The review on the following pages was written by John Pym and was published in *Semigroup Forum* **59** (1999), 310-314.

Semigroup Forum Vol. 59 (1999) 310-314

BOOK REVIEW

Algebra in the Stone-Čech Compactification by Neil Hindman and Dona Strauss

De Gruyter Expositions in Mathematics 27
de Gruyter, Berlin, New York, 1998, xiii + 485 pp.
ISBN 3-11-015420-X

John Pym

Communicated by Boris M. Schein

This is an excellent book. This review is an attempt to convince the reader that this verdict is not the prejudice of an enthusiast but a sober, sound judgement.

The title might suggest that the subject matter of the book is rather esoteric. The Stone-Čech compactification is a remote object. Its elements are ultrafilters and the existence of non-trivial ultrafilters depends on the axiom of choice. This means that an interesting ultrafilter has never been seen, and although analysts may not be troubled by objects which can only be imagined, a down-to-earth algebraist or number theorist might wonder why time should be wasted reading about them. In fact ultrafilters have a very desirable property: if the positive integers $\mathbb{N}$ are partitioned into a finite number of sets, then any ultrafilter on $\mathbb{N}$ will pick out exactly one set from the partition. From this observation, a really wild speculation would be that maybe ultrafilters could be used to prove van der Waerden's Theorem, that given an integer ℓ and a finite partition of $\mathbb{N}$, there is a set in the partition which contains an arithmetic progression of length ℓ. A proof of this kind was published in 1989 by V Bergelson, H Furstenberg, Y Katznelson and N Hindman. It is essentially easy. I myself have presented it to a general audience in a one hour lecture, including background theory semigroup theorists already know.

The Stone-Čech compactification $\beta\mathbb{N}$– the set of all ultrafilters on $\mathbb{N}$ with a natural topology which happens to be compact – is an essential ingredient in this proof. Whole books have been devoted to this compactification as a topological object since its first appearance in 1937. But for applications to number theory, the addition of $\mathbb{N}$ has to be extended to $\beta\mathbb{N}$. Opinions differ about when this was first done. M M Day certainly could without difficulty have added it to his 1957 paper on amenable semigroups, but he did not. In 1963, P Civin and B Yood mentioned that as a consequence of their study of second dual Banach algebras $\beta\mathbb{Z}$ has a natural semigroup structure but they did nothing with it. The real beginnings of the theory may therefore be ascribed to R Ellis [3]. He showed that βS is a semigroup for any discrete group S using an ultrafilter approach, and he did so because he needed that semigroup structure in topological dynamics. There is good reason even for number theorists to consider general semigroups S : the cases $(\mathbb{N},+)$, $(\mathbb{N},\cdot)$, $(\mathbb{Z},+)$ etc can all be dealt with at the same time. Moreover if S is a semigroup under two operations $+$ and $\cdot$ simultaneously then both extend to βS. That is why the present book is about *algebra* in Stone-Čech compactifications and not simply semigroup structure.

This book aims at, and succeeds in, taking the subject from its beginnings to the frontiers of research. Long-standing readers of *Semigroup Forum* will probably omit the first chapter, which is a good account of (a small fraction of) the elementary algebraic structure of semigroups, in particular of the smallest ideal, but their students will welcome it. Hard-line algebraists can discover from Chapter 2 how simple the basic theory of compact semigroups is. The topological Stone-Čech compactification of a discrete space is presented in the third chapter as far as it is needed for the rest of the book.

The key construction is presented in Chapter 4. If a discrete space S is a semigroup then so is βS. The properties of βS are at first sight discouraging, for even if S is commutative, βS is rarely so. Indeed there is a topological version of this observation which is even less promising; the multiplication in βS is continuous in one variable but in nearly all cases not in the other. However βS does have a smallest ideal, and semigroup properties of βS can often be given combinatorial characterisations in terms of ultrafilters.

Next come indications of the spectacular achievements which can arise from the use of βS. One fundamental example involves finite sums or products. Using the product formulation, given a sequence $\langle x_n \rangle$ in S write

$$\mathrm{FP}(\langle x_n \rangle) = \{x_{n_1} x_{n_2} \ldots x_{n_k} : k \text{ is an integer and } n_1 < n_2 < \ldots < n_k\}.$$

Then if $A_1, \ldots, A_r$ is a finite partition of S there exist a sequence $\langle x_n \rangle$ in S and i such that $\mathrm{FP}(\langle x_n \rangle) \subseteq A_i$. The proof is very simple. The closures $\overline{A_1}, \ldots, \overline{A_r}$ form a partition of βS, and since any compact semigroup contains an idempotent one of the $\overline{A_i}$ must contain an idempotent; the interpretation of 'idempotent' in terms of ultrafilters gives the finite product result. This proof, due to F Galvin and S Glazer, marks the first success of βS in this area. Neil Hindman had previously established the conclusion for $\mathbb{N}$ by a hard combinatorial argument. Now $\beta\mathbb{N}$ has two semigroup operations, one coming from addition, and one from multiplication, in $\mathbb{N}$. If we take a partition of $\mathbb{N}$ there exist sequences $\langle x_n \rangle$, $\langle y_n \rangle$ and a set in the partition which contains both $\mathrm{FP}(\langle x_n \rangle)$ and also the similarly-defined set of finite sums $\mathrm{FS}(\langle y_n \rangle)$. This result was first established using $\beta\mathbb{N}$ by an argument a little more subtle than the one above, and only afterwards was an elementary proof discovered. Later in this book it is shown that it is not possible to take $\langle x_n \rangle = \langle y_n \rangle$ in this conclusion, a result related to the impossibility of solving the equation $p + p = p.p$ in $\beta\mathbb{N}$.

Neil Hindman began investigating $\beta\mathbb{N}$ as an algebraic object with a compact topology in the 1970s with a view to its applications in Ramsey Theory (a particular kind of study of partitions). It was inevitable that a fascination with the intrinsic properties of this structure would follow. Over a third of this book is devoted to βS in its own right. An early result showed that the non-commutativity in $\beta\mathbb{N}$ is extreme: $(\beta\mathbb{N}, +)$ contains $2^{\mathfrak{c}}$ copies of the free group on $2^{\mathfrak{c}}$ generators. It also has $2^{\mathfrak{c}}$ minimal left ideals and the same number of minimal right ideals. Of course, $2^{\mathfrak{c}}$ is also the number of points in $\beta\mathbb{N}$, so these numbers are as large as possible in this context.

One of the most remarkable features of the theory (and one which I personally still find mysterious) is the ubiquity in semigroups of the form βS of a subsemigroup which the authors denote by $\mathbb{H}$, a notation which appears to hint at a structure whose importance may rival that of $\mathbb{R}$ or $\mathbb{Z}$. It is simple to define:

$$\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}},$$

with the closures taken in $\beta\mathbb{N}$. If $\langle x_n \rangle$ is a sequence with distinct finite products (that is, the expression of any element in the form $x_{n_1}x_{n_2}\ldots x_{n_k}$ with $n_1 < n_2 < \ldots < n_k$ is unique) in any semigroup S then the closure $\overline{\mathrm{FP}(\langle x_n \rangle)}$ in βS contains a copy of $\mathbb{H}$. Moreover, there is a copy of $\mathbb{H}$ near any right cancellable element in βG for any discrete group G. The results about the complexity of $\beta\mathbb{N}$ mentioned above are proved using $\mathbb{H}$. $\mathbb{H}$ also contains an infinite decreasing sequence of idempotents in the usual order ($e \leq f$ if and only if $e = ef = fe$). But simple questions about $\mathbb{H}$ remain unanswered; for example, does it contain an infinite increasing sequence of idempotents?

Topologists know that $\beta\mathbb{N}$ contains many – $2^{\mathfrak{c}}$ – topological copies of itself. Does $(\beta\mathbb{N}, +)$ contain any algebraic and topological copies of itself? The solution was given in 1992 by Dona Strauss in the most dramatic of her many contributions to the subject. Any continuous homomorphism from $\beta\mathbb{N}$ into $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ must have *finite* image (so in particular cannot be injective). This raises immediately another question: which finite subsemigroups can be found in $\mathbb{N}^*$? Most of the answer was provided by E Zelenjuk in 1996: the only finite groups in $\beta\mathbb{N}$ are singletons, and this book contains the best version available in English of his remarkable proof. Igor Protasov has shown that any finite subgroup of βG for a countable group G is isomorphic with a quotient of a finite subgroup of G, but this result is too recent to have been included. The eager reader will now want to know whether $\beta\mathbb{N}$ contains the semigroup with two elements and all products zero; this is presently the most annoying unsolved problem in the area.

The reviewer must curb his enthusiasm for $\beta\mathbb{N}$ and not mention all the excitement generated by idempotents, cancellation, chains of ideals, relationships with the Rudin-Keisler order, consequences of Martin's Axiom, and whether sums can be equal to products in $\mathbb{N}^*$, and describe the other vistas which the authors lay before the readers of their book. The principal applications presented, as mentioned above, are to Ramsey theory, with the Finite Sums Theorem and van der Waerden's Theorem being the best known examples. Precise formulations of results here tend to be lengthy, so the reviewer can only drop names – Ramsey's Theorem, Rado's 1933 Theorem on partition regular matrices, the Hales-Jewett Theorem (a much generalised van der Waerden's Theorem), and the even more powerful Central Sets Theorem. (A set $A \subseteq S$ is *central* if $\overline{A}$ contains a minimal idempotent in βS.) The latter has the immediate consequence that in the van der Waerden Theorem the differences between consecutive terms in the arithmetic progressions obtained can be chosen from the finite sums of any preassigned sequence.

Chapter 19 of the book is devoted to relationships with topological dynamics. Of course, a treatment on the scale of de Vries's compendious [5] is not possible in 15 pages. Discrete dynamical systems are obtained when a semigroup – or more usually, a group – S can be indentified with a semigroup of continuous mappings on a compact space X. The pointwise closure of S in the set of all mappings from X to itself is a semigroup with a compact topology called the *enveloping semigroup.* A remarkable example here was given by W Ruppert: a simple action of $\mathbb{N}$ on the circle has $\beta\mathbb{N}$ as its enveloping semigroup. The main results presented are that some combinatorial notions – such as of central set – are the same as ones used in topological dynamics but formulated in different ways there.

Another chapter is devoted to Szemeredi type theorems. The typical result here says that if a subset of the integers is 'large' in some sense then it contains abitrarily long arithmetic progressions. A proof of a main step in a establishing this result (Furstenberg's Correspondence Principle) is presented using $\beta\mathbb{N}$, but in a very rare appeal to other work, the authors do not give a complete proof of the main result.

A final chapter takes a brief look at other compactifications of semigroups. Again the reader who wishes to study them will need other sources, such as [1] or [4]. These compactifications are constructed efficiently by the method in [2], but in keeping with the spirit of this volume, alternative approaches using semigroups of filters are described. Almost the last result in the book shows that the largest natural semigroup compactification of $\mathbb{R}$ can be obtained from $\beta\mathbb{Z}$ by affixing an interval $(0, 1)$ between the points p and $p + 1$ for each point $p \in \beta\mathbb{Z}$ (a conclusion due to M Filali in 1990), another indication of the important roles played by Stone-Čech compactifications with algebraic structure.

This is a book written by the world's two leading experts in the field which is readily accessible to beginning graduate students yet takes them rapidly to the frontiers of the subject. It is remarkably up to date, including results from papers which have not appeared at the time this review was written (in addition to original results by the authors themselves). It is, however, not quite a complete encyclopaedia of knowledge as it stood in mid 1998. For example, the reviewer's obsession with the topological centre $\{p \in \beta S : q \mapsto pq \text{ is continuous } \}$ is not indulged except for commutative groups S. But I cannot complain about the selection of topics, and can only marvel at the amount the authors do include. The material is presented very clearly and precisely (the name 'Hindman' on a publication is a guarantee of accuracy, though a short list of minor errors can be found on the website `www.degruyter.com/highlights/hindman.html`). The publishers have done a superb job, and I found the book a delight to use and handle. Every respectable library should have a copy. Every semigroup theorist should at least browse through the first five chapters – these in themselves form a minibook, one that could form the basis of a graduate course – to wonder at these applications of semigroup theory. And we should all find ourselves inspired to tackle some apparently simple questions which urgently need answers.

References

[1] Berglund, J F, H D Junghenn and P Milnes, "Analysis on Semigroups," Wiley, New York, 1989.

[2] Bourbaki, N, "Théorie des Ensembles, Chap. IV, Structures," Hermann, Paris, 1957.

[3] Ellis, R, "Lectures on Topological Dynamics," Benjamin, New York, 1969.

[4] Ruppert, W A F, "Compact Semitopological Semigroups," Lecture Notes in Mathematics 1079, Springer, Berlin, 1984.

[5] de Vries, J, "Elements of Topological Dynamics," Kluwer, Dordrecht, 1993.

School of Mathematics and Statistics,
The University of Sheffield,
Sheffield S3 7RH,
UK

Received January 11, 1999