Monochromatic x, xy, and $x + y$ in $\mathbb{N}$

Our aim is to prove the following theorem due to Joel Moreira [M].

Theorem 1. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. There exist $i \in \{1, 2, \ldots, r\}$ and infinitely many y such that $\{x \in \mathbb{N} : \{x, xy, x + y\} \subseteq C_i\}$ is piecewise syndetic.*

To prove this we need the following lemma.

Lemma 2. *Let $F \in \mathcal{P}_f(\mathbb{N})$ and let A be a piecewise syndetic subset of $\mathbb{N}$. There is an infinite set $G \subseteq \mathbb{N}$ such that for each $n \in G$, $A \cap \bigcap_{m \in F}(-mn + A)$ is piecewise syndetic. In fact, $\{n \in \mathbb{N} : A \cap \bigcap_{m \in F}(-mn + A)$ is piecewise syndetic$\}$ is an IP*-set.*

This is a consequence of a general result due to Bergelson and Leibman [BL] and uses the fact that if $m \in \mathbb{N}$ and $P(x) = mx$ then P is a polynomial which takes on integer values on the integers and has zero constant term. Their result applies to any finite number of such polynomials. A simplified, but still quite complicated, proof of their result was presented in [H].

I present below as Theorem 7 a further simplification due to John Johnson.

We write ${}^{\mathbb{N}}S$ for the set of (infinite) sequences in S.

Definition 3. Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$. Then A is a *J-set* provided whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + \sum_{t \in H} f(t) \in A$.

We note that J-sets satisfy a statement that is superficially stronger in two respects; that is one can get $a \in A$ and may assume $\min H$ is arbitrarily large.

Lemma 4. *Let $(S, +)$ be a commutative semigroup, let A be a J-set in S, and let $m \in \mathbb{N}$. Then whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in A$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > m$ and for each $f \in F$, $a + \sum_{t \in H} f(t) \in A$.*

Proof. Pick $b \in S$. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$. For $f \in F$ define $g_f \in {}^{\mathbb{N}}S$ by for $t \in \mathbb{N}$, $g_f(t) = b + f(m + t)$ and let $\overline{b}$ be the constant sequence. Let $G = \{\overline{b}\} \cup \{g_f : f \in F\}$. Pick $d \in S$ and $K \in \mathcal{P}_f(\mathbb{N})$ such that for each $h \in G$, $d + \sum_{t \in K} h(t) \in A$. Let $n = |K|$, let $a = d + nb$ and let $H = m + K$. □

Theorem 5. *Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$. If A is piecewise syndetic, then A is a J-set.*

Proof. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, let $l = |F|$, and enumerate F as $\{f_1, f_2, \ldots, f_l\}$. Let $Y = \times_{t=1}^{l} \beta S$. Then by [HS, Theorem 2.22], Y is a compact right topological semigroup and if $\vec{s} \in \times_{t=1}^{l} S$, then $\lambda_{\vec{s}}$ is continuous. For $i \in \mathbb{N}$, let

$$I_i = \big\{ \big(a + \textstyle\sum_{t \in H} f_1(t), \ldots, a + \sum_{t \in H} f_l(t)\big) : a \in S\,,\ H \in \mathcal{P}_f(\mathbb{N}), \text{ and } \min H > i \big\}$$

and let $E_i = I_i \cup \{(a, a, \ldots, a) : a \in S\}$.

Let $E = \bigcap_{i=1}^{\infty} \overline{E_i}$ and let $I = \bigcap_{i=1}^{\infty} \overline{I_i}$. We claim that E is a subsemigroup of Y and I is an ideal of E. To this end, let $\vec{p}, \vec{q} \in E$. We show that $\vec{p} + \vec{q} \in E$ and if either $\vec{p} \in I$ or $\vec{q} \in I$, then $\vec{p} + \vec{q} \in I$. Let U be an open neighborhood of $\vec{p} + \vec{q}$ and let $i \in \mathbb{N}$. Since $\rho_{\vec{q}}$ is continuous, pick a neighborhood V of $\vec{p}$ such that $V + \vec{q} \subseteq U$. Pick $\vec{x} \in E_i \cap V$ with $\vec{x} \in I_i$ if $\vec{p} \in I$. If $\vec{x} \in I_i$ so that $\vec{x} = \big(a + \sum_{t \in H} f_1(t), \ldots, a + \sum_{t \in H} f_l(t)\big)$ for some $a \in S$ and some $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H > i$, let $j = \max H$. Otherwise, let $j = i$. Since $\lambda_{\vec{x}}$ is continuous, pick a neighborhood W of $\vec{q}$ such that $\vec{x} + W \subseteq U$. Pick $\vec{y} \in E_j \cap W$ with $\vec{y} \in I_j$ if $\vec{q} \in I$. Then $\vec{x} + \vec{y} \in E_i \cap U$ and if either $\vec{p} \in I$ or $\vec{q} \in I$, then $\vec{x} + \vec{y} \in I_i \cap U$.

By [HS, Theorem 2.23], $K(Y) = \times_{t=1}^{l} K(\beta S)$. Pick by [HS, Theorem 4.40] some $p \in K(\beta S) \cap \overline{A}$. Then $\overline{p} = (p, p, \ldots, p) \in K(Y)$. We claim that $\overline{p} \in E$. To see this, let U be a neighborhood of $\overline{p}$, let $i \in \mathbb{N}$, and pick $C_1, C_2, \ldots, C_l \in p$ such that $\times_{t=1}^{l} \overline{C_t} \subseteq U$. Pick $a \in \bigcap_{t=1}^{l} C_t$. Then $\overline{a} = (a, a, \ldots, a) \in U \cap E_i$. Thus $\overline{p} \in K(Y) \cap E$ and consequently $K(Y) \cap E \neq \emptyset$. Then by [HS, Theorem 1.65], we have that $K(E) = K(Y) \cap E$ and so $\overline{p} \in K(E) \subseteq I$. Then $I_1 \cap \times_{t=1}^{l} \overline{A} \neq \emptyset$ so pick $z \in I_1 \cap \times_{t=1}^{l} \overline{A}$ and pick $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $z = \big(a + \sum_{t \in H} f_1(t), \ldots, a + \sum_{t \in H} f_l(t)\big)$. □

Lemma 6. *Let $A \subseteq \mathbb{N}$, let L be a minimal left ideal of $\beta\mathbb{N}$, and assume that $\overline{A} \cap L \neq \emptyset$. There exists a syndetic set $E \subseteq \mathbb{N}$ such that for all $F \in \mathcal{P}_f(E)$ there exists $X \subseteq \mathbb{N}$ such that $\overline{X} \cap L \neq \emptyset$ and $F + X \subseteq A$.*

Proof. Pick $q \in \overline{A} \cap L$ and let $E = \{x \in \mathbb{N} : -x + A \in q\}$. By [HS, Theorem 4.39], E is syndetic. Let $F \in \mathcal{P}_f(E)$ and let $X = \bigcap_{f \in F}(-f + A)$. Then $F + X \subseteq A$ and since $X \in q$, $\overline{X} \cap L \neq \emptyset$. □

Theorem 7. *Let $v = v + v \in \beta\mathbb{N}$, let L be a minimal left ideal of $\beta\mathbb{N}$, let $k \in \mathbb{N}$, and let $A \subseteq \mathbb{N}$ such that $\overline{A} \cap L \neq \emptyset$. Then $\{n \in \mathbb{N} : \overline{A} \cap L \cap \bigcap_{i=1}^{k} \overline{(-in + A)} \neq \emptyset\} \in v$.*

Proof. Let $C = \{n \in \mathbb{N} : \overline{A} \cap L \cap \bigcap_{i=1}^{k} \overline{(-in + A)} \neq \emptyset\}$. To show that $C \in v$ it suffices to show that for every $B \in v$, $C \cap B \neq \emptyset$, so let $B \in v$. Since v is an idempotent, pick a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq B$.

We claim that

($*$) if $n \in \mathbb{N}$ and there exists $X \subseteq \mathbb{N}$ such that $\overline{X} \cap L \neq \emptyset$
and $\{0\} \cup \{n, 2n, \ldots, kn\} + X \subseteq A$, then $n \in C$.

To establish ($*$), let $n \in \mathbb{N}$ and assume we have $X \subseteq \mathbb{N}$ such that $\overline{X} \cap L \neq \emptyset$ and $\{0\} \cup \{n, 2n, \ldots, kn\} + X \subseteq A$. Pick $r \in \overline{X} \cap L$. Since $X \subseteq A$, we have that $r \in \overline{A} \cap L$. To see that $n \in C$ we show that for $i \in \{1, 2, \ldots, k\}$, $(-in + A) \in r$. Given $i \in \{1, 2, \ldots, k\}$, we have $in + X \subseteq A$ so $X \subseteq (-in + A)$ so $in + A \in r$.

Pick by Lemma 6 a syndetic set $E \subseteq \mathbb{N}$ such that for all $F \in \mathcal{P}_f(E)$ there exists $X \subseteq \mathbb{N}$ such that $\overline{X} \cap L \neq \emptyset$ and $F + X \subseteq A$.

For $i \in \{1, 2, \ldots, k\}$ define $f_i \in {}^{\mathbb{N}}S$ by $f_i(t) = i x_t$. By Theorem 5 E is a J-set so pick by Lemma 4 some $a \in E$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for $i \in \{1, 2, \ldots, k\}$, $a + \sum_{t \in H} f_i(t) \in E$. Let $F = \{a\} \cup \big\{a + \sum_{t \in H} f_i(t) : i \in \{1, 2, \ldots, k\}\big\}$. Pick $X \subseteq \mathbb{N}$ such that $\overline{X} \cap L \neq \emptyset$ and $F + X \subseteq A$.

We claim that $\sum_{t \in H} x_t \in C$, so that $B \cap C \neq \emptyset$ as required. We have that $\{0\} \cup \{\sum_{t \in H} x_t, 2\sum_{t \in H} x_t, \ldots, k\sum_{t \in H} x_t\} + (a + X) \subseteq A$ so by ($*$) it suffices to show that $\overline{a + X} \cap L \neq \emptyset$. By the continuity of λ_a, $\overline{a + X} = a + \overline{X}$. Pick $r \in \overline{X} \cap L$. Then $a + r \in L$ and $a + r \in a + \overline{X} = \overline{a + X}$. □

We are now in a position to prove Lemma 2.

Lemma 2. *Let $F \in \mathcal{P}_f(\mathbb{N})$ and let A be a piecewise syndetic subset of $\mathbb{N}$. There is an infinite set $G \subseteq \mathbb{N}$ such that for each $n \in G$, $A \cap \bigcap_{m \in F}(-mn + A)$ is piecewise syndetic. In fact, $\{n \in \mathbb{N} : A \cap \bigcap_{m \in F}(-mn + A)$ is piecewise syndetic$\}$ is an IP*-set.*

Proof. Pick a minimal left ideal L of $(\beta\mathbb{N}, +)$ such that $\overline{A} \cap L \neq \emptyset$. Let $v = v + v \in \beta\mathbb{N}$ and let $G = \{n \in \mathbb{N} : A \cap \bigcap_{m \in F}(-mn + A)$ is piecewise syndetic$\}$. We shall show that $G \in v$. Let $k = \max F$ and let $E = \{n \in \mathbb{N} : \overline{A} \cap L \cap \bigcap_{i=1}^{k} \overline{(-in + A)} \neq \emptyset\}$. By Theorem 7, $E \in v$.

We claim that $E \subseteq G$. To see this, let $n \in E$. Pick $r \in \overline{A} \cap L \cap \bigcap_{i=1}^{k} \overline{(-in + A)}$. Then $A \cap \bigcap_{m \in F}(-mn + A) \in r$ so $A \cap \bigcap_{m \in F}(-mn + A)$ is piecewise syndetic. □

Lemma 8. *Let A be a piecewise syndetic subset of $\mathbb{N}$ and let $y \in \mathbb{N}$. Then Ay is piecewise syndetic.*

Proof. Pick $k \in \mathbb{N}$ such that for all $F \in \mathcal{P}_f(\mathbb{N})$ there exists $x \in \mathbb{N}$ such that $F + x \subseteq \bigcup_{t=1}^{k}(-t + A)$.

I claim that for all $F \in \mathcal{P}_f(\mathbb{N})$ there exists $x \in \mathbb{N}$ such that $F + x \subseteq \bigcup_{t=1}^{ky+y}(-t + Ay)$. So let $F \in \mathcal{P}_f(\mathbb{N})$ and pick $l \in \mathbb{N}$ such that $F \subseteq \{1, 2, \ldots, ly\}$. Pick $x \in \mathbb{N}$ such that $\{1, 2, \ldots, l\} + x \subseteq \bigcup_{t=1}^{k}(-t + A)$. We claim that $F + xy \subseteq \bigcup_{t=1}^{ky+y}(-t + Ay)$. Let $z \in F$, pick $a \in \{1, 2, \ldots, l\}$ such that $(a-1)y < z \leq ay$, and let $u = ay - z$. Pick $t \in \{1, 2, \ldots, k\}$ such that $t + a + x \in A$. Then $ty + ay + xy \in Ay$ so $(ty + u) + z + xy \in Ay$ and $1 \leq ty + u < ky + y$. $\square$

Lemma 9. *Let $y \in \mathbb{N}$ and let $A \subseteq \mathbb{N}$. Then A is piecewise syndetic if and only if ther exists $k \in \mathbb{N}$ such that for all $F \in \mathcal{P}_f(\mathbb{N})$ there exists $x \in \mathbb{N}$ such that $F + x \subseteq \bigcup_{t=y+1}^{k}(-t + A)$.*

Proof. The sufficiency is trivial. For the necessity, pick $G \in \mathcal{P}_f(\mathbb{N})$ such that for all $F \in \mathcal{P}_f(\mathbb{N})$ there exists $x \in \mathbb{N}$ such that $F + x \subseteq \bigcup_{t \in G}(-t + A)$. Let $m = \max G$ and let $k = y + m$. Let $F \in \mathcal{P}_f(\mathbb{N})$ be given. Then $F + y \in \mathcal{P}_f(\mathbb{N})$ so pick $x \in \mathbb{N}$ such that $F + y + x \subseteq \bigcup_{t \in G}(-t + A)$. We claim that $F + x \subseteq \bigcup_{t=y+1}^{k}(-t + A)$, so let $v \in F$. Pick $t \in G$ such that $t + v + y + x \in A$. Then $(t + y) + v + x \in A$ and $y + 1 \leq t + y \leq m + y = k$. $\square$

Lemma 10. *Let $y \in \mathbb{N}$ and let A be a piecewise syndetic subset of $\mathbb{N}$ such that $A \subseteq \mathbb{N}y$. Then A/y is piecewise syndetic.*

Proof. By Lemma 9, pick $k \in \mathbb{N}$ such that for all $F \in \mathcal{P}_f(\mathbb{N})$ there exists $x \in \mathbb{N}$ such that $F + x \subseteq \bigcup_{t=y+1}^{k}(-t + A)$. We claim that for all $F \in \mathcal{P}_f(\mathbb{N})$ there exists $x \in \mathbb{N}$ such that $F + x \subseteq \bigcup_{t=1}^{k}(-t + A/y)$, so let $F \in \mathcal{P}_f(\mathbb{N})$. Then $Fy \in \mathcal{P}_f(\mathbb{N})$ so pick $z \in \mathbb{N}$ such that $Fy + z \subseteq \bigcup_{t=y+1}^{k}(-t + A)$. Pick $a \in \mathbb{N}$ such that $(a-1)y < z \leq ay$ and let $u = ay - z$. (Then $u \in \{0, 1, \ldots, y-1\}$.) We claim $F + a \subseteq \bigcup_{s=1}^{k}(-s + A/y)$. So let $v \in F$. Then $vy + z \in \bigcup_{t=y+1}^{k}(-t + A)$ so pick $t \in \{y+1, y+2, \ldots, k\}$ such that $t + vy + z \in A$. Since $A \subseteq \mathbb{N}y$, there is some $b \in \mathbb{N}$ such that $t + vy + ay - u = t + vy + z = by$ so $\frac{t-u}{y} + v + a = b$ and thus $\frac{t-u}{y} \in \mathbb{Z}$ and $t - u \geq y + 1 - (y-1) > 0$ so $\frac{t-u}{y} \in \mathbb{N}$. Thus $\frac{t-u}{y} \in \{1, 2, \ldots, k\}$. $\square$

The proof of the next theorem uses the fact that if the union of finitely many subsets of $\mathbb{N}$ is piecewise syndetic, then one of them is. The easiest way for us to see that is to note that A is piecewise syndetic iff A is a member of an ultrafilter in $K(\beta\mathbb{N})$.

Theorem 11. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. There exist $i \in \{1, 2, \ldots, r\}$, an increasing sequence $\langle z_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$, and a sequence $\langle E_n \rangle_{n=1}^{\infty}$ of piecewise syndetic subsets of $\mathbb{N}$ such that for each n, $E_n \subseteq \mathbb{N}z_n$ and if $w \in E_n$ and $x = \frac{w}{z_n}$, then $\{x, xz_n, x + z_n\} \subseteq C_i$.*

Proof. Choose $t_0 \in \{1, 2, \ldots, r\}$ such that C_{t_0} is piecewise syndetic and let $B_0 = C_{t_0}$. By Lemma 2 with $F = \{1\}$, pick $y_1 \in \mathbb{N}$ such that $B_0 \cap (B_0 - y_1)$ is piecewise syndetic and let $D_1 = B_0 \cap (B_0 - y_1)$. By Lemma 4, $y_1 D_1$ is piecewise syndetic. Since $y_1 D_1 = \bigcup_{i=1}^{r} (y_1 D_1 \cap C_i)$, pick $t_1 \in \{1, 2, \ldots, r\}$ such that $y_1 D_1 \cap C_{t_1}$ is piecewise syndetic and let $B_1 = (y_1 D_1 \cap C_{t_1})$.

Let $k \in \mathbb{N}$ and assume we have chosen $\langle y_j \rangle_{j=1}^{k}$, $\langle B_j \rangle_{j=0}^{k}$, $\langle t_j \rangle_{j=0}^{k}$, and $\langle D_j \rangle_{j=1}^{k}$ satisfying the following induction hypotheses.

(1) For $j \in \{1, 2, \ldots, k\}$, $y_j \in \mathbb{N}$ and if $j > 1$, $y_j > y_{j-1}$.

(2) For $j \in \{1, 2, \ldots, k\}$, D_j is a piecewise syndetic subset of $\mathbb{N}$.

(3) For $j \in \{0, 1, \ldots, k\}$, $t_j \in \{1, 2, \ldots, r\}$.

(4) For $j \in \{0, 1, \ldots, k\}$, B_j is a piecewise syndetic subset of $\mathbb{N}$.

(5) For $j \in \{0, 1, \ldots, k\}$, $B_j \subseteq C_{t_j}$.

(6) For $j \in \{1, 2, \ldots, k\}$, $B_j \subseteq y_j D_j$.

(7) For $j < m$ in $\{0, 1, \ldots, k\}$, $B_m \subseteq y_m y_{m-1} \cdots y_{j+1} B_j$.

(8) For $m \in \{1, 2, \ldots, k\}$, $D_m \subseteq B_{m-1} \cap (B_{m-1} - y_m)$ and, if $m > 1$, then $D_m \subseteq \bigcap_{j=1}^{m-1} (B_{m-1} - (y_{m-1} y_{m-2} \cdots y_j)^2 y_m)$.

All hypotheses hold for $k = 1$.

For $j \in \{1, 2, \ldots, k\}$, let $u_j = y_k y_{k-1} \cdots y_j$ and let $F = \{1, u_1^2, u_2^2, \ldots, u_k^2\}$. By Lemma 2 pick $y_{k+1} > y_k$ such that $B_k \cap (B_k - y_{k+1}) \cap \bigcap_{j=1}^{k} (B_k - u_j^2 y_{k+1})$ is piecewise syndetic and let $D_{k+1} = B_k \cap (B_k - y_{k+1}) \cap \bigcap_{j=1}^{k} (B_k - u_j^2 y_{k+1})$. Note that hypotheses (1), (2), and (8) hold at $k+1$.

By Lemma 8, $y_{k+1} D_{k+1}$ is piecewise syndetic and $y_{k+1} D_{k+1} = \bigcup_{i=1}^{r} (y_{k+1} D_{k+1} \cap C_i)$ so pick $t_{k+1} \in \{1, 2, \ldots, r\}$ such that $y_{k+1} D_{k+1} \cap C_{t_{k+1}}$ is piecewise syndetic and let $B_{k+1} = y_{k+1} D_{k+1} \cap C_{t_{k+1}}$. Note that hypotheses (3), (4), (5), and (6) hold for $k+1$.

We need to verify hypothesis (7) so let $j < m$ in $\{0, 1, \ldots, k+1\}$ be given. If $m \leq k$, then (7) holds by assumption so assume that $m = k+1$. We have $B_{k+1} \subseteq y_{k+1} D_{k+1} \subseteq y_{k+1} B_k$. If $j = k$, we are done, so assume that $j < k$ in which case by (7) at k we have $B_k \subseteq y_k y_{k-1} \cdots y_{j+1} B_j$ so $B_{k+1} \subseteq y_{k+1} y_k \cdots y_{j+1} B_j$ as required.

The construction is complete. Pick $i \in \{1, 2, \ldots, r\}$ such that $\{k \in \mathbb{N} : t_k = i\}$ is infinite and let $G = \{k \in \mathbb{N} : t_k = i\}$. We then choose a sequence $\langle k(n) \rangle_{n=0}^{\infty}$ in G so that, letting $z_n = y_{k(n)} y_{k(n)-1} \cdots y_{k(n-1)+1}$ for $n \in \mathbb{N}$, we have $z_{n+1} > z_n$. For $n \in \mathbb{N}$, let $E_n = B_{k(n)}$. Then each E_n is piecewise syndetic. Also,

$$E_n = B_{k(n)} \subseteq y_{k(n)} y_{k(n)-1} \cdots y_{k(n-1)+1} B_{k(n-1)} = z_n B_{k(n-1)} \subseteq z_n \mathbb{N}\,.$$

Let $w \in E_n$ and let $x = \frac{w}{z_n}$. We need to show that $\{x, x z_n, x + z_n\} \subseteq C_i$. Now

$xz_n = w \in E_n = B_{k(n)} \subseteq C_{t_{k(n)}} = C_i$. Also $xz_n \in E_n \subseteq z_n B_{k(n-1)}$ so $x \in B_{k(n-1)} \subseteq C_{t_{k(n-1)}} = C_i$. It remains to show that $x + z_n \in C_i$. Now

$$\begin{aligned}
&z_n(x+z_n)\\
&= w + z_n^2 \in B_{k(n)} + z_n^2 \subseteq y_{k(n)} D_{k(n)} + z_n^2\\
&\subseteq y_{k(n)}(B_{k(n)-1} - y_{k(n)} y_{k(n)-1}^2 y_{k(n)-2}^2 \cdots y_{k(n-1)+1}^2) + z_n^2\\
&\subseteq y_{k(n)}(y_{k(n)-1} y_{k(n)-2} \cdots y_{k(n-1)+1} B_{k(n-1)} - y_{k(n)} y_{k(n)-1}^2 y_{k(n)-2}^2 \cdots y_{k(n-1)+1}^2) + z_n^2\\
&= y_{k(n)} y_{k(n)-1} y_{k(n)-2} \cdots y_{k(n-1)+1} B_{k(n-1)} - y_{k(n)}^2 y_{k(n)-1}^2 y_{k(n)-2}^2 \cdots y_{k(n-1)+1}^2 + z_n^2\\
&= z_n B_{k(n-1)}\,.
\end{aligned}$$

So $x + z_n \in B_{k(n-1)} \subseteq C_{t_{k(n-1)}} = C_i$. □

We are now ready to prove Theorem 1 (as a corollary to Theorem 11).

Theorem 1. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^r C_i$. There exist $i \in \{1, 2, \ldots, r\}$ and infinitely many y such that $\{x \in \mathbb{N} : \{x, xy, x+y\} \subseteq C_i\}$ is piecewise syndetic.*

Proof. Pick i, $\langle z_n \rangle_{n=1}^\infty$, and $\langle E_n \rangle_{n=1}^\infty$ as guaranteed by Theorem 11. Given $n \in \mathbb{N}$, if $y = z_n$, then $E_n/y \subseteq \{x \in \mathbb{N} : \{x, xy, x+y\} \subseteq C_i\}$ and by Lemma 10, E_n/y is piecewise syndetic. □

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