**IP* Sets in Product Spaces**

**Vitaly Bergelson**

and

**Neil Hindman**

**Abstract.** An IP* set in a semigroup \((S, \cdot)\) is a set which meets every set of the form \(FP(\langle x_n \rangle_{n=1}^\infty) = \{\Pi_{n \in F} x_n : F \text{ is a finite nonempty subset of } \mathbb{N}\}\), where the products are taken in increasing order of indices. We show here, using the Stone-Čech compactification of the product space \(S_1 \times S_2 \times \ldots \times S_\ell\), that if each \(S_i\) is commutative, then whenever \(C\) is an IP* set in \(S_1 \times S_2 \times \ldots \times S_\ell\) and for each \(i \in \{1, 2, \ldots, \ell\}\), \(\langle x_i, n \rangle_{n=1}^\infty\) is a sequence in \(S_i\), \(C\) contains cartesian products of arbitrarily large finite substructures of \(FP(\langle x_1, n \rangle_{n=1}^\infty) \times FP(\langle x_2, n \rangle_{n=1}^\infty) \times \ldots \times FP(\langle x_\ell, n \rangle_{n=1}^\infty)\). (The notion of “substructure” is made precise in Definition 2.4.) We show further that \(C\) need not contain any product of infinite substructures and that the commutativity hypothesis may not be omitted. Similar results apply to arbitrary finite products of semigroups.

By way of contrast, we show in Theorem 2.3 that a much stronger conclusion holds for some cell of any finite partition of \(S_1 \times S_2 \times \ldots \times S_\ell\) without even any commutativity assumptions.

**1. Introduction.**

In a semigroup \((S, \cdot)\), we write \(\Pi_{n \in F} x_n\) for the product written in increasing order of indices. (Thus \(\Pi_{n \in \{1,5,7\}} x_n = x_1 \cdot x_5 \cdot x_7\).) We further write \(FP(\langle x_n \rangle_{n=1}^\infty) = \{\Pi_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}\) where \(\mathcal{P}_f(\mathbb{N}) = \{A : A \text{ is a finite nonempty subset of } \mathbb{N}\}\) and \(\mathbb{N}\) is the set of positive integers. (Similarly, if the operation of the semigroup is denoted by +, we write \(FS(\langle x_n \rangle_{n=1}^\infty) = \{\Sigma_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}\).) Loosely following Furstenberg [5] we say that a set \(A \subseteq S\) is an IP set if and only if there is a sequence \(\langle x_n \rangle_{n=1}^\infty\) in \(S\) with \(FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A\). A set \(C \subseteq S\) is then an IP* set if and only if \(C \cap A \neq \emptyset\) for every IP set \(A\) (equivalently if and only if \(C \cap FP(\langle x_n \rangle_{n=1}^\infty) \neq \emptyset\) for every sequence \(\langle x_n \rangle_{n=1}^\infty\) in \(S\)).

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Now IP* sets not only must meet every IP set; a much stronger statement is true. This statement involves the notion of a product subsystem which we pause now to define.

1.1 Definition. Let \((S, \cdot)\) be a semigroup and let \(\langle y_n \rangle_{n=1}^{\infty}\) be a sequence in \(S\). The sequence \(\langle x_n \rangle_{n=1}^{\infty}\) is a product subsystem of \(\langle y_n \rangle_{n=1}^{\infty}\) if and only if there is a sequence \(\langle H_n \rangle_{n=1}^{\infty}\) in \(\mathcal{P}_f(\mathbb{N})\) such that

(a) for each \(n \in \mathbb{N}\), \(\max H_n < \min H_{n+1}\) and

(b) for each \(n \in \mathbb{N}\), \(x_n = \prod_{t \in H_n} y_t\).

Note that if \(\langle x_n \rangle_{n=1}^{\infty}\) is a product subsystem of \(\langle y_n \rangle_{n=1}^{\infty}\), then \(FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq FP(\langle y_n \rangle_{n=1}^{\infty})\). (If requirement (a) of the definition is replaced by the requirement that \(H_n \cap H_m = \emptyset\) when \(n \neq m\), this conclusion may fail if \(S\) is not commutative. For example, if \(x_1 = y_1 \cdot y_3\) and \(x_2 = y_2\), then \(x_1 \cdot x_2\) need not be in \(FP(\langle y_n \rangle_{n=1}^{\infty})\).) In the event the operation in \(S\) is denoted by +, we change products to sums and refer to a sum subsystem.

The much stronger statement to which we referred is that given any IP* set \(C \subseteq S\) and any sequence \(\langle y_n \rangle_{n=1}^{\infty}\) in \(S\), there is a product subsystem \(\langle x_n \rangle_{n=1}^{\infty}\) of \(\langle y_n \rangle_{n=1}^{\infty}\) such that \(FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C\). This statement will be proved as Corollary 1.7 below. (It is well known, but we don’t have a convenient reference for it.)

It was shown in [2] that IP* sets may be large in unexpected ways. For example [2, Theorem 2.6] an IP* set in \((\mathbb{N}, +)\) must contain an infinite sequence with all of its sums (as expected) and its products as well. We investigate in this paper the extent to which IP* sets in product semigroups must be large in terms of sequences in the coordinates. One may ask for example, given semigroups \(S_1\) and \(S_2\), an IP* set \(C\) in \(S_1 \times S_2\), and sequences \(\langle w_n \rangle_{n=1}^{\infty}\) in \(S_1\) and \(\langle z_n \rangle_{n=1}^{\infty}\) in \(S_2\), whether there must be product subsystems \(\langle x_n \rangle_{n=1}^{\infty}\) of \(\langle w_n \rangle_{n=1}^{\infty}\) and \(\langle z_n \rangle_{n=1}^{\infty}\) of \(\langle z_n \rangle_{n=1}^{\infty}\) with \(FP(\langle x_n \rangle_{n=1}^{\infty}) \times FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq C\).

We give a strong negative answer to this question in Section 2. One does find however that if \(S_1\) and \(S_2\) are commutative semigroups, then any IP* set in \(S_1 \times S_2\) must contain the product of arbitrarily large finite substructures of \(\langle w_n \rangle_{n=1}^{\infty}\) and \(\langle z_n \rangle_{n=1}^{\infty}\) and a similar statement applies to any finite product of semigroups (Theorem 2.6).

We also see a surprising turning of the tables. One is accustomed to finding prop-
erties that must hold for some cell of any finite partition and automatically expecting at least such a conclusion for IP* sets. For example, as we have already mentioned, while one cell of a partition of \(N\) must contain \(FS((x_n)_{n=1}^{\infty})\) for some sequence \((x_n)_{n=1}^{\infty}\), any IP* set must contain \(FS((x_n)_{n=1}^{\infty}) \cup FP((x_n)_{n=1}^{\infty})\). Similarly, we will see below that given any discrete semigroup \(S\) and any idempotent \(p\) in \(\beta S\), while trivially \(p\) must belong to the closure of some cell of any partition of \(S\), \(p\) must belong also to the closure of every IP* set.

By contrast we will see in Theorems 2.2 and 2.7 that one can guarantee a much stronger conclusion for one cell of any partition of the product of two semigroups than can be guaranteed for IP* sets in the same product.

A special semigroup is of significant interest for these problems, namely the semigroup \((P_f(N), \cup)\) because a version of Theorem 2.6 for this semigroup is sufficient to imply the validity of Theorem 2.6 for all semigroups. We present this derivation in Section 3.

Our proofs utilize the algebraic structure of the Stone-Čech compactification \(\beta S\) of a discrete semigroup \((S, \cdot)\). We take \(\beta S\) to be the set of all ultrafilters on \(S\), identifying the principal ultrafilters with the points of \(S\). We denote also by \(\cdot\) the operation on \(\beta S\) making \((\beta S, \cdot)\) a right topological semigroup with \(S\) contained in its topological center. That is, for all \(p \in \beta S\), the function \(\rho_p : \beta S \rightarrow \beta S\) defined by \(\rho_p(q) = q \cdot p\) is continuous and for all \(x \in S\), the function \(\lambda_x : \beta S \rightarrow \beta S\) defined by \(\lambda_x(q) = x \cdot q\) is continuous. The reader is referred to [6] and [7] for an elementary introduction to this operation, with the caution that there \((\beta S, \cdot)\) is taken to be left rather than right topological. (We have made the switch to conform to majority usage, at least among our collaborators.) The basic fact characterizing the right continuous operation on \(\beta S\) is, given \(p, q \in \beta S\) and \(A \subseteq S\), \(A \in p \cdot q\) if and only if \(\{x \in S : x^{-1}A \in q\} \in p\) where \(x^{-1}A = \{y \in S : x \cdot y \in A\}\). In the event that the operation is denoted by +, the characterization above becomes \(A \in p + q\) if and only if \(\{x \in S : -x + A \in q\} \in p\) where \(-x + A = \{y \in S : x + y \in A\}\).

We will use only a few basic facts about \((\beta S, \cdot)\) which we present here.
1.2 Theorem. *Any compact right topological semigroup has an idempotent.*

**Proof.** [4, Corollary 2.10].

The proof of the following theorem is the Galvin-Glazer proof of the Finite Sum Theorem, which was never published by them, though it has appeared in surveys. Those places where it has appeared (to our knowledge) however use left continuity so that the resulting products are in decreasing order of indices. To minimize confusion, we present the proof here.

1.3 Theorem. *Let \((S, \cdot)\) be a semigroup, let \(p\) be an idempotent in \(\beta S\), and let \(A \in p\). There is a sequence \(\langle x_n \rangle_{n=1}^\infty\) in \(S\) such that \(FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A\).*

**Proof.** Let \(A_1 = A\) and let \(B_1 = \{x \in S : x^{-1}A_1 \subseteq p\}\). Since \(A_1 \subseteq p = p \cdot p, B_1 \subseteq p\). Pick \(x_1 \in B_1 \cap A_1\), let \(A_2 = A_1 \cap (x_1^{-1}A_1)\), and note that \(A_2 \subseteq p\). Inductively given \(A_n \subseteq p\), let \(B_n = \{x \in S : x^{-1}A_n \subseteq p\}\). Since \(A_n \subseteq p \cdot p, B_n \subseteq p\). Pick \(x_n \in B_n \cap A_n\) and let \(A_{n+1} = A_n \cap (x_n^{-1}A_n)\).

To see for example why \(x_2 \cdot x_4 \cdot x_5 \cdot x_7 \in A\), note that \(x_7 \in A_7 \subseteq A_6 \subseteq x_5^{-1}A_5\) so that \(x_5 \cdot x_7 \in A_5 \subseteq x_4^{-1}A_4\). Thus \(x_4 \cdot x_5 \cdot x_7 \in A_4 \subseteq A_3 \subseteq x_2^{-1}A_2\) so that \(x_2 \cdot x_4 \cdot x_5 \cdot x_7 \in A_2 \subseteq A_1 = A\).

More formally, we show by induction on \(|F|\) that if \(F \in \mathcal{P}_f(\mathbb{N})\) and \(m = \min F\) then \(\Pi_{n \in F} x_n \in A_m\). If \(|F| = 1\), then \(\Pi_{n \in F} x_n = x_m \in A_m\). Assume \(|F| > 1\), let \(G = F \setminus \{m\}\), and let \(k = \min G\). Note that since \(k > m\), \(A_k \subseteq A_{m+1}\). Then by the induction hypothesis, \(\Pi_{n \in G} x_n \in A_k \subseteq A_{m+1} \subseteq x_m^{-1}A_m\) so \(\Pi_{n \in F} x_n = x_m \cdot \Pi_{n \in G} x_n \in A_m\).

1.4 Theorem. *Let \(S\) be a semigroup and let \(\langle x_n \rangle_{n=1}^\infty\) be a sequence in \(S\). Then \(\bigcap_{m=1}^\infty \text{cl}FP(\langle x_n \rangle_{n=m}^\infty)\) is a subsemigroup of \(\beta S\). In particular, there is an idempotent \(p\) in \(\beta S\) such that for each \(m \in \mathbb{N}\), \(FP(\langle x_n \rangle_{n=m}^\infty) \subseteq p\).*

**Proof.** Let \(T = \bigcap_{m=1}^\infty \text{cl}FP(\langle x_n \rangle_{n=m}^\infty)\). Then \(T\) is the intersection of a nested collection of closed nonempty subsets of \(\beta S\) so \(T \neq \emptyset\). To see that \(T\) is a semigroup, let \(p, q \in T\) be given and let \(m \in \mathbb{N}\). To see that \(FP(\langle x_n \rangle_{n=m}^\infty) \subseteq p \cdot q\) we show that \(FP(\langle x_n \rangle_{n=m}^\infty) \subseteq \{s \in S : s^{-1}FP(\langle x_n \rangle_{n=m}^\infty) \subseteq q\}\). (Since \(FP(\langle x_n \rangle_{n=m}^\infty) \subseteq p\), this will suffice.) To this end, let \(s \in FP(\langle x_n \rangle_{n=m}^\infty)\) be given and pick \(F \in \mathcal{P}_f(\mathbb{N})\) with \(\min F \geq m\) such
that \( s = \Pi_{n \in F} x_n \). Let \( k = \max F + 1 \). Then \( FP(\langle x_n \rangle_{n=k}^\infty) \in q \) so it suffices to show that \( FP(\langle x_n \rangle_{n=k}^\infty) \subseteq s^{-1}FP(\langle x_n \rangle_{n=m}^\infty) \). So, let \( t \in FP(\langle x_n \rangle_{n=k}^\infty) \) be given and pick \( G \in \mathcal{P}_f(\mathbb{N}) \) with \( \min G \geq k \) such that \( t = \Pi_{n \in G} x_t \). Then \( \max F < \min G \) so \( st = \Pi_{n \in F \cup G} x_n \in FP(\langle x_n \rangle_{n=m}^\infty) \).

For the “in particular” conclusion note that by Theorem 1.2, \( T \) has an idempotent.

\[
\square
\]

The following simple characterization of IP* sets also provides some explanation of the origin of “IP”.

1.5 Theorem. Let \((S, \cdot)\) be a semigroup and let \( C \subseteq S \). Then \( C \) is an IP* set if and only if for every idempotent \( p \in \beta S, C \in p \).

Proof. Necessity. Let \( p \cdot p = p \in \beta S \) and suppose \( C \notin p \). Then \( S \setminus C \in p \) so by Theorem 1.3 there is a sequence \( \langle x_n \rangle_{n=1}^\infty \) with \( FP(\langle x_n \rangle_{n=1}^\infty) \subseteq S \setminus C \), a contradiction.

Sufficiency. Let \( \langle x_n \rangle_{n=1}^\infty \) be a sequence in \( S \) and pick by Theorem 1.4 an idempotent \( p \) with \( FP(\langle x_n \rangle_{n=1}^\infty) \in p \). Then \( C \cap FP(\langle x_n \rangle_{n=1}^\infty) \in p \) so \( C \cap FP(\langle x_n \rangle_{n=1}^\infty) \neq \emptyset \). \( \square \)

1.6 Theorem. Let \((S, \cdot)\) be a semigroup, let \( \langle y_n \rangle_{n=1}^\infty \) be a sequence in \( S \), and let \( p \) be an idempotent in \( \bigcap_{m=1}^\infty c(FP(\langle y_n \rangle_{n=m}^\infty)) \). If \( A \in p \), then there is a product subsystem \( \langle x_n \rangle_{n=1}^\infty \) of \( \langle y_n \rangle_{n=1}^\infty \) with \( FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A \).

Proof. Pick by Theorem 1.4 an idempotent \( p \in \beta S \) such that for each \( m \in \mathbb{N} \), \( FP(\langle x_n \rangle_{n=m}^\infty) \in p \). (The rest of the proof is now a modification of the proof of Theorem 1.3.)

Let \( C_1 = A \), let \( B_1 = \{ y \in S : y^{-1}C_1 \in p \} \), and pick \( y_1 \in B_1 \cap C_1 \). Pick \( H_1 \in \mathcal{P}_f(\mathbb{N}) \) such that \( y_1 = \Pi_{t \in H_1} x_t \). Let \( k_1 = \max H_1 + 1 \) and let \( C_2 = C_1 \cap y_1^{-1}C_1 \cap FP(\langle x_t \rangle_{t=k_1}^\infty) \).

Inductively, given \( n > 1 \) and \( k_{n-1} \in \mathbb{N} \) and \( C_n \in p \) with \( C_n \subseteq FP(\langle x_t \rangle_{t=k_{n-1}}^\infty) \), let \( B_n = \{ y \in S : y^{-1}C_n \in p \} \). Pick \( y_n \in B_n \cap C_n \) and pick \( H_n \in \mathcal{P}_f(\mathbb{N}) \) with \( \max H_n \geq k_{n-1} \) such that \( y_n = \Pi_{t \in H_n} x_t \). Let \( k_n = \max H_n + 1 \) and let \( C_n+1 = C_n \cap y_n^{-1}C_n \cap FP(\langle x_t \rangle_{t=k_n}^\infty) \).

Then one immediately sees that \( \langle y_n \rangle_{n=1}^\infty \) is a product subsystem of \( \langle x_n \rangle_{n=1}^\infty \). Just as in the proof of Theorem 1.3 one sees that \( FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A \). \( \square \)
The following corollary will be used in the proof of Theorem 2.9.

1.7 Corollary. Given any IP* set $C \subseteq S$ and any sequence $\langle y_n \rangle_{n=1}^\infty$ in $S$, there is a product subsystem $\langle x_n \rangle_{n=1}^\infty$ of $\langle y_n \rangle_{n=1}^\infty$ such that $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq C$.

Proof. Let $T = \bigcap_{m=1}^\infty \ell FP(\langle y_n \rangle_{n=m}^\infty)$. By Theorem 1.4 $T$ is a compact subsemigroup of $\beta S$, so pick by Theorem 1.2 an idempotent $p \in T$. By Theorem 1.5 $C \in p$, so by Theorem 1.6 there is a product subsystem $\langle x_n \rangle_{n=1}^\infty$ of $\langle y_n \rangle_{n=1}^\infty$ such that $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq C$. □

1.8 Theorem. Let $X$ be a set and let $\mathcal{G} \subseteq \mathcal{P}(X)$. The following statements are equivalent.

(a) For each $r \in \mathbb{N}$, if $X = \bigcup_{i=1}^r A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $G \in \mathcal{G}$ with $G \subseteq A_i$.

(b) There is an ultrafilter $p$ on $X$ such that for each $A \in p$, there exists $G \in \mathcal{G}$ with $G \subseteq A$.

Proof. [7, Theorem 6.7]. □

We write $\mathbb{N}$ for the set of positive integers and $\omega$ for the set of nonnegative integers.

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2. IP* Sets in Products.

We show in this section that IP* sets in any finite product of commutative semigroups contain products of arbitrarily large finite subsystems of sequences in the coordinates. We first modify the notion of product subsystems (Definition 1.1) to apply to finite sequences. (Requirement (a) below is of course a triviality, but see the discussion following Definition 1.1 for the fact that requirement (b) can be important.)

2.1 Definition. Let $\langle y_n \rangle_{n=1}^\infty$ be a sequence in a semigroup $S$ and let $k, m \in \mathbb{N}$. Then $\langle x_n \rangle_{n=1}^m$ is a product subsystem of $\langle y_n \rangle_{n=k}^\infty$ if and only if there exists a sequence $\langle H_n \rangle_{n=1}^m$ in $\mathcal{P}_f(\mathbb{N})$ such that
(a) \(\min H_1 \geq k\),
(b) \(\max H_n < \min H_{n+1}\) for each \(n \in \{1, 2, \ldots, m - 1\}\), and
(c) \(x_n = \Pi_{t \in H_n} y_t\) for each \(n \in \{1, 2, \ldots, m\}\).

We establish, using an argument that we have used before [1], a purely Ramsey
Theoretic property in the product of arbitrary semigroups which is of interest in its own
right, independent of IP* sets. It is similar in flavor to the Milliken-Taylor Theorem [8, 9]. We first present the two dimensional version, partly because it is simpler but also to
present a stronger conclusion than will be needed here which will contrast sharply with
Theorem 2.7 below.

2.2 Theorem. Let \(S_1\) and \(S_2\) be semigroups and let \((y_{1,n})_{n=1}^\infty\) and \((y_{2,n})_{n=1}^\infty\) be
sequences in \(S_1\) and \(S_2\) respectively. Let \(k, r \in \mathbb{N}\) and let \(S_1 \times S_2 = \bigcup_{j=1}^r C_j\). There
exist \(j \in \{1, 2, \ldots, r\}\) and a product subsystem \((x_{1,n})_{n=1}^\infty\) of \((y_{1,n})_{n=k}^\infty\) such that for each
\(m \in \mathbb{N}\) there is a product subsystem \((x_{2,n})_{n=1}^\infty\) of \((y_{2,n})_{n=k}^\infty\) such that

\[
FP((x_{1,n})_{n=1}^m) \times FP((x_{2,n})_{n=1}^\infty) \subseteq C_i.
\]

Proof. Given \(i \in \{1, 2\}\), pick by Theorem 1.4 an idempotent

\[
p_i \in \bigcap_{k=1}^\infty \cl_{\beta S_i} FP((y_{i,n})_{n=k}^\infty).
\]

For \(x \in S_1\) and \(j \in \{1, 2, \ldots, r\}\), define \(B_2(x, j) = \{y \in S_2 : (x, y) \in C_j\}\), and for
\(j \in \{1, 2, \ldots, r\}\), let \(B_1(j) = \{x \in S_1 : B_2(x, j) \in p_2\}\).

Now for each \(x \in S_1\), \(S_2 = \bigcup_{j=1}^r B_2(x, j)\) so there exists \(j \in \{1, 2, \ldots, r\}\) such
that \(B_2(x, j) \in p_2\). Consequently, \(S_1 = \bigcup_{j=1}^r B_1(j)\) so pick \(j \in \{1, 2, \ldots, r\}\) such that
\(B_1(j) \in p_1\). Pick by Theorem 1.6 a product subsystem \((x_{1,n})_{n=1}^\infty\) of \((y_{1,n})_{n=k}^\infty\) such that

\[
FP((x_{1,n})_{n=1}^\infty) \subseteq B_1(j).
\]

Let \(m \in \mathbb{N}\). Then \(FP((x_{1,n})_{n=1}^m) \subseteq B_1(j)\) so

\[
\bigcap\{B_2(a, j) : a \in FP((x_{1,n})_{n=1}^m)\} \in p_2.
\]

Pick by Theorem 1.6 a product subsystem \((x_{2,n})_{n=1}^\infty\) of \((y_{2,n})_{n=k}^\infty\) such that

\[
FP((x_{2,n})_{n=1}^\infty) \subseteq \bigcap\{B_2(a, j) : a \in FP((x_{1,n})_{n=1}^m)\}.
\]
Then $FP((x_{1,n})_{n=1}^{m}) \times FP((x_{2,n})_{n=1}^{\infty}) \subseteq C_i$. 

It is a consequence of Theorem 2.7 below that one cannot get a product of full infinite subsystems, even if the cells of the partition $\{C_1, C_2, \ldots, C_r\}$ are required to be symmetric. (Recall that a subset $C$ of a cartesian product is symmetric provided $(a, b) \in C$ implies $(b, a) \in C$.)

At first glance it is not clear how to generalize the proof of Theorem 2.2 to higher dimensions, because the two coordinates are treated so differently. It turns out that intermediate coordinates are treated more like the first than the last. One can in fact replace the last coordinate $FP((x_{\ell,n})_{n=1}^{m})$ in the following lemma by $FP((x_{\ell,n})_{n=1}^{\infty})$.

2.3 Theorem. Let $\ell \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, \ell\}$ let $S_i$ be a semigroup and let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $S_i$. Let $m, k, r \in \mathbb{N}$ and let $\times_{i=1}^{\ell} S_i = \bigcup_{j=1}^{r} C_j$. There exists $j \in \{1, 2, \ldots, r\}$ and for each $i \in \{1, 2, \ldots, \ell\}$, there exists a product subsystem $\langle x_{i,n} \rangle_{n=1}^{m}$ of $\langle y_{i,n} \rangle_{n=k}^{\infty}$ such that

$$\times_{i=1}^{\ell} FP((x_{i,n})_{n=1}^{m}) \subseteq C_j.$$

Proof. Given $i \in \{1, 2, \ldots, \ell\}$, pick by Theorem 1.4 an idempotent $p_i \in \bigcap_{k=1}^{\infty} \ell_{\beta S_i} FP((y_{i,n})_{n=k}^{\infty})$.

For $(x_1, x_2, \ldots, x_{\ell-1}) \in \times_{i=1}^{\ell-1} S_i$ and $j \in \{1, 2, \ldots, r\}$, let

$$B_{\ell}(x_1, x_2, \ldots, x_{\ell-1}, j) = \{ y \in S_\ell : (x_1, x_2, \ldots, x_{\ell-1}, y) \in C_j \}.$$

Now given $t \in \{2, 3, \ldots, \ell - 1\}$, assume that $B_{t+1}(x_1, x_2, \ldots, x_t, j)$ has been defined for each $(x_1, x_2, \ldots, x_t) \in \times_{i=1}^{t} S_i$ and each $j \in \{1, 2, \ldots, r\}$. Given $(x_1, x_2, \ldots, x_{t-1}) \in \times_{i=1}^{t-1} S_i$ and $j \in \{1, 2, \ldots, r\}$, let

$$B_{t}(x_1, x_2, \ldots, x_{t-1}, j) = \{ y \in S_t : B_{t+1}(x_1, x_2, \ldots, x_{t-1}, y, j) \in p_{t+1} \}.$$ 

Finally, given that $B_2(x, j)$ has been defined for each $x \in S_1$ and each $j \in \{1, 2, \ldots, r\}$, let $B_1(j) = \{ x \in S_1 : B_2(x, j) \in p_2 \}$.

We show by downward induction on $t$ that for each $t \in \{2, 3, \ldots, \ell\}$ and each $(x_1, x_2, \ldots, x_{t-1}) \in \times_{i=1}^{t-1} S_i$, $S_t = \bigcup_{j=1}^{r} B_t(x_1, x_2, \ldots, x_{t-1}, j)$. This is trivially true
for $t = \ell$. Assume $t \in \{2, 3, \ldots, \ell - 1\}$ and the statement is true for $t + 1$. Let $(x_1, x_2, \ldots, x_{t-1}) \in \prod_{i=1}^{t-1} S_i$. Given $y \in S_t$, one has that

$$S_{t+1} = \bigcup_{j=1}^{r} B_{t+1}(x_1, x_2, \ldots, x_{t-1}, y, j)$$

so one may pick $j \in \{1, 2, \ldots, r\}$ such that $B_{t+1}(x_1, x_2, \ldots, x_{t-1}, y, j) \in \mathcal{P}_{t+1}$. Then $y \in B_t \cap (x_1, x_2, \ldots, x_{t-1}, j)$.

Since for each $x \in S_1$, $S_2 = \bigcup_{j=1}^{r} B_2(x, j)$, one sees similarly that $S_1 = \bigcup_{j=1}^{r} B_1(j)$.

Pick $j \in \{1, 2, \ldots, r\}$ such that $B_1(j) \in \mathcal{P}_1$.

Pick by Theorem 1.6 a product subsystem $\langle x_{1,n} \rangle_{n=1}^{\infty}$ of $\langle y_{1,n} \rangle_{n=1}^{\infty}$ such that $\mathcal{P}(\langle x_{1,n} \rangle_{n=1}^{\infty}) \subseteq B_1(j)$. Let

$$D_2 = \bigcap \{ B_2(a, j) : a \in \mathcal{P}(\langle x_{1,n} \rangle_{n=1}^{m}) \}.$$

Since $\mathcal{P}(\langle x_{1,n} \rangle_{n=1}^{m})$ is finite, we have $D_2 \in \mathcal{P}_2$ so pick by Theorem 1.6 a product subsystem $\langle x_{2,n} \rangle_{n=1}^{\infty}$ of $\langle y_{2,n} \rangle_{n=1}^{\infty}$ such that $\mathcal{P}(\langle x_{2,n} \rangle_{n=1}^{\infty}) \subseteq D_2$.

Let $t \in \{2, 3, \ldots, \ell - 1\}$ and assume $\langle x_{t,n} \rangle_{n=1}^{\infty}$ has been chosen. Let

$$D_{t+1} = \bigcap \{ B_{t+1}(a_1, a_2, \ldots, a_t, j) : (a_1, a_2, \ldots, a_t) \in \prod_{i=1}^{t} \mathcal{P}(\langle x_{i,n} \rangle_{n=1}^{m}) \}.$$

Then $D_{t+1} \in \mathcal{P}_{t+1}$ so pick by Theorem 1.6 a product subsystem $\langle x_{t+1,n} \rangle_{n=1}^{\infty}$ of $\langle y_{t+1,n} \rangle_{n=1}^{\infty}$ such that $\mathcal{P}(\langle x_{t+1,n} \rangle_{n=1}^{\infty}) \subseteq D_{t+1}$.

Then $\prod_{t=1}^{\ell} \mathcal{P}(\langle x_{t,n} \rangle_{n=1}^{m}) \subseteq C_j$, as required. \hfill \Box

In contrast with Theorem 2.3, where the partition conclusion applied to products of arbitrary semigroups, we now restrict ourselves to products of commutative semigroups. We will see in Theorem 2.8 that without this restriction one is not guaranteed products of any subsystems at all.

**2.4 Definition.** Let $S$ be a semigroup, let $\langle y_n \rangle_{n=1}^{\infty}$ be a sequence in $S$, and let $m \in \mathbb{N}$. The sequence $\langle x_n \rangle_{n=1}^{m}$ is a weak product subsystem of $\langle y_n \rangle_{n=1}^{\infty}$ if and only if there exists a sequence $\langle H_n \rangle_{n=1}^{m}$ in $\mathcal{P}_f(\mathbb{N})$ such that $H_n \cap H_k = \emptyset$ when $n \neq k$ in $\{1, 2, \ldots, m\}$ and $x_n = \Pi_{\ell \in H_n} y_\ell$ for each $n \in \{1, 2, \ldots, m\}$.

Recall by way of contrast, that in a product subsystem one requires that $\max H_n < \min H_{n+1}$.
2.5 Lemma. Let \( \ell \in \mathbb{N} \) and for each \( i \in \{1, 2, \ldots, \ell\} \), let \( S_i \) be a commutative semigroup and let \( \langle y_{i,n} \rangle_{n=1}^{\infty} \) be a sequence in \( S_i \). Let

\[
\mathcal{L} = \{ p \in \beta(\times_{i=1}^{\ell} S_i) : \text{for each } A \in p \text{ and each } m, k \in \mathbb{N} \text{ there exist for each } i \in \{1, 2, \ldots, \ell\} \text{ a weak product subsystem} \]

\[
\langle x_{i,n} \rangle_{n=1}^{m} \text{ of } \langle y_{i,n} \rangle_{n=k}^{\infty} \text{ such that } \times_{i=1}^{\ell} \text{FP}(\langle x_{i,n} \rangle_{n=1}^{m}) \subseteq A \}.
\]

Then \( \mathcal{L} \) is a compact subsemigroup of \( \beta(\times_{i=1}^{\ell} \mathbb{N}) \).

Proof. Since any product subsystem is also a weak product subsystem, we have by Theorems 2.3 and 1.8 that \( \mathcal{L} \neq \emptyset \). Since \( \mathcal{L} \) is defined as the set of ultrafilters all of whose members satisfy a given property, \( \mathcal{L} \) is closed, hence compact. To see that \( \mathcal{L} \) is a semigroup, let \( p, q \in \mathcal{L} \), let \( A \in p \cdot q \) and let \( m, k \in \mathbb{N} \). Then \( \{ \vec{a} \in \times_{i=1}^{\ell} S_i : \vec{a}^{-1} A \in q \} \in p \) (where, recall, \( \vec{a}^{-1} A = \{ \vec{b} : \vec{a} \cdot \vec{b} \in A \} \)) so choose for each \( i \in \{1, 2, \ldots, \ell\} \) a weak product subsystem \( \langle x_{i,n} \rangle_{n=1}^{m} \) of \( \langle y_{i,n} \rangle_{n=k}^{\infty} \) such that

\[
\times_{i=1}^{\ell} \text{FP}(\langle x_{i,n} \rangle_{n=1}^{m}) \subseteq \{ \vec{a} \in \times_{i=1}^{\ell} S_i : \vec{a}^{-1} A \in q \}.
\]

Given \( i \in \{1, 2, \ldots, \ell\} \) and \( n \in \{1, 2, \ldots, m\} \), pick \( H_{i,n} \in \mathcal{P}(\mathbb{N}) \) with \( \min H_{i,n} \geq k \) such that \( x_{i,n} = \prod_{t \in H_{i,n}} \langle y_{i,t} \rangle \) and if \( 1 \leq n < s \leq m \), then \( H_{i,n} \cap H_{i,s} = \emptyset \).

Let \( r = \max(\bigcup_{i=1}^{\ell} \bigcup_{n=1}^{m} H_{i,n}) + 1 \) and let

\[
B = \bigcap \{ \vec{a}^{-1} A : \vec{a} \in \times_{i=1}^{\ell} \text{FP}(\langle x_{i,n} \rangle_{n=1}^{m}) \}.
\]

Then \( B \in q \) so choose for each \( i \in \{1, 2, \ldots, \ell\} \) a weak product subsystem \( \langle z_{i,n} \rangle_{n=1}^{m} \) of \( \langle y_{i,n} \rangle_{n=r}^{\infty} \) such that \( \times_{i=1}^{\ell} \text{FP}(\langle z_{i,n} \rangle_{n=1}^{m}) \subseteq B \). Given \( i \in \{1, 2, \ldots, \ell\} \) and \( n \in \{1, 2, \ldots, m\} \), pick \( K_{i,n} \in \mathcal{P}(\mathbb{N}) \) with \( \min K_{i,n} \geq r \) such that \( z_{i,n} = \Pi_{t \in K_{i,n}} \langle y_{i,t} \rangle \) and if \( 1 \leq n < s \leq m \), then \( K_{i,n} \cap K_{i,s} = \emptyset \).

For \( i \in \{1, 2, \ldots, \ell\} \) and \( n \in \{1, 2, \ldots, m\} \), let \( L_{i,n} = H_{i,n} \cup K_{i,n} \). Then \( \Pi_{t \in L_{i,n}} \langle y_{i,t} \rangle = \Pi_{t \in H_{i,n}} \langle y_{i,t} \rangle \cdot \Pi_{t \in K_{i,n}} \langle y_{i,t} \rangle = x_{i,t} \cdot z_{i,t} \) and if \( 1 \leq n < s \leq m \), then \( L_{i,n} \cap L_{i,s} = \emptyset \). Thus for each \( i \in \{1, 2, \ldots, \ell\} \), \( \langle x_{i,t} \cdot z_{i,t} \rangle_{n=1}^{m} \) is a weak product subsystem of \( \langle y_{i,n} \rangle_{n=k}^{\infty} \).

Finally we claim that

\[
\times_{i=1}^{\ell} \text{FP}(\langle x_{i,n} \cdot z_{i,n} \rangle_{n=1}^{m}) \subseteq A.
\]
To this end let $\vec{c} \in \times_{i=1}^\ell FP(\langle x_{i,n} \cdot z_{i,n} \rangle_{n=1}^m)$ be given. For each $i \in \{1,2,\ldots,\ell\}$, pick $F_i \subseteq \{1,2,\ldots,m\}$ such that $c_i = \Pi_{t \in F_i} (x_{i,t} \cdot z_{i,t})$ and let $a_i = \Pi_{t \in F_i} x_{i,t}$ and $b_i = \Pi_{t \in F_i} z_{i,t}$. Then $\vec{b} \in \times_{i=1}^\ell FP(\langle z_{i,n} \rangle_{n=1}^m)$ so $\vec{b} \in B$. Since $\vec{a} \in \times_{i=1}^\ell FP(\langle x_{i,n} \rangle_{n=1}^m)$ one has that $\vec{b} \in \vec{a}^{-1}A$ so that $\vec{a} \cdot \vec{b} \in A$. Since each $S_i$ is commutative, we have for each $i \in \{1,2,\ldots,\ell\}$ that $\Pi_{t \in F_i} (x_{i,t} \cdot z_{i,t}) = (\Pi_{t \in F_i} x_{i,t}) \cdot (\Pi_{t \in F_i} z_{i,t})$ so that $\vec{c} = \vec{a} \cdot \vec{b}$ as required.

2.6 Theorem. Let $\ell \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,\ell\}$, let $S_i$ be a commutative semigroup and let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $S_i$. Let $C$ be an IP* set in $\times_{i=1}^\ell S_i$ and let $m \in \mathbb{N}$. Then for each $i \in \{1,2,\ldots,\ell\}$ there is a weak product subsystem $\langle x_{i,n} \rangle_{n=1}^m$ of $\langle y_{i,n} \rangle_{n=1}^{\infty}$ such that $\times_{i=1}^\ell FP(\langle x_{i,n} \rangle_{n=1}^m) \subseteq C$.

Proof. Let $\mathcal{L}$ be as in Lemma 2.5. Then $\mathcal{L}$ is a compact subsemigroup of $\times_{i=1}^\ell S_i$ so by Theorem 1.2 there is an idempotent $p \in \mathcal{L}$. By Theorem 1.5, $C \in p$. Thus by the definition of $\mathcal{L}$ for each $i \in \{1,2,\ldots,\ell\}$ there is a weak product subsystem $\langle x_{i,n} \rangle_{n=1}^m$ of $\langle y_{i,n} \rangle_{n=1}^{\infty}$ such that $\times_{i=1}^\ell FP(\langle x_{i,n} \rangle_{n=1}^m) \subseteq C$.

Three natural questions are raised by Theorem 2.6. (1) Can one obtain infinite weak product subsystems (defined in the obvious fashion) such that $\times_{i=1}^\ell FP(\langle x_{i,n} \rangle_{n=1}^{\infty}) \subseteq C$? (2) Can one replace “weak product subsystems” with “product subsystems”? (3) Can one omit the requirement that the semigroups $S_i$ be commutative? We answer all three of these questions in the negative.

The first two questions are answered in Theorem 2.7 using the semigroup $(\mathbb{N},+)$. Since the operation is addition we refer to “sum subsystems” rather than “product subsystems”.

2.7 Theorem. There is an IP* set $C$ in $\mathbb{N} \times \mathbb{N}$ such that:

(a) There do not exist $z \in \mathbb{N}$ and a sequence $\langle x_{n} \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that either $\{z\} \times FS(\langle x_{n} \rangle_{n=1}^{\infty}) \subseteq C$ or $FS(\langle x_{n} \rangle_{n=1}^{\infty}) \times \{z\} \subseteq C$. (In particular there do not exist infinite weak sum subsystems $\langle x_{1,n} \rangle_{n=1}^{\infty}$ and $\langle x_{2,n} \rangle_{n=1}^{\infty}$ of $\langle 2^n \rangle_{n=1}^{\infty}$ with $FS(\langle x_{1,n} \rangle_{n=1}^{\infty}) \times FS(\langle x_{2,n} \rangle_{n=1}^{\infty}) \subseteq C$).

(b) There do not exist sum subsystems $\langle x_{n} \rangle_{n=1}^{2}$ and $\langle y_{n} \rangle_{n=1}^{2}$ of $\langle 2^n \rangle_{n=1}^{\infty}$ with
\( \{x_1, x_2\} \times \{y_1, y_2\} \subseteq C. \)

**Proof.** Let

\[
C = (\mathbb{N} \times \mathbb{N}) \setminus \{(\Sigma_{n \in F} 2^n, \Sigma_{n \in G} 2^n) : F, G \in \mathcal{P}(\omega) \text{ and } \max F < \min G \text{ or } \max G < \min F \}.
\]

To see that \( C \) is an IP* set in \( \mathbb{N} \times \mathbb{N} \), suppose instead that we have a sequence \( \langle (x_n, y_n) \rangle_{n=1}^{\infty} \) in \( \mathbb{N} \times \mathbb{N} \) with

\[
FS((\langle x_n, y_n \rangle)_{n=1}^{\infty}) \subseteq \{(\Sigma_{n \in F} 2^n, \Sigma_{n \in G} 2^n) : F, G \in \mathcal{P}(\omega) \text{ and } \max F < \min G \text{ or } \max G < \min F \}.
\]

Pick \( F_1 \) and \( G_1 \) in \( \mathcal{P}(\omega) \) such that \( x_1 = \Sigma_{t \in F_1} 2^t \) and \( y_1 = \Sigma_{t \in G_1} 2^t \). Let \( k = \max(F_1 \cup G_1) \). Choose \( H \in \mathcal{P}_f(\mathbb{N}) \) such that \( \min H > 1, \ 2^{k+1} \mid \Sigma_{n \in H} x_n \) and \( 2^{k+1} \mid \Sigma_{n \in H} y_n \).

(Consider congruence classes mod \( 2^{k+1} \) to see that one can do this.) Pick \( F', G' \in \mathcal{P}(\omega) \) such that \( \Sigma_{n \in H} x_n = \Sigma_{n \in F'} 2^n \) and \( \Sigma_{n \in H} y_n = \Sigma_{n \in G'} 2^n \). Then \( k + 1 \leq \min(F' \cup G') \), so \( x_1 + \Sigma_{n \in H} x_n = \Sigma_{n \in F_1 \cup F'} 2^n \) and \( y_1 + \Sigma_{n \in H} y_n = \Sigma_{n \in G_1 \cup G'} 2^n \). Also

\[
\Sigma_{n \in \{1\} \cup H}(x_n, y_n) = (x_1 + \Sigma_{n \in H} x_n, y_1 + \Sigma_{n \in H} y_n)
\]

\[
\in \{(\Sigma_{n \in F} 2^n, \Sigma_{n \in G} 2^n) : F, G \in \mathcal{P}(\omega) \text{ and } \max F < \min G \text{ or } \max G < \min F \}.
\]

Thus \( k + 1 \leq \max(F_1 \cup F') < \min(G_1 \cup G') \leq k \) or \( k + 1 \leq \max(G_1 \cup G') < \min(F_1 \cup F') \leq k \), a contradiction.

To establish (a), suppose that one has \( z \in \mathbb{N} \) and a sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in \( \mathbb{N} \) such that either \( \{z\} \times FS((\langle x_n \rangle_{n=1}^{\infty}) \subseteq C \) or \( FS((\langle x_n \rangle_{n=1}^{\infty}) \times \{z\} \subseteq C \) and assume without loss of generality that \( \{z\} \times FS((\langle x_n \rangle_{n=1}^{\infty}) \subseteq C \). Pick \( F \in \mathcal{P}(\omega) \) such that \( z = \Sigma_{t \in F} 2^t \) and let \( k = \max F \). Pick \( H \in \mathcal{P}_f(\mathbb{N}) \) such that \( 2^{k+1} \mid \Sigma_{n \in H} x_n \). Pick \( G \in \mathcal{P}(\omega) \) such that \( \Sigma_{n \in H} x_n = \Sigma_{t \in G} 2^t \). Then \( \max F < \min G \) so \( (z, \Sigma_{n \in H} x_n) \notin C \).

To establish (b) suppose that one has sum subsystems \( \langle x_n \rangle_{n=1}^{2^n} \) and \( \langle y_n \rangle_{n=1}^{2^n} \) of \( \langle 2^n \rangle_{n=1}^{\infty} \) with \( \{x_1, x_2\} \times \{y_1, y_2\} \subseteq C \). Pick \( F_1, G_1, F_2, G_2 \in \mathcal{P}(\omega) \) such that \( x_1 = \Sigma_{n \in F_1} 2^n \), \( x_2 = \Sigma_{n \in F_2} 2^n \), \( y_1 = \Sigma_{n \in G_1} 2^n \), \( y_2 = \Sigma_{n \in G_2} 2^n \), \( \max F_1 < \min F_2 \), and \( \max G_1 < \min G_2 \). Without loss of generality, \( \max F_1 \geq \max G_1 \). But then we have \( \max G_1 \leq \max F_1 < \min F_2 \) so \( (x_2, y_1) \notin C \), a contradiction. \( \Box \)
A striking contrast is provided by Theorems 2.2 and 2.7. It is easy to divide most semigroups into two classes, neither of which is an IP* set. Consequently it is not too surprising when one finds a property that must be satisfied by an IP* set in a semigroup $S$ (such as containing a sequence with all of its sums and products when $S = \mathbb{N}$) which need not be satisfied by any cell of a partition of $S$. In this case we have a property, namely that of containing $FS(\langle x_1, n \rangle_{n=1}^m) \times FS(\langle x_2, n \rangle_{n=1}^\infty)$ for some sum subsystems of any given sequences, which must be satisfied by some cell of a partition of $\mathbb{N} \times \mathbb{N}$, but need not be satisfied by IP* sets in $\mathbb{N} \times \mathbb{N}$.

The third question raised by Theorem 2.6 is answered with an example very similar to that used in the proof of Theorem 2.7.

**2.8 Theorem.** Let $S$ be the free semigroup on the alphabet $\{y_1, y_2, y_3, \ldots\}$. There is an IP* set $C$ in $S \times S$ such that there do not exist weak product subsystems $\langle x_n \rangle_{n=1}^{2}$ and $\langle w_n \rangle_{n=1}^{2}$ of $\langle y_n \rangle_{n=1}^{\infty}$ such that $\{x_1, x_2\} \times \{w_1, w_2\} \subseteq C$.

**Proof.** Let

$$C = (S \times S) \backslash \{(\Pi_{n \in F} y_n, \Pi_{n \in G} y_n) : F, G \in \mathcal{P}_f(\mathbb{N}) \quad \text{and} \quad \max F < \min G \text{ or } \max G < \min F\}.$$  

To see that $C$ is an IP* set, suppose one has a sequence $\langle (x_n, w_n) \rangle_{n=1}^{\infty}$ with

$$FP(\langle (x_n, w_n) \rangle_{n=1}^{\infty}) \subseteq \{(\Pi_{n \in F} y_n, \Pi_{n \in G} y_n) : F, G \in \mathcal{P}_f(\mathbb{N}) \quad \text{and} \quad \max F < \min G \text{ or } \max G < \min F\}.$$  

Given any $i < j$ in $\mathbb{N}$, pick $F_i, F_j, G_i, G_j, F_{i,j}, G_{i,j} \in \mathcal{P}_f(\mathbb{N})$ such that $x_i = \Pi_{n \in F_i} y_n$, $x_j = \Pi_{n \in F_j} y_n$, $w_i = \Pi_{n \in G_i} y_n$, $w_j = \Pi_{n \in G_j} y_n$, $x_i x_j = \Pi_{n \in F_{i,j}} y_n$, and $w_i w_j = \Pi_{n \in F_i} y_n$. Since $x_i x_j = \Pi_{n \in F_{i,j}} y_n$, we have that max $F_i < \min F_j$ and $F_{i,j} = F_i \cup F_j$ and similarly max $G_i < \min G_j$ and $G_{i,j} = G_i \cup G_j$. Then we may pick $j \in \mathbb{N}$ such that max$(F_i \cup G_1) < \min(F_j \cup G_j)$.

Then

$$(x_1 x_j, w_1 w_j) \notin \{(\Pi_{n \in F} y_n, \Pi_{n \in G} y_n) : F, G \in \mathcal{P}_f(\mathbb{N}) \quad \text{and} \quad \max F < \min G \text{ or } \max G < \min F\},$$

a contradiction.
Now suppose we have weak product subsystems $\langle x_n \rangle_{n=1}^2$ and $\langle w_n \rangle_{n=1}^2$ of $\langle y_n \rangle_{n=1}^\infty$ such that $\{x_1, x_2\} \times \{w_1, w_2\} \subseteq C$. Pick $F_1, F_2, G_1, G_2 \in \mathcal{P}_f(\mathbb{N})$ such that $x_1 = \prod_{n \in F_1} y_n$, $x_2 = \prod_{n \in F_2} y_n$, $w_1 = \prod_{n \in G_1} y_n$, and $w_2 = \prod_{n \in G_2} y_n$. Since $x_1 x_2 \in FP(\langle y_n \rangle_{n=1}^\infty)$, we must have $\max F_1 < \min F_2$ and similarly $\max G_1 < \min G_2$. Without loss of generality, $\max F_1 \geq \max G_1$. But then we have $\max G_1 \leq \max F_1 < \min F_2$ so $(x_2, w_1) \notin C$, a contradiction.  

We know that there are certain IP* sets, namely sets of returns in a dynamical system, which satisfy stronger conclusions than arbitrary IP* sets. (See, for example, [3].) It is possible to show that if $C$ is such a dynamically defined IP* set and $S$ has an identity, one can choose infinite product subsystems guaranteed by Theorem 2.6. We do have the following simple result establishing a  

2.9 Theorem. Let $S_1$ and $S_2$ be semigroups and let $C$ be an IP* set in $S_1 \times S_2$. Let $\ell, m \in \mathbb{N}$ and let $\{\langle w_i, n \rangle_{n=1}^\infty : i \in \{1, 2, \ldots, \ell\}\}$ be a set of sequences in $S_1$ and let $\{\langle z_j, n \rangle_{n=1}^\infty : j \in \{1, 2, \ldots, m\}\}$ be a set of sequences in $S_2$. Then there is a sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that:  

(a) for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and  

(b) if for each $n \in \mathbb{N}$, each $i \in \{1, 2, \ldots, \ell\}$ and each $j \in \{1, 2, \ldots, m\}$, $x_i, n = \Pi_{t \in H_n} w_{i,t}$ and $y_j, n = \Pi_{t \in H_n} z_{j,t}$, then for each $i \in \{1, 2, \ldots, \ell\}$ and each $j \in \{1, 2, \ldots, m\}$, $FP(\langle (x_i, n, y_j, n) \rangle_{n=1}^\infty) \subseteq C$.  

Proof. Enumerate $\{1, 2, \ldots, \ell\} \times \{1, 2, \ldots, m\}$ as $\langle (i(k), j(k)) \rangle_{k=1}^\ell m$. Let for each $n \in \mathbb{N}$, $a_{1,n} = (w_{i(1)}, n, z_{j(1)}, n)$. Pick by Corollary 1.7 a sequence $\langle H_{1,n} \rangle_{n=1}^\infty$ such that if $b_{1,n} = \Pi_{t \in H_{1,n}} a_{1,n}$, then $FP(\langle b_{1,n} \rangle_{n=1}^\infty) \subseteq C$. Let $a_{2,n} = (\Pi_{t \in H_{1,n}} w_{i(2), t}, \Pi_{t \in H_{1,n}} z_{j(2), t})$.  

Inductively, given $a_k, n = (\Pi_{t \in H_{k-1,n}} w_{i(k), t}, \Pi_{t \in H_{k-1,n}} z_{j(k), t})$ pick by Corollary 1.7 a
sequence \(\langle F_{k,n}\rangle_{n=1}^{\infty}\) in \(P_f(\mathbb{N})\) such that if \(b_{k,n} = \Pi_{t \in F_{k,n}} a_{k,n}\), then \(FP(\langle b_{k,n}\rangle_{n=1}^{\infty}) \subseteq C\).

For each \(n\) let \(H_{k,n} = \bigcup_{t \in F_{k,n}} H_{k-1,t}\) and let (if \(k < \ell m\))

\[a_{k+1,n} = (\Pi_{t \in H_{k,n}} w_{i(k+1),t}, \Pi_{t \in H_{k,n}} z_{j(k+1),t}).\]

The induction being complete, let \(H_n = H_{\ell m,n}\) for each \(n\). For each \(n \in \mathbb{N}\), each \(i \in \{1,2,\ldots,\ell\}\) and each \(j \in \{1,2,\ldots,m\}\), let \(x_{i,n} = \Pi_{t \in H_n} w_{i,t}\) and \(y_{j,n} = \Pi_{t \in H_n} z_{j,t}\).

Then, given \(i \in \{1,2,\ldots,\ell\}\) and \(j \in \{1,2,\ldots,m\}\), pick \(k \in \mathbb{N}\) such that \((i,j) = (i(k),j(k))\). Then \(FP(\langle (x_{i,n}, y_{j,n})\rangle_{n=1}^{\infty}) \subseteq FP(\langle b_{k,n}\rangle_{n=1}^{\infty}) \subseteq C\).

\[\square\]

3. An Alternate Derivation of Finite Substructures.

We show here that a version of Theorem 2.6 for the semigroup \((P_f(\mathbb{N}), \cup)\) suffices to derive Theorem 2.6 in its entirety. Since the operation in this semigroup is \(\cup\) we write \(FU(\langle H_n\rangle_{n=1}^{m}) = \{\bigcup_{n \in F} H_n : \emptyset \neq F \subseteq \{1,2,\ldots,m\}\}\). Similarly, given a sequence \(\langle (H_{1,n}, H_{2,n})\rangle_{n=1}^{\infty}\) in \(P_f(\mathbb{N}) \times P_f(\mathbb{N})\) we write \(FU(\langle (F_{1,n}, F_{2,n})\rangle_{n=1}^{m}) = \{\bigcup_{n \in F} H_{1,n}, \bigcup_{n \in F} H_{2,n} : \emptyset \neq F \subseteq \{1,2,\ldots,m\}\}\).

For the semigroup \((P_f(\mathbb{N}), \cup)\), Theorem 2.6 is completely trivial because the only IP* set in \(P_f(\mathbb{N}) \times P_f(\mathbb{N})\) is \(P_f(\mathbb{N}) \times P_f(\mathbb{N})\) itself. (This is because every element of \(P_f(\mathbb{N})\) is an idempotent.) We need a weaker notion of IP* set in order to obtain useful results. We will say that a sequence \(\langle H_{i,n}\rangle_{n=1}^{\infty}\) is a disjoint sequence provided \(H_{i,n} \cap H_{i,m} = \emptyset\) whenever \(n \neq m\).

3.1 Definition. Let \(\ell \in \mathbb{N}\). A set \(C \subseteq \times_{i=1}^{\ell} P_f(\mathbb{N})\) is a weak IP* set if and only if for any disjoint sequences \(\langle H_{1,n}\rangle_{n=1}^{\infty}, \langle H_{2,n}\rangle_{n=1}^{\infty}, \ldots, \langle H_{\ell,n}\rangle_{n=1}^{\infty}\) in \(P_f(\mathbb{N})\), one has \(C \cap FU(\langle (H_{1,n}, H_{2,n}, \ldots, H_{\ell,n})\rangle_{n=1}^{\infty}) \neq \emptyset\).

We will also need to concern ourselves with a restricted subsemigroup of \(\beta(\times_{i=1}^{\ell} P_f(\mathbb{N}))\).

3.2 Definition. Let \(\ell \in \mathbb{N}\). Then \(I_{\ell} = \{p \in \beta(\times_{i=1}^{\ell} P_f(\mathbb{N})) : \text{for each } n \in \mathbb{N}, \langle H_1, H_2, \ldots, H_\ell : (\bigcup_{i=1}^{\ell} H_i) \cap \{1,2,\ldots,n\} = \emptyset\} \in p\}\).

It is routine to show that \(I_{\ell}\) is a subsemigroup of \(\beta(\times_{i=1}^{\ell} P_f(\mathbb{N}))\). We modify Theorem 1.5.
3.3 Theorem. Let $\ell \in \mathbb{N}$, let $C \subseteq \times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N})$ be a weak IP* set, and let $p$ be an idempotent in $\mathcal{I}_\ell$. Then $C \in p$.

Proof. As in the proof of Theorem 1.6 one shows that given any $A \in p$ there exist disjoint sequences $(H_1,n)_{n=1}^\infty, (H_2,n)_{n=1}^\infty, \ldots, (H_\ell,n)_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$, such that

$$FU((H_1,n, H_2,n, \ldots, H_\ell,n))_{n=1}^\infty \subseteq A.$$ 

Consequently, one cannot have $\times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N})\setminus C \in p$. \hfill $\square$

One can define a strong IP set in $\times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N})$ by requiring that this set contain $FU((H_1,n, H_2,n, \ldots, H_\ell,n))_{n=1}^\infty$ where each $(H_i,n)_{n=1}^\infty$ is a disjoint sequence. One can then show in a fashion similar to the proof of Corollary 1.7 that any weak IP* set meets any strong IP set along a strong IP set.

Next we modify Lemma 2.5.

3.4 Lemma. Let $\ell \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,\ell\}$ and each $n \in \mathbb{N}$, let $Y_{i,n} = \{n\}$. Let

$$\mathcal{M} = \{p \in \mathcal{I}_\ell : \text{for each } A \in p \text{ and each } m,k \in \mathbb{N}$$

$$\text{there exist for each } i \in \{1,2,\ldots,\ell\} \text{ a weak product subsystem}$$

$$(H_i,n)_{n=1}^m \text{ of } (Y_{i,n})_{n=k}^\infty \text{ such that } \times_{i=1}^{\ell} FU((H_i,n)_{n=1}^m) \subseteq A \}. \]$$

Then $\mathcal{M}$ is a compact subsemigroup of $\beta(\times_{i=1}^{\ell} \mathbb{N})$.

Proof. Let $\mathcal{L}$ be as in Lemma 2.5. Then $\mathcal{M} = \mathcal{L} \cap \mathcal{I}_\ell$. Since both $\mathcal{L}$ and $\mathcal{I}_\ell$ are semigroups, it suffices to show that $\mathcal{M} \cap \mathcal{L} \neq \emptyset$.

For each $n \in \mathbb{N}$, let $A_n = \{(H_1,H_2,\ldots,H_\ell) \in \times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N}) : (\bigcup_{i=1}^{\ell} H_i) \cap \{1,2,\ldots,n\} = \emptyset\}$. Let $\mathcal{B} = \{C \subseteq \times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N}) : \text{for each } m,k \in \mathbb{N} \text{ and each choice of a weak product subsystem } (H_i,n)_{n=1}^m \text{ of } (Y_{i,n})_{n=k}^\infty, \times_{i=1}^{\ell} FU((H_i,n)_{n=1}^m) \cap C \neq \emptyset\}$. Now if $p \in \beta(\times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N}))$ and $\{A_k : k \in \mathbb{N}\} \cup \mathcal{B} \subseteq p$, then $p \in \mathcal{L} \cap \mathcal{I}_\ell$ (since, given any $A \in p$, $\times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N}) \setminus A \notin \mathcal{B}$). Thus it suffices to show that $\{A_k : k \in \mathbb{N}\} \cup \mathcal{B}$ has the finite intersection property.

To see this, suppose instead we have some $k \in \mathbb{N}$ and some $C_1,C_2,\ldots,C_r \in \mathcal{B}$ such that $A_k \cap \bigcap_{j=1}^r C_i = \emptyset$. Then $(\times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N}) \setminus A_k) \cup \bigcup_{j=1}^r (\times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N}) \setminus C_j)$ so by
Theorem 2.3, some one of these sets contains $\times_{i=1}^\ell FU(\langle H_i,n \rangle_{n=1}^m)$ for some choice of weak product subsystems $\langle H_i,n \rangle_{n=1}^m$ of $\langle Y_i,n \rangle_{n=1}^\infty$, which is impossible.

3.5 Theorem. Let $\ell \in \mathbb{N}$, let $C \subseteq \times_{i=1}^\ell P_f(\mathbb{N})$ be a weak IP* set, and let $m \in \mathbb{N}$. Then for each $i \in \{1,2,\ldots,\ell\}$ there is a disjoint sequence $\langle H_i,n \rangle_{n=1}^m$ such that $\times_{i=1}^\ell FU(\langle H_i,n \rangle_{n=1}^m) \subseteq C$.

Proof. Let $\mathcal{M}$ be as in Lemma 3.4. Then $\mathcal{M}$ is a compact subsemigroup of $I_\ell$ so pick an idempotent $p \in \mathcal{M}$. By Theorem 3.3 $C \in p$ so by the definition of $\mathcal{M}$ we have for each $i \in \{1,2,\ldots,\ell\}$ some disjoint sequence $\langle H_i,n \rangle_{n=1}^m$ such that $\times_{i=1}^\ell FU(\langle H_i,n \rangle_{n=1}^m) \subseteq C$.

Finally we show how Theorem 3.5 suffices to yield Theorem 2.6 for any semigroups.

(We reprint its formulation for the convenience of the reader.)

2.6 Theorem. Let $\ell \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,\ell\}$, let $S_i$ be a commutative semigroup and let $\langle y_i,n \rangle_{n=1}^\infty$ be a sequence in $S_i$. Let $C$ be an IP* set in $\times_{i=1}^\ell S_i$ and let $m \in \mathbb{N}$. Then for each $i \in \{1,2,\ldots,\ell\}$ there is a weak product subsystem $\langle x_i,n \rangle_{n=1}^m$ of $\langle y_i,n \rangle_{n=1}^\infty$ such that $\times_{i=1}^\ell FP(\langle x_i,n \rangle_{n=1}^m) \subseteq C$.

Proof. Let

$$D = \{(H_1,H_2,\ldots,H_\ell) \in \times_{i=1}^\ell P_f(\mathbb{N}) : (\Pi_{t \in H_1} y_{1,t},\Pi_{t \in H_2} y_{2,t},\ldots,\Pi_{t \in H_\ell} y_{\ell,t}) \in C\}.$$ 

Then it is easy to see that $D$ is a weak IP* set. So pick for each $i \in \{1,2,\ldots,\ell\}$ a disjoint sequence $\langle H_i,n \rangle_{n=1}^m$ as guaranteed by Theorem 3.5. For each $n \in \{1,2,\ldots,m\}$ and each $i \in \{1,2,\ldots,\ell\}$, let $x_{i,n} = \Pi_{t \in H_i} y_{i,t}$. Then $\times_{i=1}^\ell FP(\langle x_i,n \rangle_{n=1}^m) \subseteq C$. 

References


