi_{\max \alpha_m} \in v$. By hypothesis (3) and [7, Lemma 4.14], for each $l \in \{1,2,\ldots,m\}$, $-(\alpha_l \cup \alpha_{l+1} \cup \ldots \cup \alpha_m) + D^\star \in v$. Pick

$$\alpha_{m+1} \in E_m \cap \Psi_{\max \alpha_m} \cap \bigcap_{l=1}^{m} -(\alpha_l \cup \alpha_{l+1} \cup \ldots \cup \alpha_m) + D^\star$$

and pick $r_{m+1} \in \overline{C_m} \cap L \cap \bigcap_{p\in S_m} \overline{-p(\alpha_{m+1}) + C_m}$. Let $q_{m+1} = -f(\alpha_{m+1}) + r_{m+1}$ and note that $q_{m+1} \in L$. Pick $t_{m+1} \in H$ such that $-t_{m+1} + B \in q_{m+1}$.

Hypotheses (1), (2), and (3) hold directly. To verify hypothesis (4), let $l \in \{0,1,\ldots,m\}$ and let $p \in R$. Assume first that $l = m$. Then $p - f \in S_m$ so $r_{m+1} + p(\alpha_{m+1}) - f(\alpha_{m+1}) \in \overline{C_m}$. Thus $-t_m + B \in r_{m+1} + p(\alpha_{m+1}) - f(\alpha_{m+1})$ and so $-\big(t_m + p(\alpha_{m+1})\big) + B \in r_{m+1} - f(\alpha_{m+1}) = q_{m+1}$ as required.

Now assume that $l < m$, let $\beta = \alpha_{l+1} \cup \alpha_{l+2} \cup \ldots \cup \alpha_m$, and notice that $\beta \in T_m$. Then $g(p,\beta) \in S_m$ so

$$r_{m+1} + g(p,\beta)(\alpha_{m+1}) \in \overline{C_m} \subseteq \overline{-\big(t_l + p(\beta)\big) + B}$$

and so $-\big(t_l + p(\beta)\big) + B \in r_{m+1} + g(p,\beta)(\alpha_{m+1}) = r_{m+1} + p(\beta \cup \alpha_{m+1}) - p(\beta) - f(\alpha_{m+1}) = q_{m+1} + p(\beta \cup \alpha_{m+1}) - p(\beta)$ and so $-\big(t_l + p(\beta \cup \alpha_{m+1})\big) + B \in q_{m+1}$ as required.

The induction being complete, choose $l < m$ such that $t_l = t_m$, which we may do because H is finite. Let $\beta = \alpha_{l+1} \cup \alpha_{l+2} \cup \ldots \cup \alpha_m$. By hypothesis (3), $\beta \in D^\star$. We have that

$$(-t_m+B)\cap\bigcap_{p\in R}\big(-\big(t_m+p(\beta)\big)+B\big)\in q_m$$

by hypotheses (1) and (4) and the fact that $t_l=t_m$. So pick $a\in(-t_m+B)\cap\bigcap_{p\in R}\big(-\big(t_m+p(\beta)\big)+B\big)$. Let $r=a+t_m+q_0$ and notice that $r\in\overline{A}\cap L\cap\bigcap_{p\in R}\overline{-p(\beta)+A}$. (Trivially $r\in L$. Also $a+t_m\in B$ and so $r\in\overline{A}$. Given $p\in R$, $a+t_m+p(\beta)\in B$ and so $A\in r+p(\beta)$.) This contradicts the fact that $\beta\in D$. □

2.9 Corollary. *Let $(G,+)$ be an abelian group, let $R\in\mathcal{R}$, and let $\langle\alpha_n\rangle_{n=1}^\infty$ be a sequence in $\mathcal{F}$ such that $\alpha_n<\alpha_{n+1}$ for all n. If A is a piecewise syndetic subset of G, then there exist $r\in\overline{A}\cap K(\beta G)$ and $\beta\in FU(\langle\alpha_n\rangle_{n=1}^\infty)$ such that $\{r+p(\beta):p\in R\}\subseteq\overline{A}$.*

Proof. Pick by [7, Lemma 5.11] some $v=v\uplus v\in\beta\mathcal{F}$ such that for each $m\in\mathbb{N}$, $FU(\langle\alpha_n\rangle_{n=m}^\infty)\in v$ and notice that in fact $v\in\mathbb{H}$. Pick by [7, Theorems 2.8 and 4.40] a minimal left ideal L of βG such that $\overline{A}\cap L\neq\emptyset$. Let $\gamma=\max\{l(p):p\in R\}$. By Theorem 2.8

$$\{\alpha\in\Psi_\gamma:\overline{A}\cap L\cap\bigcap_{p\in R}\overline{-p(\alpha)+A}\neq\emptyset\}\in v\,,$$

Since also $FU(\langle\alpha_n\rangle_{n=1}^\infty)\in v$, pick $\beta\in FU(\langle\alpha_n\rangle_{n=1}^\infty)\cap\Psi_\gamma$ such that

$$\overline{A}\cap L\cap\bigcap_{p\in R}\overline{-p(\beta)+A}\neq\emptyset$$

and pick $r\in\overline{A}\cap L\cap\bigcap_{p\in R}\overline{-p(\beta)+A}$. □

Since, given any finite partition of G, one cell must be piecewise syndetic, the following corollary tells us that in any such partition there must be one cell for which there are many β's and, for each such β, a large set of a's with $\{a\}\cup\{a+p(\beta):p\in R\}$ contained in that one cell. In particular, Theorem B from the introduction holds.

2.10 Corollary. *Let $(G,+)$ be an abelian group, let $R\in\mathcal{R}$, and let $\langle\alpha_n\rangle_{n=1}^\infty$ be a sequence in $\mathcal{F}$ such that $\alpha_n<\alpha_{n+1}$ for all n. If A is a piecewise syndetic subset of G, then there exists $\beta\in FU(\langle\alpha_n\rangle_{n=1}^\infty)$ such that $\{a\in A:\{a+p(\beta):p\in R\}\subseteq A\}$ is piecewise syndetic.*

Proof. Pick by Corollary 2.9, some $r\in\overline{A}\cap K(\beta G)$ and $\beta\in FU(\langle\alpha_n\rangle_{n=1}^\infty)$ such that $\{r+p(\beta):p\in R\}\subseteq\overline{A}$. Then $A\cap\bigcap_{p\in R}(-p(\beta)+A)\in r$ and so, by [7, Theorem 4.40], $A\cap\bigcap_{p\in R}(-p(\beta)+A)$ is piecewise syndetic. If $a\in A\cap\bigcap_{p\in R}(-p(\beta)+A)$, then $\{a+p(\beta):p\in R\}\subseteq A$. □

3. Weak VIP Systems in Arbitrary Commutative Semigroups

In this section we introduce the notion of weak VIP system for arbitrary commutative semigroups, and establish that Theorem C from the introduction remains valid for such semigroups when the notion of "VIP system" is replaced by that of "weak VIP system".

3.1 Definition. Let $(S,+)$ be a commutative semigroup. Define an equivalence relation $\doteqdot$ on S by $x \doteqdot y$ if and only if there exists $z \in S$ such that $x+z=y+z$, denoting the equivalence class of x by $[x]$.

We omit the routine proof of the following lemma.

3.2 Lemma. *Let $(S,+)$ be a commutative semigroup and for $x,y \in S$, define $[x]+[y]=[x+y]$. This operation is well defined, and with this operation $S/\doteqdot$ is a commutative cancellative semigroup.*

In extending the notion of VIP system to an arbitrary commutative semigroup, we modify the characterization of Lemma 1.3. If S is a commutative cancellative semigroup and h is a VIP system in its group of quotients, one has for each $\alpha \in \Psi_{l(h)}$ some $f(\alpha)$ and $q(\alpha)$ in S such that $h(\alpha)=f(\alpha)-q(\alpha)$.

3.3 Definition. Let $(S,+)$ be a commutative semigroup. If S has an identity, denote it by 0. Otherwise let 0 be a two sided identity adjoined to S. A *weak VIP system* in S is a pair (f,q) such that there exist $l,d \in \mathbb{N}$ such that $f:\Psi_l \to S$, $q:\Psi_l \to S \cup \{0\}$, and whenever $\alpha_0,\alpha_1,\ldots,\alpha_d$ are pairwise disjoint members of Ψ_l, one has

$$\sum_{\substack{B\subseteq\{\alpha_0,\ldots,\alpha_d\}\\ B\neq\emptyset,\ |B|\text{ even}}} f(\textstyle\bigcup B) + \sum_{\substack{B\subseteq\{\alpha_0,\ldots,\alpha_d\}\\ |B|\text{ odd}}} q(\textstyle\bigcup B) \doteqdot \sum_{\substack{B\subseteq\{\alpha_0,\ldots,\alpha_d\}\\ |B|\text{ odd}}} f(\textstyle\bigcup B) + \sum_{\substack{B\subseteq\{\alpha_0,\ldots,\alpha_d\}\\ B\neq\emptyset,\ |B|\text{ even}}} q(\textstyle\bigcup B)\,.$$

Notice in particular that if $l,d \in \mathbb{N}$, $f:\Psi_l \to S$, and whenever $\alpha_0,\alpha_1,\ldots,\alpha_d$ are pairwise disjoint members of Ψ_l one has

$$\sum_{\substack{B\subseteq\{\alpha_0,\ldots,\alpha_d\}\\ B\neq\emptyset,\ |B|\text{ even}}} f(\textstyle\bigcup B) \doteqdot \sum_{\substack{B\subseteq\{\alpha_0,\ldots,\alpha_d\}\\ |B|\text{ odd}}} f(\textstyle\bigcup B)\,,$$

then $(f,\overline{0})$ is a weak VIP system in S, where $\overline{0}$ is the function with domain Ψ_l which is constantly equal to 0.

We now see that the notion which we have defined is indeed a natural extension of the notion of a VIP system.

3.4 Lemma. *Let $(S,+)$ be a commutative semigroup, let G be the group of quotients of $S/\doteqdot$, let $l \in \mathbb{N}$, let $f : \Psi_l \to S$ and let $q : \Psi_l \to S \cup \{0\}$. Then (f,q) is a weak VIP system in S if and only if the function $h : \Psi_l \to G$ defined by $h(\alpha) = [f(\alpha)] - [q(\alpha)]$ is a VIP system in G.*

Proof. Necessity. By assumption (and Lemma 3.2) we have some $d \in \mathbb{N}$ such that for any pairwise disjoint $\alpha_0, \alpha_1, \ldots, \alpha_d$ in Ψ_l one has

$$\sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ B \neq \emptyset,\ |B| \text{ even}}} [f(\bigcup B)] + \sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ |B| \text{ odd}}} [q(\bigcup B)] =$$
$$\sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ |B| \text{ odd}}} [f(\bigcup B)] + \sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ B \neq \emptyset,\ |B| \text{ even}}} [q(\bigcup B)],$$

so that

$$\sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ B \neq \emptyset,\ |B| \text{ even}}} h(\bigcup B) \ = \sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ |B| \text{ odd}}} h(\bigcup B)$$

and hence, by Lemma 1.3, h is a VIP system in G.

Sufficiency. Since h is a VIP system in G, we have by Lemma 1.3 that there is some $d \in \mathbb{N}$ such that for any pairwise disjoint $\alpha_0, \alpha_1, \ldots, \alpha_d$ in $\Psi_{l(h)}$ one has

$$\sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ B \neq \emptyset,\ |B| \text{ even}}} h(\bigcup B) \ = \sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ |B| \text{ odd}}} h(\bigcup B)$$

and consequently

$$\sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ B \neq \emptyset,\ |B| \text{ even}}} [f(\bigcup B)] + \sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ |B| \text{ odd}}} [q(\bigcup B)] =$$
$$\sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ |B| \text{ odd}}} [f(\bigcup B)] + \sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ B \neq \emptyset,\ |B| \text{ even}}} [q(\bigcup B)]$$

so that

$$\sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ B \neq \emptyset,\ |B| \text{ even}}} f(\bigcup B) + \sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ |B| \text{ odd}}} q(\bigcup B) \doteqdot \sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ |B| \text{ odd}}} f(\bigcup B) + \sum_{\substack{B \subseteq \{\alpha_0,\ldots,\alpha_d\} \\ B \neq \emptyset,\ |B| \text{ even}}} q(\bigcup B)$$

as required. $\square$

In [6] we defined a weak VIP system in a commutative cancellative semgroup S as a function $f : \Psi_l \to S$ for some l which is a VIP system in the group of quotients of

S. If one has such f, then by Lemma 3.4, the pair $(f,\overline{0})$ satisfies Definition 3.3. Our current definition is strictly more general for such semigroups because the function h of Lemma 3.4 need not take values in $S/\doteqdot$.

We now turn our attention to some preliminary lemmas involving the algebra of βS.

3.5 Lemma. *Let $(S,+)$ be a commutative semigroup, let $a,b,\in S$, and let $p\in c\ell K(\beta S)$. If $a\doteqdot b$, then $a+p=b+p$.*

Proof. Let $I=\{x\in S: a+x=b+x\}$. Then I is an ideal of S and so by [7, Corollary 4.18], $\overline{I}$ is an ideal of βS so that $c\ell K(\beta S)\subseteq\overline{I}$. Thus $I\in p$. Since λ_a and λ_b agree on a member of p, they agree at p. □

Notice that $\pi:S\to S/\doteqdot$ defined by $\pi(a)=[a]$ has a continuous extension $\widetilde{\pi}:\beta S\to\beta(S/\doteqdot)$.

3.6 Lemma. *Let $(S,+)$ be a commutative semigroup. If S has an identity denote it by 0, and otherwise, let 0 be a two sided identity adjoined to S. Let $p+p=p\in K(\beta S)$, and let $C\in p$. Let $C^\star=\{x\in C:-x+C\in p\}$, let $D=\{[a]:a\in C^\star\}$, let $D^\star=\{x\in D:-x+D\in\widetilde{\pi}(p)\}$, let $b\in S$, and let $c\in S\cup\{0\}$.*

(a) If $[b]-[c]\in D$, then $-b+(c+C^\star)\in p$.

(b) If $b\in C^\star$, then $[b]\in D^\star$.

Proof. (a). Since $[b]-[c]\in D$, pick $a\in C^\star$ such that $[b]-[c]=[a]$. Since $a\in C^\star$, we have by [7, Lemma 4.14] that $-a+C^\star\in p$. That is, $C^\star\in a+p$ and consequently $c+C^\star\in c+a+p$. Also $[b]=[c+a]$ so that, by Lemma 3.5, we have $c+C^\star\in b+p$ so that $-b+(c+C^\star)\in p$.

(b). Since $b\in C^\star$, $[b]\in D$. Also by [7, Lemma 4.14], $-b+C^\star\in p$ and so $\pi[-b+C^\star]\in\widetilde{\pi}(p)$ by [7, Lemma 3.30]. We claim that $\pi[-b+C^\star]\subseteq-[b]+D$ so that $-[b]+D\in\widetilde{\pi}(p)$ as required. Let $z\in\pi[-b+C^\star]$ and pick $y\in-b+C^\star$ such that $z=[y]$. Then $b+y\in C^\star$ so $[b]+[y]=[b+y]\in D$ and thus $z=[y]\in-[b]+D$ as required. □

3.7 Lemma. *Let $(S,+)$ be a commutative semigroup and let $p+p=p\in K(\beta S)$. Let G be the group of quotients of $S/\doteqdot$. Then $\widetilde{\pi}(p)\in K(\beta G)$ and $\widetilde{\pi}(p)+\widetilde{\pi}(p)=\widetilde{\pi}(p)$.*

Proof. The second conclusion holds because $\widetilde{\pi}$ is a homomorphism by [7, Corollary 4.22]. To establish that $\widetilde{\pi}(p)\in K(\beta G)$, we show first that $S/\doteqdot$ is piecewise syndetic in G, its group of quotients. Indeed, let $H=\{0\}$ and let $F\in\mathcal{P}_f(G)$ be given. For each $y\in F$, pick a_y and b_y in $S/\doteqdot$ such that $y=a_y-b_y$. Let $x=\sum_{y\in F}b_y$. Then $F+x\subseteq(S/\doteqdot)=\bigcup_{t\in H}(-t+S/\doteqdot)$.

Since $S/\doteqdot$ is piecewise syndetic in G, we have by [7, Theorem 4.40] that $\overline{S/\doteqdot} \cap K(\beta G) \neq \emptyset$ and thus by [7, Theorem 1.65] $K\big(\beta(S/\doteqdot)\big) = S/\doteqdot \cap K(\beta G)$. By [7, Exercise 1.7.3], $\widetilde{\pi}[K(\beta S)] = K\big(\beta(S/\doteqdot)\big)$ and so $\widetilde{\pi}(p) \in K\big(\beta(S/\doteqdot)\big) \subseteq K(\beta G)$. $\square$

We shall see that the configurations that we obtain can be chosen with the arguments α taken from the set of finite unions of any prespecified increasing sequence in $\mathcal{F}$.

3.8 Definition. Let $\langle \delta_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$. The sequence $\langle \alpha_n \rangle_{n=1}^{\infty}$ is a *union subsystem* of $\langle \delta_n \rangle_{n=1}^{\infty}$ if and only if there exists a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{F}$ such that for each $n \in \mathbb{N}$, $H_n < H_{n+1}$ and $\alpha_n = \bigcup_{t \in H_n} \delta_t$.

The following is the main result of this paper. We remind the reader that we have been aiming to generalize Theorem C in the introduction. We shall show after the proof of Theorem 3.9 that it does indeed generalize Theorem C.

3.9 Theorem. *Let $(S, +)$ be a commutative semigroup, let $\langle \delta_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$ with $\delta_n < \delta_{n+1}$ for each n, let $k \in \mathbb{N}$, and for each $i \in \{1, 2, \ldots, k\}$, let $(f^{(i)}, q^{(i)})$ be a weak VIP system in S. If C is a central set in S, then there exist a union subsystem $\langle \alpha_n \rangle_{n=1}^{\infty}$ of $\langle \delta_n \rangle_{n=1}^{\infty}$, and functions $y^{(i)} : FU(\langle \alpha_n \rangle_{n=1}^{\infty}) \to C$ for each $i \in \{0, 1, \ldots, k\}$ such that (1) $y^{(0)}(\gamma \cup \beta) = y^{(0)}(\gamma) + y^{(0)}(\beta)$ for $\gamma, \beta \in FU\big(\langle \alpha_n \rangle_{n=1}^{\infty}\big)$ with $\gamma \cap \beta = \emptyset$, and (2) $y^{(i)}(\gamma) + q^{(i)}(\gamma) = y^{(0)}(\gamma) + f^{(i)}(\gamma)$ for all $\gamma \in FU\big(\langle \alpha_n \rangle_{n=1}^{\infty}\big)$ and all $i \in \{1, 2, \ldots, k\}$.*

Proof. Let G be the group of quotients of $S/\doteqdot$. Pick $p + p = p \in K(\beta S)$ such that $C \in p$. Let $C^\star = \{x \in C : -x + C \in p\}$, let $D = \{[a] : a \in C^\star\}$, and let $D^\star = \{x \in D : -x + D \in \widetilde{\pi}(p)\}$. By Lemma 3.7, $\widetilde{\pi}(p)$ is an idempotent in $K(\beta G)$, and thus $D^\star$ is central in G.

Pick $l \in \mathbb{N}$ such that for each $i \in \{1, 2, \ldots, k\}$, $\Psi_l \subseteq \text{domain}(f^{(i)}) = \text{domain}(q^{(i)})$. By Lemma 3.4 we have for each $i \in \{1, 2, \ldots, k\}$, the function $h^{(i)} : \Psi_l \to G$ defined by $h^{(i)}(\alpha) = [f^{(i)}(\alpha)] - [q^{(i)}(\alpha)]$ is a VIP system in G. By restricting the domains, we may presume that for each $i \in \{1, 2, \ldots, k\}$, $\text{stdeg}(h^{(i)}) = \deg(h^{(i)})$.

Since $D^\star$ is central in G, it is piecewise syndetic in G. Thus, by Corollary 2.10, pick $b_1 \in G$ and $\alpha_1 \in FU(\langle \delta_n \rangle_{n=1}^{\infty})$ such that

$$\{b_1, b_1 + h^{(1)}(\alpha_1), b_1 + h^{(2)}(\alpha_1), \ldots, b_1 + h^{(k)}(\alpha_1)\} \subseteq D^\star .$$

Pick $H_1 \in \mathcal{F}$ such that $\alpha_1 = \bigcup_{t \in H_1} \delta_t$. Since $b_1 \in D$, pick $a_1 \in C^\star$ such that $b_1 = [a_1]$. Given $i \in \{1, 2, \ldots, k\}$ we have

$$[a_1 + f^{(i)}(\alpha_1)] - [q^{(i)}(\alpha_1)] = b_1 + h^{(i)}(\alpha_1) \in D^\star$$

so, by Lemma 3.6(a), $-\big(a_1+f^{(i)}(\alpha_1)\big)+(q^{(i)}(\alpha_1)+C^\star)\in p$. Since $a_1\in C^\star$, $-a_1+C^\star\in p$ by [7, Lemma 4.14]. Pick

$$x_1\in(-a_1+C^\star)\cap\textstyle\bigcap_{i=1}^k\big(-\big(a_1+f^{(i)}(\alpha_1)\big)+(q^{(i)}(\alpha_1)+C^\star)\big)\,.$$

For each $i\in\{1,2,\ldots,k\}$, pick $y^{(i)}(\alpha_1)\in C^\star$ such that $x_1+a_1+f^{(i)}(\alpha_1)=q^{(i)}(\alpha_1)+y^{(i)}(\alpha_1)$. Then $\{a_1+x_1,y^{(1)}(\alpha_1),y^{(2)}(\alpha_1),\ldots,y^{(k)}(\alpha_1)\}\subseteq C^\star$.

Inductively, let $n\in\mathbb{N}$ and assume that we have chosen sequences $\langle a_t\rangle_{t=1}^n$ and $\langle x_t\rangle_{t=1}^n$ in S and $\langle \alpha_t\rangle_{t=1}^n$ and $\langle H_t\rangle_{t=1}^n$ in $\mathcal{F}$ such that for each $t\in\{1,2,\ldots,n-1\}$ (if any) $H_t<H_{t+1}$, and for each $t\in\{1,2,\ldots,n\}$, $\alpha_t=\bigcup_{j\in H_t}\delta_j$. Also assume that for each $i\in\{1,2,\ldots,k\}$, we have a function $y^{(i)}:FU(\langle\alpha_t\rangle_{t=1}^n)\to C^\star$ such that whenever $\emptyset\neq F\subseteq\{1,2,\ldots,n\}$ and $\gamma=\bigcup_{t\in F}\alpha_t$, one has $\sum_{t\in F}(a_t+x_t)\in C^\star$ and for each $i\in\{1,2,\ldots,k\}$, $\sum_{t\in F}(a_t+x_t)+f^{(i)}(\gamma)=y^{(i)}(\gamma)+q^{(i)}(\gamma)$.

For each $\gamma\in FU(\langle\alpha_t\rangle_{t=1}^n)$, each $i\in\{1,2,\ldots,k\}$ and each $\beta\in\mathcal{F}$ with $\beta>\alpha_n$, let $g^{(i,\gamma)}(\beta)=h^{(i)}(\gamma\cup\beta)-h^{(i)}(\gamma)$. By Lemma 2.3, each $g^{(i,\gamma)}$ is a VIP system in G. Note that by Lemma 3.6(b) we have that for each F with $\emptyset\neq F\subseteq\{1,2,\ldots,n\}$, $[\sum_{t\in F}(a_t+x_t)]\in D^\star$ and, if $\gamma=\bigcup_{t\in F}\alpha_t$ and $i\in\{1,2,\ldots,k\}$, then $[y^{(i)}(\gamma)]\in D^\star$, and consequently $-[\sum_{t\in F}(a_t+x_t)]+D^\star\in\widetilde{\pi}(p)$ and $-[y^{(i)}(\gamma)]+D^\star\in\widetilde{\pi}(p)$.

Let $m=\max H_n+1$ and let

$$\begin{aligned}E\;&=\;D^\star\cap\textstyle\bigcap\big\{-[\sum_{t\in F}(a_t+x_t)]+D^\star:\emptyset\neq F\subseteq\{1,2,\ldots,n\}\big\}\cap\\ &\textstyle\bigcap\{-[y^{(i)}(\gamma)]+D^\star:i\in\{1,2,\ldots,k\},\text{ and }\gamma\in FU(\langle\alpha_t\rangle_{t=1}^n)\,.\end{aligned}$$

Then $E\in\widetilde{\pi}(p)$ and so E is piecewise syndetic in G. Pick by Corollary 2.10 some $b_{n+1}\in G$ and $\alpha_{n+1}\in FU(\langle\delta_t\rangle_{t=m}^\infty)$ such that

$$\begin{aligned}&\{b_{n+1}\}\cup\big\{b_{n+1}+h^{(i)}(\alpha_{n+1}):i\in\{1,2,\ldots,k\}\big\}\cup\\ &\{b_{n+1}+g^{(i,\gamma)}(\alpha_{n+1}):i\in\{1,2,\ldots,k\}\text{ and }\gamma\in FU(\langle\alpha_t\rangle_{t=1}^n)\}\subseteq E\,.\end{aligned}$$

Pick $H_{n+1}\subseteq\{m,m+1,m+2,\ldots\}$ such that $\alpha_{n+1}=\bigcup_{j\in H_{n+1}}\delta_j$.

Since $b_{n+1}\in D$, pick $a_{n+1}\in C^\star$ such that $b_{n+1}=[a_{n+1}]$. Since $a_{n+1}\in C^\star$, we have that $-a_{n+1}+C^\star\in p$. Also, given $\emptyset\neq F\subseteq\{1,2,\ldots,n\}$, we have $[a_{n+1}]\in-[\sum_{t\in F}(a_t+x_t)]+D^\star$ so that $[a_{n+1}+\sum_{t\in F}(a_t+x_t)]\in D^\star$ and hence, by Lemma 3.6(a), $-\big(a_{n+1}+\sum_{t\in F}(a_t+x_t)\big)+C^\star\in p$. Now, given $i\in\{1,2,\ldots,k\}$,

$$[a_{n+1}+f^{(i)}(\alpha_{n+1})]-[q^{(i)}(\alpha_{n+1})]=b_{n+1}+h^{(i)}(\alpha_{n+1})\in D^\star$$

so by Lemma 3.6(a), $-\big(a_{n+1}+f^{(i)}(\alpha_{n+1})\big)+(q^{(i)}(\alpha_{n+1})+C^\star)\in p$.

We now claim that for each F with $\emptyset\neq F\subseteq\{1,2,\ldots,n\}$ and each $i\in\{1,2,\ldots,k\}$, if $\mu=\alpha_{n+1}\cup\bigcup_{t\in F}\alpha_t$, then

$$[a_{n+1}+\textstyle\sum_{t\in F}(a_t+x_t)+f^{(i)}(\mu)]-[q^{(i)}(\mu)]\in D^\star$$

and thus $-\big(a_{n+1}+\sum_{t\in F}(a_t+x_t)+f^{(i)}(\mu)\big)+(q^{(i)}(\mu)+C^\star)\in p$. So let such F, i, and μ be given and let $\gamma=\bigcup_{t\in F}\alpha_t$. Then

$$[a_{n+1}]+g^{(i,\gamma)}(\alpha_{n+1})\in E\subseteq -[y^{(i)}(\gamma)]+D^\star$$

and so $[a_{n+1}+y^{(i)}(\gamma)]+g^{(i,\gamma)}(\alpha_{n+1})\in D^\star$. Now $g^{(i,\gamma)}(\alpha_{n+1})+h^{(i)}(\gamma)=h^{(i)}(\gamma\cup\alpha_{n+1})$. That is, $g^{(i,\gamma)}(\alpha_{n+1})+[f^{(i)}(\gamma)]-[q^{(i)}(\gamma]=[f^{(i)}(\mu)]-[q^{(i)}(\mu)]$. Thus $[a_{n+1}+y^{(i)}(\gamma)+f^{(i)}(\mu)+q^{(i)}(\gamma)]-[f^{(i)}(\gamma)+q^{(i)}(\mu)]\in D^\star$. Since $y^{(i)}(\gamma)+q^{(i)}(\gamma)=\sum_{t\in F}(a_t+x_t)+f^{(i)}(\gamma)$ we have that $[a_{n+1}+\sum_{t\in F}(a_t+x_t)+f^{(i)}(\mu)]-[q^{(i)}(\mu)]\in D^\star$ as claimed.

Pick

$$\begin{aligned}x_{n+1}\in\quad &(-a_{n+1}+C^\star)\cap\bigcap\big\{-\big(a_{n+1}+\textstyle\sum_{t\in F}(a_t+x_t)\big)+C^\star:\emptyset\neq F\subseteq\{1,2,\ldots,n\}\big\}\\ &\cap\bigcap\big\{-\big(a_{n+1}+f^{(i)}(\alpha_{n+1})\big)+(q^{(i)}(\alpha_{n+1})+C^\star):i\in\{1,2,\ldots,k\}\big\}\\ &\cap\bigcap\{-\big(a_{n+1}+\textstyle\sum_{t\in F}(a_t+x_t)+f^{(i)}(\mu)\big)+(q^{(i)}(\mu)+C^\star):\\ &\emptyset\neq F\subseteq\{1,2,\ldots,n\}\,,i\in\{1,2,\ldots,k\},\text{ and }\mu=\alpha_{n+1}\cup\textstyle\bigcup_{t\in F}\alpha_t\}\,.\end{aligned}$$

Given $i\in\{1,2,\ldots,k\}$, pick $y^{(i)}(\alpha_{n+1})\in C^\star$ such that $y^{(i)}(\alpha_{n+1})+q^{(i)}(\alpha_{n+1})=x_{n+1}+a_{n+1}+f^{(i)}(\alpha_{n+1})$. Given $i\in\{1,2,\ldots,k\}$, $\emptyset\neq F\subseteq\{1,2,\ldots,n\}$, and $\mu=\alpha_{n+1}\cup\bigcup_{t\in F}\alpha_t$, pick $y^{(i)}(\mu)\in C^\star$ such that $y^{(i)}(\mu)+q^{(i)}(\mu)=x_{n+1}+a_{n+1}+\sum_{t\in F}(a_t+x_t)+f^{(i)}(\mu)$.

One can routinely verify that the induction hypotheses are satisfied. To complete the proof, we only need to define the function $y^{(0)}:FU(\langle\alpha_n\rangle_{n=1}^\infty)\to C$. Given $\gamma\in FU(\langle\alpha_n\rangle_{n=1}^\infty)$, pick the unique $F\in\mathcal{P}_f(\mathbb{N})$ such that $\gamma=\bigcup_{t\in F}\alpha_t$ and let $y^{(0)}(\gamma)=\sum_{t\in F}(x_t+a_t)$. □

In the event that all of the weak VIP systems are of the form $(f,\overline{0})$, Theorem 3.9 takes the following simpler form.

3.10 Corollary. *Let $(S,+)$ be a commutative semigroup, let $\langle\delta_n\rangle_{n=1}^\infty$ be a sequence in $\mathcal{F}$ with $\delta_n<\delta_{n+1}$ for each n, let $k\in\mathbb{N}$, and for each $i\in\{1,2,\ldots,k\}$, let $(f^{(i)},\overline{0})$ be a weak VIP system in S. If C is a central set in S, then there exist a sequence $\langle a_n\rangle_{n=1}^\infty$ in S and a union subsystem $\langle\alpha_n\rangle_{n=1}^\infty$ of $\langle\delta_n\rangle_{n=1}^\infty$ such that whenever $F\in\mathcal{P}_f(\mathbb{N})$ and $\gamma=\bigcup_{n\in F}\alpha_n$ one has*

$$\{\textstyle\sum_{n\in F}a_n\}\cup\big\{\textstyle\sum_{n\in F}a_n+f^{(i)}(\gamma):i\in\{1,2,\ldots,k\}\big\}\subseteq C\,.$$

Proof. Pick $\langle\alpha_n\rangle_{n=1}^\infty$ and for each $i\in\{0,1,\ldots,k\}$ pick $y^{(i)}:FU(\langle\alpha_n\rangle_{n=1}^\infty)\to C$ as guaranteed by Theorem 3.9. For $n\in\mathbb{N}$, let $a_n=y^{(0)}(\alpha_n)$ and notice that if $F\in\mathcal{P}_f(\mathbb{N})$ and $\gamma=\bigcup_{n\in F}\alpha_n$, then $\sum_{n\in F}a_n=y^{(0)}(\gamma)\in C$ by condition (1). Then by (2), for $i\in\{1,2,\ldots,k\}$, $\sum_{t\in F}a_t+f^{(i)}(\gamma)=y^{(i)}(\gamma)\in C$. □

Finally, as promised, we show that Theorem C of the introduction is indeed a consequence of Theorem 3.9.

3.11 Corollary. *Let $(S,+)$ be a commutative cancellative semigroup, let $k \in \mathbb{N}$, for each $i \in \{1,2,\ldots,k\}$, let $h^{(i)}$ be a VIP system in the group of quotients of S, and let C be a central set in S. Then there exist sequences $\langle a_n\rangle_{n=1}^\infty$ in S and $\langle \alpha_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that $\alpha_n < \alpha_{n+1}$ for each n and for every $F \in \mathcal{F}$ and every $i \in \{1,2,\ldots,k\}$, if $\gamma = \bigcup_{t\in F}\alpha_t$, then $\sum_{t\in F} a_t \in C$ and $\sum_{t\in F} a_t + h^{(i)}(\gamma) \in C$.*

Proof. For each $i \in \{1,2,\ldots,k\}$ and each $\alpha \in \Psi_{l(h^{(i)})}$, pick $f^{(i)}(\alpha)$ and $q^{(i)}(\alpha)$ in S such that $h^{(i)}(\alpha) = f^{(i)}(\alpha) - q^{(i)}(\alpha)$. Then by Lemma 3.4, each $(f^{(i)}, q^{(i)})$ is a weak VIP system in S. Pick $\langle\alpha_n\rangle_{n=1}^\infty$ and for each $i \in \{0,1,\ldots,k\}$ a function $y^{(i)} : FU(\langle\alpha_n\rangle_{n=1}^\infty) \to C$ as guaranteed by Theorem 3.9. For $n \in \mathbb{N}$, let $a_n = y^{(0)}(\alpha_n)$ so that if $F \in \mathcal{P}_f(\mathbb{N})$ and $\gamma = \bigcup_{n\in F}\alpha_n$, then $\sum_{n\in F} a_n = y^{(0)}(\gamma) \in C$. Also, given $i \in \{1,2,\ldots,k\}$,

$$\textstyle\sum_{t\in F} a_t + h^{(i)}(\gamma) = \sum_{t\in F} a_t + f^{(i)}(\gamma) - q^{(i)}(\gamma) = y^{(i)}(\gamma) \in C\,.$$

□

References

[1] V. Bergelson, H. Furstenberg, and R. McCutcheon, *IP-sets and polynomial recurrence*, Ergodic Theory and Dynamical Systems **16** (1996), 963-974.

[2] V. Bergelson and A. Leibman, *Polynomial extensions of van der Waerden's and Szemerédi's theorems*, Journal Amer. Math. Soc. **9** (1996) 725-753.

[3] V. Bergelson and A. Leibman, *Set polynomials and a polynomial extension of Hales-Jewett theorem*, Annals of Math. **150** (1999), 33-75.

[4] H. Furstenberg, *Recurrence in ergodic theory and combinatorical number theory*, Princeton University Press, Princeton, 1981.

[5] N. Hindman, *Problems and new results in the algebra of βS and Ramsey Theory*, in <u>Unsolved problems on mathematics for the 21st Century</u>, J. Abe and S. Tanaka (eds.), IOS Press, Amsterdam, to appear.

[6] N. Hindman and R. McCutcheon, *VIP systems in partial semigroups*, manuscript.

[7] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.

[8] R. McCutcheon, *An infinitary polynomial van der Waerden theorem*, J. Combinatorial Theory (Series A) **86** (1999), 214-231.

Neil Hindman
Department of Mathematics
Howard University
Washington, DC 20059
nhindman@howard.edu
nhindman@aol.com

Randall McCutcheon
Department of Mathematics
University of Maryland
College Park, MD 20742
randall@math.umd.edu