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Research of Wis Comfort on ultrafilters and the Stone-Čech compactification

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Dedication. Wis showed me how to teach, he taught me how to prove theorems, and he demonstrated how to deal honestly with everyone – but no so honestly as to cause unnecessary hurt.

Abstract

I survey the publications of Wis Comfort whose main subject was either the Stone-Čech compactification of a completely regular Hausdorff space or the set of ultrafilters on a given set. These areas are tied together by the fact that if X is a discrete space, its Stone-Čech compactification can be viewed as the set of ultrafilters on X.

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1. Introduction

When I was his student, from 1966 to 1969, Wis was primarily a topologist, with a special interest in the Stone-Čech compactification and ultrafilters. His specialization in the theory of topological groups came later. (All but two of the publications reviewed in this article were published by 1980. By that time, Wis had only published four papers about topological groups.)

In Section 2 we will present some of Wis' results on set theory and the theory of ultrafilters. Section 3 will deal with Stone-Čech remainders, and Section 4 will consist of other results about the topology of βX.

I will presume that the reader is familiar with the basic facts about ultrafilters and the Stone-Čech compactification as presented in Chapters 1, 2, and

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3 of [19]. Also, as in that book, all hypothesized spaces will be assumed to be completely regular Hausdorff spaces.

2. Set theory and ultrafilters

If one mentions the name "Wis Comfort", especially in conjunction with "Stelios Negrepontis", to mathematicians, they are likely to think of the book *The Theory of Ultrafilters* [16]. I will ignore that book here, not because of a lack of interest, but rather because a thorough review would occupy all of the space allocated for this paper. I will also only mention the papers [6] and [7] on ultrafilters because they are themselves surveys.

During the two years that Wis was on the faculty of the University of Massachusetts, his paper *A theorem of Stone-Čech type, and a theorem of Tychonoff type, without the axiom of choice* [3], was making quite a stir, both before and after its publication. I quote from the introduction: "Experience gained presenting the content of this paper before a learned audience discloses that the wily, attentive listener will expend more energy searching for the possible hidden presence of the axiom of choice in the proofs than he will in following the positive, constructive aspects of these proofs. . . . The following worthy principle of exposition is, then, abandoned: An argument or a proof should be omitted from a manuscript, if it is more easily established, or logically less essential, than some other argument or proof which has been omitted."

In [3] Wis defined a space to be compact* if and only if whenever M is a maximal ideal in the ring $C^*(X)$ of all bounded continuous real valued functions on X, one has $\bigcap\{Z(f) : f \in M\} \neq \emptyset$, where $Z(f) = \{x \in X : f(x) = 0\}$. Using some information from [19, Exercise 2L], it is easy to see without using choice that any compact space is compact*. And, of course, in the presence of choice, the notions are equivalent. Wis proved, in several parts, without invoking the axiom of choice, the following theorems.

Theorem 2.1. *For each space X, there is a compact* space βX in which (a homeomorph of) X is dense and C^*-embedded; the space βX may be chosen homeomorphic with a closed subset of a product of closed intervals* $[0, 1]$.

Proof. [3, Theorem 3.2*] □

Theorem 2.2. *(a) If X is compact*, then $X = \beta X$. (b) Each compact* space is (homeomorphic with) a closed subset of a product of intervals* $[0, 1]$.

Proof. [3, Theorem 3.3*] □

Of course, in the absence of choice not all powers of $[0, 1]$ are compact.

Theorem 2.3. *If $\beta' X$ is a compact* space in which X is dense and C^*-embedded, then there is a homeomorphism from βX onto $\beta' X$ leaving X fixed pointwise.*

Proof. [3, Theorem 3.6*] □

Wis noted in the conclusion of [3] that the appeal of the notion of compact* is somewhat limited by the fact that it is consistent with ZF that $\mathbb{N}$ with the discrete topology is compact*.

In [14] Wis and Stelios Negrepontis considered the following relation between cardinals α and δ. They said that $\alpha\mathcal{S}\delta$ if and only if there is an α-complete filter on the discrete space X of cardinality δ which does not extend to an α-complete ultrafilter. (A filter $\mathcal{F}$ is α-complete if and only if whenever $\mathcal{G} \subseteq \mathcal{F}$ and $|\mathcal{G}| < \alpha$, one has $\bigcap \mathcal{G} \in \mathcal{F}$.) Notice that any filter on any set is ω-complete so $\omega\mathcal{S}\delta$ fails for every cardinal δ.

The authors characterized when $\alpha\mathcal{S}\delta$ fails for a measurable cardinal $\alpha > \omega$ and a cardinal $\delta \geq \alpha$ in terms of the topology of $U_\alpha(\delta)$, the space of α-uniform ultrafilters on δ, with the relative topology from $\beta(\delta)$, the Stone-Čech compactification of the set δ with the discrete topology. (From here on, if I write $\beta(\alpha)$ for some cardinal α and don't specify otherwise, I shall assume that α has the discrete topology.) The authors define $\Omega_\alpha(\delta)$ to be the set of α-complete ultrafilters on δ. (So a cardinal α is measurable if and only if $\Omega_\alpha(\alpha) \neq \emptyset$.) It is easy to see that $\Omega_\alpha(\delta) \subseteq U_\alpha(\delta)$. Given an open subset V of $U_\alpha(\delta)$, the *type* of V is the least cardinal τ such that there is a set $\mathcal{V}$ of clopen subsets of $U_\alpha(\delta)$ such that $|\mathcal{V}| = \tau$ and $\bigcup \mathcal{V} = V$.

Theorem 2.4. *If $\omega < \alpha \leq \delta$ and α is measurable, then $\alpha\mathcal{S}\delta$ fails if and only if*

(1) $\Omega_\alpha(\delta)$ is C^*-embedded in $U_\alpha(\delta)$ and

(2) each open subset of $U_\alpha(\delta)$ which contains $\Omega_\alpha(\delta)$ has an open subset which is dense in $U_\alpha(\delta)$ and has type no larger than α.

Proof. [14, Theorem 3.6] □

In [14, Theorem 3.9] the authors showed that if $\alpha^+ = 2^\alpha$ and $\delta = \alpha$, then condition (2) of Theorem 2.4 can be deleted.

In [15], again a joint effort of Wis and Stelios Negrepontis, the authors consider the notion of a family of κ-large oscillation. To avoid confusion with cardinal exponentiation, I write here ${}^\alpha\delta$ to denote the set of functions from α to δ, so that $|{}^\alpha\delta| = \delta^\alpha$.

Definition 2.5. Let α, δ, and κ be cardinals and let $\mathcal{F} \subseteq {}^\alpha\delta$. The family $\mathcal{F}$ has *κ-large oscillation* if and only if, whenever $\lambda < \kappa$, $\langle f_\zeta \rangle_{\zeta<\lambda}$ is a λ-sequence of distinct elements of $\mathcal{F}$, and $\langle \xi_\zeta \rangle_{\zeta<\lambda}$ is a λ-sequence of (not necessarily distinct) elements of α, there exists $\sigma < \alpha$ such that $f_\zeta(\sigma) = \xi_\zeta$ for each $\zeta < \lambda$.

The main theorem of [15] establishes the equivalence of seven conditions, three of which are as follows.

Theorem 2.6. *Let α and κ be cardinals for which $\omega \leq \kappa \leq \alpha$. The following statements are equivalent.*

(1) $\alpha = \sup\{\alpha^\lambda : \lambda < \kappa\}$.

(2) There exists $\mathcal{F} \subseteq {}^\alpha\alpha$ such that $|\mathcal{F}| = 2^\alpha$ and $\mathcal{F}$ has κ-large oscillation.

(3) There exists $\mathcal{F} \subseteq {}^{\alpha}\alpha$ such that $|\mathcal{F}| = \alpha^+$ and $\mathcal{F}$ has κ-large oscillation.

Proof. [15, Theorem 3.1] $\square$

As the authors remark in the introduction, the equivalence of (2) and (3) in Theorem 2.6 "may account for the use of families of κ-large oscillation in avoiding the generalized continuum hypothesis".

Recall that, given $p \in \beta(\alpha)$, the *type* of p,

$$\tau(p) = \{q \in \beta(\alpha) : (\exists h : \alpha \xrightarrow[\text{onto}]{\text{1-1}} \alpha)(h^\beta(q) = p)\},$$

where $h^\beta : \beta(\alpha) \to \beta(\alpha)$ is the continuous extension of h. Also, the Rudin-Keisler order on $\beta(\alpha)$ is defined by $p \prec q$ if and only if there exists $h : \alpha \to \alpha$ such that $h^\beta(q) = p$. Further, this order induces an order on the set of types of $\beta(\alpha)$.

One of the applications of Theorem 2.6 involves the set $\Omega_\kappa(\alpha)$ introduced in [14] and discussed above.

Theorem 2.7. *Let κ be a regular cardinal for which $\omega \le \kappa \le \alpha = \sup\{\alpha^\lambda : \lambda < \kappa\}$ and suppose that each κ-complete filter on α extends to a κ-complete ultrafilter. Then each set $\mathcal{A} \subseteq \tau[\Omega_\kappa(\alpha)]$ with $|\mathcal{A}| \le 2^\alpha$ has an upper bound with respect to the Rudin-Keisler ordering of types.*

Proof. [15, Theorem 4.3] $\square$

The last paper to be considered in this section is one of only two joint papers I wrote with Wis. (The other does not fall under the subject matters covered here.)

Given an infinite cardinal α, I will write $U(\alpha)$ for $U_\alpha(\alpha)$. Thus, for example, $U(\omega)$ is just $\beta\omega \setminus \omega$. It has been known at least since 1956 [18] that $\beta\mathbb{N} \setminus \mathbb{N}$ is an F'-space, that is disjoint cozero sets have disjoint closures. Given a cardinal λ, say that a point x of a topological space X is λ-point provided it lies in the closure of each of λ pairwise disjoint open sets. I showed in [21] that there exist 2^ω-points in $U(\omega)$ and if the continuum hypothesis is assumed, each point of $U(\omega)$ is a 2^ω-point. After the publication of [9] it was shown without invoking any special set theoretic assumptions by Balcar and Vojtáš [1] that every point of $U(\omega)$ is a 2^ω-point.

What is probably the main result of [9] is the following.

Theorem 2.8. *Let α and γ be cardinals with $\alpha \ge \omega$. If there is a γ-point in $U(\alpha)$, then for every nonempty open subset V of $U(\alpha)$, there is a subset S of V such that S is homeomorphic to $\beta(\alpha)$ and every point of S is a γ-point of $U(\alpha)$.*

Proof. [9, Theorem 3.6] $\square$

Given the result of Balcar and Vojtáš cited above, Theorem 2.8 is only of interest if $\alpha > \omega$.

3. The Stone-Čech remainder

It is an old result of Tarski [24] that if α and δ are infinite cardinals, then there is a collection $\mathcal{D}$ of infinite subsets of α such that $|\mathcal{D}| = \delta$ and each pair from $\mathcal{D}$ has finite intersection if and only if $\delta \leq \alpha^\omega$. Consequently if $\delta \leq \alpha^\omega$, then $\beta(\alpha) \setminus \alpha$ contains a collection of δ pairwise disjoint open sets. In [8], Wis and Hugh Gordon addressed the corresponding problem for $\beta X \setminus X$ where X is not necessarily discrete. They obtain the following characterization.

Theorem 3.1. *Let X be a topological space and let α be an infinite cardinal. The following statements are equivalent.*

(1) There exists a family $\mathcal{U}$ of pairwise disjoint nonempty open subsets of $\beta X \setminus X$ with $|\mathcal{U}| = \alpha$.

(2) There exists a family $\mathcal{V}$ of cozero sets in X such that $|\mathcal{V}| = \alpha$ and

(a) for all $U \in \mathcal{V}$, U contains a noncompact zero set and

(b) for all U and V in $\mathcal{V}$, if $U \neq V$, then $c\ell_X(U \cap V)$ is compact.

Proof. [8, Theorem 3.3] □

They also established a simpler sufficient condition.

Theorem 3.2. *Let X be a topological space and let α be an infinite cardinal. Assume that there is a locally finite collection $\mathcal{F}$ of nonempty open subsets of X with $|\mathcal{F}| = \alpha$ such that for each $U \in \mathcal{F}$, $c\ell_X(U)$ is compact. Then there is a family $\mathcal{U}$ of pairwise disjoint open subsets of $\beta X \setminus X$ such that $|\mathcal{U}| = \alpha^\omega$.*

Proof. [8, Theorem 3.1] □

As an easy consequence of Theorem 3.2, the authors showed that if X is locally compact and not pseudocompact (i.e., there is an unbounded continuous real valued function on X), then there is a collection of 2^ω pairwise disjoint open subsets of $\beta X \setminus X$. As a consequence one has that $\beta\mathbb{R} \setminus \mathbb{R}$ contains a collection of 2^ω pairwise disjoint open sets and we have already observed that $\beta\mathbb{N} \setminus \mathbb{N}$ does as well. By way of contrast, Wis showed in [4] that $\beta\mathbb{Q} \setminus \mathbb{Q}$ is separable and consequently there does not exist a family $\mathcal{U}$ of pairwise disjoint open subsets of $\beta\mathbb{Q} \setminus \mathbb{Q}$ with $|\mathcal{U}| > \omega$.

Recall that the *density character* of a space is the least cardinality of a dense subset. In [2] Wis, in a result "which appears to be new" established that if α is an infinite cardinal, then the density character of $\beta(\alpha) \setminus \alpha$ is α^ω. Wis then proved the following theorem.

Theorem 3.3. *Let α be an infinite cardinal with the discrete topology. There is a continuous function from $\beta(\alpha)$ onto $\beta(\alpha) \setminus \alpha$ if and only if $\alpha = \alpha^\omega$.*

Proof. [2, Theorem 4.2] □

He noted in particular that there is a continuous function from $\beta(\omega_1)$ onto $\beta(\omega_1) \setminus \omega_1$ if and only if the continuum hypothesis holds. He also established the following consequence of the continuum hypothesis.

Theorem 3.4. *Assume the continuum hypothesis holds. If there is a retraction from βX onto $\beta X \setminus X$, then X is locally compact and pseudocompact.*

Proof. [2, Theorem 2.6] $\square$

Let Ω be the set of points of $\beta(\omega_1) \setminus \omega_1$ that are in the closure of some countable subset of ω_1. Let Λ be the set of functions from ω_1 to $\{0,1\}$ with the order topology induced by the lexicographic order. Let Λ_π be the set Λ with the topology with basis consisting of all G_δ sets with respect to the original topology. In [13], Wis and Stelios Negrepontis showed, often assuming the continuum hypothesis, that each of $\beta\mathbb{N} \setminus \mathbb{N}$, Ω, and Λ_π, "has many homeomorphs, some of them familiar."

In [17] Wis and Liam O'Callaghan addressed the issue of finding spaces that are homeomorphic to their own Stone-Čech remainder. Their main results are the following two theorems. Recall that a space is realcompact if and only if it is homeomorphic to a closed subspace of a product of copies of $\mathbb{R}$. Also recall that if $X \subseteq Y \subseteq \beta X$, then $\beta Y = \beta X$.

Theorem 3.5. *Let X be a first-countable realcompact space and suppose there exists Y such that $X \subseteq Y \subseteq \beta X$ and Y is homeomorphic to $\beta Y \setminus Y$. Then X is discrete and Y is pseudocompact.*

Proof. [17, Theorem 2.6(1)] $\square$

Theorem 3.6. *If X is an infinite discrete space, then there is a pseudocompact space Y such that $X \subseteq Y \subseteq \beta X$ and Y is homeomorphic to $\beta Y \setminus Y$.*

Proof. [17, Theorem 2.6(2)] $\square$

They also prove that if there is an Ulam-measurable cardinal (i.e., a cardinal supporting a countably additive $\{0,1\}$-valued measure), then there exist a nondiscrete metric space X and a space Y such that $X \subseteq Y \subseteq \beta X$ and Y is homeomorphic to $\beta Y \setminus Y$.

In [10], Wis and Akio Kato showed that for certain cardinals α and δ, $\beta(\omega)\setminus\omega$ can be written as the union of δ pairwise disjoint and pairwise nonhomeomorphic sets, often satisfying additional conditions. For example, they established the following theorem.

Theorem 3.7. *Let $1 \leq \alpha \leq 2^{\mathfrak{c}}$. The space $\omega^* = \beta(\omega) \setminus \omega$ can be written as $\bigcup_{\xi<\alpha} C_\xi$ where for each $\xi < \alpha$, C_ξ dense in ω^*, C_ξ has cardinality $2^{\mathfrak{c}}$, and there is no one-to-one continuous function from C_ξ to $\omega^* \setminus C_\xi$. (In particular, the spaces C_ξ are pairwise nonhomeomorphic.)*

Proof. [10, Theorem 4.1] $\square$

In [11], Wis, Akio Kato, and Saharon Shelah considered the relation $\omega^* \to (Y)^1_2$, which means that whenever $\omega^* = \beta(\omega) \setminus \omega$ is partitioned into two cells, one of these cells has a subset which is homeomorphic to Y. They show that this relation fails if $|Y| \geq 2^{\mathfrak{c}}$, fails if Y is infinite and countably compact, and fails if $Y = \omega \cup \{p\}$ for some $p \in \omega^*$. They also show that the relation holds for Y discrete if and only if $|Y| \leq \mathfrak{c}$ and that there are certain nondiscrete P-spaces Y for which it holds.

4. Additional results on the topology of βX

In [12], Wis and Stelios Negrepontis considered the following notion.

Definition 4.1. A pair of spaces (X, Y) is called a *C^*-pair* if and only if $X \times Y$ is C^*-embedded in $\beta X \times Y$ and in $X \times \beta Y$. It is a *proper C^*-pair* if, in addition, $X \times Y$ is not C^*-embedded in $\beta X \times \beta Y$.

The motivation for studying this notion was Glicksberg's Theorem [20] which says that $X \times Y$ is C^*-embedded in $\beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact. (Equivalently, $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact.)

The authors give an example of a nondiscrete topological group G for which (G, G) is a proper C^*-pair. Among other results, they prove the following theorem.

Theorem 4.2. *Let D be a discrete space. Then (D, Y) is a proper C^*-pair for each infinite P-space Y if and only if $|D| = \omega$.*

Proof. [12, Theorem 4.2] □

They then conjectured that Theorem 4.2 could be strengthened by showing that if (X, Y) is a proper C^*-pair for each P-space Y, then X is the countable discrete space. In a note added in proof they remark that their conjecture had been proved by Tony Hager. Tony never published his proof, but the result was generalized by Norman Noble in [22, Theorem 3.5] who noted that the relevant part of that theorem was due to Tony.

The *Baire sets* in a space X are defined to be the members of the smallest σ-algebra containing the set of zero sets of X. So, trivially, zero sets are Baire sets. In [5], Wis proved the following theorem and went on to show how this extended an earlier result of Kenneth Ross and Karl Stromberg [23].

Theorem 4.3. *Assume that X is a Baire set in βX and let A be a closed Baire set in X. Then A is a zero set in X.*

Proof. [5, Theorem 1.2] □

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